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Conserved quantities, optimal system and explicit solutions of a (1 + 1)-dimensional generalised coupled mKdV-type system

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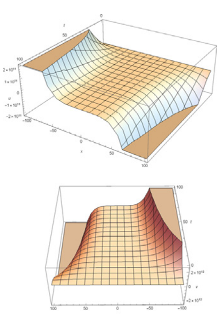
GRAPHICAL ABSTRACT



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ABSTRACT

Introduction: The purpose of this paper is to study, a (1 + 1)-dimensional generalised coupled modified Korteweg-de Vries-type system from Lie group analysis point of view. This system is studied in the literature for the first time. The authors found this system to be interesting since it is non-decouplable and possesses higher generalised symmetries.
Objectives: We look for the closed-form solutions and conservation laws of the system.
Methods: Optimal system of one-dimensional subalgebras for the system was obtained and then used to perform symmetry reductions and construct group invariant solutions. Power series solutions for the system were also obtained. The system has no variational principle and as such, we employed the multiplier method and used a homotopy integral formula to derive the conserved quantities.
Results: Group invariant solutions and power series solutions were constructed and three conserved vectors for the system were derived.
Conclusion: The paper studies the (1 + 1)-dimensional generalised coupled modified Korteweg-de Vries-type system for the first time and constructs its exact solutions and conservation laws.

Introduction

Nonlinear partial differential equations (NLPDEs) have rapidly become indispensable in the quest to conceptualise the world around us [1–50]. We give a few recent studies of NLPDEs presented in the literature. For instance, the numerical treatments

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to a complex order fractional nonlinear one-dimensional problem of Burgers equations was discussed in [1]. Computation of solutions to fractional order partial reaction diffusion equations was presented in [2]. Kadomtsev–Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation was investigated in [3] and exact solutions were constructed. In [4], the (2 + 1)-dimensional B-type Kadomtsev–Petviashvili equation of fluid mechanics was studied and soliton molecules and some novel interaction solutions were discussed. The (2 + 1)-dimensional modified dispersive water-wave system was considered in [5] and variable separation solutions were obtained. The authors of [6] examined the KP-BBM equation and constructed periodic, multi wave, cross-kink wave and breather wave solutions. The (4 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation was studied using the Hirota bilinear form and rational wave solutions were obtained in [7]. Lie symmetry analysis was carried out on a generalized (2 + 1)-dimensional KP equation [8] and in [9], the (2 + 1)-dimensional dispersive long wave equation was investigated via the truncated Painlevé series method. A new method was introduced in [10] to find exact solutions for NLPDEs of mathematical physics. Symmetry analysis of the Kelvin-Voigt viscoelasticity equation and a generalized (2 + 1)-dimensional double dispersion equation was discussed in [11,12].

Thus, due to the fact that several physical phenomena of the real world are modelled by NLPDEs, it is of immense importance that their exact solutions are investigated. It is in this spirit that many methods have been developed over the years to obtain closed-form solutions of NLPDEs and furthermore, that their conserved quantities are established. Lie's (1842–99) work stands out amongst the sea of literature and indeed forms the basis for the works of many brilliant modern-day scholars. Lie group analysis [13–17] is a revolutionary symmetry-based method for systematically solving differential equations. Although discovered in the late 19th century it got its popularity during the middle of the 20th century because of the availability of computer software. Other principal methods for obtaining soliton and periodic solutions of NLPDEs have been developed over the years. These include the Riccati-Bernoulli sub-ODE method [18], the homogeneous balance of undetermined coefficients method [19–22], the first integral method [23], the bifurcation technique [24], the generalised unified method [25], the multiple exp-function method [26], dynamical system approach[27,28], simplified Hirota's method [29,30], the ((G′/G))-expansion function method [31], Kudryashov's method [32], Jacobi elliptic function expansion technique [33], and the power series technique, [34,36,37].

The celebrated Noether's theorem [38,39] for determining conserved currents for systems of PDEs with variational principle is a novel idea and has been delved upon by numerous renowned scholars. However, the method of obtaining conserved currents by enlisting Noether's theorem comes with an intractable limitation, that of requiring the PDE or system of PDEs to have a variational principle. A great deal of useful mathematical models of natural phenomena do not have a variational principle. It is against this backdrop that in recent times astute mathematicians have sought a generalisation of Noether's theorem with the intent of incorporating PDEs with or without a variational principle [40–51]. One such generalisation is the aptly named multiplier approach [15]. See also [49]. In this work, we use multipliers along with a homotopy integral formula to obtain the local non-trivial conserved quantities of a non-decouplable system of NLPDEs.

In [52], Foursov performed a classification of coupled potential KdV-type and modified KdV-type equations that possess higher symmetries, and eleven new systems were presented for which Hamiltonian and bi-Hamiltonian formulations were provided for some of the equations. His work focused on coupled and symmetric systems of type $(u_t = F[u, v])$ and $(v_t = F[v, u])$. It is against this backdrop that the following previously unknown coupled system was obtained in [52]:

$$\begin{aligned} &u_t - u^2 v_x - 3uvu_x - u_{xxx} = 0,\\ &v_t - v^2 u_x - 3uvv_x - v_{xxx} = 0. \end{aligned} \tag{1}$$

In this work, however, we investigate the generalised coupled mKdV-type system

$$\begin{aligned} &E_1 \equiv u_t - u^2 v_x - \alpha uvu_x - u_{xxx} = 0,\\ &E_2 \equiv v_t - v^2 u_x - \alpha uvv_x - v_{xxx} = 0 \end{aligned} \tag{2}$$

with (α) a constant. System (1) is interesting in that it is non-decouplable and possesses higher generalised symmetries. To the best of our knowledge, the above system (2) will be studied extensively for the first time in this paper and the results are therefore new. With the application of Lie symmetry analysis we seek to derive conserved quantities and exact solutions for the system (2). For this we shall utilise multiplier approach via the homotopy integral and Lie group analysis along with power series solution method.

Conserved currents

For the generalised coupled mKdV-type system (2) with multipliers $\left(\Lambda^1\right)$ and $\left(\Lambda^2\right)$ we determine the corresponding conserved quantities by using the homotopy integral method, but first we find the multipliers.

To compute all multipliers of (2), we invoke the determining condition

$$\frac{\delta}{\delta u}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0,\ \frac{\delta}{\delta v}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0. \tag{3}$$

Here the operators $(\delta/\delta u)$ and $(\delta/\delta v)$ are the well known Euler operators specified as

$$\begin{aligned} &\tfrac{\delta}{\delta u} = \tfrac{\partial}{\partial u} - D_t \tfrac{\partial}{\partial u_t} - D_x \tfrac{\partial}{\partial u_x} + D_x^2 \tfrac{\partial}{\partial u_{xx}} - D_x^3 \tfrac{\partial}{\partial u_{xxx}},\\ &\tfrac{\delta}{\delta v} = \tfrac{\partial}{\partial v} - D_t \tfrac{\partial}{\partial v_t} - D_x \tfrac{\partial}{\partial v_x} + D_x^2 \tfrac{\partial}{\partial v_{xx}} - D_x^3 \tfrac{\partial}{\partial v_{xxx}} \end{aligned} \tag{4}$$

with (D_t) and (D_x) being the total derivatives given by

$$\begin{aligned} &D_t = \tfrac{\partial}{\partial t} + u_t \tfrac{\partial}{\partial u} + v_t \tfrac{\partial}{\partial v} + u_{tt} \tfrac{\partial}{\partial u_t} + v_{tt} \tfrac{\partial}{\partial v_t} + u_{tx} \tfrac{\partial}{\partial u_x} + v_{tx} \tfrac{\partial}{\partial v_x} + \cdots,\\ &D_x = \tfrac{\partial}{\partial x} + u_x \tfrac{\partial}{\partial u} + v_x \tfrac{\partial}{\partial v} + u_{xx} \tfrac{\partial}{\partial u_x} + v_{xx} \tfrac{\partial}{\partial v_x} + u_{xt} \tfrac{\partial}{\partial u_t} + v_{xt} \tfrac{\partial}{\partial v_t} + \cdots. \end{aligned}$$

Here we seek to compute second order multipliers

$$[\Lambda^m = \Lambda^m(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}).\ \ m = 1, 2.]$$

Expanding (3) and comparing powers of derivatives of $(u), (v)$, we acquire a system of thirty-six PDEs. The solution to this system of PDEs is akin to the algorithm for obtaining Lie point symmetries. Solving this system of PDEs gives

$$\begin{aligned} \Lambda^1 = \ & C_1\left[\{(\alpha+1)tuv^2 + xv + 3tv_{xx}\}\right]\\ & + \tfrac{1}{\alpha+1}[C_2\{(\alpha+1)uv^2 + 3v_{xx}\} + C_3(\alpha+1)v],\ \alpha \neq -1,\\ \Lambda^2 = \ & C_1\left[xu + \{(\alpha+1)vu^2 + 3u_{xx}\}t\right]\\ & + \tfrac{1}{\alpha+1}[C_2\{((\alpha+1)u^2 v + 3u_{xx})\}] + C_3 u,\ \alpha \neq -1 \end{aligned} \tag{5}$$

with $(C_1, C_2\,\text{and}\,C_3)$ constants. Thus, we obtain three multipliers given by

$$\begin{aligned} \Lambda_1^1 = \ & (\alpha+1)tuv^2 + xv + 3tv_{xx},\ \Lambda_1^2 = xu + ((\alpha+1)vu^2 + 3u_{xx})t;\\ \Lambda_2^1 = \ & \tfrac{1}{\alpha+1}\{(\alpha+1)uv^2 + 3v_{xx}\},\ \Lambda_2^2 = \tfrac{1}{\alpha+1}\{(\alpha+1)u^2 v + 3u_{xx}\};\\ \Lambda_3^1 = \ & v,\ \Lambda_3^2 = u \end{aligned} \tag{6}$$

From these three sets of multipliers, conserved densities $\left(T^i\right)$ and spatial fluxes $\left(X^i\right)$ for $(i = 1, 2, 3)$ [49] can be obtained in several ways. In this work, we use the homotopy integral:

$$\begin{aligned} T = & \int_0^1 \left(\partial_\lambda u_\lambda \sum_{l=1}^{k} (-D)^{l-1} \cdot \left(\frac{\partial E\Lambda}{\partial(\partial^{l-1}\partial_t u)}\right)|_{u=u_{(\lambda)}} \right. \\ & + \partial_\lambda \partial u_\lambda \sum_{l=2}^{k} (-D)^{l-2} \cdot \left(\frac{\partial E\Lambda}{\partial(\partial^{l-1}\partial_t u)}\right)|_{u=u_{(\lambda)}} \\ & \left. + \cdots + \partial_\lambda \partial^{k-1} u_\lambda \left(\frac{\partial E\Lambda}{\partial(\partial^{k-1}\partial_t u)}\right)|_{u=u_{(\lambda)}} \right) \mathrm{d}\lambda + D_x \cdot \Theta \\ X = & \int_0^1 \left(\partial_\lambda u_\lambda \sum_{l=1}^{k} (-D)^{l-1} \cdot \left(\frac{\partial E\Lambda}{\partial(\partial^{l-1}\partial_x u)}\right)|_{u=u_{(\lambda)}} \right. \\ & + \partial_\lambda \partial u_\lambda \sum_{l=2}^{k} (-D)^{l-2} \cdot \left(\frac{\partial E\Lambda}{\partial(\partial^{l-1}\partial_x u)}\right)|_{u=u_{(\lambda)}} \\ & \left. + \cdots + \partial_\lambda \partial^{k-1} u_\lambda \left(\frac{\partial E\Lambda}{\partial(\partial^{k-1}\partial_x u)}\right)|_{u=u_{(\lambda)}} \right) \mathrm{d}\lambda - D_t \cdot \Theta + D_x \cdot \Gamma, \end{aligned} \tag{7}$$

where (k) is the differential order of system (E), (Θ) is some vector function and (Γ) is an antisymmetric tensor. The homotopy integral (7) simplifies to [49]

$$\Phi = \int_0^1 \sum_{j=1}^{k} \partial_\lambda \partial^{j-1} u^m_{(\lambda)} \left(\sum_{l=j}^{k} (-D)^{l-j} \cdot \left(\frac{\partial E_m \Lambda^m}{\partial u} \right) |_{u^m = u^m_{(\lambda)}} \right) \mathrm{d}\lambda,$$

where $(m = 1, \cdots, n)$, and (n) is the number of dependent variables. Also, $(\Phi = (T, X))$ is a conserved quantity composed of conserved density (T) and spatial flux (X). Thus, for (2) and (5), we have

$$\begin{aligned} T = & \int_0^1 \left(u \left(\frac{\partial E_1 \Lambda_1}{\partial u_t}\right)|_{u=u_{(\lambda)}} + v \frac{\partial E_2 \Lambda_2}{\partial v_t}|_{v=v_{(\lambda)}} \right) \mathrm{d}\lambda, \\ X = & \int_0^1 \left[u \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial u_x}\right)|_{u=u_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxx}}\right)|_{u=u_{(\lambda)}} \right\} \right. \\ & + u_x \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} \right\} + u_{xx} \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxx}}\right)|_{u=u_{(\lambda)}} \\ & + u \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial u_x}\right)|_{u=u_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xxx}}\right)|_{u=u_{(\lambda)}} \right\} \\ & + u_x \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xx}}\right)|_{u=u_{(\lambda)}} \right\} + u_{xx} \left(\frac{\partial E_2 \Lambda_2}{\partial u_{xxx}}\right)|_{u=u_{(\lambda)}} \\ & + v \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial v_x}\right)|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xxx}}\right)|_{v=v_{(\lambda)}} \right\} \\ & + v_x \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} \right\} + v_{xx} \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xxx}}\right)|_{v=v_{(\lambda)}} \\ & v \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial v_x}\right)|_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xxx}}\right)|_{v=v_{(\lambda)}} \right\} \\ & \left. + v_x \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} + D_x^2 \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}}\right)|_{v=v_{(\lambda)}} \right\} + v_{xx} \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xxx}}\right)|_{v=v_{(\lambda)}} \right] \mathrm{d}\lambda. \end{aligned} \tag{8}$$

Choosing the homotopy $(u_{(\lambda)} = \lambda u)$, for $\left(\Lambda_1^1\right)$ and $\left(\Lambda_1^2\right)$ we have

$$\begin{aligned} T^1 &= \int_0^1 (2u^2\lambda^3 t v^2 \alpha + 2u^2\lambda^3 v^2 t + 2\lambda u v x + 3\lambda u v_{xx} t + 3 t v \lambda u_{xx}) \mathrm{d}\lambda \\ &= \tfrac{1}{2}\alpha t v^2 u^2 + \tfrac{1}{2} t v^2 u^2 + x v u + \tfrac{3}{2} t u v_{xx} + \tfrac{3}{2} t v u_{xx} \end{aligned}$$

and

$$\begin{aligned} X^1 = \int_0^1 \Big(& -v_x\lambda u - \lambda v u_x - 4\alpha\lambda^3 t v u v_x u_x - 2\alpha x u^2 \lambda^3 v^2 - 2\lambda^5\alpha^2 t v^3 u^3 - 4\lambda^5 \alpha t u^3 v^3 \\ & -6\lambda t v_{xx} u_{xx} - 2\lambda x u_{xx} v - 3\lambda t v u_{xt} - 2\lambda^5 t u^3 v^3 - 3\lambda t u v_{tx} - 2\lambda u x v_{xx} - 4\lambda^3 t v^2 u_x^2 \\ & -4\lambda^3 t v_x^2 u^2 + 2\lambda x v_x u_x + 3\lambda t u_x v_t + 3\lambda t u_t v_x + 2\lambda^3 \alpha t u^2 v_x^2 - 4 v^2 t \lambda^3 u u_{xx} \\ & +2\alpha\lambda^3 t v^2 u_x^2 - 4 v_{xx} t \lambda^3 v (u)^2 - 2u^2\lambda^3 v^2 x + 8 v\lambda^3 u v_x u_x t - 4 v^2 \alpha \lambda^3 t u u_{xx} \\ & -4\lambda^3 \alpha t v u^2 v_{xx} \Big) \mathrm{d}\lambda \\ = & -\tfrac{1}{2}\alpha x u^2 v^2 + \tfrac{1}{2}\alpha t u_x^2 v^2 - \tfrac{1}{3}\alpha^2 t u^3 v^3 + \tfrac{1}{2}\alpha t u^2 v_x^2 - \tfrac{2}{3}\alpha t \alpha v^3 u^3 - t u_{xx} u v^2 \\ & - t v u^2 v_{xx} - \alpha t u u_x v v_x - \alpha t u u_{xx} v^2 + 2 t u u_x v v_x - \alpha t u^2 v v_{xx} - \tfrac{1}{2} v u_x - \tfrac{1}{2} u v_x \\ & - \tfrac{3}{2} t v u_{tx} - t u_x^2 v^2 - \tfrac{1}{2} x u^2 v^2 - \tfrac{1}{3} t u^3 v^3 + \tfrac{3}{2} t u_x v_t - x u v_{xx} + x u_x v_x - x v u_{xx} \\ & - 3 t u_{xx} v_{xx} + \tfrac{3}{2} t u_t v_x - t u^2 v_x^2 - \tfrac{3}{2} t u v_{tx}. \end{aligned}$$

In a similar manner, we obtain two more conserved quantities of (2) associated with the two multipliers $\left((\Lambda_2^1, \Lambda_2^2)\right)$ and $\left((\Lambda_3^1, \Lambda_3^2)\right)$. These are

$$\begin{aligned} T^2 &= \tfrac{1}{2(\alpha+1)} \{\alpha u^2 v^2 + v^2 u^2 + 3 u v_{xx} + 3 u_{xx} v\}, \\ X^2 &= -\tfrac{1}{6(\alpha+1)} \{2\alpha^2 u^3 v^3 + 4\alpha u^3 v^3 + 2u^3 v^3 + 6\alpha u^2 v v_{xx} - 3\alpha u^2 v_x^2 + 6\alpha u u_x v v_x \\ &\quad + 6\alpha u u_{xx} v^2 - 3\alpha u_x^2 v^2 + 6 v_{xx} v u^2 + 6 u^2 v_x^2 - 12 u u_x v v_x + 6 u u_{xx} v^2 + 6 u_x^2 v^2 \\ &\quad + 9 u v_{tx} - 9 u_t v_x + 9 u_{tx} v - 9 u_x v_t + 18 u_{xx} v_{xx}\}; \end{aligned}$$

$$\begin{aligned} T^3 &= u v, \\ X^3 &= v_x u_x - u v_{xx} - u_{xx} v - \tfrac{1}{2}(\alpha+1) u^2 v^2. \end{aligned}$$

Analytical solutions of (2)

Here we utilise point symmetries of the generalised coupled mKdV-type system (2) and build up an optimal system of one-dimensional subalgebras in order to achieve symmetry reductions as well as symmetry invariant solutions.

Optimal system of one-dimensional subalgebras for (2)

The symmetry group of (2) can be obtained by using MAPLE and it consists of

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t},\ X_3 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},\ X_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u},$$

which are space and time translations and scaling symmetries, respectively. We now seek to exploit one elementary facet of Lie algebras, that is, the bilinear product property. The commutators of these Lie symmetries are tabulated in Table 1, where the entry at the intersection of (i) th row with $(j)th$ column is a reckoning of Lie bracket $([X_i, X_j])$ [15,16].

We now enlist Lie series [15]

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\mathrm{ad}\, X_i)^n (X_j),$$

together with bilinear products in Table 1 to obtain adjoint representation. The results are tabulated in Table 2 below.

With the aid of Table 1 and Table 2 and by prudently applying adjoint maps, we see that the optimal system of 1-dimensional subalgebras is spanned by

$$\begin{aligned} & X_1 = \tfrac{\partial}{\partial x}, \\ & X_2 = \tfrac{\partial}{\partial t}, \\ & X_1 + X_2 = \tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial t}, \\ & X_2 - X_1 = \tfrac{\partial}{\partial t} - \tfrac{\partial}{\partial x} \\ & cX_3 + X_4 = x\tfrac{\partial}{\partial x} + 3t\tfrac{\partial}{\partial t} + (c-2)u\tfrac{\partial}{\partial u} - cv\tfrac{\partial}{\partial v}, \\ & aX_1 + bX_2 + X_3 = a\tfrac{\partial}{\partial x} + b\tfrac{\partial}{\partial x} + u\tfrac{\partial}{\partial u} - v\tfrac{\partial}{\partial v} \end{aligned}$$

with (a,b,c) constants. The symmetry $((x, t, u, v))\ (\mapsto)((-x, t, u, v))$ is discrete and maps $(X_2 - X_1)$ onto $(X_1 + X_2)$, thus reducing our optimal system to five inequivalent subgroups, viz.,

$$\{X_1, X_2, X_1 + X_2, cX_3 + X_4, aX_1 + bX_2 + cX_3\}. \tag{9}$$

Symmetry reductions and explicit solutions of (2)

Table 1
Commutation relations of 4-dimensional Lie algebra of (2).

[,]	(X_1)	(X_2)	(X_3)	(X_4)
(X_1)	(0)	(0)	(0)	(X_1)
(X_2)	(0)	(0)	(0)	$(3X_2)$
(X_3)	(0)	(0)	(0)	(0)
(X_4)	$(-X_1)$	$(-3X_2)$	(0)	(0)

Table 2
Adjoint table of Lie algebra of (2).

Ad	(X_1)	(X_2)	(X_3)	(X_4)
(X_1)	(X_1)	(X_2)	(X_3)	$(X_4\text{-}\varepsilon X_1)$
(X_2)	(X_1)	(X_2)	(X_3)	$(X_4\text{-}3\varepsilon X_2)$
(X_3)	(X_1)	(X_2)	(X_3)	(X_4)
(X_4)	$(e^{\varepsilon}X_1)$	$(e^{3\varepsilon}X_2)$	(X_3)	(X_4)

We now present symmetry reductions and some explicit solutions of (2) according to the optimal system (9).

Symmetry reductions

We proceed to compute the invariant solutions of each of the five cases presented in (9) and utilise firstly to transform the NLPDE system (2) into several systems of nonlinear ordinary differential equations (NLODEs). Furthermore, we present the solutions for (2).

Case 1. (X_1)

For vector field (X_1), we obtain the invariant $(\xi = x)$ and the group-invariant solutions $(u(t,x) = U(\xi))$ and $(v(t,x) = V(\xi))$. Substitution of these values of $(u), (v)$ in (2) gives the NLODEs

$$\begin{aligned} &U''' + 3UVU' + V'U^2 = 0,\\ &V''' + 3VUV' + V^2U' = 0. \end{aligned} \tag{10}$$

Case 2. (X_2)

Without giving detail, it can be readily seen that this case leads to the obvious constant solutions

$$U = k_1,\ \ V = k_2 \tag{11}$$

with (k_1, k_2) constants.

Case 3. $(X_1 + X_2)$

The third member of (9), namely $(X_1 + X_2)$ provides us with the invariant $(\xi = x - t)$ and group-invariant solutions $(u = U(\xi))$ and $(v = V(\xi))$. Consequently, $(U), (V)$ satisfy

$$\begin{aligned} &U''' + \alpha UU'V + V'U^2 + U' = 0,\\ &V''' + \alpha VUV' + U'V^2 + V' = 0. \end{aligned} \tag{12}$$

Case 4. $(cX_3 + X_4)$

In this case, the invariants $\left(\xi = xt^{1/3}\right), \left(u = t^{(c-2)/3}U(\xi)\right)$ and $\left(v = t^{c/3}V(\xi)\right)$ are apparent, with functions $(U), (V)$ conforming to the system

$$\begin{aligned} &3\,U''' + 3\alpha UVU' + 3\,U^2V' + \xi U' - (c-2)U = 0,\\ &3\,V''' + 3\alpha UVV' + 3\,V^2U' + \xi V' + cV = 0. \end{aligned} \tag{13}$$

Case 5. $(aX_1 + bX_2 + X_3)$

In this final instance, we obtain the invariants $(\xi = bx - at), \left(u = e^{t/b}U(\xi)\right)$ and $\left(v = e^{-t/b}V(\xi)\right)$. By substituting these invariant solutions into system (2) we find the following system of nonlinear ODEs:

$$\begin{aligned} &b^2U^2V' + \alpha b^2UVU' + b^4U''' + abU' - U = 0,\\ &b^2V^2U' + \alpha b^2UVV' + b^4V''' + abV' + V = 0. \end{aligned} \tag{14}$$

Explicit solutions of (2)

In this subsection, we determine exact power series solutions [34–37] for the ODEs (10) and (12)–(14). For the ODE (10), we have the power series form

$$U(\xi) = \sum_{j=0}^{\infty} p_j\xi^j \ \text{ and } \ V(\xi) = \sum_{j=0}^{\infty} q_j\xi^j, \tag{15}$$

where (p_j) and (q_j) for $(j = 1, 2, \cdots)$ are undetermined constants. From (15), we obtain

$$\begin{aligned} &U'(\xi) = \sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j,\ \ V'(\xi) = \sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j,\\ &U''(\xi) = \sum_{j=0}^{\infty}(j+1)(j+2)p_{j+2}\xi^j,\ \ V''(\xi) = \sum_{j=0}^{\infty}(j+1)(j+2)q_{j+2}\xi^j,\\ &U'''(\xi) = \sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j,\\ &V'''(\xi) = \sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j. \end{aligned} \tag{16}$$

Substituting (15) and (16) into (10) we have

$$\begin{aligned} &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right)\\ &\qquad + \left(\sum_{j=0}^{\infty}p_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) = 0,\\ &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right)\\ &\qquad + \left(\sum_{j=0}^{\infty}q_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right) = 0, \end{aligned}$$

which simplifies to

$$\begin{aligned} &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\xi^j\\ &+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\xi^j = 0,\\ &\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\xi^j\\ &+\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\xi^j = 0. \end{aligned} \tag{17}$$

From (17) the following recursion formulae are understood:

$$\begin{aligned} p_{j+3} = &-\tfrac{1}{(j+1)(j+2)(j+3)}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\right.\\ &\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\right),\\ q_{j+3} = &-\tfrac{1}{(j+1)(j+2)(j+3)}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\right.\\ &\left.+\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\right), \end{aligned} \tag{18}$$

for $(j = 0, 1, 2, \cdots)$ Thus, by choosing the constants (p_m) and (q_m), $(m = 0, 1, 2,)$ successive coefficients can be obtained, uniquely, from (18). We have, for instance,

$$\begin{aligned} p_3 &= -\tfrac{1}{6}p_0^2q_1 - \tfrac{1}{2}p_0q_0p_1,\\ q_3 &= -\tfrac{1}{6}q_0^2p_1 - \tfrac{1}{2}p_0q_0q_1;\\ p_4 &= -\tfrac{1}{12}p_0^2q_2 - \tfrac{1}{4}p_2p_0q_0 - \tfrac{5}{24}p_1p_0q_1 - \tfrac{1}{8}p_1^2q_0,\\ q_4 &= -\tfrac{1}{12}p_2q_0^2 - \tfrac{1}{4}p_0q_2q_0 - \tfrac{5}{24}p_1q_1q_0 - \tfrac{1}{8}p_0q_1^2;\\ p_5 &= \tfrac{1}{20}p_0^3q_0q_1 + \tfrac{1}{12}p_1p_0^2q_0^2 - \tfrac{2}{15}p_2p_0q_1 - \tfrac{7}{60}p_1p_0q_2 - \tfrac{3}{20}p_1p_2q_0 - \tfrac{1}{15}p_1^2q_1,\\ q_5 &= \tfrac{1}{20}p_0p_1q_0^3 + \tfrac{1}{12}p_0^2q_1q_0^2 - \tfrac{7}{60}p_2q_1q_0 - \tfrac{2}{15}p_1q_2q_0 - \tfrac{1}{15}p_1q_1^2 - \tfrac{3}{20}p_0q_1q_2; \end{aligned}$$

and so on. Consequently, the exact power series solution of (10) or in fact (2), is given by (see Figs. 1–3)

$$
\begin{aligned}
u(t,x) = {} & p_0 + p_1\xi + p_2\xi^2 - \left(\tfrac{1}{6}p_0^2q_1 + \tfrac{1}{2}p_0q_0p_1\right)\xi^3 - \left(\tfrac{1}{12}p_0^2q_2 + \tfrac{1}{4}p_2p_0q_0\right. \\
& \left. + \tfrac{5}{24}p_1p_0q_1 + \tfrac{1}{8}p_1^2q_0\right)\xi^4 + \left(\tfrac{1}{20}p_0^3q_0q_1 + \tfrac{1}{12}p_1p_0^2q_0^2 - \tfrac{2}{15}p_2p_0q_1 - \tfrac{7}{60}p_1p_0q_2\right. \\
& \left. - \tfrac{3}{20}p_1p_2q_0 - \tfrac{1}{15}p_1^2q_1\right)\xi^5 - \sum_{j=3}^{\infty}\tfrac{j!}{(j-3)!}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\right. \\
& \left. + \sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\right)\xi^{j+3},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
v(t,x) = {} & q_0 + q_1\xi + q_2\xi^2 - \left(\tfrac{1}{6}q_0^2p_1 + \tfrac{1}{2}p_0q_0q_1\right)\xi^3 - \left(\tfrac{1}{12}p_2q_0^2 + \tfrac{1}{4}p_0q_2q_0 + \tfrac{5}{24}p_1q_1q_0\right. \\
& \left. + \tfrac{1}{8}p_0q_1^2\right)\xi^4 + \left(\tfrac{1}{20}p_0p_1q_0^3 + \tfrac{1}{12}p_0^2q_1q_0^2 - \tfrac{7}{60}p_2q_1q_0 - \tfrac{2}{15}p_1q_2q_0 - \tfrac{1}{15}p_1q_1^2\right. \\
& \left. - \tfrac{3}{20}p_0q_1q_2\right)\xi^5 - \sum_{j=3}^{\infty}\tfrac{j!}{(j-3)!}\left(3\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\right. \\
& \left. + \sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\right)\xi^{j+3}.
\end{aligned}
\tag{20}
$$

Indications from Figs. 4 and 5 are that solution (19) is convergent. Below is a consolidation of the above solution profiles, further illustrating convergence.

We now employ the same procedure to solve system (13). Substituting (16) into (13), we have

$$
\begin{aligned}
& 3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\alpha\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)p_{j+1}\xi^j\right) \\
& +3\left(\sum_{j=0}^{\infty}p_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) + \xi\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) - (c-2)\left(\sum_{j=0}^{\infty}p_j\xi^j\right) = 0, \\
& 3\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\alpha\left(\sum_{j=0}^{\infty}p_j\xi^j\right)\left(\sum_{j=0}^{\infty}q_j\xi^j\right)\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) \\
& +3\left(\sum_{j=0}^{\infty}q_j\xi^j\right)^2\left(\sum_{j=0}^{\infty}(j+1)p_j\xi^j\right) + \xi\left(\sum_{j=0}^{\infty}(j+1)q_{j+1}\xi^j\right) + c\left(\sum_{j=0}^{\infty}q_j\xi^j\right) = 0.
\end{aligned}
$$

Simplifying the above system we now have

$$
\begin{aligned}
& 3\textstyle\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)p_{j+3}\xi^j + 3\alpha\sum_{j=0}^{\infty}\displaystyle\sum_{k=0}^{j}\textstyle\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\xi^j \\
& +\textstyle\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1}\xi^j + \sum_{j=0}^{\infty}(j-c+2)p_j\xi^j = 0, \\
& 3\textstyle\sum_{j=0}^{\infty}(j+1)(j+2)(j+3)q_{j+3}\xi^j + 3\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\displaystyle\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\xi^j \\
& +\textstyle\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1}\xi^j + \sum_{j=0}^{\infty}(j+c)q_j\xi^j = 0.
\end{aligned}
\tag{21}
$$

The system (21) is true if the coefficients of $\left(\xi^j\right), (j \in \mathbb{Z}^+)$ are equal to (0):

$$
\begin{aligned}
p_{j+3} = {} & -\tfrac{1}{3(j+1)(j+2)(j+3)}\left(3\alpha\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}p_{j-k+1}\right. \\
& \left. +\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_ip_{k-i}q_{j-k+1} + j - c + 2\right) \\
q_{j+3} = {} & -\tfrac{1}{3(j+1)(j+2)(j+3)}\left(3\alpha\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)p_iq_{k-i}q_{j-k+1}\right. \\
& \left. +\sum_{k=0}^{j}\sum_{i=0}^{k}(j-k+1)q_iq_{k-i}p_{j-k+1} + j + c\right)
\end{aligned}
\tag{22}
$$

for $(j = 0, 1, 2, \cdots)$ Thus, for arbitrary constants (p_m) and (q_m), $(m = 0, 1, 2)$, we have

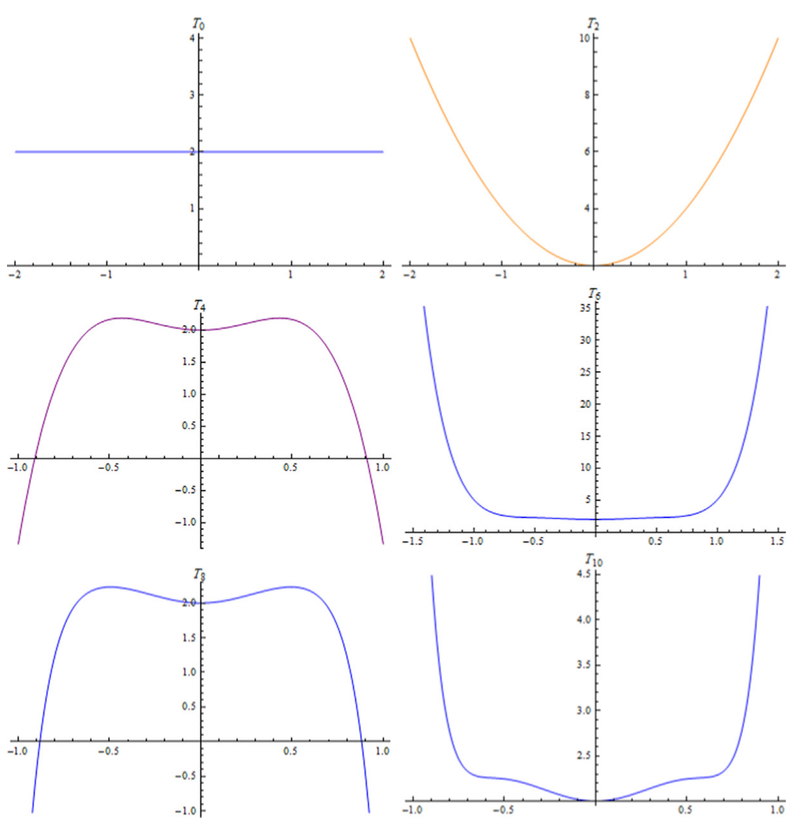

Fig. 1. Profiles of even partial sums for solution (19).

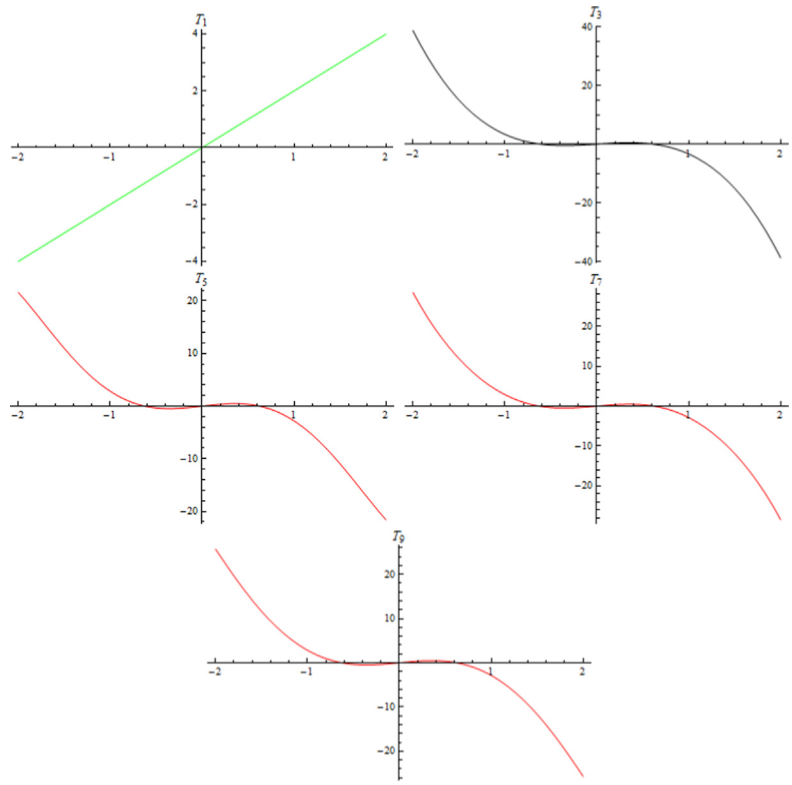

Fig. 2. Profiles of odd partial sums for solution (19).

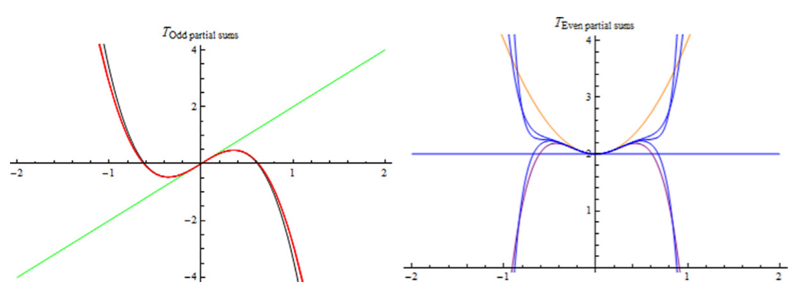

Fig. 3. Consolidated profiles of partial sums $(T_0)(\cdots)(T_{10})$ for solution (19).

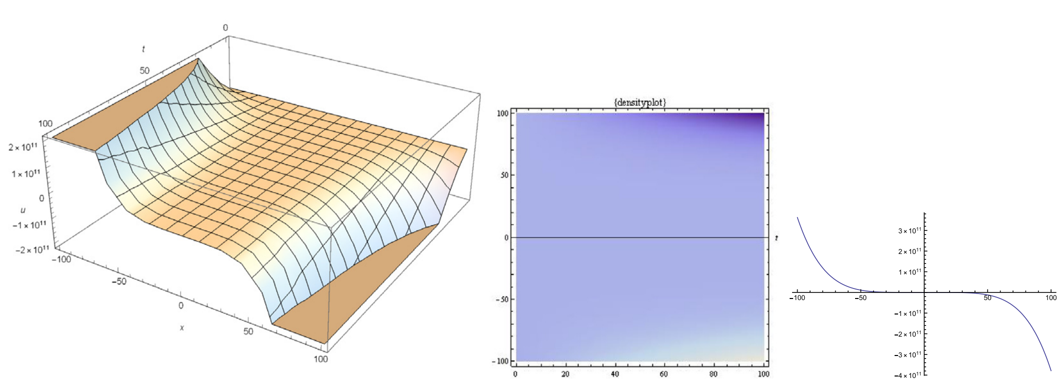

Fig. 4. Profiles of solution (24).

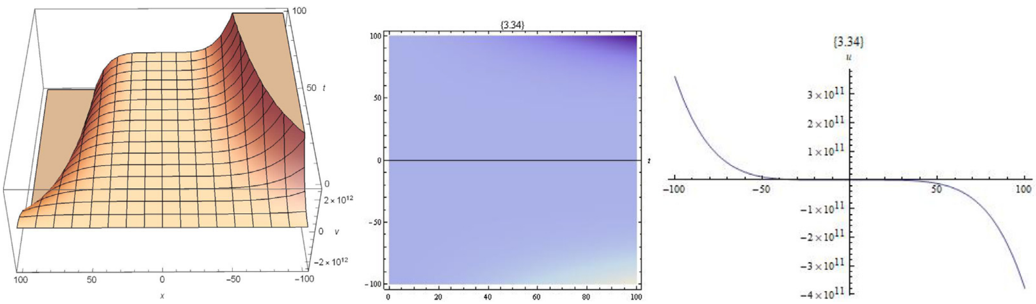

Fig. 5. Profiles of solution (25).

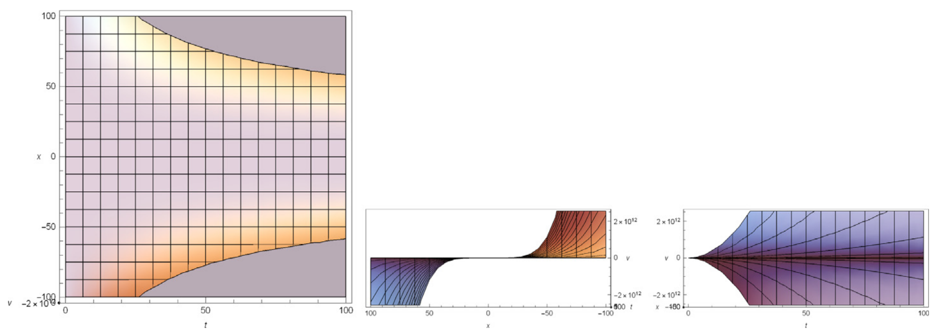

Fig. 6. Profiles of (25) from different vantage points.

$$\begin{aligned}
p_3 &= -\tfrac{1}{6}\alpha p_0 q_0 p_1 - \tfrac{1}{18} p_0^2 q_1 + \tfrac{1}{18} c - \tfrac{1}{9},\\
q_3 &= -\tfrac{1}{6}\alpha p_0 q_0 q_1 - \tfrac{1}{18} q_0^2 p_1 - \tfrac{1}{18} c;\\
p_4 &= -\tfrac{1}{12}\alpha p_0 q_0 p_2 - \tfrac{1}{36} p_0^2 q_2 - \tfrac{1}{72}(3\alpha+2) p_0 q_1 p_1 - \tfrac{1}{24}\alpha p_1^2 q_0 + \tfrac{1}{72} c - \tfrac{1}{24},\\
q_4 &= -\tfrac{1}{12}\alpha p_0 q_0 q_2 - \tfrac{1}{36} q_0^2 p_2 - \tfrac{1}{72}(3\alpha+2) q_0 q_1 p_1 - \tfrac{1}{24}\alpha q_1^2 p_0 - \tfrac{1}{72} c - \tfrac{1}{72};\\
p_5 &= \tfrac{1}{120}\alpha^2 p_0^2 q_0^2 p_1 + \tfrac{1}{180}\alpha p_0^3 q_0 q_1 - \tfrac{1}{360}\alpha c p_0 q_0 + \tfrac{1}{180}\alpha p_0 q_0 + \tfrac{1}{1080} p_0^2 q_0^2 p_1\\
&\quad + \tfrac{1}{1080} p_0^2 c - \tfrac{1}{30} p_0 q_1 p_2 \alpha - \tfrac{1}{90} p_0 q_1 p_2 - \tfrac{1}{60} p_0 p_1 q_2 \alpha - \tfrac{1}{45} p_0 p_1 q_2\\
&\quad - \tfrac{1}{20}\alpha p_1 q_0 p_2 - \tfrac{1}{60}\alpha p_1^2 q_1 - \tfrac{1}{180} p_1^2 q_1 + \tfrac{1}{180} c - \tfrac{1}{45},\\
q_5 &= \tfrac{1}{120}\alpha^2 p_0^2 q_0^2 q_1 + \tfrac{1}{180}\alpha p_0 q_0^3 p_1 + \tfrac{1}{360}\alpha c p_0 q_0 + \tfrac{1}{1080} q_0^2 p_0^2 q_1 - \tfrac{1}{1080} c q_0^2\\
&\quad + \tfrac{1}{540} q_0^2 - \tfrac{1}{30} q_0 p_1 q_2 \alpha - \tfrac{1}{90} q_0 p_1 q_2 - \tfrac{1}{60} q_0 q_1 p_2 \alpha - \tfrac{1}{45} q_0 q_1 p_2\\
&\quad - \tfrac{1}{20}\alpha p_0 q_1 q_2 - \tfrac{1}{60} q_1^2 p_1 \alpha - \tfrac{1}{180} q_1^2 p_1 + \tfrac{1}{180} c - \tfrac{1}{90}.
\end{aligned} \tag{23}$$

Hence the exact power series solution to (13) is

$$\begin{aligned}
u(t,x) &= t^{(c-2)/3}\Big\{p_0 + p_1\xi + p_2\xi^2 - \big(\tfrac{1}{6}\alpha p_0 q_0 p_1 + \tfrac{1}{18} p_0^2 q_1 - \tfrac{1}{18} c + \tfrac{1}{9}\big)\xi^3\\
&\quad - \big(\tfrac{1}{12}\alpha p_0 q_0 p_2 + \tfrac{1}{36} p_0^2 q_2 + \tfrac{1}{72}(3\alpha+2) p_0 q_1 p_1 + \tfrac{1}{24}\alpha p_1^2 q_0\\
&\quad - \tfrac{1}{72} c + \tfrac{1}{24}\big)\xi^4 + \big(\tfrac{1}{120}\alpha^2 p_0^2 q_0^2 p_1 + \tfrac{1}{180}\alpha p_0^3 q_0 q_1 - \tfrac{1}{360}\alpha c p_0 q_0\\
&\quad + \tfrac{1}{180}\alpha p_0 q_0 + \tfrac{1}{1080} p_0^2 q_0^2 p_1 + \tfrac{1}{1080} c p_0^2 - \tfrac{1}{30} p_0 q_1 p_2 \alpha - \tfrac{1}{90} p_0 q_1 p_2\\
&\quad - \tfrac{1}{60} p_0 p_1 q_2 \alpha - \tfrac{1}{45} p_0 p_1 q_2 - \tfrac{1}{20}\alpha p_1 q_0 p_2 - \tfrac{1}{60}\alpha p_1^2 q_1 - \tfrac{1}{180} p_1^2 q_1\\
&\quad + \tfrac{1}{180} c - \tfrac{1}{45}\big)\xi^5 + \sum_{n=3}^{\infty} \tfrac{3n!}{(n-3)!}\left(3\alpha \sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) p_i q_{k-i} p_{n-k+1}\right.\\
&\quad \left. + \sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) p_i p_{k-i} q_{n-k+1} + n - c + 2\right)\xi^{n+3}\Big\},
\end{aligned} \tag{24}$$

$$\begin{aligned}
v(t,x) &= t^{c/3}\Big\{q_0 + q_1\xi + q_2\xi^2 - \big(\tfrac{1}{6}\alpha p_0 q_0 q_1 + \tfrac{1}{18} q_0^2 p_1 + \tfrac{1}{18} c\big)\xi^3 - \big(\tfrac{1}{12}\alpha p_0 q_0 q_2\\
&\quad + \tfrac{1}{36} q_0^2 p_2 + \tfrac{1}{72}(3\alpha+2) q_0 q_1 p_1 + \tfrac{1}{24}\alpha q_1^2 p_0 + \tfrac{1}{72} c + \tfrac{1}{72}\big)\xi^4\\
&\quad + \big(\tfrac{1}{120}\alpha^2 p_0^2 q_0^2 q_1 + \tfrac{1}{180}\alpha p_0 q_0^3 p_1 + \tfrac{1}{360}\alpha c p_0 q_0 + \tfrac{1}{1080} q_0^2 p_0^2 q_1\\
&\quad - \tfrac{1}{1080} c q_0^2 + \tfrac{1}{540} q_0^2 - \tfrac{1}{30} q_0 p_1 q_2 \alpha - \tfrac{1}{90} q_0 p_1 q_2 - \tfrac{1}{60} q_0 q_1 p_2 \alpha\\
&\quad - \tfrac{1}{45} q_0 q_1 p_2 - \tfrac{1}{20}\alpha p_0 q_1 q_2 - \tfrac{1}{60} q_1^2 p_1 \alpha - \tfrac{1}{180} q_1^2 p_1 + \tfrac{1}{180} c - \tfrac{1}{90}\big)\xi^5\\
&\quad - \sum_{n=3}^{\infty} \tfrac{3n!}{(n-3)!}\left(3\alpha \sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) p_i q_{k-i} q_{n-k+1}\right.\\
&\quad \left. + \sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1) q_i q_{k-i} p_{n-k+1} + n + c\right)\xi^{n+3}\Big\}.
\end{aligned} \tag{25}$$

We now provide $(3D)$ renderings of (24) and (25) for $(n = 0, 1, 2)$ and for arbitrary values of $(p_0, q_0, p_1, q_1, p_2, q_2)$ and (c) in Figs. 4–6.

Concluding remarks

In this paper, we considered a (1 + 1)-dimensional generalised coupled modified KdV-type system. This system is studied for the first time in this paper. It has no variational principle and as such, we employed the multiplier method and used a homotopy integral formula to derive the conserved quantities to which it conforms. As already seen, the homotopy integral approach is relatively simple, concise and elegant compared to other traditional avenues of computing conserved quantities. In this work, we provided a step-by-step illustrative example of this algorithm with the aid of a system which has not been previously studied. Herein lies the novelty of our work. Again we sought the optimal system of one-dimensional algebras for this system by invoking its four-dimensional Lie algebras. This enabled us to transform the system into several systems of NLODEs. With the aid of the power series solution method, we solved some of the systems of NLODEs and obtained its exact solutions which are by extension also solutions of system (2).

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

Declaration of Competing Interest

The authors declare no conflict of interest.

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