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Correspondence and requests for materials should be addressed to J.R.C. (choiardor@hanmail.net)

Novel characteristics of energy spectrum for 3D Dirac oscillator analyzed via Lorentz covariant deformed algebra

Malika Betrouche[1], Mustapha Maamache[2] & Jeong Ryeol Choi[3]

[1]Laboratoire de Physique Mathématique et Subatomique (lpmps), Départment de Physique, Faculté des Sciences Exactes, Université Constantine1, Constantine 25000, Algeria, [2]Laboratoire de Physique Quantique et Systèmes Dynamiques, Département de Physique, Faculté des Sciences, Université Ferhat Abbas Setif 1, Setif 19000, Algeria, [3]Department of Radiologic Technology, Daegu Health College, Yeongsong-ro 15, Daegu 702-722, Republic of Korea.

We investigate the Lorentz-covariant deformed algebra for Dirac oscillator problem, which is a generalization of Kempf deformed algebra in 3 + 1 dimension of space-time, where Lorentz symmetry are preserved. The energy spectrum of the system is analyzed by taking advantage of the corresponding wave functions with explicit spin state. We obtained entirely new results from our development based on Kempf algebra in comparison to the studies carried out with the non-Lorentz-covariant deformed one. A novel result of this research is that the quantized relativistic energy of the system in the presence of minimal length cannot grow indefinitely as quantum number n increases, but converges to a finite value, $c/\sqrt{\beta}$ where c is the speed of light and β is a parameter that determines the scale of noncommutativity in space. If we consider the fact that the energy levels of ordinary oscillator is equally spaced, which leads to monotonic growth of quantized energy with the increment of n, this result is very interesting. The physical meaning of this consequence is discussed in detail.

Large momenta tied to large spatial dimensions, that appears in string theoretic considerations, may shed light to the existence of a minimal length scale. The coordinates in D-dimensional space no longer commutable with one another under the presence of minimal length scale. The interest for studying the effects of noncommutativity on characteristics of quantum systems has been gradually increasing over the past decade. Several works in this direction suggest the existence of natural ultra violet (UV) cut-off at the planck scale. This reflects the existence of non-zero uncertainties in position and/or in momentum, giving the concept of the generalized uncertainty principal (GUP) that involves some correction terms in its expression. Of course, the consideration of GUP in a certain quantum system leads to a modification of both wave functions and the corresponding energy spectrum[1–4].

Perturbative string theory in sufficiently high energy regime such as black hole may give rise to disturbance of space-time implying the appearance of non-negligible effects of the minimal length[5]. The consideration of minimal length concept is crucial for precise description of broad physical fields such as non-commutative geometry[6], non-commutative field theories[7–9], black hole physics[10,11], loop quantum gravity[12], string theory[13–17], and others[18–21]. For nonrelativistic case, the solution of Schrödinger equation in momentum space and the corresponding energy spectrum has been studied in arbitrary dimensions considering minimal length, mainly for basic systems involving harmonic oscillator[19,20,22], coulomb potential system[23–26], and one-dimensional box problems[27]. By the way, the only relativistic problem that can be solved without approximation is Dirac oscillator (DO) established via the substitution $\mathbf{P} \to \mathbf{P} - i\beta m\omega \mathbf{X}$ in relativistic Dirac equation[28]. Dirac oscillator has attracted great attention thanks to its essential applicability in particle physics and quantum gravity.

Dirac oscillator has been investigated in momentum space representation by Green's function technique[29,30] and by coherent states approach[31]. It turned out that Kempf algebra[32] is useful for determining wave functions and energy spectrum in 1D with consideration of its thermodynamic properties in the presence of minimal length. This approach has been extended to 3D by Quesne and Tkachuk[33] using supersymmetric quantum mechanics based on shape-invariance methods.

In this work, we plan to solve the DO problem in a somewhat different context, suggesting a new covariant deformed algebra in 3 + 1 space-time, which preserves Lorentz symmetry. Due to several difference of our research from Kempf's one based on his deformed algebra, our result is somewhat different and cannot be reduced to a simple one in nonrelativistic limit. In particular, we are interested in investigating the effects of space deformation, characterized by the presence of minimal length scale, on quantized energy spectrum for a relativistic Dirac oscillator. The behavior of energy spectrum in high quantum number limit ($n \to \infty$) will be analyzed in order to promote deep understanding for intrinsic quantum nature of the system.

We introduce the Lorentz covariant deformed algebra and it will be used to solve the quantum problem of Dirac oscillator with minimal length in momentum representation. Dirac wave functions will be derived and the DO energy spectrum will be determined. Its asymptotic behavior will be estimated in both non-relativistic and non-deformed cases. An interesting discrepancy between our energy spectrum and the one obtained by Kempf deformed algebra[34] will be addressed.

Results

Let us start with a brief review of deformed quantum mechanics in 3 dimensions. According to Refs. 22, 33, we introduce deformed formula of position and momentum operators represented in terms of momentum variable such that

$$X_i = i\hbar\left[\left(1+\beta P^2\right)\frac{\partial}{\partial p_i}+\beta' p_i p_j\frac{\partial}{\partial p_j}+\gamma p_i\right], \qquad (1)$$

$$P_i = p_i, \qquad (2)$$

where β, β', and γ are some non-negative parameters which are very small. These operators yield non-covariant Kempf algebra that reads to

$$\left[X_i,P_j\right] = i\hbar\left[\delta_{ij}\left(1+\beta P^2\right)+\beta' p_i p_j\right], \qquad (3)$$

$$\left[X_i,X_j\right] = -i\hbar\left[2\beta-\beta'+(2\beta+\beta')\beta P^2\right]\epsilon_{ijk}L_k, \qquad (4)$$

$$\left[P_i,P_j\right] = 0. \qquad (5)$$

This algebra with the parameters $\beta = \gamma = 0$ in 3D is initiated by Snyder[35] in 1940's. After on, the attempt for investigating Dirac oscillator with this algebraic formulation was realized by Quesne et al.[36,37] in 1 + 1 dimensions in the case $\beta' = \gamma = 0$. They derived wave functions of the system and estimated the bound-state energy.

The components of the angular momentum are given by

$$L_i = \left(1+\beta P^2\right)^{-1}\epsilon_{ijk}X_j P_k, \quad i=1,2,3. \qquad (6)$$

These satisfy the usual commutation relations of the form

$$\left[L_i,X_j\right] = i\hbar\epsilon_{ijk}X_k, \quad \left[L_i,P_j\right] = i\hbar\epsilon_{ijk}P_j. \qquad (7)$$

If we consider physical states with $\langle \mathbf{P}\rangle = 0$ and the fact that the momentum uncertainties ΔP_i are isotropic, canonical variables represented in Eqs. (1) and (2) no longer gives the Heisenberg uncertainty principle. Instead, we can express its modified form (GUP) as

$$\Delta X_i\, \Delta P_i \geq \frac{\hbar}{2}\left[1+3\beta(\Delta P_i)^2+\beta'(\Delta P_i)^2\right]. \qquad (8)$$

By taking the saturation of GUP and minimizing it with respect to ΔP_i, we have an isotropic minimal length which is

$$\Delta X_{\min} = \hbar\sqrt{3\beta+\beta'}. \qquad (9)$$

The parameter γ in Eq. (1) does not affect the commutation relations and only modify the squeezing factor of the momentum space measure. In fact, the inner product is now defined by

$$\int\frac{d^3p}{\left[1+(\beta+\beta')p^2\right]^{1-\frac{\gamma-\beta'}{\beta+\beta'}}}|p\rangle\langle p| = 1. \qquad (10)$$

Now, let us introduce the (3 + 1) dimensional Lorentz-covariant algebra[33,34]. In this case we have to make the following substitution in Eqs. (3)–(5): $p^2 \to p_0^2 - p^2 = p_\nu p^\nu$, $p_i x_i \to p_0 x_0 - p_i x^i = p_\nu x^\nu$, where $p_\nu = (p_0, p_i)$ and $x_\nu = (x_0, x_i)$. If we consider in this scheme that

$$X^\mu = (1-\beta p_\nu p^\nu)x^\mu - \beta' p^\mu p_\nu x^\nu + i\hbar\gamma p^\mu, \quad P^\mu = p^\mu, \qquad (11)$$

we easily show that the Lorentz covariant commutation relations can be deduced to be

$$[X^\mu,P^\nu] = -i\hbar\left[\left(1-\beta\left(\left(P^0\right)^2-P^2\right)\right)g^{\mu\nu}-\beta' P^\mu P^\nu\right], \qquad (12)$$

$$[X^\mu,X^\nu] = -i\hbar\left[2\beta-\beta'-(2\beta+\beta')\beta\left(\left(P^0\right)^2-P^2\right)\right]L^{\mu\nu}, \qquad (13)$$

$$[P^\mu,P^\nu] = 0, \qquad (14)$$

where $g^{\mu\nu} = Diag(1, -1, -1, -1)$ and

$$L^{\mu\nu} = \left[1-\beta\left(\left(P^0\right)^2-P^2\right)\right]^{-1}(P^\nu X^\mu - P^\mu X^\nu). \qquad (15)$$

Notice that this algebra collapses to the Snyder's algebra[35] for D = 3, $\beta = \gamma = 0$ and $\beta' = (a/\hbar)^2$. Some algebras that can be fulfilled with $L^{\mu\nu}$ are given by

$$\left[X^\lambda,L^{\mu\nu}\right] = 2i\hbar\beta' P^\lambda L^{\mu\nu}, \qquad (16)$$

$$\left[P^\lambda,L^{\mu\nu}\right] = i\hbar\left(P^\nu g^{\lambda\mu}-P^\mu g^{\lambda\nu}\right), \qquad (17)$$

$$\left[L^{\mu\nu},L^{\alpha\lambda}\right] = i\hbar\left(g^{\mu\lambda}L^{\nu\alpha}+g^{\nu\alpha}L^{\mu\lambda}+g^{\mu\alpha}L^{\lambda\nu}+g^{\nu\lambda}L^{\alpha\mu}\right). \qquad (18)$$

Finally the inner product in momentum space, Eq. (10), becomes

$$\int\frac{d^3p}{\left[1-(\beta+\beta')p_\nu p^\nu\right]^{1-\frac{2\gamma-3\beta'}{2(\beta+\beta')}}}|p\rangle\langle p| = 1. \qquad (19)$$

The Dirac equation in (3 + 1) dimensions for a free spinor reads $(\gamma^\mu p_\mu - m)\,\psi = 0$, where m is the rest mass of the particle and ψ is the four component spinor wave function and γ^μ are four square matrices. In general, the standard representation of γ^μ has the form

$$\gamma = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathbf{I}_{(2\times 2)} & 0 \\ 0 & -\mathbf{I}_{(2\times 2)} \end{pmatrix}, \qquad (20)$$

where σ is (2×2) hermitian Pauli matrix and $\mathbf{I}$ is the (2×2) unit matrix. The relativistic Dirac oscillator introduced by other researchers[28] can now be obtained using the non-minimal coupling $\mathbf{P} - im\omega\mathbf{X}$ in the free particle Dirac equation, where ω is the frequency of the oscillator. This coupling gives

$$\left(c\,\gamma\left(\mathbf{P}-\mathbf{i}\gamma^0 m\omega\mathbf{X}\right)+\gamma^0 mc^2\right)\psi = W\psi. \qquad (21)$$

From the substitution of $\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$ in Eq. (21), we obtain the following coupled differential equations

$$W\,\psi_a = c\,\sigma(\mathbf{P}+im\omega\mathbf{X})\psi_b + mc^2\psi_a, \qquad (22)$$

$$W\,\psi_b = c\,\sigma(\mathbf{P}-im\omega\mathbf{X})\psi_a - mc^2\psi_b. \qquad (23)$$

Because $\psi_b(p, p^0)$ approaches zero as the system become nonrelativistic case ($v \ll c$), $\psi_b(p, p^0)$ is usually called the small wave function[38]. However, $\psi_a(p, p^0)$ is relatively large in most cases so that we can call it as the large wave function. Let us rearrange Eq. (23) in terms of ψ_b. Then, by inserting it in Eq. (22), we obtain the operator equation for the large component such that

$$(W^2-m^2c^4)\psi_a=c^2\{\mathbf{P}^2+m^2\omega^2\mathbf{X}^2 + im\omega[\sigma\mathbf{X},\sigma\mathbf{P}]+i\, m^2\omega^2\sigma(\mathbf{X}\wedge\mathbf{X})\}\psi_a. \quad (24)$$

Wave functions. At this stage, let us decompose the wave function into a radial part and spin angular part as

$$\psi(\mathbf{P})=\begin{pmatrix} F(p)\mathcal{H}_\kappa^{m_j}(\hat{\mathbf{u}}) \\ G(p)\mathcal{H}_{-\kappa}^{m_j}(\hat{\mathbf{u}}) \end{pmatrix}, \quad (25)$$

where $\hat{\mathbf{u}}=\dfrac{\mathbf{P}}{|\mathbf{P}|}$ is a unit vector.

For the case of a simple situation $\beta'=\gamma=0$, the use of the algebra of Eq. (12) leads to

$$[\sigma\mathbf{X},\sigma\mathbf{P}]=i\hbar\left[1-\beta\left((p^0)^2-p^2\right)\right]\left(\frac{2\sigma\mathbf{L}}{\hbar}+3\right), \quad (26)$$

$$\mathbf{X}\wedge\mathbf{X}=-i\hbar\alpha\mathbf{L}, \quad (27)$$

where $\alpha=2\beta\ (1-\beta)((p^0)^2-p^2)$ and $\mathbf{L}$ is the orbital angular momentum. Now we consider the action of $\sigma\mathbf{L}$ on the spin-angular function $\mathcal{H}_\kappa^{m_j}(\hat{\mathbf{u}})$:

$$\sigma\mathbf{L}\mathcal{H}_\kappa^{m_j}(\hat{\mathbf{u}})=\hbar\kappa\mathcal{H}_\kappa^{m_j}(\hat{\mathbf{u}}), \quad (28)$$

Here, the quantum number κ is equal to $s(2j+1)-1$ where s is the spin, $j=l+s$, and l is the angular momentum. Then, using the first of Eq. (25) with Eqs. (26)–(28), Eq. (24) becomes

$$\frac{\left(c^2(p^0)^2-m^2c^4\right)}{c^2}F(p,p^0)=\{\mathbf{P}^2+m^2\omega^2\mathbf{X}^2 + \left[1-\beta\left((p^0)^2-p^2\right)\right](2\hbar m^2\omega^2\beta-2m\omega)\hbar\kappa -3m\omega\hbar\left[1-\beta\left((p^0)^2-p^2\right)\right]\}F(p,p^0). \quad (29)$$

Here, we used the relation $W=cp^0$. If we consider Eq. (11), the momentum space representation of $\mathbf{X}^2$ is given by

$$X^2=-\hbar^2\left(\left[1-\beta\left((p^0)^2-p^2\right)\right]\frac{\partial}{\partial p}\right)^2-\hbar^2\left[1-\beta\left((p^0)^2-p^2\right)\right]^2 \times\left(\frac{2}{p}\frac{\partial}{\partial p}-\frac{L^2}{p^2}\right). \quad (30)$$

From the use of this relation in Eq. (29), we obtain the following differential equation

$$\begin{aligned}&\left\{\left[m\omega\hbar(2\kappa+3)-2m^2\omega^2\hbar^2\beta\kappa\right]\left(1-\beta(p^0)^2\right)-m^2c^2+(p^0)^2\right\}F(p,p^0)\\ &=-m^2\omega^2\hbar^2\left\{\left[\left[1-\beta\left((p^0)^2-p^2\right)\right]\frac{\partial}{\partial p}\right]^2+\left[1-\beta\left((p^0)^2-p^2\right)\right]^2\frac{2}{p}\frac{\partial}{\partial p}\right.\\ &-\left(1-\beta(p^0)^2\right)^2\frac{L^2}{p^2}-2\beta\left(1-\beta(p^0)^2\right)L^2\\ &\left.+\left[\frac{-1}{m^2\omega^2\hbar^2}-\beta^2L^2+\frac{(2\kappa+3)\beta-2\beta^2m\omega\hbar\kappa}{m\omega\hbar}\right]p^2\right\}F(p,p^0).\end{aligned} \quad (31)$$

To simplify this equation, we define a new deformation parameter of the form

$$\theta=\frac{\beta}{1-\beta(p^0)^2}. \quad (32)$$

This allows us to rewrite Eq. (31) as

$$\begin{aligned}&\left[\frac{m\omega\hbar(2\kappa+3)-2m^2\omega^2\hbar^2\beta\kappa}{1-\beta(p^0)^2}+\frac{(p^0)^2-m^2c^2}{\left(1-\beta(p^0)^2\right)^2}\right]F(p,p^0)\\ &=-m^2\omega^2\hbar^2\left\{\left[(1+\theta p^2)\frac{\partial}{\partial p}\right]^2+(1+\theta p^2)^2\frac{2}{p}\frac{\partial}{\partial p}-\frac{L^2}{p^2}\right.\\ &-\frac{2\beta}{1-\beta(p^0)^2}L^2+\frac{1}{\left(1-\beta(p^0)^2\right)^2}\left[\frac{-1}{m^2\omega^2\hbar^2}-\beta^2L^2\right.\\ &\left.\left.+\frac{(2\kappa+3)\beta-2\beta^2m\omega\hbar\kappa}{m\omega\hbar}\right]p^2\right\}F(p,p^0).\end{aligned} \quad (33)$$

Introducing new variables such that

$$\rho=\frac{1}{\sqrt{\theta}}\arctan p\sqrt{\theta},\quad k_c=\sqrt{m\omega\hbar\theta}, \quad (34)$$

we cast Eq. (33) in the form

$$\begin{aligned}-\frac{\zeta_c}{k_c^2}F(p,p^0)=&\left\{\frac{m\omega\hbar}{k_c^2}\frac{\partial^2}{\partial\rho^2}+\left[\frac{2m\omega\hbar\sqrt{\theta}}{k_c^2\tan\left(\rho\sqrt{\theta}\right)}+\frac{2m\omega\hbar\sqrt{\theta}}{k_c^2}\tan\left(\rho\sqrt{\theta}\right)\right]\frac{\partial}{\partial\rho}\right.\\ &-\frac{m\omega\hbar\theta}{k_c^2}L^2\left[\cot\left(\rho\sqrt{\theta}\right)\right]^2-\frac{2m\omega\hbar}{k_c^2}\theta L^2+\frac{m\omega\hbar}{k_c^2}\frac{\theta}{\beta^2}\\ &\times\left[\frac{-1}{m^2\omega^2\hbar^2}-\beta^2L^2+\frac{(2\kappa+3)\beta-2\beta^2m\omega\hbar\kappa}{m\omega\hbar}\right]\\ &\left.\times\left[\tan\left(\rho\sqrt{\theta}\right)\right]^2\right\}F(p,p^0),\end{aligned} \quad (35)$$

where

$$\zeta_c=-\left[\frac{2m\omega\hbar\beta\kappa-(2\kappa+3)}{1-\beta(p^0)^2}-\frac{1}{m\omega\hbar}\frac{(p^0)^2-m^2c^2}{\left(1-\beta(p^0)^2\right)^2}\right]. \quad (36)$$

Let us make further change of variables, $S=\sin\left(\dfrac{k_c\rho}{\sqrt{m\omega\hbar}}\right)$ and $C=\cos\left(\dfrac{k_c\rho}{\sqrt{m\omega\hbar}}\right)$, along with

$$F=C^{\lambda_c}f(C), \quad (37)$$

where λ_c is a constant that will be determined later. Then, we have

$$\begin{aligned}&(1-S^2)f''-\left(2\lambda_c+1-\frac{2}{S}\right)f'+\left[\left(\frac{\zeta_c}{k_c^2}-L^2-3\lambda_c\right)-\frac{L^2}{S^2}\right.\\ &\left.+\left(\lambda_c^2-3\lambda_c-L^2+\frac{\theta}{\beta}\frac{(2\kappa+3)}{k_c^2}-2\kappa-\frac{\theta^2}{\beta^2}\frac{1}{k_c^4}\right)\frac{S^2}{C^2}\right]f=0.\end{aligned} \quad (38)$$

At this stage we eliminate the term proportional to $\dfrac{S^2}{C^2}$ by choosing λ_c to be the solution of the following differential equation

$$\lambda_c^2-3\lambda_c-L^2+\frac{\theta}{\beta}\frac{(2\kappa+3)}{k_c^2}-2\kappa-\frac{\theta^2}{\beta^2}\frac{1}{k_c^4}=0. \quad (39)$$

Through a straightforward calculation, we easily have

$$\lambda_{c,\pm}=\frac{3}{2}\pm\sqrt{\frac{9}{4}+L^2-\frac{\theta}{\beta}\frac{(2\kappa+3)}{k_c^2}+2\kappa+\left(\frac{\theta}{\beta}\right)^2+\frac{1}{k_c^4}}. \quad (40)$$

Among these two solutions, the physically acceptable one is only $\lambda_{c,+}$. This can be verified by examining them with reference of the GUP using the same method given in Ref. 34 for a similar problem.

A further simplification can be fulfilled by eliminating the centrifugal barrier term in Eq. (38) by setting $f(S)=S^l g(S)$ and $z=2S^2-1$:

$$\begin{aligned}&(1-z^2)g''(z)+[(b-a)-(a+b+2)z]g'(z)\\&+\frac{1}{4}\left[\frac{\zeta}{k_c^2}-2L^2-(2l+3)\lambda_c+l\right]g(z)=0.\end{aligned} \tag{41}$$

We now introduce the following new parameters

$$a=\lambda_{c,+}-\frac{3}{2},\quad b=\frac{1}{2}+l,\quad n'=\frac{n-l}{2}, \tag{42}$$

where n' is non-negative integer. Then, by imposing the following constraint

$$\frac{1}{4}\left[\frac{\zeta}{k_c^2}-2L^2-(2l+3)\lambda_{c,+}+l\right]=n'(n'+a+b+1), \tag{43}$$

we can rewrite Eq. (41) in the form

$$\begin{aligned}&(1-z^2)g''(z)+[(b-a)-(a+b+2)z]g'(z)\\&+n'(n'+a+b+1)g(z)=0.\end{aligned} \tag{44}$$

We see that the solutions of this equation are expressed in terms of Jacobi polynomials

$$g(z)=N\,P_{n'}^{(a,b)}(z), \tag{45}$$

where N is a normalization constant. Then, the large radial component $F(z)$ is given by

$$F(z)=N\,2^{-\frac{a+b+1}{2}}(1-z)^{\frac{a+3/2}{2}}(1+z)^{\frac{b-1/2}{2}}P_{n'}^{(a,b)}(z). \tag{46}$$

By returning to the old variable p, we immediately have

$$F(p,p^0)=N\sqrt{\theta}^{\,b-\frac{1}{2}}\left(1+\theta p^2\right)^{-\frac{a+b+1}{2}}p^{b-\frac{1}{2}}P_{n'}^{(a,b)}\left(\frac{\theta p^2-1}{\theta p^2+1}\right). \tag{47}$$

We now calculate the small component of the DO wave function $\psi_b(p, p^0)$ using

$$\psi_b(p,p^0)=\frac{c\,\sigma(\mathbf{P}-im\omega\mathbf{X})}{W+mc^2}\psi_a(p,p^0). \tag{48}$$

Using the Lorentz-covariant operator algebra (see **Methods** section), we confirm that this equation yields

$$\begin{aligned}\psi_b(p,p^0)&=\frac{c}{W+mc^2}\sigma_p\\&\times\left[p+m\omega\hbar\left[1-\beta\left(\left(p^0\right)^2-p^2\right)\right]\left(\frac{\partial}{\partial p}-\frac{\sigma\mathbf{L}}{p}\right)\right]\psi_a(p,p^0),\end{aligned} \tag{49}$$

where $\sigma_p=\frac{\sigma\mathbf{p}}{p}$.

It is important to use the action of $\sigma\mathbf{L}$ and σ_p on $\mathcal{Y}_\kappa^{m_j}(\hat{\mathbf{u}})$ function, where

$$\sigma_p\mathcal{Y}_\kappa^{m_j}(\hat{\mathbf{u}})=-\mathcal{Y}_{-\kappa}^{m_j}(\hat{\mathbf{u}}), \tag{50}$$

and after some simplifications on $\psi_b(p,p^0)$ formula, we can express the small radial wave function $G(p, p^0)$ as

$$\begin{aligned}G(p,p^0)=&\frac{-m\omega\hbar c}{W+mc^2}\left[\left(\frac{1}{m\omega\hbar}-\beta\hbar\kappa\right)p\right.\\&\left.+\left(1-\beta\left(\left(p^0\right)^2-p^2\right)\right)\frac{\partial}{\partial p}-\frac{\left(1-\beta\left(p^0\right)^2\right)\hbar\kappa}{p}\right]F(p,p^0).\end{aligned} \tag{51}$$

Notice that this is equivalent to the one that appears in Kempf non-covariant deformed algebra[34]. By using ω' instead of ω, where

$$\omega'=\left(1-\beta\left(p^0\right)^2\right)\omega=\frac{\beta}{\theta}\omega, \tag{52}$$

Eq. (51) can be rewritten as

$$\begin{aligned}G(p,p^0)=&\frac{-m\omega'\hbar c}{W+mc^2}\left[\left(\frac{1}{m\omega'\hbar}-\theta\hbar\kappa\right)p\right.\\&\left.+\left(1+\theta p^2\right)\frac{\partial}{\partial p}-\frac{\hbar\kappa}{p}\right]F(p,p^0).\end{aligned} \tag{53}$$

This change makes it easy to normalize the wave function of the relativistic Dirac oscillator on large and small radial components:

$$F(p,p^0)=\frac{1}{p}R_1(p,p^0), \tag{54}$$

$$G(p,p^0)=\frac{1}{p}R_2(p,p^0). \tag{55}$$

The large radial component $R_1(p, p^0)$ is given by

$$R_1(p,p^0)=N\sqrt{\theta}^{\,b-\frac{1}{2}}p^{b+\frac{1}{2}}f^{-\frac{a+b+1}{2}}P_{n'}^{(a,b)}(z), \tag{56}$$

where $f(p)$ is defined as $f(p) = 1 + \theta p^2$.

For the case of the small component $R_2(p, p^0)$, it is necessary to distinguish spin up and spin down states. For $s=\frac{1}{2}$, we have

$$\begin{aligned}R_2(p,p^0)=&\frac{-m\omega'\hbar cN\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}\left[f\frac{\partial}{\partial p}+\left(\frac{1}{m\omega'\hbar}-\theta\hbar\kappa\right)p-\frac{\hbar\kappa}{p}\right]\\&\times p^{b+\frac{1}{2}}f^{-\frac{a+b+1}{2}}P_{n'}^{(a,b)}(z)\\=&\frac{-2m\omega'\hbar c\,N\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}p^{b-\frac{1}{2}}f^{-\frac{a+b+1}{2}}(1+z)\frac{d}{dz}P_{n'}^{(a,b)}(z)\\=&\frac{-2m\omega'\hbar c\theta\,(a+b+n'+1)N\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}p^{b+\frac{3}{2}}f^{-\frac{a+b+3}{2}}\\&\times P_{n'-1}^{(a+1,b+1)}(z),\end{aligned} \tag{57}$$

whereas, for $s=-\frac{1}{2}$, it yields

$$\begin{aligned}R_2(p,p^0)=&\frac{-m\omega'\hbar c\,N\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}\left[f\frac{\partial}{\partial p}+\left(\frac{1}{m\omega\hbar}-\theta\hbar(\kappa+1)\right)p\right.\\&\left.-\frac{\hbar(\kappa+1)}{p}\right]p^{b+\frac{1}{2}}f^{-\frac{a+b+1}{2}}P_{n'}^{(a,b)}(z)\\=&\frac{-2m\omega'\hbar c\,N\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}p^{b-\frac{1}{2}}f^{-\frac{a+b+1}{2}}\left[(1+z)\frac{d}{dz}+b\right]P_{n'}^{(a,b)}(z)\\=&\frac{-2m\omega'\hbar c\,(b+n')N\sqrt{\theta}^{\,b-\frac{1}{2}}}{W+mc^2}p^{b-\frac{1}{2}}f^{-\frac{a+b+1}{2}}P_{n'}^{(a+1,b-1)}(z).\end{aligned} \tag{58}$$

When we derive these two equations, some properties relevant to the Jacobi polynomials, which have appeared in Ref. 39, are used.

Now, let us determine the normalization constant N via the relation

$$\frac{\theta}{\beta}\int_0^\infty\frac{d^3p}{f(p,p^0)}\left(\left|R_1(p,p^0)\right|^2+\left|R_2(p,p^0)\right|^2\right)=1. \tag{59}$$

If we consider this, the normalized components of the wave function $F(p,p^0)$ and $G(p,p^0)$ deduced from Eqs. (54) and (55) become

$$F\left(p,p^0\right)=\left(\frac{W_c+mc^2}{2W_c}\right)^{\frac{1}{2}}A^{(n')}(a,b)p^{b-\frac{1}{2}}f^{-\frac{1}{2}(a+b+1)}P_{n'}^{(a,b)}(z), \tag{60}$$

$$G\left(p,p^0\right)=-\sigma\left(\frac{W_c-mc^2}{2W_c}\right)^{\frac{1}{2}} \times A^{(N)}\left(\tilde{a},\tilde{b}\right)p^{\tilde{b}-\frac{1}{2}}f^{-\frac{1}{2}(\tilde{a}+\tilde{b}+1)}P_N^{(\tilde{a},\tilde{b})}(z), \tag{61}$$

where

$$A^{(n')}(a,b)=\left(\frac{2\theta^{b+1}(a+b+2n'+1)n'!\Gamma(a+b+n'+1)}{\Gamma(a+n'+1)\Gamma(b+n'+1)}\right)^{\frac{1}{2}}$$
$$\tilde{a}=a+1 \quad \tilde{b}=b+2s \quad N=n'-s-\frac{1}{2} \quad \sigma=\frac{W_c}{|W_c|}, \tag{62}$$

with

$$\begin{cases} n'=1,2,3,\cdots & \text{for } s=\frac{1}{2} \text{ and } \sigma=-1 \\ n'=0,1,2,\cdots & \text{otherwise.} \end{cases} \tag{63}$$

Energy spectrum. Using the expressions of n', a, b, $\lambda_{c,+}$ and ζ given in Eq. (43), it is possible to find the energy spectrum of Dirac oscillator with Lorentz-covariant deformed algebra. A straightforward calculation leads to.

$$W_c^2=c^2\left(p^0\right)^2=m^2c^4\left(\frac{1+\frac{\hbar\omega}{mc^2}\Delta}{1+\hbar\omega m\beta\Delta}\right), \tag{64}$$

where we have set

$$\Delta=2\left(n+\frac{3}{2}\right)[(\hbar\omega m\beta)^2\left(L^2+9/4\right)-(3+2\kappa)\hbar\omega m\beta +2\hbar^2\omega^2m^2\beta^2\kappa+1]^{1/2}+\hbar\omega m\beta\left(n^2+L^2+3n+\frac{9}{2}\right) +2\kappa(\hbar\omega m\beta-1)-3. \tag{65}$$

It is interesting to link Eq. (64) with the energy spectrum of 3D Dirac oscillator obtained with the non-covariant Kempf deformed algebra[34]. In fact, by inspection, we easily see that Eq. (64) can be rewritten as

$$W_c^2=\frac{W_{nc}^2}{1+\hbar\omega m\beta\Delta}, \tag{66}$$

where W_{nc} is the energy spectrum of Dirac oscillator developed with non-covariant deformed algebra[34]:

$$W_{nc}^2=m^2c^4\left(1+\frac{\hbar\omega}{mc^2}\Delta\right). \tag{67}$$

From Fig. 1, we see that W_c is lower than W_{nc} except when $\beta=0$. The difference between them becomes large as quantum number n and parameter β increase.

The presence of the additional factor $(1+\hbar\omega m\beta\Delta)^{-1}$ makes the energy spectrum bounded. In fact, by considering $\underset{n\to\infty}{\Delta}\to\hbar\omega m\beta n^2$, we obtain

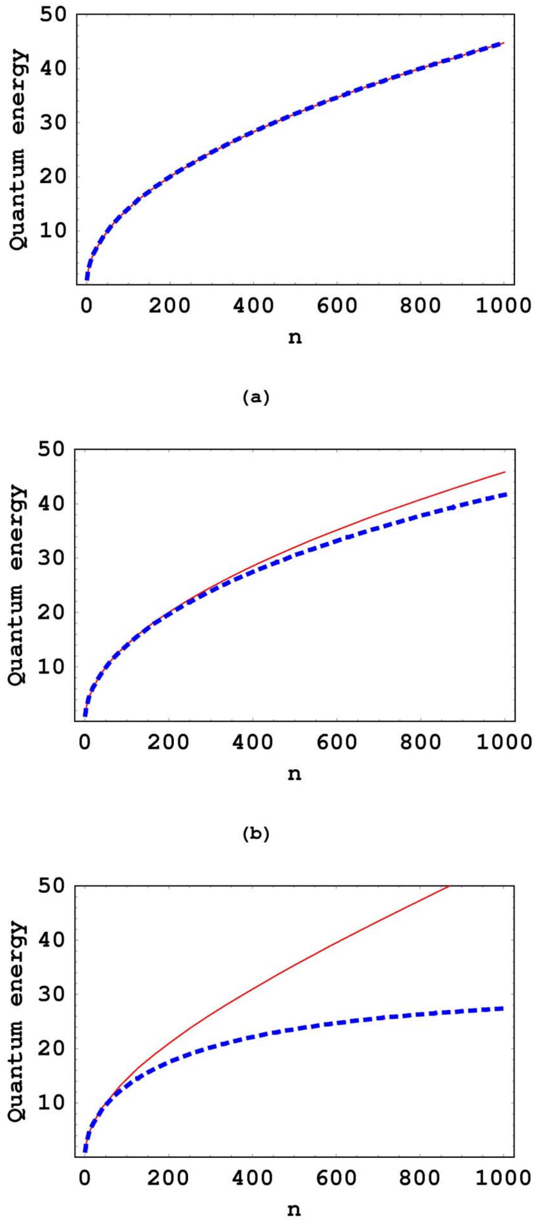
$$\lim_{n\to\infty} W_c=\frac{c}{\sqrt{\beta}}. \tag{68}$$

Thus, the energy of the relativistic Dirac oscillator in deformed space is not allowed to increase indefinitely, but approaches to a finite value. This is the main result of our report. However, if we remove the deformation of the space by setting $\beta\to 0$, the energy in the large n limit becomes $\lim_{n\to\infty} W_c=\infty$, as expected. In more detail, in order to get increasing values of the energy without upper bound when n increases, we see from Eq. (64) that a necessary condition is $1+\frac{\hbar\omega}{mc^2}\Delta>1+\hbar\omega m\beta\Delta$, yielding the following constraint

$$\beta m^2c^2<1. \tag{69}$$

It is shown in second part of **Methods** section that how to map the solution of Dirac oscillator with Lorentz-covariant deformed algebra to the solution with the non-covariant deformed algebra of Kempf et al.[1–4] through (alternative) derivation of the energy spectrum. Now, let's analyze the results in some limiting cases.

i) *The case* $\beta=0$



(a)

(b)

(c)

Figure 1 | Comparison of W_c (dotted line) with W_{nc} (solid line), where $\beta=0$ (a), $\beta=0.0001$ (b), and $\beta=0.001$ (c). We have taken $c=1$, $m=1$, $\omega=1$, $L=1$, $\kappa=1$, and $\hbar=1$ and all these values are chosen dimensionlessly for convenience.

In this case we have $W_c = W_{nc}$ and one obtains

$$W_c^2 - m^2c^4 = 2\hbar\omega mc^2(n-\kappa), \quad (70)$$

or

$$W_c^2 - m^2c^4 = 2m\omega\hbar c^2\left(n-j+\frac{1}{2}\right) \quad \text{for } s=\frac{1}{2}, \quad (71)$$

$$W_c^2 - m^2c^4 = 2m\omega\hbar c^2\left(n+j+\frac{3}{2}\right) \quad \text{for } s=-\frac{1}{2}. \quad (72)$$

These are exactly the energy levels of the usual Dirac oscillator.

ii) *The non-relativistic limit*

The non-relativistic limit is obtained by setting $W_c = mc^2 + E_c$ with the assumption that $E_c \ll mc^2$, where E_c being the non-relativistic energy. Indeed, from Eq. (66), we obtain the following formula

$$E_c = \frac{(1-\beta m^2c^2)}{(1+\beta m\omega\hbar\Delta)}E_{nc}, \quad (73)$$

where E_{nc} is the non-relativistic energy of the system analyzed with non-covariant algebra. This relation shows that the non-covariant deformed algebra does not equivalent to the non-relativistic limit of the Lorentz-covariant deformed algebra. This conclusion coincide with the previous reports[36,37] developed for the case of the one-dimensional Dirac oscillator with Lorentz covariant deformed algebra. We see from Eq. (73) that the non-bound condition, Eq. (69), also holds for the non-relativistic case.

Before closing this section, let us see the allowed condition associated with the parameter θ. By rewriting Eq. (32) as a function of θ, we have

$$\beta = \frac{\theta}{1+\theta(p^0)^2}. \quad (74)$$

Using Eq. (69) one easily get the following bound condition for the norm of 3D momentum

$$\mathbf{p}.\mathbf{p} < \frac{1}{\theta}. \quad (75)$$

Using again Eq. (69) we further obtain

$$p^\mu p_\mu < \frac{1}{\beta}, \quad (76)$$

which means that an UV cut-off naturally implemented in the Lorentz-covariant deformed algebra. By combining Eqs. (75) and (76), one obtains the condition

$$\beta\left(p^0\right)^2 < 1, \quad (77)$$

for the physically acceptable states. This can also be obtained from the deformed inner product, by demanding that the weight function in Eq. (19) is free from singularities[36,37].

Discussion

In this paper, we have investigated the problem of relativistic Dirac oscillator with minimal length in 3 + 1 dimensional space-time on the basis of Lorentz-covariant algebra introduced using particular variable transformations (in the case $\beta' = \gamma = 0$). The wave functions of the system and the corresponding energy spectrum are derived considering the Lorentz covariant commutation relations given in Eqs. (12)–(14). We confirmed that the energy spectrum is different from the one obtained from Kempf non-covariant algebra[34] by the presence of the factor $(1 + \beta m\omega\hbar\Delta)^{-1}$ [see Eq. (66)], which proves the novelty of the algebra used.

It was important to remark that the energy spectrum we obtained is bounded as shown in Eq. (68) whereas the energy spectrum for the DO problem based on the Kempf algebra is not. If we compare this with the familiar result that the energy levels of the ordinary oscillator is equally spaced leading monotonic growth of the energy with the increment of the quantum number n[40], this result is very surprising. Apparently, Eq. (68) implies that the spacing of the energy levels asymptotically approaches to zero for sufficiently large n. We can conclude that the deformation of space restricts the total (relativistic) quantum energy for a mode and the allowed energy become small with the increase of deformation factor β. In the meantime, in case that the deformation of the space disappears, this effect vanishes and the energy spectrum recovers to previously known one as expected.

We have found that Kempf deformed algebra is not a non-relativistic limit of Lorentz-covariant deformed algebra. This outcome is in good agreement with the discussions in Refs. 36, 37. In the limit $\beta \to 0$, the usual relativistic DO eigenvalues are recovered. The two components of wave function take the form of DO wave functions with Kempf algebra in 3 dimensions[33,34], but the normalization constant and the quantities a and b are dependent on p^0 which manifest themselves in the small parameter θ.

Methods

Lorentz-covariant operator algebra. To manage Eq. (48) in the text, it is useful to consider some relations of Ref. 33 with appropriate modifications considering the Lorentz-covariant algebra, which are

$$\sigma\mathbf{X} = i\hbar\sigma_i\left(\left[1-\beta\left(\left(p^0\right)^2-p^2\right)\right]\frac{\partial}{\partial p_i}\right), \quad (78)$$

$$\sigma_i\frac{\partial}{\partial p_i} = \left(\frac{\partial}{\partial p}+\frac{\sigma\mathbf{L}+2}{p}\right)\sigma_p = \sigma_p\left(\frac{\partial}{\partial p}-\frac{\sigma\mathbf{L}}{p}\right), \quad (79)$$

$$\frac{\partial}{\partial p}\sigma_p = \sigma_p\frac{\partial}{\partial p}, \quad (80)$$

where $\sigma_p = \frac{\sigma\mathbf{p}}{p}$.

Derivation of the energy spectrum. We show how to map the solution of Dirac oscillator with Lorentz-covariant deformed algebra to the solution with the non-covariant deformed algebra of Kempf et al.[1–4]. The key quantities are k_c and ζ_c, and the corresponding ones in the case of the non-covariant deformed algebra are given by[34]

$$k_{nc} = \sqrt{\hbar\omega m\beta}, \quad (81)$$

$$\zeta_{nc} = \frac{W^2 - m^2c^4 + 2\hbar\omega mc^2\kappa + 3\hbar\omega mc^2 - 2m^2\hbar^2\omega^2c^2\beta\kappa}{\hbar\omega mc^2}. \quad (82)$$

It is easy to show that

$$k_c = \sqrt{\frac{\theta}{\beta}}k_{nc}, \quad (83)$$

$$\zeta_c = \zeta_{nc}\frac{\theta}{\beta}, \quad (84)$$

$$\lambda_c = \lambda_{nc}. \quad (85)$$

Now, by comparing Eqs. (81) and (82) with Eqs. (83) and (84), and using Eq. (43), we obtain

$$W_c^2 = \frac{W_{nc}^2}{1+\hbar\omega m\beta\Delta}, \quad (86)$$

which is exactly the same as Eq. (66). On the other hand the wave functions given in Eqs. (60) and (61) can be obtained from the ones obtained in the setup with non-covariant deformed algebra by using Eq. (32) and $p_{nc} = \sqrt{\frac{\theta}{\beta}}p$ and taking into account an extra factor $(\theta/\beta)^{3/2}$ in the normalization constant.

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Author contributions

M.B. performed scientific calculation. M.B., M.M. and J.R.C. wrote the paper.

Additional information

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