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Noisy propagation of Gaussian states in optical media with finite bandwidth

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We address propagation and entanglement of Gaussian states in optical media characterised by nontrivial spectral densities. In particular, we consider environments with a finite bandwidth $J(\omega) = J_0[\theta(\omega - \Omega) - \theta(\omega - \Omega - \delta)]$, and show that in the low temperature regime $T \ll \Omega^{-1}$: (i) secular terms in the master equation may be neglected; (ii) attenuation (damping) is strongly suppressed; (iii) the overall diffusion process may be described as a Gaussian noise channel with variance depending only on the bandwidth. We find several regimes where propagation is not much detrimental and entanglement may be protected from decoherence.

Engineering and control of quantum systems in the presence of noise is a crucial step in the development of quantum technology^{1–10}. In turn, much attention has been devoted to describe the dynamics of open quantum systems for different kind of environments, i.e. different sources of damping and decoherence^{11–14}. It is often challenging to obtain the exact dynamics of an open quantum system and different kinds of approaches have been developed to derive an approximate description at different levels of accuracy. Besides assuming a weak coupling, the approximations usually employed to obtain analytic master (or Langevin) equations¹⁵ for the system under investigation include neglecting memory effects (Markovian approximation), using time coarse graining (not accounting for potential contributions coming from the short-time dynamics), and assuming some simplified form for the spectral density of the environment, e.g. a Lorentzian one or an Ohmic one with a large frequency cut-off.

On the other hand, there are several systems where nontrivial spectral densities appear, e.g. polarons in metals and semi-conductors, photonic crystals and micro-mechanical oscillators, and a question arises on the effect of the interaction with this kind of media on the quantum properties of a given system^{16–19}. Moreover, memory effects, backflow of information and short-time dynamics can play a significant role in these scenarios and a non-Markovian approach should be employed^{20–22}. Structured environments have been thoroughly studied and characterized from the point of view of quantum probing, both in the continuous- and discrete-variables regimes^{23–25}, in the context of bath engineering for controlled systems. In particular, we focus attention on composite systems where the propagation of frequencies in a limited range is forbidden, owing to the spectral properties of their constituents. A relevant example is that of photonics crystals made of materials with very different optical properties (e.g. different refraction indices), overall resulting in the creation of photonic band gaps^{26–30}.

In this paper, motivated by some recent experimental implementations in photonic crystal wave guides^{31–33}, we address the propagation of Gaussian states of light through a medium characterized by a finite bandwidth δ , i.e. the spectral density $J(\omega)$ of the structured environment is non-vanishing only in a given interval $[\Omega, \Omega + \delta]$. Though our analysis is based on a simple model, it shows that these spectral features result in a peculiar short-time dynamics of the system, where the secular terms may be neglected, pure attenuation is strongly suppressed, and the overall dynamics may be described as a Gaussian noise channel with variance depending only on the bandwidth parameters δ and J_0 and not on the natural frequency ω_0 of the mode, nor on the location Ω of the bandwidth in the spectrum. We then use our results to study the propagation of Gaussian entangled states in media with finite bandwidth.

The paper is structured as follows: in section “[The interaction model](#)”, we describe our model and present the non-Markovian master equation. Section “[Dynamical evolution of Gaussian states](#)” illustrates the solution of the master equation with an initial Gaussian state. In addition, we review how to quantify entanglement for two-mode continuous-variable (CV) Gaussian state. In section “[Validity of the secular approximation and](#)

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“entanglement dynamics” we investigate the validity of the secular approximation, and discuss the dynamics of entanglement. Section “Conclusions” closes the paper with some concluding remarks.

The interaction model

Let us consider a single-mode field at natural frequency ω_0 , the system, interacting with an environment that we assume at thermal equilibrium. The interaction Hamiltonian in natural units may be written as

$$H = \frac{\omega_0}{2} (P^2 + X^2) + \sum_n \frac{\omega_n}{2} (P_n^2 + X_n^2) - \alpha X \otimes \sum_j \gamma_j X_j, \quad (1)$$

where α (dimensionless) is the overall coupling strength between the system and the environment, X and P are the canonical operators of the system mode, X_j and P_j are operators of the environmental modes. We remind that in terms of the field mode, the quadrature operators are given by $X(\varphi) = (a e^{-i\varphi} + a^\dagger e^{i\varphi})/\sqrt{2}$, with $X \equiv X(0)$ and $P \equiv X(\pi/2)$. Finally, the quantities ω_j denote the frequencies of the environmental modes, and γ_j are the (dimensional) couplings between the system and the j -th environmental mode. At $t = 0$ we assume a factorized state $\varrho_0 \otimes \mathcal{E}$, where ϱ_0 is the initial state of the system and \mathcal{E} an equilibrium (thermal) state of the environment, i.e. $\mathcal{E} = e^{-\beta H_B}/\mathcal{Z}$, where $\beta = T^{-1}$ is the inverse temperature, H_B the free energy of the environment and \mathcal{Z} the partition function. This is a Gaussian state too. We use natural units and thus besides $\hbar = 1$ we also have the Boltzmann constant $k_B = 1$.

Upon evolving the overall system and tracing out the environmental degrees of freedom we obtain a time-local master equation, which describes the noisy evolution of the system mode^{11,34,35}

$$\dot{\varrho}(t) = -i[H_0, \varrho(t)] + i r(t)[X^2, \varrho(t)] - i \gamma(t)[X, \{P, \varrho(t)\}] - \Delta(t)[X, [X, \varrho(t)]] + \Pi(t)[X, [P, \varrho(t)]], \quad (2)$$

where H_0 is the free Hamiltonian (first term in Eq. (1)), and $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote commutators and anticommutators, respectively. The first term in Eq. (2) is due to the unitary part of the time evolution, whereas the second one induces a time-dependent energy-shift. The third term is a damping term and the last two are responsible for diffusion. The different time-dependent coefficients link the non-Markovian features of the dynamics with the spectral structure of the environment and its thermal excitations. Up to second order in α we have

$$\gamma(\tau) = \int_0^\tau ds \sin(\omega_0 s) \int_0^\infty d\omega J(\omega) \sin(\omega s), \quad \Delta(\tau) = \int_0^\tau ds \cos(\omega_0 s) \int_0^\infty d\omega \coth \frac{\beta\omega}{2} J(\omega) \cos(\omega s), \quad (3)$$

$$r(\tau) = \int_0^\tau ds \cos(\omega_0 s) \int_0^\infty d\omega J(\omega) \sin(\omega s), \quad \Pi(\tau) = \int_0^\tau ds \sin(\omega_0 s) \int_0^\infty d\omega \coth \frac{\beta\omega}{2} J(\omega) \cos(\omega s), \quad (4)$$

where $J(\omega) = \alpha^2 \sum_j \frac{\gamma_j}{2} \delta(\omega - \omega_j)$ is the spectral density of the environment. The average number of thermal excitations for the mode at frequency ω is given by $N(\omega) = (e^{\beta\omega} - 1)^{-1} = \frac{1}{2}(\coth \frac{\beta\omega}{2} - 1)$, i.e. $\coth \frac{\beta\omega}{2} = 2N(\omega) + 1$.

A general solution of the master equation (2) can be found through the quantum characteristic approach³⁶ in terms of the canonical variables $\mathbf{z} = (x, p)$, assuming a weak coupling regime which corresponds to fulfill the condition $\alpha \ll 1$:

$$\chi[\mathbf{z}](t) = e^{-\mathbf{z}^T \overline{W}(t) \mathbf{z}} \chi \left[e^{-\frac{\Gamma(t)}{2}} R^{-1}(t) \mathbf{z} \right] (0), \quad (5)$$

where

$$\Gamma(t) = 2 \int_0^t \gamma(\tau) d\tau \quad (6)$$

$$R(t) \simeq \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \quad M(\tau) = \begin{pmatrix} \frac{\Delta(\tau)}{2} & -\frac{\Pi(\tau)}{2} \\ -\frac{\Pi(\tau)}{2} & 0 \end{pmatrix} \quad (7)$$

$$\overline{W}(t) = e^{-\Gamma(t)} [R^{-1}(t)]^T W(t) R^{-1}(t) \quad W(t) = \int_0^t e^{\Gamma(\tau)} R^T(\tau) M(\tau) R(\tau) d\tau. \quad (8)$$

Since the Hamiltonian (1) is at most bilinear in the system quadrature operators, it is easy to prove that it induces a Gaussian evolution map, i.e. a map which preserves the Gaussian character of any initial Gaussian state³⁷. For this reason, the resulting dynamics is usually referred to as a Gaussian channel.

Dynamical evolution of Gaussian states

In this section we review the solution of the master equation for Gaussian states³⁶. In particular, we focus on two-mode Gaussian states (each one interacting with its own environment) in order to analyze the dynamics of entanglement. On the other hand, the conclusions about the features of the channel are general, and apply to signals with any number of modes.

Let us thus consider a single two-mode Gaussian state, with characteristic function at time $t = 0$ $\chi_0(\mathbf{z}) = \exp\{-\frac{1}{2} \mathbf{z}^T \sigma_0 \mathbf{z} - i \mathbf{z}^T \overline{X}_m\}$. The initial covariance matrix σ_0 is a 4×4 matrix

$$\sigma_0 = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 \\ \mathbf{C}_0^T & \mathbf{B}_0 \end{pmatrix}, \tag{9}$$

where $\mathbf{A}_0 = a \mathbb{I}$, $\mathbf{B}_0 = b \mathbb{I}$, $\mathbf{C}_0 = \text{Diag}(c_1, c_2)$, with $a, b > 0$ and c_1, c_2 real numbers, and \mathbb{I} the 2×2 identity matrix. We remind that the system-environment interaction, and thus the time evolution, maintains the Gaussian character^{34,38-40}. The evolved state is a two-mode Gaussian state with mean and covariance matrix described by

$$\bar{\mathbf{X}}_t = e^{-\Gamma(t)/2} [R(t) \oplus R(t)] \bar{\mathbf{X}}_{in} \tag{10}$$

$$\sigma_t = e^{-\Gamma(t)} [R(t) \oplus R(t)] \sigma_0 [R(t) \oplus R(t)]^T + 2(\bar{W}(t) \oplus \bar{W}(t)). \tag{11}$$

Upon substituting the expression of the coefficients obtained in the weak-coupling approximation, into Eqs. (10) and (11), the covariance matrix at time t may be written as

$$\sigma_t = \begin{pmatrix} \mathbf{A}_t & \mathbf{C}_t \\ \mathbf{C}_t^T & \mathbf{A}_t \end{pmatrix}, \tag{12}$$

where

$$\mathbf{A}_t = \mathbf{A}_0 e^{-\Gamma(t)} + \begin{pmatrix} \Delta_\Gamma(t) + [\Delta_{co}(t) - \Pi_{si}(t)] & -[\Delta_{si}(t) - \Pi_{co}(t)] \\ -[\Delta_{si}(t) - \Pi_{co}(t)] & \Delta_\Gamma(t) - [\Delta_{co}(t) - \Pi_{si}(t)] \end{pmatrix} \tag{13}$$

and

$$\mathbf{C}_t = \begin{pmatrix} c e^{-\Gamma(t)} \cos(2\omega_0 t) & c e^{-\Gamma(t)} \sin(2\omega_0 t) \\ c e^{-\Gamma(t)} \sin(2\omega_0 t) & -c e^{-\Gamma(t)} \cos(2\omega_0 t) \end{pmatrix}. \tag{14}$$

In order to obtain the compact forms (13) and (14), we have introduced the following expression

$$\Delta_\Gamma(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Delta(s) ds \tag{15}$$

and the secular coefficients

$$\Delta_{co}(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Delta(s) \cos[2\omega_0(t-s)] ds \quad \Delta_{si}(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Delta(s) \sin[2\omega_0(t-s)] ds \tag{16a}$$

$$\Pi_{co}(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Pi(s) \cos[2\omega_0(t-s)] ds \quad \Pi_{si}(t) = e^{-\Gamma(t)} \int_0^t e^{\Gamma(s)} \Pi(s) \sin[2\omega_0(t-s)] ds. \tag{16b}$$

In situations where the secular terms may be neglected (as we will see, this is our case) the diagonal blocks of the covariance matrix evolve as

$$\mathbf{A}_t = \mathbf{A}_0 e^{-\Gamma(t)} + \Delta_\Gamma(t) \mathbb{I}. \tag{17}$$

Let us now introduce the main ingredient of our analysis, i.e. a specification of the properties of the bosonic reservoirs through the form of the spectral density. In order to describe the presence of a finite bandwidth, i.e. the fact that the propagation of certain frequencies, or a certain range of frequencies, is forbidden, we consider a spectral density of the form

$$J(\omega) = J_0[\theta(\omega - \Omega) - \theta(\omega - \Omega - \delta)].$$

The parameter δ represents the bandwidth of the distribution, Ω specifies its location within the spectrum and J_0 is the amplitude of the spectral density. In the low temperature regime, i.e. $T \ll \Omega^{-1}$, we may safely assume $N(\omega) \approx 0$ and $\coth \frac{\omega}{2T} \approx 1$ in Eq. (3), and obtain an analytic expression for the relevant coefficients. In particular, we get a simplified form of the master equation coefficients

$$\Delta(t) = J_0 \delta t, \quad \gamma(t) = \frac{1}{3} J_0 \delta \omega_0 \Omega t^3. \tag{18}$$

Using these expressions, it is straightforward to obtain the time-integrated functions. We have $\Gamma(t) = \frac{1}{6} J_0 \delta \omega_0 \Omega t^4$, and $\Delta_\Gamma(t) = \frac{1}{2} J_0 \delta t^2$. In the following, we express all the quantities in units of ω_0 , i.e. we make the replacements $t \rightarrow \tau = \omega_0 t$, $J_0 \rightarrow J_0/\omega_0$, $\Omega \rightarrow \Omega/\omega_0$, and $\delta \rightarrow \delta/\omega_0$. In this way, we may write the time-integrated functions in terms of the dimensionless time τ and the dimensionless parameters δ , J_0 , and Ω

$$\Gamma(\tau) = \frac{1}{6} J_0 \delta \Omega \tau^4, \quad \Delta_\Gamma(\tau) = \frac{1}{2} J_0 \delta \tau^2. \tag{19}$$

Notice that the damping term depends on all the parameters, whereas the diffusion one does not depend on Ω , i.e. on the location of the bandwidth. Since the memory effects due to the non-Markovian nature of the environment are taking place over short times, the above results imply that attenuation (damping) is strongly suppressed

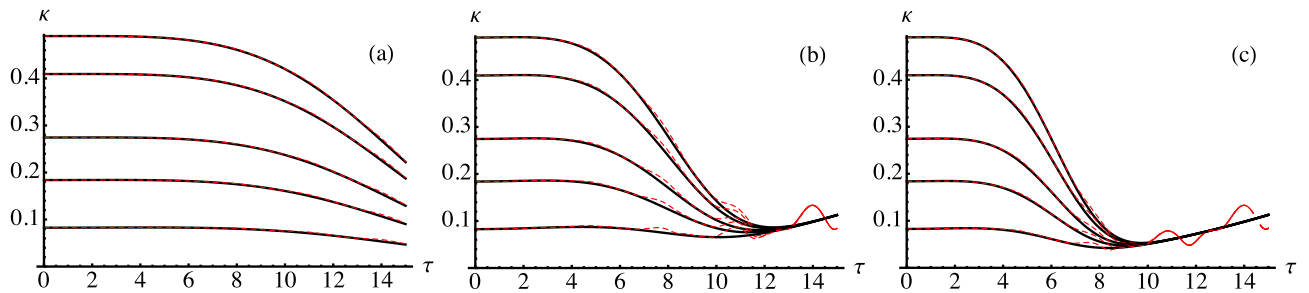


Figure 1. Comparison between the results obtained with and without the secular terms. The three panels show the symplectic eigenvalue κ as a function of the dimensionless time τ for different values of the other (dimensionless) parameters. The solid black curves denote results obtained with the secular approximation (i.e. dropping the secular terms) whereas the red dashed curves denote the full expression including the secular terms. In the left panel [panel (a)] we show results for $\delta = 10^{-4}$, $\Omega = J_0 = 1$ and different values of the TWB parameter r (from top to bottom $r = 0.01, 0.1, 0.3, 0.5$, and 0.9). In panel (b), we show results for $\delta = 10^{-3}$ and the same values of the other parameters as in panel (a). In panel (c), we show results for $\delta = 10^{-3}$, $J_0 = 1$, $\Omega = 3$ and for different values of the TWB parameter [as in panel (a)].

in the propagation through media with a finite bandwidth. Moreover, if the secular terms may be neglected, the overall diffusion process is governed by Eq. (17) and by the expression of $\Delta_\Gamma(\tau)$. In other words, in this case the dynamics corresponds to a Gaussian channel where gain and loss compensate each other and the resulting effect is that of a diffusion³⁸. This situation is usually referred to as a *Gaussian noise channel*. In our case, the variance of the Gaussian noise is depending only on the bandwidth parameters δ and J_0 , and not on the location Ω .

Quantification of entanglement. The entanglement of a two-mode CV system may be quantified by different entanglement monotones, including entanglement negativity⁴¹ and entanglement of formation^{42–44}. Both may be computed starting from the covariance matrix of the system. Here, for analytic convenience, we adopt the negativity E_N as an entanglement quantifier,

$$E_N = \max(0, -2 \log \kappa) \quad (20)$$

where κ is the minimum symplectic eigenvalue of the partially transposed Gaussian state under investigation. In terms of the the covariance matrix σ_t in Eq. (12), and exploiting the fact that we are dealing with symmetric states we may write κ as

$$\kappa = \sqrt{2} \sqrt{I_1 - I_3 - \sqrt{(I_1 - I_3)^2 - I_4}}. \quad (21)$$

The quantities I_k are the symplectic invariants of the covariance matrix, i.e. $I_1 = \det[\mathbf{A}_t]$, $I_3 = \det[\mathbf{C}_t]$ and $I_4 = \det[\sigma_t]$.

Validity of the secular approximation and entanglement dynamics

Protecting entanglement during evolution and avoiding entangled sudden death^{9,45–53} is a major task in different areas of quantum information science. In this framework, it has been shown⁵⁴ that some beneficial effects may be obtained by engineering structured environment such as a photonic bandgap materials^{55,56}, and this motivates us to analyze in some details the dependence of the dynamics on the parameters of spectral density, and their interplay with the natural frequency of the involved modes.

The evolution induced by the master equation Eq. (2) corresponds to a Gaussian map, i.e., an initial Gaussian state maintains its character. It is thus possible to obtain the expression of the covariance matrix at time and then evaluate entanglement at any time for the two modes initially excited in any entangled Gaussian state⁵⁷. To be specific, we address situations where the system of two modes is initially prepared in a twin-beam (TWB) state, i.e. a maximally entangled Gaussian state having covariance matrix coefficients $a = b = \cosh 2r$, $c = \sinh 2r$ with $r > 0$ (see Eq. (9)) and analyze the state propagation in a reservoir with a finite bandwidth spectral density.

At first, we investigate the validity of the secular approximation, i.e. we check when the secular terms may be dropped. To this aim, we calculate the symplectic eigenvalue κ with and without the secular terms for an initial TWB state with parameter r and compare results for different values of the involved parameters. Using the secular approximation, the expression of κ reads as follows

$$\kappa = \frac{1}{2} \left(\tau^2 J_0 \delta + e^{-2r - \frac{1}{\delta} \tau^4 J_0 \delta \Omega} \right) \quad (22)$$

whereas the full expression including the secular terms is rather cumbersome and it is not reported here. Notice also that κ depends on the product $J_0 \delta$ rather than on the two parameters independently. We will thus set $J_0 = 1$ (which means $J_0 = \omega_0$ in the original dimensional system) in the following.

The comparison between the secular and non secular solution is illustrated in Fig. 1. More specifically, in the left panel of Fig. 1a we show κ as a function of τ for $\delta = 10^{-4}$, $\Omega = J_0 = 1$ and for different values of the TWB parameter r . In the center panel, κ as a function of τ is shown for $\delta = 10^{-3}$ and the same values of the other

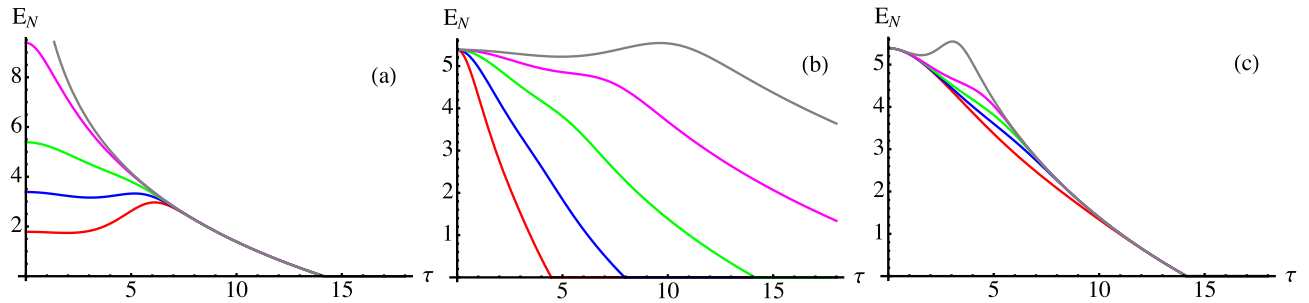


Figure 2. Entanglement negativity E_N as a function of the dimensionless time τ for different values of the involved (dimensionless) parameters. In panel (a) we show results for $\Omega = 1$, $J_0\delta = 0.01$ and different values of r , from top to bottom $r = 10, 2, 1, 0.5, 0.1$, corresponding to gray, magenta, green, blue and red curve, respectively. In panel (b) we show results for $\Omega = 1$, $r = 1$, and different values of the product $J_0\delta$, from top to bottom $J_0\delta = 10^{-3}, 10^{-5/2}, 10^{-2}, 10^{-3/2}, 10^{-1}$ [same colors as in panel (a)]. In panel (c) we show results for $J_0\delta = 0.01$, $r = 1$, and different values of Ω , from top to bottom $\Omega = 10, 2, 1, 0.5, 0.1$ [same colors as in panel (a)].

parameters. Finally, in the right panel of Fig. 1c we show κ as a function of τ for $\delta = 10^{-3}$, $J_0 = 1$, $\Omega = 3$ and for different values of the TWB parameter r . In all the plots, the solid black lines denote the results without the secular terms and the red dashed lines with the secular terms. We remind that all the involved parameters are in units of ω_0 .

The first observation is that the validity of the secular approximation is a property of the channel, i.e. it is almost independent on the TWB parameter. Moreover, we see that for short times the secular approximation is always valid, independently on the other parameters. How “short” should be the time depends instead on the property of the environment. More precisely, the secular approximation holds for longer times if the bandwidth δ and the location Ω are smaller. We also notice that the region of validity of the secular approximation coincides with the region where the dynamics for different values of r differ: when the function $\kappa(\tau)$ no longer depends on the initial value of r the secular approximation starts to fail (though the behaviour of the different functions may be closer also before that time).

Having established that the secular approximation holds in a rather wide of the parameter range and for a rather long period of time, we now proceed by studying the dynamics of entanglement, i.e. we study the behaviour of E_N [calculated according to Eqs. (20) and (22)] as a function of time for different values of the involved parameters. Results are reported in Fig. 2, where we show E_N as function of τ for: $\Omega = 1$, $J_0\delta = 0.01$ and different values of r [panel (a)], $\Omega = 1$, $r = 1$, and different values of the product $J_0\delta$ [panel (b)], $J_0\delta = 0.01$, $r = 1$, and different values of Ω [panel (c)].

The first observation is that the E_N is not monotone in time, a clear signature of non-Markovianity^{53,58}. Moreover, we observe the phenomenon of entanglement sudden death^{45,46} which occurs at a time determined only by the bandwidth [compare results in panels (a) and (c), where δ is fixed, to those in panel (b) where δ is varying].

Overall, the results of Fig. 2 may be summarised as follows. The initial entanglement (determined by r) influences the dynamics at short times, but then the behaviour becomes universal and determined by the properties of the environment. In particular, sudden death occurs at a time determined by the bandwidth. The location of the bandwidth influences entanglement to a lesser extent. On the other hand, it may be fruitfully exploited, since increasing Ω leads to an increase of non-Markovianity and, in turn, to a temporary increase of entanglement [see panel (c)], at least for large Ω .

Conclusions

In this work, we have analyzed the entanglement dynamics in optical media characterized by a finite bandwidth. Upon assuming weak coupling and low temperature, we have obtained an exact analytic solution for the time dependent two-mode covariance matrix describing a Gaussian state of our system in the short time non-Markovian limit. Our results show that attenuation (damping) is strongly suppressed whereas the diffusion term depends only on the bandwidth.

We have investigated the entanglement dynamics as a function of the bandwidth, the natural frequency and the initial amount of entanglement and show that there exist a wide range of situations where decoherence is not much detrimental and entanglement may persist for a longer time. We have also proved that secular terms may be neglected in the short time non-Markovian limit. Our results are encouraging and show that materials with a photonic bandgap may provide a reliable way to transmit entanglement over long distance.

Data availability

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

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Author contributions

All the authors contributed equally to design and perform the research, as well as writing the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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