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# Fixed point results for weak contractions in partially ordered *b*-metric space

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## Abstract

**Objectives:** We explore the existence of a fixed point as well as the uniqueness of a mapping in an ordered *b*-metric space using a generalized $(\check{\psi}, \hat{\eta})$-weak contraction. In addition, some results are posed on a coincidence point and a coupled coincidence point of two mappings under the same contraction condition. These findings generalize and build on a few recent studies in the literature. At the end, we provided some examples to back up our findings.

**Result:** In partially ordered *b*-metric spaces, it is discussed how to obtain a fixed point and its uniqueness of a mapping, and also investigated the existence of a coincidence point and a coupled coincidence point for two mappings that satisfying generalized weak contraction conditions.


**Keywords:** $(\check{\psi}, \hat{\eta})$-weak contraction, Fixed point, Coincidence and coupled coincidence points, Ordered *b*-metric space

**Mathematics Subject Classification:** 54H25, 47H10

## Introduction

In a wide range of pure and applied mathematics problems, fixed points of mappings that satisfy contractive conditions in extended metric spaces are extremely useful. First, Ran and Reuings [32] described the existence of fixed points in this direction for certain maps in ordered metric space and exhibited matrix linear equations applications. Following that, Nieto et al. [28, 29] expanded the result of [32] to nondecreasing mappings and used their findings to obtain differential equations solutions. Agarwal et al. [3] and O'Regan et al. [30] examined the influence of generalized contractions in ordered spaces at the same time. Bhaskar and Lakshmikantham [11] first developed coupled fixed point theory for some maps, then used the results to find a unique solution to periodic boundary value problems. Following that, Lakshmikantham and Ćirić [22], which were the extensions of [11] involving monotone property to a function in the space, pioneered the idea of coupled coincidence, common fixed point results. [19, 25, 34–37] provide additional information on coupled fixed point effects in various spaces under various contractive conditions.

A *b*-metric space is one of several generalizations of a standard metric space proposed by Bakhtin in his work [9], and widely used by Czerwik in his work [14, 15]. Following that, a lot of progress was made in acquiring the results of fixed points to single valued as well as multi-valued operators in the space, as evidenced by [1, 2, 4–8, 10, 13, 16–18, 20, 21, 23, 24, 26, 27, 31, 38–41].

We demonstrate some fixed points results for mappings in ordered *b*-metric space that satisfy a generalized weak contraction in this paper. The results from [10, 11, 19, 22, 33] are expanded here as well as some examples noted to support the findings at the end of our work.

## Preliminaries

The following definitions are subsequently used in our study.

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**Definition 2.1** [15] A $b$-metric is a mapping $\eth : \mathscr{E} \times \mathscr{E} \to [0, +\infty)$ that satisfies the properties below for all $\varepsilon, \wp, \zeta$ in $\mathscr{E}$ and some $\mathrm{s} \geq 1$,

(a) $\eth(\varepsilon, \wp) = 0$ if and if $\varepsilon = \wp$,
(b) $\eth(\varepsilon, \wp) = \eth(\wp, \varepsilon)$,
(c) $\eth(\varepsilon, \wp) \leq \mathrm{s}(\eth(\varepsilon, \zeta) + \eth(\zeta, \wp))$.

A $b$-metric space is specified as $(\mathscr{E}, \eth, \mathrm{s})$.

***Example 2.2*** The space $L_q[0,1]$, where $0 < q < 1$ of all real functions $f(t), t \in [0,1]$ such that $\int_0^1 |f(t)|^q dt < \infty$ is a $b$-metric space if we take $\eth(\varepsilon, \wp) = \int_0^1 (|f(t) - g(t)|^q dt)^{\frac{1}{q}}$, for all $\varepsilon, \wp \in L_q[0,1]$.

**Note 2.3** Every metric space is a $b$-metric space with $\mathrm{s} = 1$, but in general a $b$-metric space need not necessarily be a metric space, as in below example 2.4 is $b$-metric space but not a metric space. Thus, the class of $b$-metric spaces is larger than the class of metric spaces.

***Example 2.4*** Let $\mathscr{E} = \mathbb{R}$ and define the mapping $\eth : \mathscr{E} \times \mathscr{E} \to \mathbb{R}^+$ by $\eth(\varepsilon, \wp) = |\varepsilon - \wp|^2$, for all $\varepsilon, \wp \in \mathscr{E}$. Then $(\mathscr{E}, \eth)$ is a $b$-metric space with coefficient $\mathrm{s} = 2$.

The generalization of the above Example 2.4 is as follows:

***Example 2.5*** Let $(\mathscr{E}, d)$ be a metric space and $q \geq 1$ be a given real number. Then $\eth(\varepsilon, \wp) = [d(\varepsilon, \wp)]^q$ is a $b$-metric on $\mathscr{E}$ with parameter $\mathrm{s} \leq 2^{\mathrm{q}-1}$.

**Definition 2.6** [10, 15] In a $b$-metric space,

(1) if $\eth(\varepsilon_n, \varepsilon) \to 0$ as $n \to +\infty$ then $\{\varepsilon_n\}$ is said to be convergent to $\varepsilon$.
(2) if $\eth(\varepsilon_n, \varepsilon_m) \to 0$ as $n, m \to +\infty$ then $\{\varepsilon_n\}$ is a Cauchy sequence.
(3) if $(\mathscr{E}, \eth, \mathrm{s})$ is a complete $b$-metric space then very Cauchy sequence is convergent.

**Definition 2.7** [15, 33] If $\mathscr{E}$ is a partial ordered set with respect to an ordered relation $\preceq$ and $\eth$ is a metric on it, then $(\mathscr{E}, \eth, \preceq)$ is a partially ordered metric space. $(\mathscr{E}, \eth, \preceq)$ is complete partially ordered $b$-metric space, despite the fact that $\eth$ is complete.

**Definition 2.8** [33] Let $\hbar : \mathscr{E} \to \mathscr{E}$ be a mapping. If $\hbar(\varepsilon) \preceq \hbar(\wp)$ for all $\varepsilon, \wp \in \mathscr{E}$ with $\varepsilon \preceq \wp$, then $\hbar$ is called monotone nondecreasing mapping.

**Definition 2.9** [12] Let $\hbar, \mathscr{I} : \mathscr{A} \to \mathscr{A}$ be two mappings, and $\mathscr{A} \neq \emptyset \subseteq \mathscr{E}$. If $\hbar\varepsilon = \mathscr{I}\varepsilon = \varepsilon$ $(\hbar\varepsilon = \mathscr{I}\varepsilon)$ for $\varepsilon \in \mathscr{A}$, then $\varepsilon$ is called a common fixed point (coincidence point) of $\hbar$ and $\mathscr{I}$.

**Definition 2.10** [12] If $\hbar\mathscr{I}\varepsilon = \mathscr{I}\hbar\varepsilon$ for all $\varepsilon \in \mathscr{A}$, then $\hbar$ and $\mathscr{I}$ are commuting.

**Definition 2.11** [12, 33] The two self mappings $\hbar$ and $\mathscr{I}$ are known to be compatible, if $\lim_{n\to+\infty} d(\mathscr{I}\hbar\varepsilon_n, \hbar\mathscr{I}\varepsilon_n) = 0$ for every sequence $\{\varepsilon_n\}$ in $\mathscr{E}$ such that $\lim_{n\to+\infty} \hbar\varepsilon_n = \lim_{n\to+\infty} \mathscr{I}\mathfrak{F}_n = \mu$, for some $\mu \in \mathscr{A}$.

**Definition 2.12** [12, 33] If $\hbar\varepsilon = \mathscr{I}\varepsilon$ for some $\varepsilon \in \mathscr{A}$, then $\hbar\mathscr{I}\varepsilon = \mathscr{I}\hbar\varepsilon$, the mappings $\hbar$ and $\mathscr{I}$ are called weakly compatible.

**Definition 2.13** [33] If $\hbar\varepsilon \preceq \hbar\wp$ implies $\mathscr{I}\varepsilon \preceq \mathscr{I}\wp$ for each $\varepsilon, \wp \in \mathscr{E}$, then the mapping $\mathscr{I}$ is called monotone $\hbar$ -nondecreasing.

**Definition 2.14** [11] Let $\mathscr{I} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and $\hbar : \mathscr{E} \to \mathscr{E}$ are two mappings,

(a) a point $(\varepsilon, \wp) \in \mathscr{E} \times \mathscr{E}$ is coupled coincidence point of $\mathscr{I}$ and $\hbar$, if $\mathscr{I}(\varepsilon, \wp) = \hbar\varepsilon$ and $\mathscr{I}(\wp, \varepsilon) = \hbar\wp$. In particular, if $\hbar$ is an identity mapping, then $(\varepsilon, \wp)$ is a coupled fixed point of $\mathscr{I}$.
(b) a point $\varepsilon \in \mathscr{E}$ is a common fixed point of $\mathscr{I}$ and $\hbar$, if $\mathscr{I}(\varepsilon, \varepsilon) = \hbar\varepsilon = \varepsilon$.
(c) if $\mathscr{I}(\hbar\varepsilon, \hbar\wp) = \hbar(\mathscr{I}\varepsilon, \mathscr{I}\wp)$ for all $\varepsilon, \wp \in \mathscr{E}$, then $\mathscr{I}$ and $\hbar$ are commuting each other.
(d) If every two elements of $\mathscr{A} \subseteq \mathscr{E}$ are comparable, then the set $\mathscr{A}$ is called a well ordered set.

**Definition 2.15** A self mapping $\check{\psi}$ on $[0, +\infty)$ that meets the conditions below is known as an altering distance function:

(a) $\check{\psi}$ is a non-decreasing and continuous function,
(b) $\check{\psi}(\ell) = 0$ if and only if $\ell = 0$.

As seen above, the symbol $\hat{\Phi}$ represents the set of all altering distance functions.

Similarly, $\hat{\Psi} : \{\hat{\eta} | \hat{\eta}$ *is a lower semi-continuous self mapping on* $[0, +\infty)$ and, $\hat{\eta}(\ell) = 0$ if and only if $\ell = 0\}$.

The presented lemmas under here are frequently used in our main results.

**Lemma 2.16** [27] *Let* $\hbar : \mathscr{E} \to \mathscr{E}$ *be a mapping, and* $\mathscr{E} \neq \emptyset$. *Then* $\mathscr{M} \subseteq \mathscr{E}$ *occurs, resulting in* $\hbar \mathscr{M} = \hbar \mathscr{E}$, *where* $\hbar : \mathscr{M} \to \mathscr{E}$ *is one-to-one.*

**Lemma 2.17** [4] *Let* $\{\varepsilon_n\}$ *and* $\{\wp_n\}$ *be two sequences and b-convergent to* $\varepsilon$ *and* $\wp$ *in a b-metric space* $(\mathscr{E}, \eth, \mathrm{s}, \preceq)$, *where* $\mathrm{s} > 1$. *Then*

$$\frac{1}{\mathrm{s}^2}\eth(\varepsilon, \wp) \leq \lim_{n\to+\infty} \inf \eth(\varepsilon_n, \wp_n) \leq \lim_{n\to+\infty} \sup \eth(\varepsilon_n, \wp_n) \leq \mathrm{s}^2\eth(\varepsilon, \wp).$$

*In particular, if* $\varepsilon = \wp$, *then* $\lim_{n\to+\infty} \eth(\varepsilon_n, \wp_n) = 0$. *In addition, for every* $\tau \in \mathscr{E}$, *we get*

$$\frac{1}{\mathrm{s}}\eth(\varepsilon, \tau) \leq \lim_{n\to+\infty} \inf \eth(\varepsilon_n, \tau) \leq \lim_{n\to+\infty} \sup \eth(\varepsilon_n, \tau) \leq \mathrm{s}d(\varepsilon, \tau).$$

## Main results

We start this section with the following fixed point theorem in an ordered *b*-metric space.

**Theorem 3.1** *Suppose* $(\mathscr{E}, \eth, \mathrm{s}, \preceq)$ *is a complete partially ordered b-metric space with* $\mathrm{s} > 1$. *A mapping* $\mathscr{I} : \mathscr{E} \to \mathscr{E}$ *is continuous and nondecreasing with respect to* $\preceq$. *If* $\varepsilon_0 \in \mathscr{E}$ *is such that* $\varepsilon_0 \preceq \mathscr{I}\varepsilon_0$ *and the following contraction condition is fulfilled, then* $\mathscr{I}$ *has a fixed point in* $\mathscr{E}$.

$$\check{\psi}(\mathrm{s}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)) \leq \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)), \tag{1}$$

*for* $\check{\psi} \in \hat{\Phi}, \hat{\eta} \in \hat{\Psi}$ *and for any* $\varepsilon, \wp \in \mathscr{E}$ *such that* $\varepsilon \preceq \wp$ *and where*

$$\mathscr{P}(\varepsilon, \wp) = \max\left\{\frac{\eth(\wp, \mathscr{I}\wp)\left[1 + \eth(\varepsilon, \mathscr{I}\varepsilon)\right]}{1 + \eth(\varepsilon, \wp)}, \frac{\eth(\varepsilon, \mathscr{I}\wp) + \eth(\wp, \mathscr{I}\varepsilon)}{2s}, \eth(\varepsilon, \mathscr{I}\varepsilon), \eth(\wp, \mathscr{I}\wp), \eth(\varepsilon, \wp)\right\}. \tag{2}$$

***Proof*** For some $\varepsilon_0 \in \mathscr{E}$ with $\mathscr{I}\varepsilon_0 = \varepsilon_0$, then the result is trivial. Assuming that $\varepsilon_0 \prec \mathscr{I}\varepsilon_0$, we describe a sequence $\{\varepsilon_n\} \subset \mathscr{E}$ by $\varepsilon_{n+1} = \mathscr{I}\varepsilon_n$ for all $n \geq 0$. However, we can deduce the following as $\mathscr{I}$ is nondecreasing,

$$\varepsilon_0 \prec \mathscr{I}\varepsilon_0 = \varepsilon_1 \preceq \mathscr{I}\varepsilon_1 = \varepsilon_2 \preceq ... \preceq \mathscr{I}\varepsilon_{n-1} = \varepsilon_n \preceq \mathscr{I}\varepsilon_n = \varepsilon_{n+1} \preceq ..... . \tag{3}$$

If $\varepsilon_{n_0} = \varepsilon_{n_0+1}$ for $n_0 \in \mathbb{N}$, then $\varepsilon_{n_0}$ is a fixed point of $\mathscr{I}$ from (3). Otherwise, for all $n \geq 1$, $\varepsilon_n \neq \varepsilon_{n+1}$. For $n \geq 1$, let $D_n = \eth(\varepsilon_{n+1}, \varepsilon_n)$. We know that for every $n \geq 1$, $\varepsilon_{n-1} \prec \varepsilon_n$ and, then the equation (1) becomes

$$\begin{aligned}\check{\psi}(D_n) = \check{\psi}(\eth(\varepsilon_n, \varepsilon_{n+1})) &= \check{\psi}(\eth(\mathscr{I}\varepsilon_{n-1}, \mathscr{I}\varepsilon_n)) \\ &\leq \check{\psi}(\mathrm{s}\eth(\mathscr{I}\varepsilon_{n-1}, \mathscr{I}\varepsilon_n)) \\ &\leq \check{\psi}(\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n)) - \hat{\eta}(\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n)).\end{aligned} \tag{4}$$

From (4), we get

$$\eth(\varepsilon_n, \varepsilon_{n+1}) = \eth(\mathscr{I}\varepsilon_{n-1}, \mathscr{I}\varepsilon_n) \leq \frac{1}{\mathscr{s}}\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n), \tag{5}$$

where

$$\begin{aligned}\mathscr{P}(\varepsilon_{n-1}, \varepsilon_n) &= \max\left\{\frac{\eth(\varepsilon_n, \mathscr{I}\varepsilon_n)\left[1 + \eth(\varepsilon_{n-1}, \mathscr{I}\varepsilon_{n-1})\right]}{1 + \eth(\varepsilon_{n-1}, \varepsilon_n)}, \frac{\eth(\varepsilon_{n-1}, \mathscr{I}\varepsilon_n) + \eth(\varepsilon_n, \mathscr{I}\varepsilon_{n-1})}{2s}, \eth(\varepsilon_{n-1}, \mathscr{I}\varepsilon_{n-1}), \eth(\varepsilon_n, \mathscr{I}\varepsilon_n), \eth(\varepsilon_{n-1}, \varepsilon_n)\right\} \\ &\leqslant \max\left\{\eth(\varepsilon_n, \varepsilon_{n+1}), \frac{\eth(\varepsilon_{n-1}, \varepsilon_n) + \eth(\varepsilon_n, \varepsilon_{n+1})}{2}, \eth(\varepsilon_{n-1}, \varepsilon_n)\right\} \leqslant \max\{D_n, D_{n-1}\}.\end{aligned} \tag{6}$$

If $\max\{D_n, D_{n-1}\} = D_n$ for certain $n \geq 1$, equation (5) is then accompanied by

$$\eth(\varepsilon_n, \varepsilon_{n+1}) \leq \frac{1}{\mathscr{s}}\eth(\varepsilon_n, \varepsilon_{n+1}),$$

this is a contradiction. Thus, $\max\{D_n, D_{n-1}\} = D_{n-1}$ for $n \geq 1$. Hence, equation (5) becomes

$$\eth(\varepsilon_n, \varepsilon_{n+1}) \leq \frac{1}{\mathscr{s}}\eth(\varepsilon_n, \varepsilon_{n-1}).$$

Since $\frac{1}{\mathscr{s}} \in (0, 1)$, then $\{\varepsilon_n\}$ is a Cauchy sequence from [1, 6, 8, 18]. Also, the completeness of $\mathscr{E}$ gives that $\varepsilon_n \to \mu \in \mathscr{E}$.

We may also deduce the following from the continuity of $\mathscr{I}$,

$$\mathscr{I}\mu = \mathscr{I}(\lim_{n\to+\infty}\varepsilon_n) = \lim_{n\to+\infty}\mathscr{I}\varepsilon_n = \lim_{n\to+\infty}\varepsilon_{n+1} = \mu. \tag{7}$$

As a result, $\mathscr{I}$ in $\mathscr{E}$ has a fixed point $\mu$. □

The continuity assumption on $\mathscr{I}$ is extracted from Theorem 3.1 and used to derive the following theorem.

**Theorem 3.2** *In Theorem 3.1, if $\mathscr{E}$ satisfies below condition, then $\mathscr{I}$ has a fixed point.*

$$\begin{aligned}&\text{If a non-decreasing sequence } \{\varepsilon_n\}\\ &\subseteq \mathscr{E} \quad \text{and } \varepsilon_n \to \sigma \text{ then } \varepsilon_n \leq \sigma,\\ &\text{for each } n \in \mathbb{N}, \text{ i.e., } \sigma = \sup \varepsilon_n.\end{aligned} \tag{8}$$

***Proof*** We have an increasing sequence $\{\varepsilon_n\} \subseteq \mathscr{E}$ that eventually converges to some $\sigma \in \mathscr{E}$ as a result of Theorem 3.1. But by the hypotheses for all $n$, $\varepsilon_n \preceq \sigma$, which means that $\sigma = \sup \varepsilon_n$.

We can now assert that $\sigma$ is a fixed point of $\mathscr{I}$. Assume that $\mathscr{I}\sigma \neq \sigma$. Let

$$\mathscr{P}(\varepsilon_n, \sigma) = \max\left\{\frac{\eth(\sigma, \mathscr{I}\sigma)\left[1 + \eth(\varepsilon_n, \mathscr{I}\varepsilon_n)\right]}{1 + \eth(\varepsilon_n, \sigma)}, \frac{\eth(\varepsilon_n, \mathscr{I}\sigma) + \eth(\sigma, \mathscr{I}\varepsilon_n)}{2s}, \eth(\varepsilon_n, \mathscr{I}\varepsilon_n), \eth(\sigma, \mathscr{I}\sigma), \eth(\varepsilon_n, \sigma)\right\} \tag{9}$$

then taking limit as $n \to +\infty$ in the equation (9) and making use of $\lim_{n\to+\infty}\varepsilon_n = \sigma$, we get

$$\lim_{n\to+\infty}\mathscr{P}(\varepsilon_n, \sigma) = \max\{\eth(\sigma, \mathscr{I}\sigma), 0\} = \eth(\sigma, \mathscr{I}\sigma). \tag{10}$$

Since, $\varepsilon_n \preceq \sigma$ for each $n$, then we obtain the following from equations (1) and (9)

$$\begin{aligned}\check{\psi}(\eth(\varepsilon_{n+1}, \mathscr{I}\sigma)) &= \check{\psi}(\eth(\mathscr{I}\varepsilon_n, \mathscr{I}\sigma)) \leq \check{\psi}(s\eth(\mathscr{I}\varepsilon_n, \mathscr{I}\sigma))\\ &\leq \check{\psi}(\mathscr{P}(\varepsilon_n, \sigma)) - \hat{\eta}(\mathscr{P}(\varepsilon_n, \sigma)).\end{aligned} \tag{11}$$

Take limit as $n \to +\infty$ in (11) and from equation (10) as well as the properties of $\check{\psi}, \hat{\eta}$, we have

$$\check{\psi}(\eth(\sigma, \mathscr{I}\sigma)) \leq \check{\psi}(\eth(\sigma, \mathscr{I}\sigma)) - \hat{\eta}(\eth(\sigma, \mathscr{I}\sigma)) < \check{\psi}(\eth(\sigma, \mathscr{I}\sigma)). \tag{12}$$

This is a contradiction to $\mathscr{I}\sigma \neq \sigma$. Hence, $\mathscr{I}\sigma = \sigma$. □

In the above theorems, the fixed point is unique if $\mathscr{E}$ meets the following condition.

$$\begin{aligned}&\text{There exists a } \sigma \text{ in } \mathscr{E} \text{ that is comparable to}\\ &\varepsilon \text{ and } \wp, \text{ for each } \varepsilon, \wp \in \mathscr{E}.\end{aligned} \tag{13}$$

**Theorem 3.3** *If $\mathscr{E}$ assumes the condition (13) in Theorem 3.1 & 3.2, then $\mathscr{I}$ has a unique fixed point in $\mathscr{E}$.*

***Proof*** Theorems 3.1 & 3.2 show that the set of fixed points of $\mathscr{I}$ is nonempty. Assume $\varepsilon^* \neq \wp^*$ are fixed points of $\mathscr{I}$ to ensure uniqueness. Following that,

$$\begin{aligned}\check{\psi}(\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*)) &\leq \check{\psi}(s\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*))\\ &\leq \check{\psi}(\mathscr{P}(\varepsilon^*, \wp^*)) - \hat{\eta}(\mathscr{P}(\varepsilon^*, \wp^*)),\end{aligned} \tag{14}$$

where

$$\mathscr{P}(\varepsilon^*, \wp^*) = \max\left\{\frac{\eth(\wp^*, \mathscr{I}\wp^*)\left[1 + \eth(\varepsilon^*, \mathscr{I}\varepsilon^*)\right]}{1 + \eth(\varepsilon^*, \wp^*)}, \frac{\eth(\varepsilon^*, \mathscr{I}\wp^*) + \eth(\wp^*, \mathscr{I}\varepsilon^*)}{2s}, \eth(\varepsilon^*, \mathscr{I}\varepsilon^*), \eth(\wp^*, \mathscr{I}\wp^*), \eth(\varepsilon^*, \wp^*)\right\}. \tag{15}$$

Therefore from equations (14) and (15), we have

$$\begin{aligned}\check{\psi}(\eth(\varepsilon^*, \wp^*)) &= \check{\psi}(\eth(\mathscr{I}\varepsilon^*, \mathscr{I}\wp^*)) \leq \check{\psi}(\eth(\varepsilon^*, \wp^*))\\ &\quad - \hat{\eta}(\eth(\varepsilon^*, \wp^*)) < \check{\psi}(\eth(\varepsilon^*, \wp^*)),\end{aligned} \tag{16}$$

this contradicts to $\varepsilon^* \neq \wp^*$. Hence, $\varepsilon^* = \wp^*$. □

Now, we have the below corollary from Theorems 3.1 to 3.3.

**Corollary 3.4** *Let $(\mathscr{E}, \eth, \preceq)$ be a partially ordered b-metric space. Suppose the mappings $\mathscr{I}, \mathscr{h} : \mathscr{E} \to \mathscr{E}$ are continuous such that*

$(C_1)$.

$$\check{\psi}(s\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) \leq \check{\psi}(\mathscr{P}_{\mathscr{h}}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_{\mathscr{h}}(\varepsilon,\wp)), \quad (17)$$

*for every* $\varepsilon, \wp \in \mathscr{E}$ *with* $\mathscr{h}\varepsilon \preceq \mathscr{h}\wp$, $s > 1$, $\check{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ *and, where*

$$\eth(\mathscr{I}\mathscr{v}, \mathscr{h}\mathscr{v}) \leq s\eth(\mathscr{I}\mathscr{v}, \mathscr{I}(\mathscr{h}\varepsilon_n)) + s^2\eth(\mathscr{I}(\mathscr{h}\varepsilon_n), \mathscr{h}(\mathscr{I}\varepsilon_n)) + s^2\eth(\mathscr{h}(\mathscr{I}\varepsilon_n), \mathscr{h}\mathscr{v}). \quad (22)$$

$$\mathscr{P}_h(\varepsilon,\wp) = \max\left\{\frac{\eth(h\wp,\mathscr{I}\wp)\left[1+\eth(h\varepsilon,\mathscr{I}\varepsilon)\right]}{1+\eth(h\varepsilon,h\wp)}, \frac{\eth(h\varepsilon,\mathscr{I}\wp)+\eth(h\wp,\mathscr{I}\varepsilon)}{2s}, \eth(h\varepsilon,\mathscr{I}\varepsilon), \eth(h\wp,\mathscr{I}\wp), \eth(h\varepsilon,h\wp)\right\}. \quad (18)$$

$(C_2)$. $\mathscr{I}\mathscr{E} \subset \mathscr{h}\mathscr{E}$ and $\mathscr{h}\mathscr{E} \subseteq \mathscr{E}$ is complete,
$(C_3)$. $\mathscr{I}$ is monotone $\mathscr{h}$-non-decreasing and
$(C_4)$. $\mathscr{I}$ and $\mathscr{h}$ are compatible.

If for some $\varepsilon_0 \in \mathscr{E}$ such that $\mathscr{h}\varepsilon_0 \preceq \mathscr{I}\varepsilon_0$, then a pair of mappings $(\mathscr{I}, \mathscr{h})$ has a coincidence point in $\mathscr{E}$.

***Proof*** By Lemma 2.16, there exists $\mathscr{M} \subset \mathscr{E}$ such that $\mathscr{h}\mathscr{M} = \mathscr{h}\mathscr{E}$ and $\mathscr{h} : \mathscr{M} \to \mathscr{E}$ is one-to-one. Now define a map $\mathscr{k} : \mathscr{h}\mathscr{M} \to \mathscr{h}\mathscr{M}$ by $\mathscr{k}(\mathscr{h}\varepsilon) = \mathscr{I}\varepsilon$, $\varepsilon \in \mathscr{M}$. Since $\mathscr{h}$ is one-to-one on $\mathscr{M}$, $\mathscr{k}$ is well defined. Then, $\mathscr{h}\mathscr{M} = \mathscr{h}\mathscr{E}$ is complete and then (17) becomes

$$\check{\psi}(s\eth(\mathscr{k}(\mathscr{h}\varepsilon), \mathscr{k}(\mathscr{h}\wp))) \leq \check{\psi}(\mathscr{P}_{\mathscr{h}}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_{\mathscr{h}}(\varepsilon,\wp)), \quad (19)$$

for every $\varepsilon,\wp \in \mathscr{E}$ with $\mathscr{h}\varepsilon \preceq \mathscr{h}\wp$ and, where

Taking $n \to +\infty$ in (22) and the continuity of $\mathscr{I}$, $\mathscr{h}$ and (21), we get $\eth(\mathscr{I}\mathscr{v}, \mathscr{h}\mathscr{v}) = 0$. That is $\mathscr{I}\mathscr{v} = \mathscr{h}\mathscr{v}$. Therefore, $\mathscr{v}$ is a coincidence point of $\mathscr{I}$, $\mathscr{h}$.

The following result can get from Corollary 3.4 by weakening its hypotheses.

**Corollary 3.5** *If* $\mathscr{E}$ *satisfies the following condition in Corollary* 3.4,

$$\text{for very nondecreasing sequence } \{\mathscr{h}\varepsilon_n\} \subseteq \mathscr{E} \text{ such that } \mathscr{h}\varepsilon_n \to \mathscr{h}\sigma, \text{ then } \mathscr{h}\varepsilon_n \leq \mathscr{h}\sigma\ (n \geq 0), \text{ i.e., } \mathscr{h}\sigma = \sup \mathscr{h}\varepsilon_n. \quad (23)$$

*then, if* $\mathscr{h}\mu \preceq \mathscr{h}(\mathscr{h}\mu)$ *for some coincidence point* $\mu$, *a coincidence point exists for the weakly compatible mappings* $(\mathscr{I}, \mathscr{h})$. *Moreover,* $(\mathscr{I}, \mathscr{h})$ *has only one common fixed*

$$\mathscr{P}_h(\varepsilon,\wp) = \max\left\{\frac{\eth(h\wp,k\eth(h\wp))[1+\eth(h\varepsilon,k\eth(h\varepsilon))]}{1+\eth(h\varepsilon,h\wp)}, \frac{\eth(h\varepsilon,k\eth(h\wp))+\eth(h\wp,k\eth(h\varepsilon))}{2s}, \eth(h\varepsilon,k\eth(h\varepsilon)), \eth(h\wp,k\eth(h\wp)), \eth(h\varepsilon,h\wp)\right\}. \quad (20)$$

Let $\varepsilon_0 \in \mathscr{M}$ such that $\mathscr{h}\varepsilon_0 \preceq \mathscr{I}\varepsilon_0 = \mathscr{k}(\mathscr{h}\varepsilon_0)$. Choose $\varepsilon_1 \in \mathscr{M}$ such that $\mathscr{h}\varepsilon_1 = \mathscr{I}\varepsilon_0 = \mathscr{k}(\mathscr{h}\varepsilon_0)$. By continuing this process, we obtain a sequence $\{\mathscr{h}\varepsilon_n\} \subset \mathscr{h}\mathscr{M}$ such that $\mathscr{h}\varepsilon_{n+1} = \mathscr{I}\varepsilon_n = \mathscr{k}(\mathscr{h}\varepsilon_n)$ for $n \geq 0$. By using the similar argument as in the proof of Theorem 3.1, we obtain that $\{\mathscr{h}\varepsilon_n\} \subset \mathscr{h}\mathscr{M}$ is a $b$-Cauchy sequence. Since $\mathscr{h}\mathscr{M}$ is complete, there exists $\mathscr{v} \in \mathscr{h}\mathscr{M}$ such that $\lim_{n\to+\infty} \mathscr{h}\varepsilon_n = \mathscr{v} \in \mathscr{h}\mathscr{E}$. Then

$$\lim_{n\to+\infty} \mathscr{h}\varepsilon_n = \lim_{n\to+\infty} \mathscr{I}\varepsilon_{n-1} = \mathscr{v}.$$

From the condition $(C_4)$, we have

$$\lim_{n\to+\infty} \eth(\mathscr{h}(\mathscr{I}\varepsilon_n), \mathscr{I}(\mathscr{h}\varepsilon_n)) = 0. \quad (21)$$

Furthermore, the triangular inequality of $b$-metric, we have

*point if and only if the set of common fixed points is well ordered.* □

***Proof*** A pair of mappings $(\mathscr{I}, \mathscr{h})$ has a coincidence point, according to Theorem 3.3 and Corollary 3.4.

Next, assume that a pair of mappings $(\mathscr{I}, \mathscr{h})$ is weakly compatible. Let $\mathscr{v} \in \mathscr{E}$ be a point with $\mathscr{v} = \mathscr{I}\mu = \mathscr{h}\mu$. Then, $\mathscr{I}\mathscr{v} = \mathscr{I}(\mathscr{h}\mu) = \mathscr{h}(\mathscr{I}\mu) = \mathscr{h}\mathscr{v}$.

Therefore,

$$\mathscr{P}_{\hbar}(\mu,\nu) = \max\left\{\frac{\eth(h\nu,I\nu)[1+\eth(h\mu,I\mu)]}{1+\eth(h\mu,h\nu)}, \frac{\eth(h\mu,I\nu)+\eth(h\nu,I\mu)}{2s}, \eth(h\mu,I\mu),\eth(h\nu,I\nu),\eth(h\mu,h\nu)\right\}$$
$$= \max\left\{0,\frac{\eth(I\mu,I\nu)}{s},\eth(I\mu,I\nu)\right\} = \eth(I\mu,I\nu). \tag{24}$$

Thus from equation (17), we get

$$\check{\psi}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)) \le \check{\psi}(\mathscr{P}_{\hbar}(\mu,\upsilon)) - \hat{\eta}(\mathscr{P}_{\hbar}(\mu,\upsilon)) \le \check{\psi}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)) - \hat{\eta}(\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)). \tag{25}$$

By the property of $\hat{\eta}$, we get $\eth(\mathscr{I}\mu,\mathscr{I}\upsilon)=0$ implies that $\mathscr{I}\upsilon=\hbar\upsilon=\upsilon$.

Finally, we can deduce from Theorem 3.3 that $(\mathscr{I},\hbar)$ has only one common fixed point if and only if the common fixed points of $(\mathscr{I},\hbar)$ is well ordered. □

***Remark 3.6*** Theorems 3.1 to 3.3 are respectively the extension of Theorems 2.1,.2.2 & 2.3 of [27].

***Remark 3.7*** Corollaries 3.4 & 3.5 are the generalizations of Corollaries 2.1 & 2.2 of [12] respectively.

**Definition 3.8** Consider a partially ordered $b$-metric space, $(\mathscr{E},\eth,\preceq)$. A mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ is known to be a generalized $(\check{\psi},\hat{\eta})$-contractive mapping with regards to $\hbar:\mathscr{E}\to\mathscr{E}$, if

$$\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im))) \le \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\zeta,\Im)) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\zeta,\Im)), \tag{26}$$

for all $\varepsilon,\wp,\zeta,\Im\in\mathscr{E}$ with $\hbar\varepsilon\preceq\hbar\zeta$ and $\hbar\wp\succeq\hbar\Im$, $k>2$, $s>1$, $\check{\psi}\in\hat{\hat{\Phi}}$, $\hat{\eta}\in\hat{\Psi}$ and where

$$\mathscr{P}_h(\varepsilon,\wp,\zeta,\Im) = \max\left\{\frac{\eth(h\zeta,\mathscr{I}\,\eth(\zeta,\Im))\left[1+\eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp))\right]}{1+\eth(h\varepsilon,h\zeta)}, \frac{\eth(h\varepsilon,\mathscr{I}\,\eth(\zeta,\Im))+\eth(h\zeta,\mathscr{I}\,\eth(\varepsilon,\wp))}{2s}, \eth(h\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp)),\eth(h\zeta,\mathscr{I}\,\eth(\zeta,\Im)),\eth(h\varepsilon,h\zeta)\right\}$$

**Theorem 3.9** *Suppose that $(\mathscr{E},\eth,\preceq)$ is a complete partially ordered b-metric space. A mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ satisfies the condition (26) and $\mathscr{I}$, $\hbar$ are continuous, $\mathscr{I}$ has mixed $\hbar$-monotone property and also commutes with $\hbar$. Assume that, if for some $(\varepsilon_0,\wp_0)\in\mathscr{E}\times\mathscr{E}$ such that $\hbar\varepsilon_0\preceq\mathscr{I}(\varepsilon_0,\wp_0)$, $\hbar\wp_0\succeq\mathscr{I}(\wp_0,\varepsilon_0)$ and $\mathscr{I}(\mathscr{E}\times\mathscr{E})\subseteq\hbar(\mathscr{E})$, then $\mathscr{I}$ and $\hbar$ have a coupled coincidence point in $\mathscr{E}$.*

***Proof*** From Theorem 2.2 of [7], there exist two sequences $\{\varepsilon_n\}$ and $\{\wp_n\}$ in $\mathscr{E}$ such that

$$\hbar\varepsilon_{n+1}=\mathscr{I}(\varepsilon_n,\wp_n),\quad \hbar\wp_{n+1}=\mathscr{I}(\wp_n,\varepsilon_n), n\ge 0.$$

In particular, the sequences $\{\hbar\varepsilon_n\}$ and $\{\hbar\wp_n\}$ are non-decreasing and non-increasing in $\mathscr{E}$. Put $\varepsilon=\varepsilon_n,\wp=\wp_n,\zeta=\varepsilon_{n+1},\Im=\wp_{n+1}$ in (26), we get

$$\check{\psi}(s^k\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})) = \check{\psi}(s^k\eth(\mathscr{I}(\varepsilon_n,\wp_n),\mathscr{I}(\varepsilon_{n+1},\wp_{n+1}))) \le \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1})) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1})), \tag{27}$$

where

$$\mathscr{P}_{\hbar}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1}) \le \max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}. \tag{28}$$

Therefore from (27), we have

$$\check{\psi}(s^k\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})) \le \check{\psi}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2})\}). \tag{29}$$

Similarly by taking $\varepsilon=\wp_{n+1},\wp=\varepsilon_{n+1},\zeta=\varepsilon_n,\Im=\varepsilon_n$ in (26), we get

$$\check{\psi}(s^k\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})) \le \check{\psi}(\max\{\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}). \tag{30}$$

We know that $\max\{\check{\psi}(l_1),\check{\psi}(l_2)\}=\check{\psi}\{\max\{l_1,l_2\}\}$ for $l_1,l_2\in[0,+\infty)$. Then by adding (29) and (30) together we get,

$$\check{\psi}(s^k\Gamma_n) \le \check{\psi}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2}),\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}) - \hat{\eta}(\max\{\eth(\hbar\varepsilon_n,\hbar\varepsilon_{n+1}),\eth(\hbar\varepsilon_{n+1},\hbar\varepsilon_{n+2}),\eth(\hbar\wp_n,\hbar\wp_{n+1}),\eth(\hbar\wp_{n+1},\hbar\wp_{n+2})\}), \tag{31}$$

where

$$\mathscr{P}(\varepsilon,\wp,\zeta,\Im)=\max\left\{\frac{\eth(\zeta,\mathscr{I}\,\eth(\zeta,\Im))\left[1+\eth(\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp))\right]}{1+\eth(\varepsilon,\zeta)},\frac{\eth(\varepsilon,\mathscr{I}\,\eth(\zeta,\Im))+\eth(\zeta,\mathscr{I}\,\eth(\varepsilon,\wp))}{2s},\right.$$
$$\left.\eth(\varepsilon,\mathscr{I}\,\eth(\varepsilon,\wp)),\eth(\zeta,\mathscr{I}\,\eth(\zeta,\Im)),\eth(\varepsilon,\zeta)\right\}.$$

$$\Gamma_n=\max\{\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2}),\eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\}. \tag{32}$$

Let us denote,

$$\varkappa_n=\max\{\eth(\hbar\,\varepsilon_n,\hbar\,\varepsilon_{n+1}),\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2}),\eth(\hbar\,\wp_n,\hbar\,\wp_{n+1}),\eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\}. \tag{33}$$

Hence from equations (29)-(32), we obtain

$$s^k\Gamma_n\le\varkappa_n. \tag{34}$$

Now to claim that

$$\Gamma_n\le\lambda\Gamma_{n-1}, \tag{35}$$

for $n\ge 1$ and $\lambda=\frac{1}{s^k}\in[0,1)$.

Suppose that if $\varkappa_n=\Gamma_n$ then from (34), we get $s^k\Gamma_n\le\Gamma_n$ this leads to $\Gamma_n=0$, since $s>1$ and thus (35) holds. Suppose $\varkappa_n=\max\{\eth(\hbar\,\varepsilon_n,\hbar\,\varepsilon_{n+1}),\eth(\hbar\,\wp_n,\hbar\,\wp_{n+1})\}$, i.e., $\varkappa_n=\Gamma_{n-1}$ then (34) follows (35).

Now from (34), we obtain that $\Gamma_n\le\lambda^n\delta_0$ and hence,

$$\eth(\hbar\,\varepsilon_{n+1},\hbar\,\varepsilon_{n+2})\le\lambda^n\Gamma_0 \text{ and } \eth(\hbar\,\wp_{n+1},\hbar\,\wp_{n+2})\le\lambda^n\Gamma_0, \tag{36}$$

which shows that $\{\hbar\,\varepsilon_n\}$ and $\{\hbar\,\wp_n\}$ in $\mathscr{E}$ are Cauchy sequences by Lemma 3.1 of [20]. Therefore, we can conclude from Theorem 2.2 of [5] that, $\mathscr{I}$ and $\hbar$ have a coincidence point in $\mathscr{E}$. □

**Corollary 3.10** *Suppose that $(\mathscr{E},\eth,\preceq)$ is a complete partially ordered b-metric space. A continuous mapping $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ has a mixed monotone property and is satisfying the below contraction conditions for all $\varepsilon,\wp,\zeta,\Im\in\mathscr{E}$ such that $\varepsilon\preceq\zeta$ and $\wp\succeq\Im$, $k>2$, $s>1$, $\check{\psi}\in\hat{\Phi}$ and $\hat{\eta}\in\hat{\Psi}$:*

(i). $$\check{\psi}(s^k\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im)))\le\check{\psi}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im))-\hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)),$$

(ii). $$\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\zeta,\Im))\le\frac{1}{s^k}\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)-\frac{1}{s^k}\hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)),$$

where

*If there exists $(\varepsilon_0,\wp_0)\in\mathscr{E}\times\mathscr{E}$ such that $\varepsilon_0\preceq\mathscr{I}(\varepsilon_0,\wp_0)$ and $\wp_0\succeq\mathscr{I}(\wp_0,\varepsilon_0)$, then $\mathscr{I}$ has a coupled fixed point in $\mathscr{E}$.*

**Theorem 3.11** *The unique coupled common fixed point for $\mathscr{I}$ and $\hbar$ exists in Theorem 3.9, if for every $(\varepsilon,\wp),(\kappa,\ell)\in\mathscr{E}\times\mathscr{E}$ there exists some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$ such that $(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$ is comparable to $(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))$ and to $(\mathscr{I}(\kappa,\mathscr{I}),\mathscr{I}(\ell,\kappa))$.*

***Proof*** The existence of a coupled coincidence point for $\mathscr{I}$ and $\hbar$ is guaranteed by the Theorem 3.9. Let $(\varepsilon,\wp),(\kappa,\ell)\in\mathscr{E}\times\mathscr{E}$ are two coupled coincidence points of $\mathscr{I}$ and $\hbar$. Now, we assert that $\hbar\,\varepsilon=\hbar\,\kappa$ and $\hbar\,\wp=\hbar\,\ell$. By the hypotheses $(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$ is comparable to $(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))$ and to $(\mathscr{I}(\kappa,\mathscr{I}),\mathscr{I}(\ell,\kappa))$ for some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$.

Now, assume the following

$$(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\wp,\varepsilon))\le(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda))$$
$$\text{and } (\mathscr{I}(\kappa,\ell),\mathscr{I}(\ell,\kappa))\le(\mathscr{I}(\Lambda,\Upsilon),\mathscr{I}(\Upsilon,\Lambda)).$$

Suppose $\Lambda_0=\Lambda$ and $\Upsilon_0=\Upsilon$ then there is a point $(\Lambda_1,\Upsilon_1)\in\mathscr{E}\times\mathscr{E}$ such that

$$\hbar\,\Lambda_1=\mathscr{I}(\Lambda_0,\Upsilon_0),\ \ \hbar\,\Upsilon_1=\mathscr{I}(\Upsilon_0,\Lambda_0)\ \ (n\ge 1).$$

As by applying the preceding argument repeatedly, we have the sequences $\{\hbar\,\Lambda_n\}$ and $\{\hbar\,\Upsilon_n\}$ in $\mathscr{E}$ such that

$$\hbar\,\Lambda_{n+1}=\mathscr{I}(\Lambda_n,\Upsilon_n),\ \ \hbar\,\Upsilon_{n+1}=\mathscr{I}(\Upsilon_n,\Lambda_n)\ \ (n\ge 0).$$

Define the sequences in the same way $\{\hbar\,\varepsilon_n\}$, $\{\hbar\,\wp_n\}$ and, $\{\hbar\,\kappa_n\}$, $\{\hbar\,\ell_n\}$ in $\mathscr{E}$ by setting $\varepsilon_0=\varepsilon$, $\wp_0=\wp$ and $\kappa_0=\kappa$, $\ell_0=\ell$. Further, we have that

$$\hbar\,\varepsilon_n\to\mathscr{I}(\varepsilon,\wp),\ \ \hbar\,\wp_n\to\mathscr{I}(\wp,\varepsilon),\ \ \hbar\,\kappa_n\to\mathscr{I}(\kappa,\ell),\ \ \hbar\,\ell_n\to\mathscr{I}(\ell,\kappa)\ \ (n\ge 1). \tag{37}$$

Thus by induction, we get

$$(\hbar\,\varepsilon_n,\hbar\,\wp_n)\le(\hbar\,\Lambda_n,\hbar\,\Upsilon_n)\ \text{ for every } n. \tag{38}$$

As a consequence of (26), we have

Hence, we have

$$\check{\psi}(\eth(\hbar\,\varepsilon,\hbar\,\Lambda_{n+1})) \le \check{\psi}(s^k\eth(\hbar\,\varepsilon,\hbar\,\Lambda_{n+1})) = \check{\psi}(s^k\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\Lambda_n,\Upsilon_n))) \le \check{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)) - \hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)), \tag{39}$$

where

$$\begin{aligned}\mathscr{P}_h(\varepsilon,\wp,\Lambda_n,\Upsilon_n) &= \max\Big\{\frac{\eth(h\Lambda_n,\mathscr{J}\,\eth(\Lambda_n,\Upsilon_n))\left[1+\eth(h\varepsilon,\mathscr{J}\,\eth(\varepsilon,\wp))\right]}{1+\eth(h\varepsilon,h\Lambda_n)}, \frac{\eth(h\varepsilon,\mathscr{J}\,\eth(\Lambda_n,\Upsilon_n))+\eth(h\Lambda_n,\mathscr{J}\,\eth(\varepsilon,\wp)}{2s}, \\ &\quad \eth(h\varepsilon,\mathscr{J}\,\eth(\varepsilon,\wp)), \eth(h\Lambda_n,\mathscr{J}\,\eth(\Lambda_n,\Upsilon_n)), \eth(h\varepsilon,h\Lambda_n)\Big\} \\ &= \max\left\{0,\frac{\eth(h\varepsilon,h\Lambda_n)}{s},\eth(h\varepsilon,h\Lambda_n)\right\} = \eth(h\varepsilon,h\Lambda_n).\end{aligned}$$

Therefore from (39), we have

$$\check{\psi}(\eth(\hbar\,\varepsilon,\hbar\,\Lambda_{n+1})) \le \check{\psi}(\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n)) - \hat{\eta}(\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n)). \tag{40}$$

As by the similar argument, we acquire that

$$\check{\psi}(\eth(\hbar\,\wp,\hbar\,\Upsilon_{n+1})) \le \check{\psi}(\eth(\hbar\,\wp,\hbar\,\Upsilon_n)) - \hat{\eta}(\eth(\hbar\,\wp,\hbar\,\Upsilon_n)). \tag{41}$$

Hence from (40) and (41), we have

$$\begin{aligned}\check{\psi}(\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_{n+1}),\eth(\hbar\,\wp,\hbar\,\Upsilon_{n+1})\}) &\le \check{\psi}(\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\}) \\ &\quad - \hat{\eta}(\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\}) \\ &< \check{\psi}(\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\}).\end{aligned} \tag{42}$$

Thus the property of $\check{\psi}$ implies,

$$\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_{n+1}),\eth(\hbar\,\wp,\hbar\,\Upsilon_{n+1})\} < \max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\}.$$

Hence, $\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\}$ is a decreasing sequence of positive reals and bounded below and by a result, we have

$$\lim_{n\to+\infty}\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\} = \Gamma,\ \Gamma \ge 0.$$

Therefore as $n \to +\infty$ in equation (42), we get

$$\check{\psi}(\Gamma) \le \check{\psi}(\Gamma) - \hat{\eta}(\Gamma), \tag{43}$$

from which we get $\hat{\eta}(\Gamma) = 0$, this implies that $\Gamma = 0$. Therefore,

$$\lim_{n\to+\infty}\max\{\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n),\eth(\hbar\,\wp,\hbar\,\Upsilon_n)\} = 0.$$

$$\lim_{n\to+\infty}\eth(\hbar\,\varepsilon,\hbar\,\Lambda_n) = 0 \text{ and } \lim_{n\to+\infty}\eth(\hbar\,\wp,\hbar\,\Upsilon_n) = 0. \tag{44}$$

From the similar argument as above, we obtain that

$$\lim_{n\to+\infty}\eth(\hbar\,\Bbbk,\hbar\,\Lambda_n) = 0 \text{ and } \lim_{n\to+\infty}\eth(\hbar\,\mathscr{J},\hbar\,\Upsilon_n) = 0. \tag{45}$$

Therefore from (44) and (45), we get $\hbar\,\varepsilon = \hbar\,\Bbbk$ and $\hbar\,\wp = \hbar\,\mathscr{J}$. Since $\hbar\,\varepsilon = \mathscr{J}(\varepsilon,\wp)$ and $\hbar\,\wp = \mathscr{J}(\wp,\varepsilon)$ and, the commutative property of $\mathscr{J}$ and $\hbar$ implies that

$$\hbar\,(\hbar\,\varepsilon) = \hbar\,(\mathscr{J}(\varepsilon,\wp)) = \mathscr{J}(\hbar\,\varepsilon,\hbar\,\wp) \text{ and } \hbar\,(\hbar\,\wp) = \hbar\,(\mathscr{J}(\wp,\varepsilon)) = \mathscr{J}(\hbar\,\wp,\hbar\,\varepsilon). \tag{46}$$

If $\hbar\,\varepsilon = \Lambda^*$ and $\hbar\,\wp = \Upsilon^*$, then from (46), we get

$$\hbar\,(\Lambda) = \mathscr{J}(\Lambda^*,\Upsilon^*) \text{ and } \hbar\,(\Upsilon^*) = \mathscr{J}(\Upsilon^*,\Lambda^*), \tag{47}$$

which exhibits that $(\Lambda^*,\Upsilon^*)$ is a coupled coincidence point of $\mathscr{J}$, $\hbar$. Hence, $\hbar\,(\Lambda^*) = \hbar\,\Bbbk$ and $\hbar\,(\Upsilon^*) = \hbar\,\mathscr{J}$ which in turn gives that $\hbar\,(\Lambda) = \Lambda^*$ and $\hbar\,(\Upsilon^*) = \Upsilon^*$. Therefore from (47), $(\Lambda^*,\Upsilon^*)$ is a coupled common fixed point of $\mathscr{J}$, $\hbar$.

Let $(\Lambda_1^*,\Upsilon_1^*)$ be another coupled common fixed point of $\mathscr{J}$, $\hbar$. Then, $\Lambda_1^* = \hbar\,\Lambda_1^* = \mathscr{J}(\Lambda_1^*,\Upsilon_1^*)$ and $\Upsilon_1^* = \hbar\,\Upsilon_1^* = \mathscr{J}(\Upsilon_1^*,\Lambda_1^*)$. But $(\Lambda_1^*,\Upsilon_1^*)$ is a coupled common fixed point of $\mathscr{J}$ and $\hbar$ then, $\hbar\,\Lambda_1^* = \hbar\,\varepsilon = \Lambda$ and $\hbar\,\Upsilon_1^* = \hbar\,\wp = \Upsilon^*$. Therefore, $\Lambda_1^* = \hbar\,\Lambda_1^* = \hbar\,\Lambda = \Lambda$ and $\Upsilon_1^* = \hbar\,\Upsilon_1^* = \hbar\,\Upsilon^* = \Upsilon^*$. Hence the uniqueness. □

**Theorem 3.12** *In Theorem 3.11, if $\hslash\,\varepsilon_0 \preceq \hslash\,\wp_0$ or $\hslash\,\varepsilon_0 \succeq \hslash\,\wp_0$, then a unique common fixed point of $\mathscr{I}$ and $\hslash$ can be found.*

***Proof*** Assume that $(\varepsilon, \wp) \in \mathscr{E}$ is a unique coupled common fixed point of $\mathscr{I}$ and $\hslash$. Then to demonstrate that $\varepsilon = \wp$. Suppose that $\hslash\,\varepsilon_0 \preceq \hslash\,\wp_0$, then we get by induction that, $\hslash\,\varepsilon_n \preceq \hslash\,\wp_n$ for $n \geq 0$. From Lemma 2 of [21], we have

$$\begin{aligned}
\check{\psi}(s^{k-2}\eth(\varepsilon,\wp)) &= \check{\psi}(s^k \frac{1}{s^2}\eth(\varepsilon,\wp)) \leq \lim_{n\to+\infty} \sup \check{\psi}(s^k \eth(\varepsilon_{n+1},\wp_{n+1})) \\
&= \lim_{n\to+\infty} \sup \check{\psi}(s^k \eth(\mathscr{I}(\varepsilon_n,\wp_n),\mathscr{I}(\wp_n,\varepsilon_n))) \\
&\leq \lim_{n\to+\infty} \sup \check{\psi}(\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) - \lim_{n\to+\infty} \inf \hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) \\
&\leq \check{\psi}(\eth(\varepsilon,\wp)) - \lim_{n\to+\infty} \inf \hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) \\
&< \check{\psi}(\eth(\varepsilon,\wp)),
\end{aligned}$$

a contradiction. Hence, $\varepsilon = \wp$.

The result can also be similar in the case of $\hslash\,\varepsilon_0 \succeq \hslash\,\wp_0$. □

***Remark 3.13*** While $s = 1$ and the result of [19], the condition

$$\check{\psi}(\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\eth,\Im))) \leq \check{\psi}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\eth),\eth(\hslash\,\wp,\hslash\,\Im)\}) - \hat{\eta}(\max\{\eth(\hslash\,\varepsilon,\hslash\,\eth),\eth(\hslash\,\wp,\hslash\,\Im)\})$$

is equivalent to,

$$\eth(\mathscr{I}(\varepsilon,\wp),\mathscr{I}(\eth,\Im)) \leq \varphi(\max\{\eth(\hslash\,\varepsilon,\hslash\,\eth),\eth(\hslash\,\wp,\hslash\,\Im)\}),$$

where $\check{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ and $\varphi$ is a continuous self mapping on $[0,+\infty)$ with $\varphi(y) < y$ for every $y > 0$ with $\varphi(y) = 0$ if and only if $y = 0$. Hence the results found here are generalized and extended the results of [11, 18, 22, 25, 27] and several comparable results.

Now depending on the type of a metric, some examples are shown here under.

***Example 3.14*** Let $\mathscr{E} = \{e_1,e_2,e_3,e_4,e_5,e_6\}$ and $\eth : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ be a metric defined by

$$\begin{aligned}
&(\varepsilon,\wp) = (\wp,\varepsilon) = 0,\ \text{if } \varepsilon = \wp = \{e_1,e_2,e_3,e_4,e_5,e_6\} \\
&\quad \text{and } \varepsilon = \wp,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 3,\ \textit{if}\ \varepsilon = \wp = \{e_1,e_2,e_3,e_4,e_5\} \\
&\quad \text{and } \varepsilon \neq \wp,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 12,\ \textit{if}\ \varepsilon = \{e_1,e_2,e_3,e_4\} \\
&\quad \text{and } \wp = e_6,\ (\varepsilon,\wp) = (\wp,\varepsilon) = 20,\ \textit{if}\ \varepsilon = e_5 \text{ and } \wp = e_6,\ \text{with usual order} \leq .
\end{aligned}$$

A self-mapping $\mathscr{I}$ on $\mathscr{E}$ defined by $\mathscr{I}e_1 = \mathscr{I}e_2 = \mathscr{I}e_3 = \mathscr{I}e_4 = \mathscr{I}e_5 = 1, \mathscr{I}e_6 = 2$ has a fixed point with $\check{\psi}(y) = \frac{y}{2}$ and $\hat{\eta}(y) = \frac{y}{4}$ where $y \in [0,+\infty)$.

***Proof*** When $s = 2$, $(\mathscr{E},\eth,\leq)$ is a complete partially ordered $b$-metric space. Let $\varepsilon,\wp \in \mathscr{E}$ such that $\varepsilon < \wp$ then we'll look at the cases below.

**Case 1.** If $\varepsilon,\wp \in \{e_1,e_2,e_3,e_4,e_5\}$ then $\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp) = \eth(e_1,e_1) = 0$. Hence,

$$\check{\psi}(2\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) = 0 \leq \check{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

**Case 2.** If $\varepsilon \in \{e_1,e_2,e_3,e_4,e_5\}$ and $\wp = e_6$, then $\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp) = \eth(e_1,e_2) = 3$, $\mathscr{P}(e_6,e_5) = 20$ and $\mathscr{P}(\varepsilon,e_6) = 12$, for $\varepsilon \in \{e_1,e_2,e_3,e_4\}$. Hence,

$$\check{\psi}(2\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)) \leq \frac{\mathscr{P}(\varepsilon,\wp)}{4} = \check{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

As a result, all of the conditions of Theorem 3.1 are met, and hence $\mathscr{I}$ has a fixed point. □

***Example 3.15*** Let us define a metric $\eth$ with usual order $\leq$ by

$$\eth(\varepsilon,\wp) = \begin{cases} 0 & ,\ \textit{if}\ \varepsilon = \wp \\ 1 & ,\ \textit{if}\ \varepsilon \neq \wp \in \{0,1\} \\ |\varepsilon - \wp| & ,\ \textit{if}\ \varepsilon,\wp \in \{0,\frac{1}{2n},\frac{1}{2m} : n \neq m \geq 1\} \\ 6 & ,\ \textit{otherwise.} \end{cases}$$

where $\mathscr{E} = \{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots,\frac{1}{n},\ldots\}$. A self-mapping $\mathscr{I}$ on $\mathscr{E}$ by $\mathscr{I}0 = 0, \mathscr{I}\frac{1}{n} = \frac{1}{12n}(n \geq 1)$ has a fixed point with $\check{\psi}(y) = y$ and $\hat{\eta}(y) = \frac{4y}{5}$ for $y \in [0,+\infty)$.

***Proof*** $\eth$ is clearly discontinuous, and $(\mathscr{E}, \eth, \preceq)$ is a complete partially ordered $b$-metric space for $s = \frac{12}{5}$. Now we'll look at the following cases for $\varepsilon, \wp \in \mathscr{E}$ with $\varepsilon < \wp$.

**Case 1**. Suppose $\varepsilon = 0$ and $\wp = \frac{1}{n}$ $(n > 0)$, then $\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(0, \frac{1}{12n}) = \frac{1}{12n}$ and $\mathscr{P}(\varepsilon, \wp) = \frac{1}{n}$ and $\mathscr{P}(\varepsilon, \wp) = \{1, 6\}$. Thus,

$$\check{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5} = \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

**Case 2**. Let $\varepsilon = \frac{1}{m}$ and $\wp = \frac{1}{n}$ where $m > n \geq 1$, then

$$\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(\frac{1}{12m}, \frac{1}{12n}), \mathscr{P}(\varepsilon, \wp) \geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathscr{P}(\varepsilon, \wp) = 6.$$

Thus,

$$\check{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5} = \check{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

Hence, we have the conclusion from Theorem 3.1 as all assumptions are fulfilled. □

***Example 3.16*** Define a metric $d : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$, where $\mathscr{E} = \{\tilde{\ell}/\tilde{\ell} : [a_1, a_2] \to [a_1, a_2] \text{ is continuous}\}$ by

$$\eth(\tilde{\ell}_1, \tilde{\ell}_2) = \sup_{y \in [a_1, a_2]} \{|\tilde{\ell}_1(y) - \tilde{\ell}_2(y)|^2\}$$

for any $\tilde{\ell}_1, \tilde{\ell}_2 \in \mathscr{E}$, $0 \leq a_1 < a_2$ with $\tilde{\ell}_1 \preceq \tilde{\ell}_2$ implies $a_1 \leq \tilde{\ell}_1(y) \leq \tilde{\ell}_2(y) \leq a_2, y \in [a_1, a_2]$. A self-mapping $\mathscr{I}$ on $\mathscr{E}$ defined by $\mathscr{I}\tilde{\ell} = \frac{\tilde{\ell}}{5}, \tilde{\ell} \in \mathscr{E}$ has a unique fixed point with $\check{\psi}(y) = y$ and $\hat{\eta}(y) = \frac{y}{3}$ for any $y \in [0, +\infty]$.

***Proof*** As $\min(\tilde{\ell}_1, \tilde{\ell}_2)(y) = \min\{\tilde{\ell}_1(y), \tilde{\ell}_2(y)\}$ is continuous and all other assumptions of Theorem 3.3 are fulfilled for $s = 2$. Hence, $0 \in \mathscr{E}$ is a unique fixed point of $\mathscr{I}$. □

## Limitations

We examined a fixed point, a coincidence point and a couple coincidence point for mappings that are satisfying generalized $(\check{\psi}, \hat{\eta})$-weak contractions in a partially ordered $b$-metric space. The findings in this paper are generalized and extended a few well-known results in the current literature. Some examples are shown at the end to support the results obtained here.

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