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Research article

A NEW flexible exponent power family of distributions with biomedical data analysis

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ABSTRACT

Probability distributions are widely utilized in applied sciences, especially in the field of biomedical science. Biomedical data typically exhibit positive skewness, necessitating the use of flexible, skewed distributions to effectively model such phenomena. In this study, we introduce a novel approach to characterize new lifetime distributions, known as the New Flexible Exponent Power (NFEP) Family of distributions. This involves the addition of a new parameter to existing distributions. A specific sub-model within the proposed class, known as the New Flexible Exponent Power Weibull (NFEP-Wei), is derived to illustrate the concept of flexibility. We employ the well-established Maximum Likelihood Estimation (MLE) method to estimate the unknown parameters in this family of distributions. A simulation study is conducted to assess the behavior of the estimators in various scenarios. To gauge the flexibility and effectiveness of the NFEP-Wei distribution, we compare it with the AP-Wei (alpha power Weibull), MO-Wei (Marshal Olkin Weibull), classical Wei (Weibull), NEP-Wei (new exponent power Weibull), FRLog-Wei (flexible reduced logarithmic Weibull), and Kum-Wei (Kumaraswamy Weibull) distributions by analyzing four distinct biomedical datasets. The results demonstrate that the NFEP-Wei distribution outperforms the compared distributions.

1. Introduction

In applied science, particularly in biomedical science, probability distributions are indispensable tools. Various parametric continuous probability distributions have been introduced in the literature for the statistical analysis and modeling of lifetime datasets. These distributions include the exponential (Exp), gamma (Gam), log-normal (Log-nor), Rayleigh (Ray), beta (Beta), and Weibull (Wei) distributions. For more details, please refer to Zichuan et al. [1]. Among these distributions, the Exponential, Rayleigh, and Weibull distributions are more popular than the Gamma, Beta, and Log-normal distributions. This popularity is due to the latter group's lack of

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a closed version of the cumulative distribution function (CDF), which makes parameter estimation challenging and necessitates numerical integration. The aforementioned distributions are suitable and frequently used to model things that occur across a lifetime phenomenon. Unfortunately, the current distributions are frequently too rigid to accurately represent complex lifetime phenomena. For instance, the widely used one-parameter exponential (Exp) distribution can only represent data with a fixed Failure Rate Function (FRF). The Rayleigh (Ray) distribution, on the other hand, is used to simulate data with a strictly rising FRF. On the other hand, the two-parameter Weibull (Wei) distribution is a more versatile choice, as it combines the characteristics of both the Exp and Ray distributions, allowing it to model data with monotonically increasing, decreasing, or constant FRF.

However, in biomedical science, datasets often exhibit unimodal, modified unimodal, or bathtub-shaped (U-shaped) FRF patterns. For more details, please refer to Almalki and Yuan [2], Zajicek [3], Lai and Xie [4], and Demicheli et al. [5]. Therefore, in such a situation, the Weibull distribution may not be a suitable choice, and a flexible distribution is required to accommodate the phenomenon. In this context, substantial efforts have been made and continue to evolve rapidly to introduce a new flexible variant of the Weibull distribution. In recent times, researchers have made significant contributions to the development of new families of distributions by incorporating one or more additional parameters into these existing distributions. Here, we refer to a few generalized families of distributions in the literature, for example, Gleaton and Lynch [6] introduced the OLL-G (odd log-logistic-G) family of distributions, Shaw and Buckley [7] proposed the distribution function of the transmuted family, Mahdavi and Kundu [8] introduced a new method for introducing statistical distributions, Bourguignon et al. [9] introduced Weibull-G family, Korkmaz [10] proposed the Ex-Wei-G (extended Weibull-G) family, Morshedy and Eliwa [11] derived the odd flexible Weibull-H family of distributions, Hussain et al. [12] proposed kum-G (Kumaraswamy Generalized) family, Huo et al. [13] proposed NL-Exp-X (new lifetime exponential-X) family, Zichuan et al. [1] introduced the NEx-F (new extended-Family) of distributions, Eghwerido et al. [14] proposed the TAP-G (transmuted alpha power-G) family, Alzaatreh et al. [15] proposed the Truncated family of distributions, Tung et al. [16] proposed Arcsine-X family of distributions, Shah et al. [17] introduced an NMEPA (new modified exponent power alpha) family, Hussain et al. [18] proposed GAEP (generalized alpha exponent power) family of distributions, Kilai et al. [19] derived GGAP (a generalization of Gull alpha power) family, Eghwerido et al. [20] proposed SExpo-G (shifted exponential-G) family of distributions, and Klakattawi et al. [21] introduced the MO-Wei-G (Marshall-Olkin Weibull Generated) family of distributions.

Recently, Xin et al. [22] suggested a new technique called, a NG-X (new generalized-X) family of distributions. The $K(x;\theta,\delta,\vartheta)$ CDF of the NG-X is given by

$$K(x;\theta,\delta,\vartheta)=1-\left(1-\frac{(1-\theta)^2A(x;\vartheta)}{[1-\theta A(x;\vartheta)]^2}\right)^{\delta},x\in R,$$

where $\delta\in R^+$, $\theta\in(0,1)$, and $A(x;\vartheta)$ is the CDF of any baseline distributions.

Hussein et al. [23] introduced another novel method of adding two extra parameters to the baseline distribution. The proposed method is used for obtaining the updated version of the existing and other modified distributions. The proposed method is called the MAPT (modified alpha power transformations) family of distributions. The $K(x;\alpha,\beta,\vartheta)$ CDF of the MAPT is given by

$$K(x;\alpha,\beta,\vartheta)=\frac{\beta^{A(x;\vartheta)^2}\alpha^{A(x;\vartheta)}-1}{\alpha\beta-1},\alpha\beta\neq1,x\in R,$$

where $\alpha\in R^+$ and $\beta\in R^+$ are the extra shape parameter and $A(x;\vartheta)$ is the CDF of baseline distribution which may depend on the parameters vector $\vartheta\in R$.

Similarly, Shah et al. [24] applied the power transformation technique and proposed a new approach to enhance the flexibility of probability distributions. They named their method the NGLog-X (new generalized logarithmic-X) approach. The $K(x;\theta,\vartheta)$ CDF (cumulative distribution function) of the NGLog-X method is provided by

$$K(x;\theta,\vartheta)=\frac{e^{\theta}A(x;\vartheta)}{(e-\log A(x;\vartheta))^{\theta}},x\in R,$$

where $\theta\in R^+$ is the shape parameter, and $A(x;\vartheta)$ is the CDF of baseline distribution depending on vector parameter $\vartheta\in R$.

The primary motivation behind this research is to leverage the strengths of the aforementioned distributions by introducing a novel approach to probability distributions called the transformed-transformer (T-X) family approach. This newly proposed distribution family is highly adaptable and offers a good fit for biological datasets. The following sections outline the content of this paper: Section 2 introduces the readers to the newly suggested family of lifetime distributions. Section 3 is made up of the prospective family's IP (identifiability property). Section 4, derives a sub-model of the proposed class, the New Flexible Exponent Power Weibull (NFEP-Wei) distribution, and shows CDF, SF (survival function), PDF (probability density function), and HF (hazard function) graphically. Section 5 derives numerous mathematical features of the proposed family. Section 6 discusses estimation approach for estimating model parameters of the proposed family, and in the same section a brief MCS (Monte Carlo simulation) study is performed to evaluate the estimator's behavior. Section 7 discusses the implementations of the suggested algorithms on four distinct biomedical datasets. Section 8 concludes with several observations and helpful insights.

2. NFEP family of distributions

The proposed method combines the Exp distribution's PDF with the transformed-transformer (T-X) family approach introduced by Alzaatreh et al. [25]. Let T be a random variable (RV) that belongs to ω_1 and ω_2 for $-\infty \leq \omega_1 < \omega_2 < \infty$, and let suppose, X be a RV having CDF $A(x;\vartheta)$, and $Z[A(x;\vartheta)]$ be a function of CDF $A(x;\vartheta)$, which fulfill three conditions, as under:

i. $Z[A(x;\vartheta)] \in (\omega_1, \omega_2)$.
ii. $Z[A(x;\vartheta)]$ is differentiable and monotonically increasing function.
iii. $x \to -\infty \Rightarrow Z[A(x;\vartheta)] \to \omega_1$ and $x \to \infty \Rightarrow Z[A(x;\vartheta)] \to \omega_2$.

The T-X family having the CDF $K(x)$, is defined by

$$K(x) = \int_{\omega_1}^{Z[A(x;\vartheta)]} r(t)dt, x \in R, \tag{1}$$

where, $Z[A(x;\vartheta)]$ satisfies the above (i)-(iii) conditions. The PDF corresponding to Eq. (1), is given by

$$k(x) = \left\{\frac{d}{dx} Z[A(x;\vartheta)]\right\} r\{Z[A(x;\vartheta)]\}, x \in R.$$

Setting $Z[A(x;\vartheta)] = -\log\left(1 - \frac{A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}}\right)$, and $r(t) = e^{-t}$ in Eq. (1), we define the CD $K(x;\delta,\vartheta)$ of the New Flexible Exponent power (NFEP) family of distributions as

$$K(x;\delta,\vartheta) = \frac{A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}}, x \in R, \tag{2}$$

where, $\delta \in R^+$ is an additional shape parameter, and $A(x;\vartheta)$ is the CDF of any baseline classical distribution, which may depend on the parameter vector $\vartheta \in R$. Adding an extra parameter to an existing distribution can result in a better fit for biomedical data. To confirm the validity of the proposed method as a CDF, we have two main prepositions:

Proposition 1. *For the CDF $K(x;\delta,\vartheta)$ in* Eq. (2)*, we must prove that*

$$\lim_{x\to-\infty} K(x;\delta,\vartheta) = 0 \text{ and } \lim_{x\to\infty} K(x;\delta,\vartheta) = 1.$$

Proof. From Eq. (2), we have

$$\lim_{x\to-\infty} K(x;\delta,\vartheta) = \lim_{x\to-\infty}\left\{\frac{A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}}\right\}. \tag{3}$$

Where, $A(x;\vartheta)$ is a valid CDF of any baseline distribution. So, we have

$$\lim_{x\to-\infty} A(x;\vartheta) = A(-\infty;\vartheta) = 0.$$

Now, from Eq. (3), we have

$$\lim_{x\to-\infty} K(x;\delta,\vartheta) = \frac{A(-\infty;\vartheta)e^{\delta A(-\infty;\vartheta)^2}}{e^{\delta}} = 0.$$

Again, from Eq. (2), we have

$$\lim_{x\to\infty} K(x;\delta,\vartheta) = \lim_{x\to\infty}\left\{\frac{A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}}\right\}. \tag{4}$$

Where, $A(x;\vartheta)$ is a valid CDF. So, we have

$$\lim_{x\to\infty} A(x;\vartheta) = A(\infty;\vartheta) = 1.$$

Now, from Eq. (4), we have

$$\lim_{x\to\infty} K(x;\delta,\vartheta) = \frac{A(\infty;\vartheta)e^{\delta A(\infty;\vartheta)^2}}{e^{\delta}} = 1.$$

Propositions 2. *The CDF $K(x;\delta,\vartheta)$ in* Eq. (2) *is differentiable and right continuous* (*RC*)

Proof.

$$\frac{d}{dx}K(x;\delta,\vartheta)=k(x;\delta,\vartheta).$$

From propositions 1 and 2, we observed that the proposed method in Eq. (2) is a valid CDF. For $\delta\in R^{+}$ and $x\in R$, the PDF $\frac{d}{dx}K(x;\delta,\vartheta)=k(x;\delta,\vartheta)$ of the NFEP family, is given by

$$k(x;\delta,\vartheta)=\frac{\delta(x;\vartheta)e^{\delta A(x;\vartheta)^2}\left\{2\delta A(x;\vartheta)^2+1\right\}}{e^{\delta}},x\in R, \tag{5}$$

where $\frac{d}{dx}A(x;\vartheta)=a(x;\vartheta)$.

Furthermore, in link to Eq. (2), and Eq. (5), the HF $h(x;\delta,\vartheta)=\frac{k(x;\delta,\vartheta)}{1-K(x;\delta,\vartheta)}$, SF (survival function) $S(x;\delta,\vartheta)=1-K(x;\delta,\vartheta)$, RHF (reverse HF) $\tau(x;\delta,\vartheta)=\frac{k(x;\delta,\vartheta)}{K(x;\delta,\vartheta)}$, and CHF (cumulative HF) $H(x;\delta,\vartheta)=-\log(S(x;\delta,\vartheta))$ of the NFEP family, are given by

$$h(x;\delta,\vartheta)=\frac{a(x;\vartheta)e^{\delta A(x;\vartheta)^2}\left\{2\delta A(x;\vartheta)^2+1\right\}}{e^{\delta}-A(x;\vartheta)e^{\delta A(x;\vartheta)^2}},x\in R,$$

$$S(x;\delta,\vartheta)=\frac{e^{\delta}-A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}},x\in R,$$

$$\tau(x;\delta,\vartheta)=\frac{a(x;\vartheta)e^{\delta A(x;\vartheta)^2}\left\{2\delta A(x;\vartheta)^2+1\right\}}{A(x;\vartheta)e^{\delta A(x;\vartheta)^2}},x\in R,$$

and

$$H(x;\delta,\vartheta)=-\log\left(\frac{e^{\delta}-A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}{e^{\delta}}\right),x\in R.$$

3. Identifiability property (IP)

In this section, we derive IP of the NFEP family of distributions using extra parameter δ. Let δ_1 and δ_2 be the two additional parameters having CDFs $K(x;\delta_1,\vartheta)$ and $K(x;\delta_2,\vartheta)$. Then, the parameter δ is identifiable, if $\delta_1=\delta_2$, for

$$K(x;\delta_1,\vartheta)=K(x;\delta_2,\vartheta). \tag{6}$$

Inserting Eq. (2) in Eq. (6), we get

$$\frac{A(x;\vartheta)e^{\delta_1 A(x;\vartheta)^2}}{e^{\delta_1}}=\frac{A(x;\vartheta)e^{\delta_2 A(x;\vartheta)^2}}{e^{\delta_2}}. \tag{7}$$

After simplifying Eq. (7), we get

$$e^{\delta_1 A(x;\vartheta)^2+\delta_2}=e^{\delta_2 A(x;\vartheta)^2+\delta_1}. \tag{8}$$

By taking logarithm of Eq. (8), we get

$$\delta_1 A(x;\vartheta)^2+\delta_2=\delta_2 A(x;\vartheta)^2+\delta_1,$$

$$\delta_1 A(x;\vartheta)^2-\delta_1=\delta_2 A(x;\vartheta)^2-\delta_2,$$

$$\delta_1\left(A(x;\vartheta)^2-1\right)=\delta_2\left(A(x;\vartheta)^2-1\right),$$

$$\delta_1=\delta_2. \tag{9}$$

From Eq. (9), after some algebraic simplification, we observed that $\delta_1=\delta_2$. Hence, the parameter δ is identifiable.

4. NFEP-Wei distribution

To provide a special sub-model of the NFEP family of distributions, let consider the CDF $A(x;\vartheta)$ and PDF $a(x;\vartheta)$ of the traditional Wei distribution are given by $A(x;\vartheta)=1-e^{-\eta x^{\theta}}$ and $a(x;\vartheta)=\theta\eta x^{\theta-1}e^{-\eta x^{\theta}}$, respectively (for $\theta,\eta\in R^{+},x\in R$), where $\vartheta=(\eta,\theta)$. Then, the $K(x;\delta,\vartheta)$ CDF of the NFEP-Wei distribution is given by

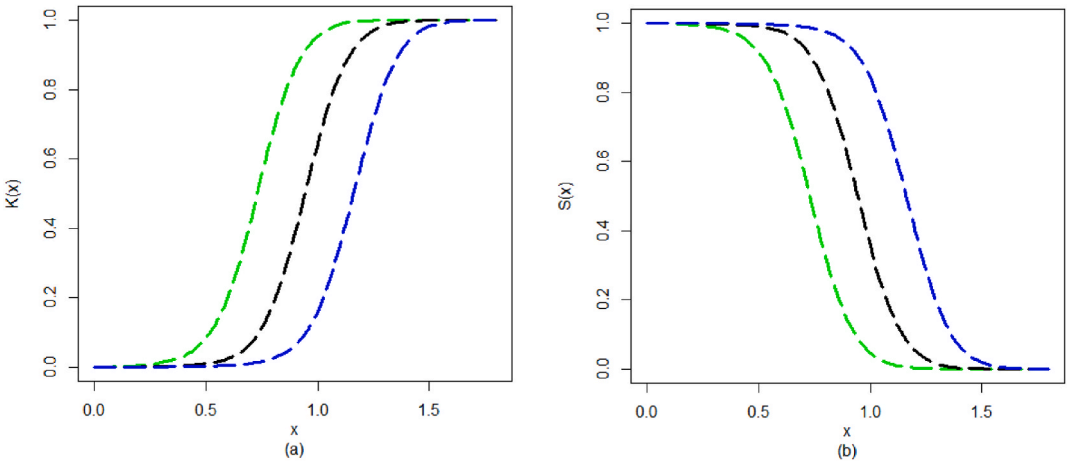

Fig. 1. Plots of (a) $K(x;\delta,\vartheta)$ CDF and (b) $S(x;\delta,\vartheta)$ SF of the NFEP-Wei model.

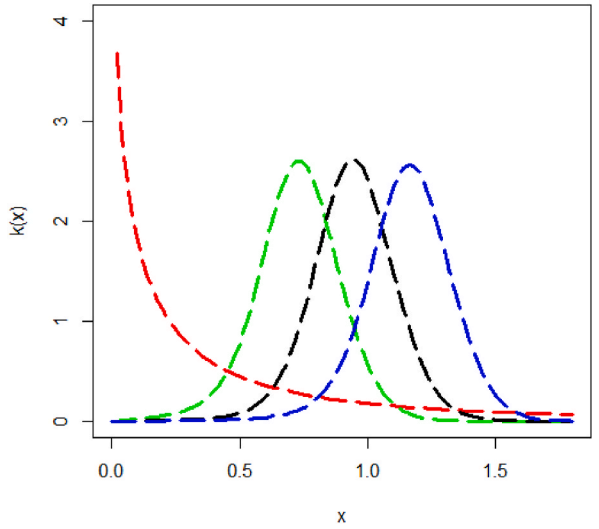

Fig. 2. Plots of $k(x;\delta,\vartheta)$ PDF with different parameters values of the NFEP-Wei model.

$$K(x;\delta,\vartheta)=\frac{\left(1-e^{-\eta x^{\theta}}\right)e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^{2}}}{e^{\delta}},\delta,\eta,\theta\in R^{+},x\in R, \tag{10}$$

Corresponding to Eq. (10), the $S(x;\delta,\vartheta)$ SF is

$$S(x;\delta,\vartheta)=\frac{e^{\delta}-\left(1-e^{-\eta x^{\theta}}\right)e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^{2}}}{e^{\delta}},x\in R.$$

Some plots of the CDF $K(x;\delta,\vartheta)$ and SF $S(x;\delta,\vartheta)$ of the NFEP-Wei model are sketched in Fig. 1(a and b). The plots are acquired for (i) $\delta=3.0,\eta=5.0,\theta=2.5$ (green curve line), (ii) $\delta=4.0,\eta=3.0,\theta=2.9$ (black curve line), and (iv) $\delta=4.0,\eta=1.5,\theta=3.5$ (blue curve line).

Link to CDF $K(x;\delta,\vartheta)$ in Eq. (10), the PDF $k(x;\delta,\vartheta)$ is given by

$$k(x;\delta,\vartheta)=\frac{\eta\theta x^{\theta-1}e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^{2}-\eta x^{\theta}}\left\{2\delta\left(1-e^{-\eta x^{\theta}}\right)^{2}+1\right\}}{e^{\delta}},x\in R. \tag{11}$$

For graphical illustration, different plots of PDF $k(x;\delta,\vartheta)$ of the NFEP-Wei distribution at different parameters values are sketched in Fig. 2. The corresponding plots are obtained for (i) $\delta=3.0,\eta=3.5,\theta=0.3$ (red curve line), (ii) $\delta=3.0,\eta=5.0,\theta=2.5$ (green curve line), (iii) $\delta=4.0,\eta=3.0,\theta=2.9$ (black curve line), and (iv) $\delta=4.0,\eta=1.5,\theta=3.5$ (blue curve line). Fig. 2 shows four

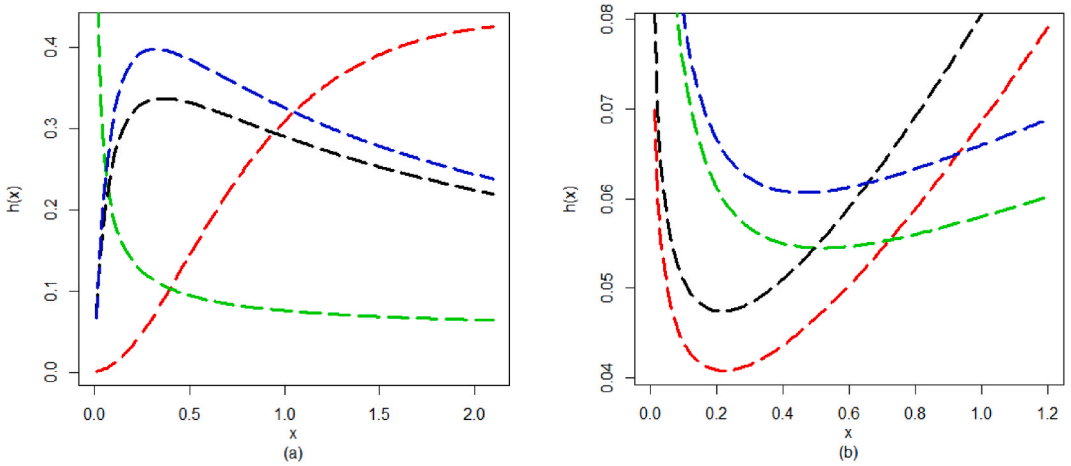

Fig. 3. Different (a) and (b) plots of $h(x;\delta,\vartheta)$ HF of the NFEP-Wei model.

possible PDF $k(x;\delta,\vartheta)$ shapes: (a) a declining or reverse-J pattern (red line), (b) a right-skewed pattern (green line), (c) a symmetrical pattern (black line), and (d) a lift-skewed pattern (blue line).

Furthermore, in link to Eq. (10), and Eq. (11), the $h(x;\delta,\vartheta)$ HF, $\tau(x;\delta,\vartheta)$ RHF, and $H(x;\delta,\vartheta)$ CHF are given, respectively, by

$$h(x;\delta,\vartheta)=\frac{\eta\theta x^{\theta-1}e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^2-\eta x^{\theta}}\left(2\delta\left(1-e^{-\eta x^{\theta}}\right)^2+1\right)}{e^{\delta}-(1-e^{-\eta x^{\theta}})e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^2}},x\in R,\ \tau(x;\delta,\vartheta)=\frac{\eta\theta x^{\theta-1}e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^2-\eta x^{\theta}}\left(2\delta\left(1-e^{-\eta x^{\theta}}\right)^2+1\right)}{e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^2}(1-e^{-\eta x^{\theta}})},x\in R,$$

and

$$H(x;\delta,\vartheta)=-\log\left(\frac{e^{\delta}-e^{\delta\left(1-e^{-\eta x^{\theta}}\right)^2}\left(1-e^{-\eta x^{\theta}}\right)}{e^{\delta}}\right),x\in R.$$

Similarly, for numerous values of parameters, some plots of $h(x;\delta,\vartheta)$ HF of the proposed NFEP-Wei model are also sketched in Fig. 3 (a and b). In Fig. 3(a and b), the plots are provided for (i) $\delta=11$, $\eta=2.4$, $\theta=0.43$ (red curve line), (ii) $\delta=1.4$, $\eta=0.5$, $\theta=0.5$ (green curve line), (iii) $\delta=15$, $\eta=3.12$, $\theta=0.2$ (black curve line), and (iv) $\delta=18$, $\eta=3.4$, $\theta=0.19$ (blue curve line). Similarly, in Fig. 3(a and b), the plots of $h(x;\delta,\vartheta)$ HF are visualized for (i) $\delta=2.9$, $\eta=0.64$, $\theta=0.8$ (red curve line), (ii) $\delta=2.5$, $\eta=0.6$, $\theta=0.6$ (green curve line), (iii) $\delta=2.8$, $\eta=0.67$, $\theta=0.8$ (black curve line), and (iv) $\delta=2.5$, $\eta=0.64$, $\theta=0.6$ (blue curve line). From Fig. 3(a and b), we can clearly see that the HF shape of the NFEP-Wib distribution can be (i) increasing, (ii) decreasing, (iii) uni-model, and (iv) bathtub form.

5. Mathematical properties of NFEP family

This section presents the main statistical/mathematical properties of the NFEP family of probability distributions.

5.1. Moments

General order moments are a set of descriptive statistics used to summarize probability distributions and their characteristics. In this context, we provide a computational representation of general order moments that correspond to the proposed NFEP family of distributions. If X is an NFEP RV, then the rth moments concerning the origin are provided by

$$E(x^r)=\int_{-\infty}^{\infty}k(x;\delta,\vartheta)dx. \tag{12}$$

when we plug the suggested family's density into Eq. (12), we obtain

$$\mu'=\frac{1}{e^{\delta}}\left(2\sum_{i=0}^{\infty}\frac{\delta^{i+1}}{i!}\int_{-\infty}^{\infty}x^r a(x;\vartheta)A(x;\vartheta)^{2i+2}dx+\sum_{0}^{\infty}\frac{\delta^{i}}{i!}\int_{-\infty}^{\infty}x^r A(x;\vartheta)^{2i}a(x;\vartheta)dx\right),$$

$$\mu'=\frac{1}{e^{\delta}}\left(2\sum\nolimits_{i=0}^{\infty}\frac{\delta^{i+1}}{i!}\varphi_{r,2i+2}+\sum\nolimits_{i=0}^{\infty}\frac{\delta^{i+1}}{i!}\varphi_{r,2i}\right),$$

where, $\varphi_{r,2i+2} = \int_{-\infty}^{\infty} x^r a(x;\vartheta)A(x;\vartheta)^{2i+2}dx$ and $\varphi_{r,2i} = \int_{-\infty}^{\infty} x^r a(x;\vartheta)A(x;\vartheta)^{2i}dx$.

Furthermore, a general expression for MGF (moments generating function) of the introduced family of density is derived as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}k(x;\delta,\vartheta)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'. \tag{13}$$

By simplifying Eq. (13), we get the MGF

$$M_X(t) = \frac{1}{e^{\delta}}\left(2\sum_{i,r=0}^{\infty}\frac{\delta^{i+1}t^r}{r!i!}\varphi_{r,2i+2} + \sum_{i,r=0}^{\infty}\frac{\delta^i t^r}{r!i!}\varphi_{r,2i}\right).$$

5.2. Residual and reverse residual life of NFEP family

The residual lifespan random variable is sometimes referred to as "time since failure" or "survival time" in survival analysis. It indicates the length of time that a person or system endures after an event, such as the time of a diagnosis, the commencement of therapy, or the beginning of an experiment. The residual lifespan random variable is frequently used to model survival data and calculate the likelihood that an event, such as deaths, failure, or recurrence, will happen at a specific period. It is a crucial tool for the study of data that has been censored, which happens when the precise survival duration for certain people or systems is unknown. The residual lifespan of the NFEP RVs, say X, is denoted by $R_{(t)}$ and is given as

$$R_{(t)}(x) = \frac{S(x+t;\delta,\vartheta)}{S(x;\delta,\vartheta)},$$

$$R_{(t)}(x) = \frac{e^{\delta} - A(x+t;\vartheta)e^{\delta A(x+t;\vartheta)^2}}{e^{\delta} - A(t;\vartheta)e^{\delta A(x;\vartheta)}}.$$

For the NFEP random variable X, the reverse residual life is $\overline{R}_{(t)}$.

$$\overline{R}_{(t)}(x) = \frac{S(x-t;\delta,\vartheta)}{S(x;\delta,\vartheta)},$$

$$\overline{R}_{(t)}(x) = \frac{e^{\delta} - A(x-t;\vartheta)e^{\delta A(x-t;\vartheta)^2}}{e^{\delta} - A(x;\vartheta)e^{\delta A(x;\vartheta)^2}}.$$

5.3. Order statistics

Let $X_1 < X_2 < X_3 < \ldots < X_n$ be a set of i.i.d RVs of size 'n' taken from NFEP family of distributions with parameters δ and ϑ, then the associated OS (order statistics) are $X_{(1:n)} \le X_{(2:n)} \le X_{(3:n)} \le \ldots \le X_{(n:n)}$ with the $K(x;\delta,\vartheta)$ CDF and $k(x;\delta,\vartheta)$ PD. From David et al. [26] the PDF of $X_{(j:n)}$, say $f_{j:n}(x)$, where $j = 1, 2, \ldots, n$, is defined by

$$f_{j:n}(x) = \frac{k(x;\delta,\vartheta)}{B(x, n-j+1)}\sum_{i=0}^{n-j}(-1)^i[K(x;\delta,\vartheta)]^{i+j-1}. \tag{14}$$

Using Eq. (2) and Eq. (5) in Eq. (14), we get the j^{th} order statistics for the proposed family of distributions.

5.4. Quantile function

Quartiles are descriptive statistic used to encapsulate a dataset's distribution. Calculating quartiles requires the use of a mathematical formula called the quartile function (QF). The function, let's say $Q(u;\delta,\vartheta)$, that satisfies the following non-linear equations is the QF of the proposed NFEP family.

$$Q(K(u;\delta,\vartheta)) = u, u \in (0,1) \tag{15}$$

By using Eq. (2) in Eq. (15), we get

$$Q(K(u;\delta,\vartheta)) = K^{-1}(u) = A^{-1}(u).$$

$$Q(u) = \delta A(x;\vartheta)^2 + \log(A(x;\vartheta) - ue^{\delta}),$$

where, u is the solution of $\delta A(x;\vartheta)^2 + \log(A(x;\vartheta) - ue^{\delta})$.

Table 1
Simulations results of the NFEP-Wei distribution for set I and set II.

		*Set*I: $\delta = 1.4, \eta = 1.0, \theta = 1.8$			Set II: $\delta = 3.2, \eta = 2.2, \theta = 2.8$		
n	Parameters	MLE	MSEs	Biases	MLE	MSEs	Biases
25	$\widehat{\delta}$	1.835226	2.890406	0.435226	3.210584	4.623899	0.010584
	$\widehat{\eta}$	1.050124	0.344733	0.050124	2.075723	0.688522	−0.124276
	$\widehat{\theta}$	1.967728	0.432582	0.167728	3.272871	1.588412	0.472871
50	$\widehat{\delta}$	1.652391	1.842478	0.252391	3.287184	2.379300	0.087184
	$\widehat{\eta}$	1.020735	0.244576	0.020735	2.115238	0.311814	−0.084761
	$\widehat{\theta}$	1.917023	0.280400	0.117022	3.091635	0.818819	0.291635
75	$\widehat{\delta}$	1.582892	1.341268	0.182891	3.248166	1.874261	0.048166
	$\widehat{\eta}$	1.017775	0.191671	0.017775	2.130770	0.246385	−0.069230
	$\widehat{\theta}$	1.880093	0.216140	0.080093	3.039680	0.642035	0.239679
100	$\widehat{\delta}$	1.493650	1.066813	0.093649	3.295353	1.602628	0.095352
	$\widehat{\eta}$	0.992794	0.163651	−0.007206	2.159513	0.194828	−0.040487
	$\widehat{\theta}$	1.891585	0.184891	0.091584	2.971380	0.459676	0.171379
200	$\widehat{\delta}$	1.416261	0.501824	0.016260	3.291924	0.976563	0.091924
	$\widehat{\eta}$	0.983430	0.093765	−0.016569	2.180209	0.112095	−0.019790
	$\widehat{\theta}$	1.866071	0.108476	0.066071	2.888326	0.250451	0.088326
300	$\widehat{\delta}$	1.414486	0.347691	0.014485	3.303093	0.617848	0.103093
	$\widehat{\eta}$	0.988928	0.066943	−0.011071	2.209978	0.059870	0.009977
	$\widehat{\theta}$	1.846026	0.078220	0.046025	2.825685	0.107226	0.025684
400	$\widehat{\delta}$	1.366821	0.243250	−0.033179	3.224899	0.427394	0.024899
	$\widehat{\eta}$	0.969982	0.051280	−0.030017	2.187224	0.046741	−0.012776
	$\widehat{\theta}$	1.859620	0.066228	0.059619	2.841787	0.084269	0.041786
500	$\widehat{\delta}$	1.390782	0.207865	−0.009217	3.257464	0.322750	0.057463
	$\widehat{\eta}$	0.984776	0.041956	−0.015223	2.204851	0.030738	0.004851
	$\widehat{\theta}$	1.836700	0.049477	0.036699	2.819982	0.048753	0.019981
600	$\widehat{\delta}$	1.397843	0.181395	−0.002156	3.255041	0.281289	0.055040
	$\widehat{\eta}$	0.989041	0.036407	−0.010958	2.205458	0.025605	0.005457
	$\widehat{\theta}$	1.829099	0.041570	0.029098	2.810966	0.040713	0.010966
700	$\widehat{\delta}$	1.382622	0.145829	−0.017378	3.239750	0.275368	0.039750
	$\widehat{\eta}$	0.986133	0.029855	−0.013866	2.200292	0.024986	0.000292
	$\widehat{\theta}$	1.828738	0.033835	0.028737	2.818897	0.039654	0.018897
800	$\widehat{\delta}$	1.383342	0.132699	−0.016657	3.232778	0.222014	0.032777
	$\widehat{\eta}$	0.984593	0.027505	−0.015406	2.201247	0.021073	0.001247
	$\widehat{\theta}$	1.829511	0.031400	0.029511	2.811853	0.033239	0.011852
900	$\widehat{\delta}$	1.362719	0.105432	−0.037280	3.230148	0.183249	0.030148
	$\widehat{\eta}$	0.978495	0.022500	−0.021504	2.201453	0.017992	0.001453
	$\widehat{\theta}$	1.831974	0.025314	0.031974	2.808406	0.027796	0.008405
1000	$\widehat{\delta}$	1.395490	0.097188	−0.004509	3.236034	0.154365	0.036033
	$\widehat{\eta}$	0.992605	0.020196	−0.007394	2.204886	0.014572	0.004885
	$\widehat{\theta}$	1.817425	0.022947	0.017425	2.801814	0.022499	0.001814

6. Estimation and simulation

Further work of this research paper presented in this section is divide into two subsections. In very first subsection, the MLEs (maximum likelihood estimators) $(\widehat{\delta}MLE, \widehat{\vartheta}MLE)$ of the parameteors (δ, ϑ) are derived. In the next subsection, a comprehensive simulation study based on NFEP-Wei distribution as a special sub-model of the NFEP family of distributions is conducted to evaluate the performance of these $(\widehat{\delta}MLE, \widehat{\vartheta}MLE)$.

6.1. MLE method

Maximum Likelihood Estimation (MLE) is a popular technique for calculating the parameters of a statistical model. The fundamental principle of MLE involves finding the set of parameter values that maximizes the likelihood of the observed data. The likelihood function assesses how well the model aligns with the data across various parameter values. It depends on both the model parameters and the observed data. The MLE method seeks the parameter values that maximize the likelihood of the observed data being generated by the model. In this section, we define the MLE method for estimating the unknown parameters of the NFEP family of distributions. Let suppose the observed values $X_1, X_2, ..., X_d$ of size d be selected randomly from the NFEP family of distributions with parameters (δ, ϑ). Then corresponding to Eq. (5), the LF $\ell(\Theta)$ is acquired as follows;

Table 2
Simulations results of the NFEP-Wei distribution for set III and set IV.

		Set III: $\delta = 2.4, \eta = 1.5, \theta = 1.9$			Set IV: $\delta = 3.0, \eta = 1.0, \theta = 4.6$		
n	Parameters	MLE	MSEs	Biases	MLE	MSEs	Biases
25	$\widehat{\delta}$	2.609820	3.205297	0.020981	3.690870	2.101958	0.690869
	$\widehat{\eta}$	1.410057	0.419553	−0.089943	1.112048	0.101021	0.112048
	$\widehat{\theta}$	2.255538	0.819189	0.355537	4.467409	0.419440	−0.132591
50	$\widehat{\delta}$	2.623254	2.316416	0.022325	3.485139	1.493573	0.485139
	$\widehat{\eta}$	1.473858	0.292558	−0.026141	1.084928	0.069390	0.084928
	$\widehat{\theta}$	2.088966	0.456476	0.188965	4.501526	0.310415	−0.098473
75	$\widehat{\delta}$	2.493442	1.870164	0.0934421	3.357265	1.206166	0.424438
	$\widehat{\eta}$	1.434147	0.252774	−0.065852	1.059008	0.055906	0.074899
	$\widehat{\theta}$	2.100369	0.389419	0.200368	4.527652	0.258406	−0.085958
100	$\widehat{\delta}$	2.563526	1.642578	0.016352	3.351537	0.994084	0.357265
	$\widehat{\eta}$	1.480091	0.205850	−0.019908	1.068215	0.047298	0.059008
	$\widehat{\theta}$	2.020522	0.296469	0.120521	4.502337	0.237131	−0.072347
200	$\widehat{\delta}$	2.414098	0.800394	0.001409	3.196913	0.588330	0.196913
	$\widehat{\eta}$	1.460252	0.113060	−0.039747	1.032180	0.029630	0.032180
	$\widehat{\theta}$	1.993773	0.153063	0.093773	4.571760	0.159532	−0.028240
300	$\widehat{\delta}$	2.425807	0.527329	0.002580	3.101236	0.404496	0.101236
	$\widehat{\eta}$	1.479430	0.075185	−0.020569	1.014651	0.021514	0.014651
	$\widehat{\theta}$	1.958515	0.097220	0.058515	4.600143	0.123264	0.000143
400	$\widehat{\delta}$	2.431273	0.388575	0.003127	3.082540	0.327923	0.082540
	$\widehat{\eta}$	1.495071	0.056090	−0.004928	1.012884	0.018184	0.012884
	$\widehat{\theta}$	1.927438	0.067249	0.027438	4.598751	0.109225	−0.001248
500	$\widehat{\delta}$	2.375952	0.288865	−0.024048	3.062336	0.267384	0.062335
	$\widehat{\eta}$	1.475582	0.040985	−0.024417	1.006813	0.015123	0.006813
	$\widehat{\theta}$	1.940625	0.048139	0.040624	4.606532	0.090850	0.006531
600	$\widehat{\delta}$	2.419211	0.231678	0.019210	3.013987	0.205556	0.013986
	$\widehat{\eta}$	1.496547	0.032660	−0.003452	0.999693	0.012317	−0.000306
	$\widehat{\theta}$	1.920273	0.034514	0.020273	4.622437	0.078337	0.022436
700	$\widehat{\delta}$	2.371514	0.184530	0.017412	3.078559	0.205900	0.078559
	$\widehat{\eta}$	1.481122	0.025949	−0.007839	1.013503	0.012350	0.013503
	$\widehat{\theta}$	1.929855	0.027495	0.016698	4.588683	0.078587	−0.011316
800	$\widehat{\delta}$	2.410285	0.142367	0.001028	3.051371	0.173223	0.051371
	$\widehat{\eta}$	1.493218	0.019086	−0.006782	1.007967	0.010747	0.007967
	$\widehat{\theta}$	1.918665	0.017738	0.018665	4.602270	0.068458	0.002270
900	$\widehat{\delta}$	2.384074	0.126248	−0.015925	3.026016	0.143721	0.026016
	$\widehat{\eta}$	1.488186	0.017785	−0.011813	1.003219	0.009000	0.003219
	$\widehat{\theta}$	1.921149	0.018141	0.021148	4.606797	0.058341	0.006797
1000	$\widehat{\delta}$	2.393682	0.118201	−0.006317	3.023297	0.139371	0.023297
	$\widehat{\eta}$	1.490034	0.015884	−0.009966	1.001446	0.008993	0.001446
	$\widehat{\theta}$	1.913514	0.014355	0.013514	4.613701	0.061406	0.013701

$$\ell(\Theta) = n\log 2 + n\log\delta + 2\sum_{c=1}^{d}\log A(x_c;\vartheta) + 2\sum_{c=1}^{d}\log a(x_c;\vartheta) + 2\delta\sum_{c=1}^{d}\log A(x_c;\vartheta)^2 - n\delta.$$

The partial derivatives of $\ell(\Theta)$ are given by

$$\frac{d}{d\delta}\ell(\Theta) = \frac{n}{\delta} + 2\sum_{c=1}^{d}\log A(x_c;\vartheta)^2 - n,$$

and

$$\frac{d}{d\vartheta}\ell(\Theta) = 2\sum_{c=1}^{d}\frac{dA(x_c;\vartheta)/d\vartheta}{A(x_c;\vartheta)} + 2\sum_{c=1}^{d}\frac{da(x_c;\vartheta)/d\vartheta}{a(x_c;\vartheta)} + 4\delta\sum_{c=1}^{d}A(x_c;\vartheta)(dA(x_c;\vartheta)/d\vartheta),$$

where $\Theta = (\delta, \vartheta)$.

Table 3
Four biomedical Datasets.

No.	Observation of the data sets	References
Data 1.	10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555	Bjerkedal [27]
Data2.	3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048	Liu et al. [28]
Data3.	12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776	Ceren et al. [29]
Data4.	1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0	Gross et al. [30]

Setting $\frac{d}{d\delta}\ell(\Theta)=0$ and $\frac{d}{d\vartheta}\ell(\Theta)=0$, solving above equations simultaneously, we will get MLEs $(\widehat{\delta}MLE, \widehat{\vartheta}MLE)$ of the parameters (δ, ϑ), respectively.

6.2. Simulation

Many techniques may be used to assess the efficacy of distribution estimators, one of which is Monte Carlo simulation with subsampling (MCS). MCS study is a resampling approach that includes producing several random samples from a known distribution, estimating the distribution's parameters using an estimator, and assessing the estimator's performance by contrasting the estimated parameters with the distribution's actual values. The effectiveness of the NFEP-Wei distribution estimators is evaluated using an MCS study. The simulation results out for four sets of parameters values; (i) *Set*I $(\delta=1.4,\eta=1.0,\theta=1.8)$, (ii) Set II $(\delta=3.2,\eta=2.2,\theta=2.8)$, (iii) Set III $(\delta=2.4,\eta=1.5,\theta=1.9)$, (iv) Set IV $(\delta=3.0,\eta=1.0,\theta=4.6)$ are calculated. For each of the aforementioned sets of parameter values, 1000 separate MCS study replicates are created using $c=25,50,\ldots,1000$. For each set of MCS study, we calculated the typical MLEs, MSEs, and biases. The new suggested model's numerical values for its biases and MSE are computed as follows:

$$Bias=\frac{1}{1000}\sum_{c=1}^{1000}(\widehat{\delta}_c-\delta),$$

and

$$MSE=\frac{1}{1000}\sum_{c=1}^{1000}(\widehat{\delta}_c-\delta)^2.$$

For numerical values of ϑ, the same method is reiterated. Table 1 displays the simulation results for *Set*I and Set II. Table 2 presents the outcomes of Sets III and IV. It is clearly observed from Tables 1 and 2 that the estimated values of $(\widehat{\delta},\widehat{\vartheta})MLE$ are steady and the MSE of $(\widehat{\delta},\widehat{\vartheta})MLE$ decreases. Biases of $(\widehat{\delta},\widehat{\vartheta})MLE$ decline or tend to zero as the sample size "c" increase or tend to infinity.

7. Application to biomedical data

Here in this section, for practical illustration, we consider four data sets from biomedical area. The first biomedical dataset (Data 1) consists of seventy-two (72) observations, depicting the survival times (ST) of guinea pigs infected with various amounts of tubercle bacilli. The second dataset (Data 2), which is also available at [https://covid19.who.int/] contains 36 observations and represents the death rate of COVID-19 patients in Canada of 36 days, from 10 April to May 15, 2020, for the purposes of numerical analysis. The third data set (Data 3) is made up of forty-four (44) observations and shows how long patients with head and neck cancer survive. Similarly, the fourth data set (Data 4) has twenty (20) observations and depicts the life periods of analgesic patients. For each data sets, the NFEP-Wei distribution is compared with different well-known distribution and observe that the proposed distribution outclasses then the other competitor. Table 3 contains all of the data sets.

The NFEP-Wei distribution is applied to all the considered data sets. The fitting results (to select the best distribution) are compared with (i) AP-Wei (alpha power Wei) model developed by Dey et al. [31], (ii) MO-Wei (Marshall-Olkin Wei) model proposed by Marshall and Olkin [32], (iii) NExpo-Wei (new exponential Wei) model proposed by Shah et al. [33], (iv) FRLog-Wei (flexible reduced logarithmic Wei) model proposed by Liu et al. [28], (v) classical Wei distribution proposed by Weibull [34], and (vi) Kum-Wei (Kumaraswamy Weibull) distribution introduced by Cordeiro et al. [35]. The CDF of these well-known compared distributions are outlined as.

- The three parameters AP-Wei model

$$G(x;a,\vartheta)=\frac{a^{\left(1-e^{-\eta x^{\vartheta}}\right)}-1}{a-1},x\in R^{+},$$

where, $a\neq 1, a\in R^{+}$.

Table 4
Summary measures of Data 1.

Min	1st Qu	Mean	Median	3rd Qu	Max
10.00	108.00	176.80	149.50	224.00	555.00

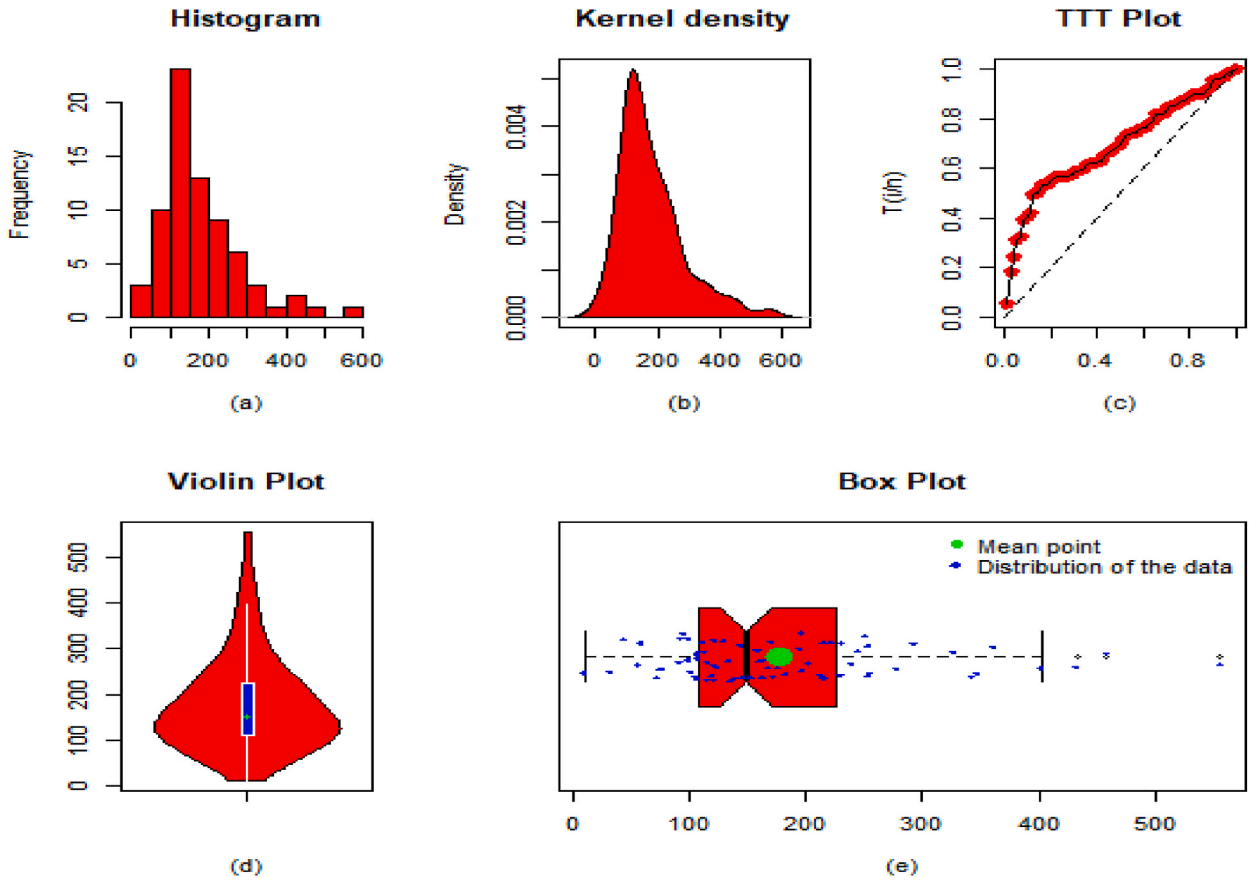

Fig. 4. Visual illustration of dataset (Data 1) using (a) Histogram plot, (b)Kernel density plot, (c)TTT plot, (d) Violin plot, and (e) Box plot.

Table 5
The $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the proposed and other competitive distributions for Data 1.

Dist.	$\widehat{\eta}MLE$	$\widehat{\theta}MLE$	$\widehat{\delta}MLE$	$\widehat{a}MLE$	$\widehat{b}MLE$	$\widehat{\beta}MLE$
NFEP-Wei	**0.048724**	**0.796168**	**4.919883**	–	–	–
AP-Wei	0.003250	1.190032	–	6.817653	–	–
MO-Wei	0.002404	1.242525	–	–	–	2.371487
Wei	0.002055	1.197816	–	–	–	–
FRL-Wei	0.003047	1.222505	–	–	–	8.160021
NExpo-Wei	0.002379	1.040409	–	–	–	–
Kum-Wei	0.013561	0.937990	–	3.760287	1.334416	–

- The three parameters MO-Wei distribution

$$G(x;\beta,\vartheta)=\frac{\left(1-e^{-\eta x^{\theta}}\right)}{(1-\beta)(1-e^{-\eta x^{\theta}})+\beta},x\in R^{+},$$

where, $\beta\in R^{+}$.

- The two parameters NExpo-Wei distribution

Table 6
The GOFMs and P-values for guinea pigs infected Dataset (Data 1).

Dist.	CM	AD	KS	P-values
NFEP-Wei	0.07059	0.40948	0.07758	0.7798
AP-Wei	0.80373	0.13802	0.17147	0.0290
MO-Wei	0.16136	0.94136	0.17256	0.0274
Wei	0.10983	0.66328	0.25928	0.0002
FRLog-Wei	0.19237	1.12405	0.18161	0.0173
NExpo-Wei	0.08960	0.52217	0.29516	0.0671
Kum-Wei	0.08173	0.52259	0.09174	0.5795

Table 7
The DMs for guinea pigs infected Dataset (Data 1).

Dist.	AIC	BIC	CAIC	HQIC
NFEP-Wei	854.90890	861.73891	855.26180	857.62791
AP-Wei	864.12672	870.95672	864.47971	866.84586
MO-Wei	865.71113	872.54111	866.06415	868.43022
Wei	877.46762	882.02180	877.64165	879.28033
FRLog-Wei	868.08501	874.91520	868.43832	870.80412
NExpo-Wei	902.70857	907.26185	902.88241	904.52120
Kum-Wei	859.49202	868.59871	860.08914	863.11743

$$G(x;\vartheta) = 1 - \left(\frac{e^{e^{-\eta x^{\theta}}} - 1}{e - e^{-\eta x^{\theta}}}\right), x \in R^{+}.$$

- The four parameters Kum-Wei distribution

$$G(x;a,b,\vartheta) = 1 - \left[1 - \left(1 - e^{-\eta x^{\theta}}\right)^{a}\right]^{b}, x \in R^{+},$$

where $a, b \in R^{+}$.

- The two parameters Wei distribution

$$G(x;\eta,\theta) = 1 - e^{-\eta x^{\theta}}, x \in R^{+}.$$

- The three parameters FRlog-Wei distribution

$$G(x;\beta,\vartheta) = 1 - \frac{\log\left(\beta + 1 - \beta\left(1 - e^{-\eta x^{\theta}}\right)\right)}{\log(1+\beta)}, x \in R^{+},$$

where $\beta \in R^{+}$.

After selecting the competing distributions, we consider AMs (analytical measures) to verify which distribution is out performed to the Data 1, Data 2, Data 3, and Data 4 among these fitted distributions. The AMs used to compare the fitted distributions are the DMs (discrimination measures), GoFMs (goodness of fit measures), and its P-values. The DMs are contained by the AIC (Akaike Information Criterion), BIC (Bayesian IC), CAIC (Consistent AIC), HQIC (Hannan Quinn IC) while the GoFMs measures are contains by the KS (Kolmogorov-Smirnov), AD (Anderson-Darling) and CM (Cramer-von-Misses). The values of GoFMs are computed as.

- The CM test statistic

$$CM = \sum_{i=1}^{n}\left[K(x_i;\delta,\vartheta) - \frac{2i-1}{2n}\right]^{2} + \frac{1}{12n}.$$

- The AD test statistics

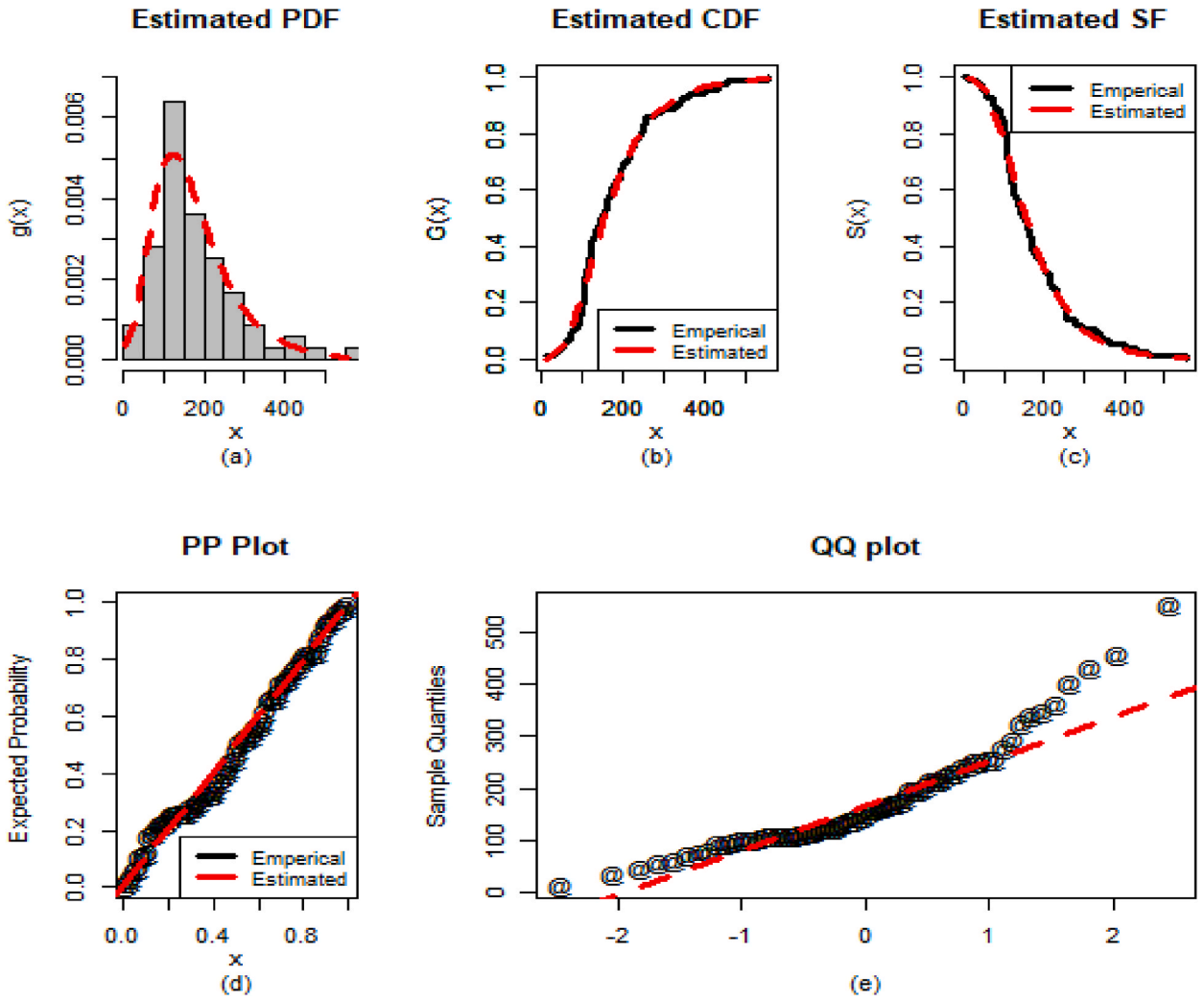

Fig. 5. The illustrations plots of (a) fitted PDF, (b)empirical CDF, (c) fitted SF, (d) PP plot, and (e) QQ plot of the NFEP-Weib distribution for Data 1.

Table 8
Summary measures of Data 2.

Mini	1st-Qu	Mean	Median	3rd-Qu	Max
1.516	2.788	3.282	3.178	3.637	6.868

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1) \times [\log K(x_i;\delta,\vartheta) + \log(1 - K(x_{i+1-n};\delta,\vartheta))].$$

- The KS test statistic

  $\sup x[K_n(x;\delta,\vartheta) - K(x;\delta,\vartheta)]$.

  While, the DMs are computed as.

- The AIC test statistics

  $AIC = 2w - 2\ell(\varphi)$.

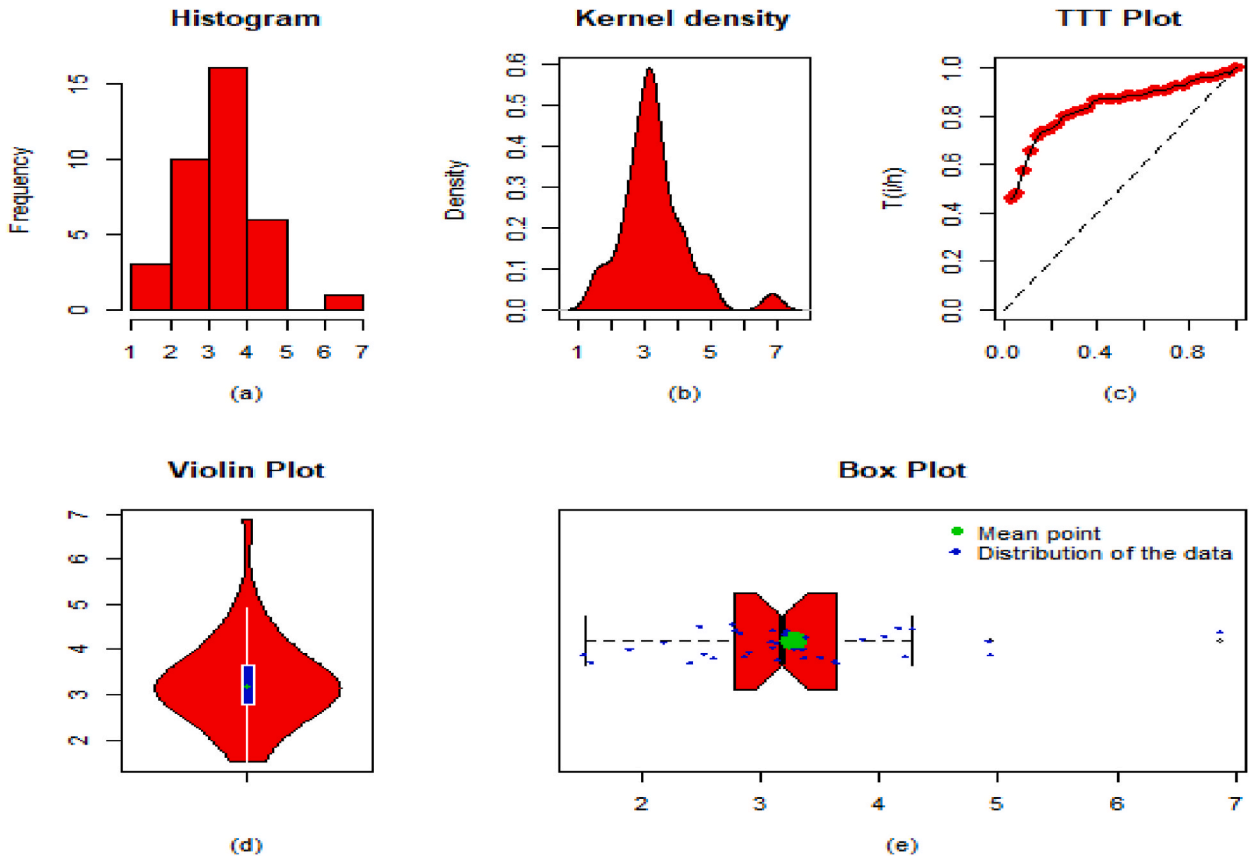

Fig. 6. Visual illustration of dataset (Data 2) using (a) Histogram plot, (b)Kernel density plot, (c)TTT plot, (d) Violin plot, and (e) Box plot.

Table 9
The $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the competitive models using COVID-19 Dataset (Dataset 1).

Dist.	$\widehat{\eta}MLE$	$\widehat{\theta}MLE$	$\widehat{\delta}MLE$	$\widehat{a}MLE$	$\widehat{b}MLE$	$\widehat{\beta}MLE$
NFEP-Weib	**0.664791**	**1.342981**	**7.695297**	–	–	–
AP-Weib	0.205920	1.953723	-	171.54246	–	–
MO-Weib	0.038697	2.832476	-	–	–	1.927467
Wei	0.014140	3.301129	-	–	–	–
FRLog-Weib	0.042111	2.814466	-	–	–	3.678752
NExpo-Weib	0.003600	3.908928	-	–	–	–
Kum-Weib	0.695443	1.063776	-	13.023971	2.111903	–

Table 10
The GOFMs and P-values for the COVID-19 dataset (data 2).

Dist.	CM	AD	KS	P-values
NFEP-Weib	0.08129	0.47084	0.10636	0.8101
AP-Wei	0.10228	0.58178	0.12286	0.6488
MO-Wei	0.18187	1.03546	0.14083	0.4732
Wei	0.17233	0.98839	0.14836	0.4065
FRL-Wei	0.19842	1.12775	0.14451	0.4398
NExpo-Wei	0.13717	0.79325	0.14381	0.4463
Kum-Wei	0.09268	0.53943	0.10583	0.7948

- The BIC test statistics

$$BIC = w\log(n) - 2\ell(\varphi).$$

Table 11
The DMs for the COVID-19 dataset (data 2).

Dist.	AIC	BIC	CAIC	HQIC
NFEP-Wei	101.65857	106.40912	102.40854	103.31662
AP-Wei	103.61564	108.36619	104.36563	105.27365
MO-Wei	109.44277	114.19336	110.19272	111.10084
Wei	106.94972	110.11680	107.31340	108.05512
FRLog-Wei	110.51631	115.26680	111.26630	112.17432
NExpo-Wei	104.29355	107.46054	104.65715	105.39890
Kum-Wei	104.00192	110.33626	105.29228	106.21276

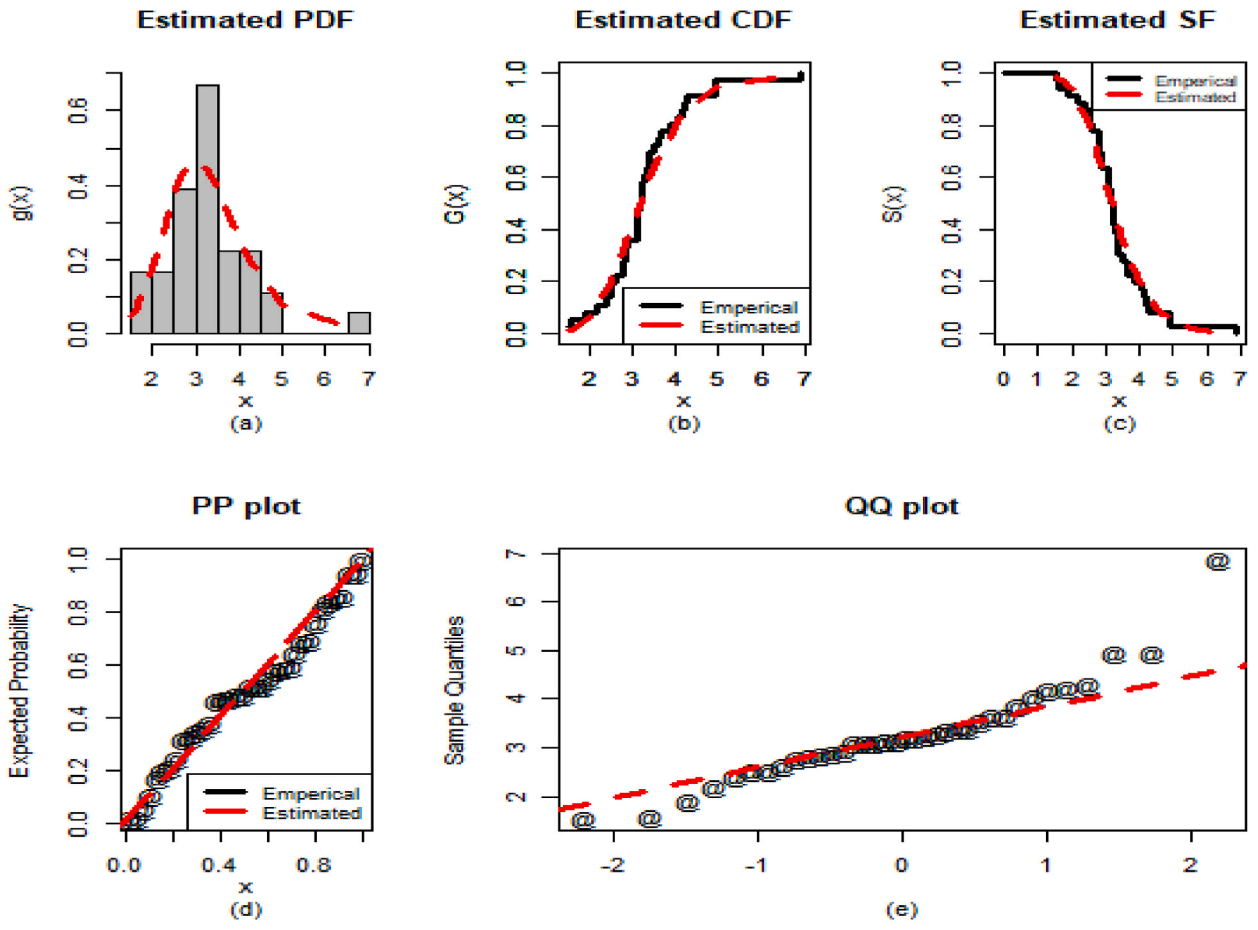

Fig. 7. The illustrations plots of (a) fitted PDF, (b)empirical CDF, (c) fitted SF, (d) PP plot, and (e) QQ plot of the NFEP-Weib distribution for Data 2.

Table 12
Summary measures of Data 3.

Min.	1st-Qu.	Mean	Median	3rd-Qu.	Max.
12.20	67.2100	223.500	128.500	219.000	1776.00

- The CAIC test statistics

$$CAIC = \frac{2nw}{n-w-1} - 2\ell(\varphi).$$

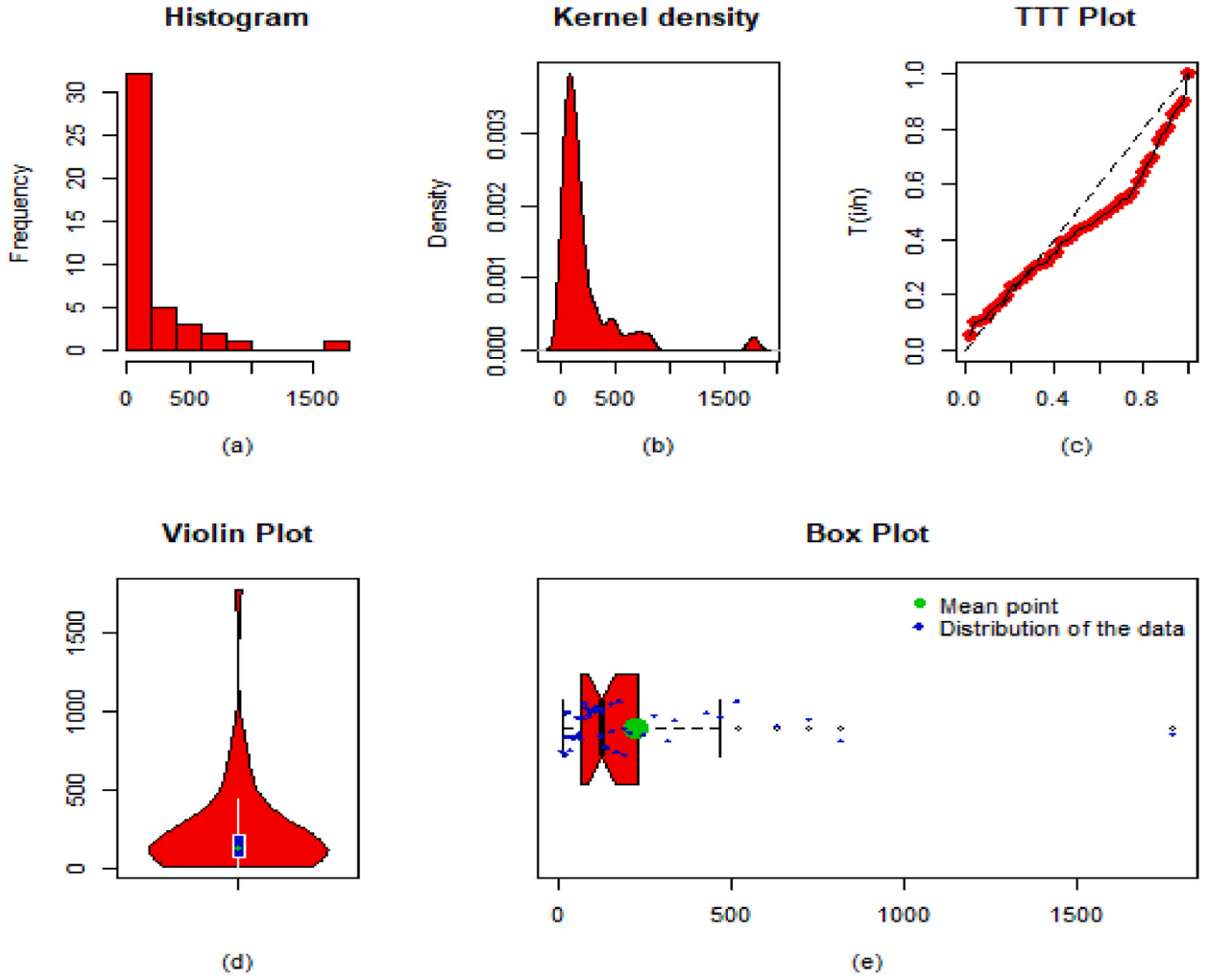

Fig. 8. Visual illustration of dataset (Data 3) using (a) Histogram plot, (b)Kernel density plot, (c)TTT plot, (d) Violin plot, and (e) Box plot.

Table 13
The $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the competitive models using head and neck cancer Dataset (Dataset 3).

Dist.	$\widehat{\eta}MLE$	$\widehat{\theta}MLE$	$\widehat{\delta}MLE$	$\widehat{a}MLE$	$\widehat{b}MLE$	$\widehat{\beta}MLE$
NEP-Wei	**1.182393**	**0.265807**	**23.957386**	–	–	–
AP-Wei	0.003265	0.992700	–	0.245030	–	–
MO-Wei	0.003033	1.001190	-	–	–	0.507524
Wei	0.007092	0.923501	-	–	–	–
FRLog-Wei	0.028596	0.761846	-	–	–	5.721750
NExpo-Wei	0.002456	1.022756	-	–	–	–
Kum-Wei	0.416466	0.459511	-	12.665165	0.490206	–

- The HQIC test statistics

$$HQIC = 2w\log(\log(n)) - 2\ell(\varphi).$$

Where, $\ell(\varphi)$ is MLF at MLEs, n is the SS (sample size), and w is the number of parameters in the model. We implement statistical R software using (AdquacyModel) package with the method of "BFGS algorithm" for the AMs of Data 1, Data 2, Data 3, and Data 4.

In general, a statistical model is deemed a better probability model for the considered data sets if it has lower GoFMs, DMs values, and a higher P-value. By using these AMs, it is discovered that the NFEP-Wei model outperforms then the other competing or fitting

Table 14
The GOFMs and P-values of the competitive models using head and neck cancer Dataset (Data 3).

Dist.	CM	AD	KS	P-values
NFEP-Wei	0.01833	0.11950	0.05692	0.9973
AP-Wei	0.09338	0.55387	0.10551	0.6723
MO-Wei	0.09492	0.56181	0.11255	0.5933
Wei	0.13834	0.80581	0.12425	0.4682
FRLog-Wei	0.19103	1.09553	0.13355	0.3789
NExpo-Wei	0.08657	0.51532	0.10064	0.7270
Kum-Wei	0.02199	0.13589	0.06546	0.9854

Table 15
The DMs of the competitive models using head and neck cancer Dataset (Data 3).

Dist.	AIC	BIC	CAIC	HQIC
NFEP-Wei	560.73223	566.08487	561.33222	562.71721
AP-Wei	567.77125	573.12383	568.37133	569.75623
MO-Wei	568.20842	573.56198	568.80847	570.19348
Wei	567.71520	571.28353	568.00783	569.03854
FRLog-Wei	572.88337	578.23595	573.48339	574.86839
NExpo-Wei	565.15683	568.72526	565.44952	566.48014
Kum-Wei	562.71149	569.84784	563.73664	565.35763

distributions for all of the studied biomedical datasets.

7.1. Analyzing data 1

Here, in this sub-section, we try to apply the NFEP-Wei distribution and compare its goodness of fit with the considered competing distribution. Corresponding to Data 1, the SMs (summary measures) are listed in Table 4, while the HP (histogram plot), KD (Kernel density), TTT-P (total time-on-test plot), VP (Violin plot), and BP (Box plot) are given in Fig. 4(a–e). Similarly, corresponding to Data 1, the $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the NFEP-Wei distribution and other competitive probability distributions are presented in Table 5. The GoFMs and P-values are provided in Table 6 and the DMs of the NFEP-Wei and other competitive distributions are presented in Table 7. The fitted PDF, CDF, SF, PP (probability probability), and QQ (quintile quintile) plots of the NFEP-Wei distribution for the analyzed data set are sketched in Fig. 5(a–e).

From the results presented in Tables 6 and 7, we can clearly see that the NFEP-Wei distribution has minimum values of these AMs and higher P-value than the other distributions applied in comparison. Similarly, from Fig. 5(a–e), it is also clear that the NFEP-Wei distribution fit estimated PDF, CDF, SF, PP (probability probability), and QQ (quintile quintile) plots very closely.

Hence, from the above discussion, we can clearly conclude that the proposed NFEP-Wei distribution is a better competitor than the other well-known probability distributions for the Data 1.

7.2. Analyzing data 2

In this sub-section, we again apply the NFEP-Wei distribution to Data 2, and compare its potentiality power to the other competitive distributions. Corresponding to Data 2, the SMs are listed in Table 8, while the HP, KD, TTT-P, VP, and BP plots are given in Fig. 6(a–e). Similarly, corresponding to Data 2, the $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the NFEP-Wei distribution and other competitive probability distributions are presented in Table 9. The GoFMs and P-values are provided in Table 10 and the DMs of the NFEP-Wei and other competitive distributions are presented in Table 11. The fitted PDF, CDF, SF, PP (probability probability), and QQ (quintile quintile) plots of the NFEP-Wei distribution for the analyzed data set are sketched in Fig. 7(a–e).

Based on the results presented in Tables 10 and 11, it is evident that the NFEP-Wei distribution exhibits the lowest values of these AMs and a higher P-value compared to the other competitive distributions used in the comparison. Additionally, Fig. 7(a–e) shows that the NFEP-Wei distribution closely matches the estimated PDF, CDF, SF, PP (probability probability), and QQ (quintile quintile) plots. Therefore, based on the above discussion, it is clear that the proposed NFEP-Wei distribution outperforms than the other well-known probability distributions for Data 2.

7.3. Analyzing data 3

In this sub-section, we again apply the NFEP-Wei distribution to Data 3, and compare its potentiality power to the other competitive distributions. Corresponding to Data 3, the SMs are listed in Table 12, while the HP, KD, TTT-P, VP, and BP plots are given in Fig. 8 (a–e). Similarly, corresponding to Data 3, the $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the NFEP-Wei distribution and other competitive probability distributions are presented in Table 13. The GoFMs and P-values are provided in Table 14 and the DMs of

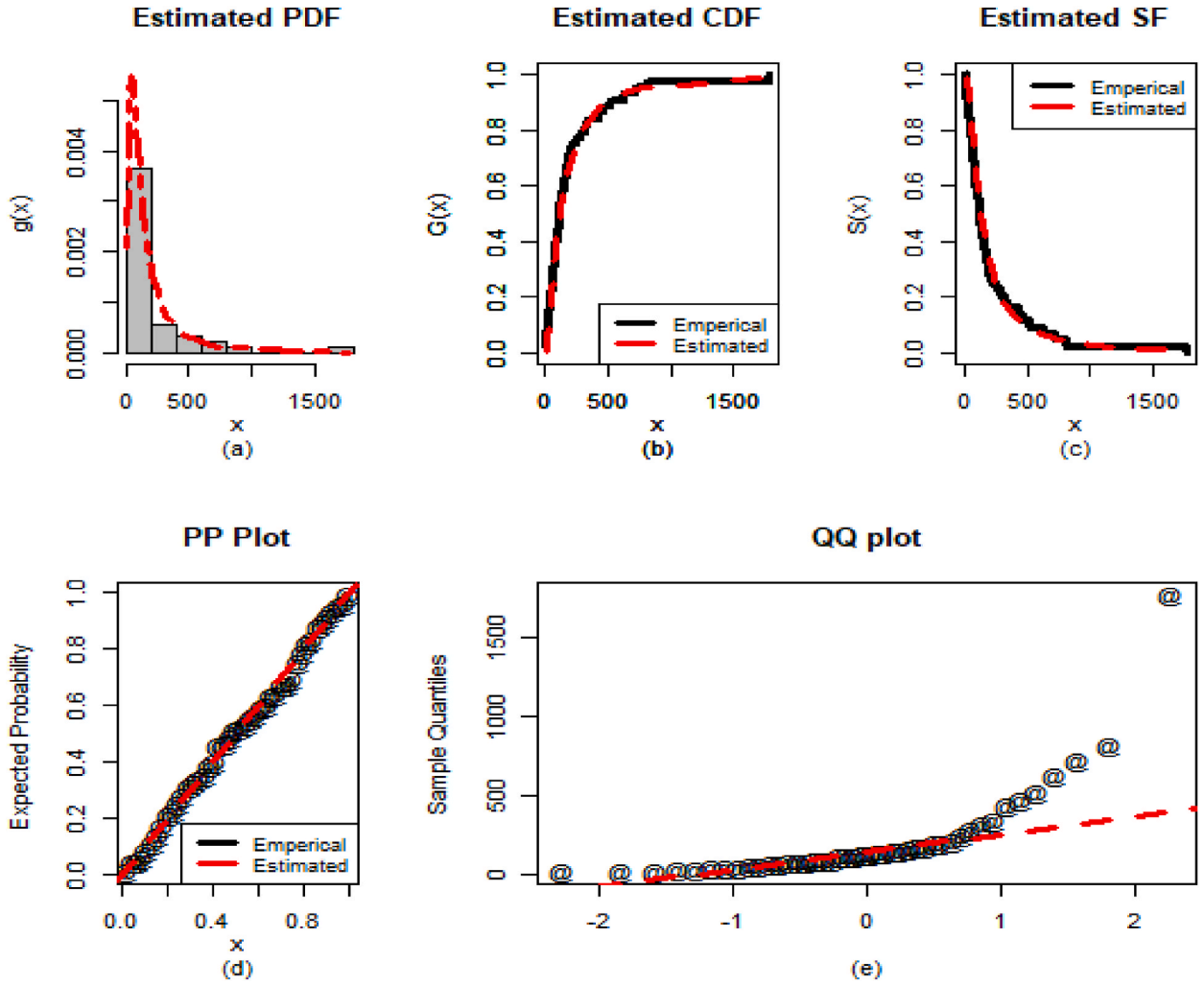

Fig. 9. The illustrations plots of (a) fitted PDF, (b)empirical CDF, (c) fitted SF, (d) PP plot, and (e) QQ plot of the NFEP-Weib distribution for Data 3.

Table 16
Summary measures of Data 4.

Min.	1st Qu.	Mean.	Median.	3rd Qu.	Max.
1.1000	1.4750	1.9000	1.7000	2.0500	1776.00

the NFEP-Wei and other competitive distributions are presented in Table 15. The fitted PDF, CDF, SF, PP (probability-probability), and QQ (quantile-quantile) plots of the NFEP-Wei distribution for the analyzed data set are sketched in Fig. 9(a–e).

From the results presented in Tables 14 and 15, we can also clearly see that the NFEP-Wei distribution has lower values of these AMs and higher P-value than the other competitive distributions applied in comparison. Similarly, from Fig. 9(a–e), it is also clear that the NFEP-Wei distribution fit estimated PDF, CDF, SF, PP (probability-probability), and QQ (quantile-quantile) plots very closely.

Hence, from the visual display and the above discussion, we can clearly conclude that the proposed NFEP-Wei distribution is a better competitor than the other well-known probability distributions for the Data 3.

7.4. Analyzing data 4

Here, we again apply the NFEP-Wei distribution to Data 4, and compare its potentiality power to the other well-known competitive distributions. Corresponding to Data 4, the SMs are listed in Table 16, while the HP, KD, TTT-P, VP, and BP plots are given in Fig. 10(a and b). Similarly, corresponding to Data 4, the $\widehat{\eta}MLE$, $\widehat{\theta}MLE$, $\widehat{\delta}MLE$, $\widehat{a}MLE$, $\widehat{b}MLE$, and $\widehat{\beta}MLE$ values of the NFEP-Wei distribution and other competitive probability distributions are presented in Table 17. The GoFMs and P-values are provided in Table 18 and the DMs of

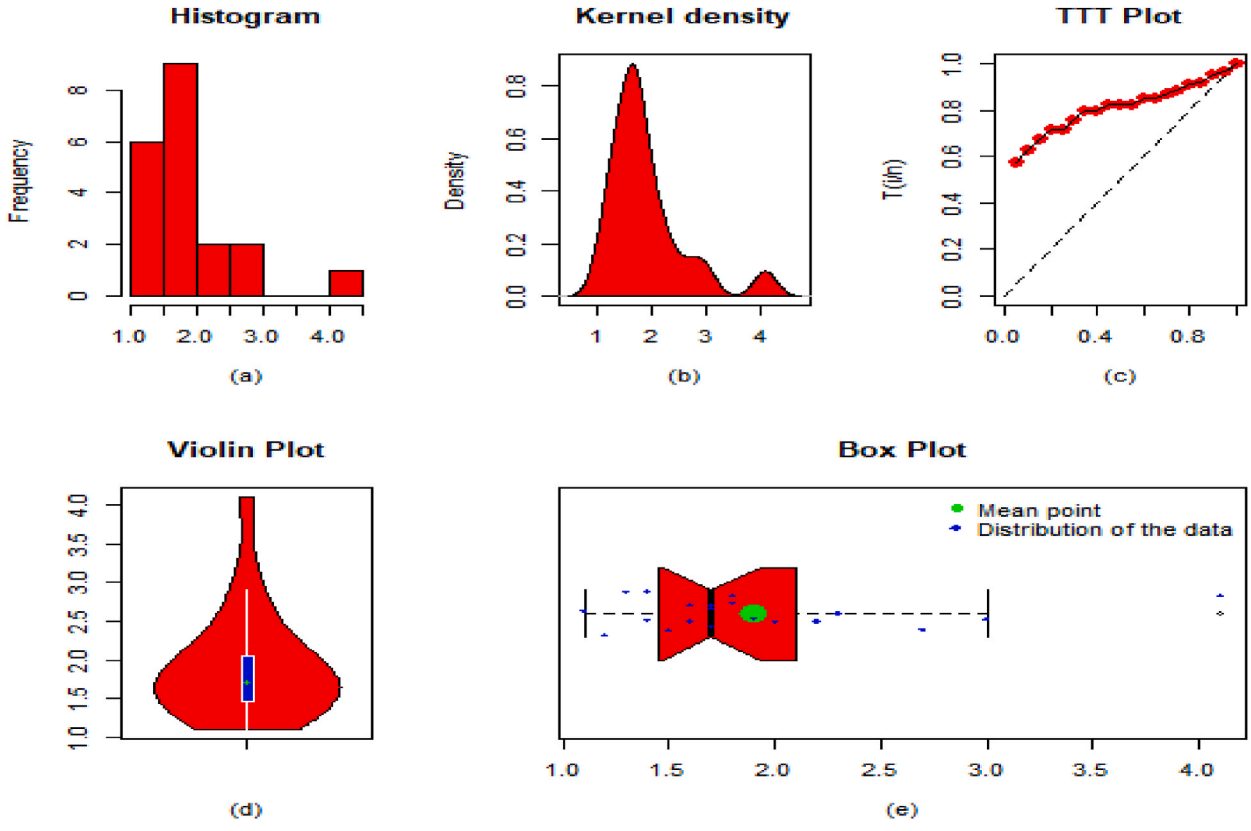

Fig. 10. Visual illustration of dataset (Data 4) using (a) Histogram plot, (b)Kernel density plot, (c)TTT plot, (d) Violin plot, and (e) Box plot.

Table 17
The $\hat{\eta}MLE$, $\hat{\theta}MLE$, $\hat{\delta}MLE$, $\hat{a}MLE$, $\hat{b}MLE$, and $\hat{\beta}MLE$ values of the competitive models using patients receiving analgisic Dataset (Dataset 4).

Dist.	$\hat{\eta}MLE$	$\hat{\theta}MLE$	$\hat{\delta}MLE$	$\hat{a}MLE$	$\hat{b}MLE$	$\hat{\beta}MLE$
NFEP-Wei	**3.29444**	**0.77505**	**57.98659**	–	–	–
AP-Wei	0.02074	3.62007	–	0.01587	–	–
MO-Wei	0.00371	4.40523	-	–	–	0.04792
Wei	0.12157	2.78699	-	–	–	–
FRL-Wei	0.19153	2.54816	-	–	–	1.363036
NExpo-Wei	0.04603	3.32338	-	–	–	–
Kum-Wei	2.74119	1.28961	-	55.89059	0.38631	–

Table 18
The GOFMs and P-values of the competitive models using patients receiving analgisic Dataset (Data 4).

Dist.	CM	AD	KS	P-values
NFEP-Wei	0.04526	0.26371	0.11608	0.9083
AP-Wei	0.13242	0.77844	0.15791	0.7009
MO-Wei	0.07829	0.46896	0.13609	0.8527
Wei	0.18571	1.09288	0.18496	0.5006
FRLog-Wei	0.14024	0.85222	0.22294	0.2732
NExpo-Wei	0.15219	0.90424	0.18129	0.5267
Kum-Wei	0.04755	0.27189	0.12915	0.8925

the NFEP-Wei and other competitive distributions are presented in Table 19. The fitted PDF, CDF, SF, PP (probability probability), and QQ (quantile quantile) plots of the NFEP-Wei distribution for the analyzed data set are sketched in Fig. 11(a–e). From visual graphical representation and the results presented in Tables 18 and 19, we can clearly see that the NFEP-Wei distribution has minimum values of these AMs (GoFMs and GMs) and higher P-value than the other competitive probability distributions applied in comparison. Similarly, from Fig. 11(a–e), it is also clear that the NFEP-Wei distribution fit estimated PDF, CDF, SF, PP (probability probability), and QQ

Table 19
The DMs of the competitive models using patients receiving analgisic Dataset (Data 4).

Dist.	AIC	BIC	CAIC	HQIC
NFEP-Wei	37.88203	40.86922	39.38203	38.46516
AP-Wei	43.41718	46.40438	44.91718	44.00031
MO-Wei	40.81298	43.80018	42.31298	41.39611
Wei	45.17281	47.16427	45.87869	45.56156
FRLog-Wei	44.70328	47.69047	46.20328	45.28641
NExpo-Wei	42.93560	44.92706	43.64148	43.32435
Kum-Wei	39.55053	43.53346	42.21719	40.32804

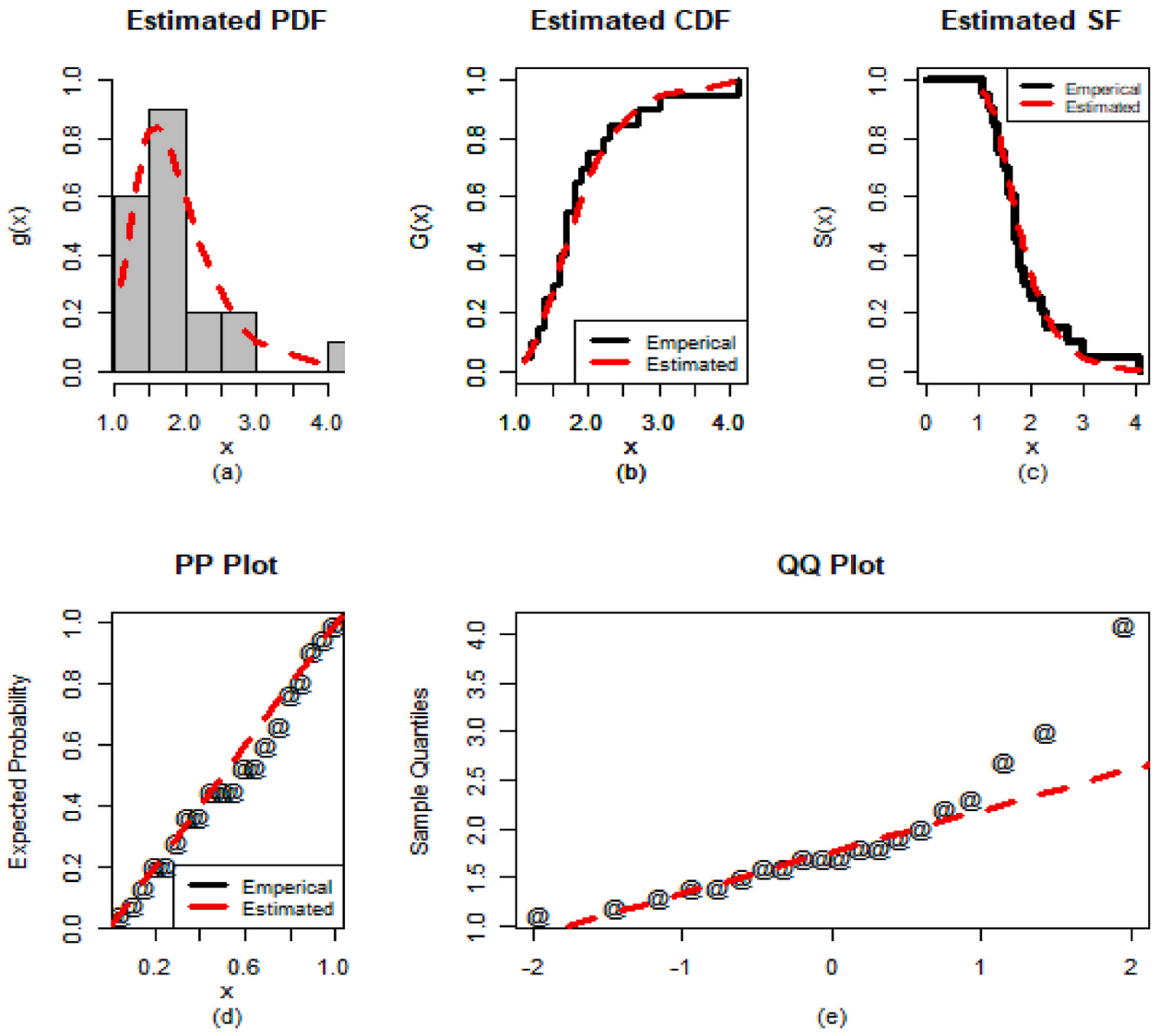

Fig. 11. The illustrations plots of (a) fitted PDF, (b)empirical CDF, (c) fitted SF, (d) PP plot, and (e) QQ plot of the NFEP-Weib distribution for Data 4.

(quintile quintile) plots very well and closely.

Hence, based on the visual graphical display and the comprehensive discussion above, it is evident that the proposed NFEP-Wei distribution outperforms other well-known probability distributions for Data 4.

8. Conclusion

In this study, we introduce and investigate a new flexible family of probability distributions known as the New Flexible Exponent Power (NFEP) family. To assess the potential of this proposed method, we focus on a specific sub-case within this family, the New Flexible Exponent Power Weibull (NFEP-Wei) distribution. We delve into various mathematical aspects, including moments, Moment Generating Function (MGF), residual life, reverse residual life, order statistics (OS), and the quantile function. To achieve this, we employ one of the most well-known estimation approaches, Maximum Likelihood Estimation, to estimate the unknown parameters of the NFEP family of distributions. A brief MCS study is carried out to investigate the efficiency of $\widehat{\eta}MLE$, $\widehat{\theta}MLE$ and $\widehat{\delta}MLE$ of the NFEP family. We applied the NFEP-Wei distribution in four different biomedical datasets and compared its goodness of fit or potentiality power with other well-known distributions, such as the AP-Wei, MO-Wei, Wei, FRLog-Wei, NExpo-Wei, and Kum-Wei distributions. In all four biomedical data sets, the proposed distribution is outclassing. Modeling medical datasets reveals that the NFEP-Wei distribution provides an excellent or optimal fit in contrast to rival distributions based on AMs and P-Values. The findings imply that the suggested approach and the distribution theory models it generates might have practical applications in the biomedical and other relevant domains.

Data availability

Data included in article/supplementary material/referenced in article.

CRediT authorship contribution statement

Zubir Shah: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Dost Muhammad Khan:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Sundus Hussain:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Nadeem Iqbal:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Jin-Taek Seong:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Sundus Naji Alaziz:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Zardad Khan:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:Jin-Taek Seong reports financial support was provided by National Research Foundation of Korea. Jin-Taek Seong reports a relationship with Graduate School of Data Science, Chonnam National University, Gwangju 61186, Republic of Korea that includes: employment. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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