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Sixth order compact multi-phase block-AGE iteration methods for computing 2D Helmholtz equation

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REVIEW HIGHLIGHTS

- Using compact nine-points, we have proposed a sixth order scheme for 2D Helmholtz equation.
- 3- and 2-phase block-AGE iteration methods are discussed to solve the large linear systems arise due to the discretization.
- The block-AGE iteration methods require less number of iterations in comparison with block-SoR method.

ARTICLE INFO

Method name:
Block-AGE Multi-stage Iteration Method

Keywords:
Multi-phase block-AGE iteration methods
9-point compact mesh
Sixth order approximation
Helmholtz equation
Block-SOR iteration method
Three-diagonal solver
Error analysis

ABSTRACT

We discuss sixth order accurate 9-point compact 2- and 3-phase block alternating group explicit (block-AGE) iteration methods for computing 2D Helmholtz equation. We use Dirichlet boundary conditions and no fictitious points are involved outside the solution region for computation. The proposed 2- and 3-phase block-AGE methods require only two and three sweeps for computation and the error analysis of the suggested approximation is analyzed. We have compared the 2- and 3-phase block-AGE iteration methods with the corresponding block successive over relaxation (block-SOR) method in three experiments, in regard to number of iterations required for convergence and cpu time, where the importance of the role performed by optimal relaxation parameters of the proposed block-AGE iteration methods become evident in stipulating the convergence and precision of the calculated results. In all cases we use the tridiagonal solver and obtain the optimal relaxation parameters through computation.

Specifications table

Subject area:	*Mathematics*
More specific subject area:	*Numerical Analysis (PDEs)*
Name of the reviewed methodology:	Multi-phase block-AGE iteration methods for the solution of 2D Helmholtz equation with the aid of sixth order compact FDM.
Keywords:	Multi-phase block-AGE iteration methods; 9-point compact stencil; sixth order approximation; Helmholtz equation; Poisson equation; block-SOR iteration method; tri-diagonal solver; error analysis.
Resource availability:	NA
Review question:	What is the order of accuracy of the suggested compact approximation for Helmholtz equation? What is the main advantage of 3-phase block-AGE method?

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Method details

Background

We consider the 2D Helmholtz equation with Dirichlet boundary conditions

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + kz = f(x,y), \quad (x,y) \in \Omega_2, \tag{1}$$

$$z(x,y) = g(x,y), (x,y) \in \partial\Omega_2, \tag{2}$$

where $\Omega_2 = \{(x,y)|0 < x, y < 1\}$ and $\partial\Omega_2$ is its boundary. If $k = \lambda^2>0$, the Eq. (1) is said to be oscillatory Helmholtz equation and λ is the wave number. For $k = 0$, the Eq. (1) is said to be Poisson equation, and if $k < 0$, the Eq. (1) is said to be monotone Helmholtz equation. The unknown quantity z(x,y) generally represents a pressure field in the frequency domain and *f(x,y)* as a source function. We assume that *z(x,y)*, *f(x,y)* and *g(x,y)* are sufficiently smooth functions. The Helmholtz equation is pivotal in describing various significant physical phenomena, encompassing the determination of potentials in time-harmonic acoustic and electromagnetic fields, the analysis of acoustic wave scattering, the reduction of noise in silencing systems, the modeling of water wave propagation, the study of membrane vibrations, and the assessment of radar scattering [1–6]. Numerous research endeavors have been directed towards achieving a more efficient and precise numerical solution for the Helmholtz and Poisson equations [7–24].

Some connected research effort done in the past on block iterative methods for elliptic boundary value problems (EBVPs) by various researchers are as follows: Evans [25,26] originally proposed group explicit iterative methods for solving large linear systems due to the discretization of EBVPs. In 1987, Evans and Yousif [27] first proposed the block alternating group explicit (BLAGE) Method for the elliptic difference equation. Evans and Mohanty [28] presented block iterative methods for 2D biharmonic equations. Mohanty and Evans [29] proposed fourth order accurate BLAGE iterative method for the solution of 2D EBVPs in polar coordinates. Later, Evans and Mohanty [30], employed SMAGE algorithms on a non-uniform mesh for the solution of nonlinear two-point boundary value problems with singularity. Mohanty [31,32], presented 3-step BLAGE iterative method for 2D EBVPs. A family of AGE iteration algorithms using compact sixth-order approximations for solving two-point nonlinear BVPs were discussed in [33–36].

As per our knowledge, no multi-phase block alternative group explicit (block-AGE) iterative method with the aid of 9-point compact sixth order approximation for the solution of 2D Helmholtz elliptic PDE has been discussed in the literature so far. In this article, we propose 9- point compact uniform mesh formulations of order of accuracy six for the solution of 2D Helmholtz equation and application of 3-phase and 2-phase block-AGE iteration technique which carry substantial amount of importance in many applied mathematical problems. Our method is cost effective and relatively fast as function evaluations at the grid points of compact cell saves the time. Our article is ordered as: We propose and formulate the sixth-order approximation for 2D Helmholtz equation. The 3-phase block-AGE iterative technique is presented followed by the corresponding error analysis. Then the 2-phase block-AGE algorithm is accorded. The numerical results in terms of number of iterations have been validated and concluding remarks are presented.

Compact formulation of method for 2D Helmholtz equation

Consider the Helmholtz equation in *x, y* coordinate system

$$z_{xx} + z_{yy} + \lambda^2 z = f(x,y), \qquad (x,y) \in \Omega_2, \tag{3}$$

Let us split the solution domain Ω_2 by mesh points (x_i, y_j), where $0 = x_0 < x_1 < \ldots < x_{N+1} = 1$; $0 = y_0 < y_1 < \ldots < y_{N+1} = 1$, with uniformly located mesh $h = x_i - x_{i-1} = y_j - y_{j-1} > 0$; $i, j = 1(1)N+1$; *i, j, N* being positive integers.

Let $z_{i,j}$ and $Z_{i,j}$ represent, the numerical and exact solutions of $z(x,y)$ at the mesh point (x_i, y_j), respectively, and $f_{i,j}=f(x_i,y_j)$, $f_{i\pm\frac{1}{2},j}=f(x_i \pm \frac{h}{2}, y_j)$, $f_{i,j\pm\frac{1}{2}}=f(x_i, y_j \pm \frac{h}{2})$. We denote $R = \frac{\lambda^2 h^2}{2}$. Then at each mesh point a 9-point compact sixth order approximation (see [24]) for the Helmholtz Eq. (3) is given by

$$\begin{aligned}\left[6\delta_x^2 + 6\delta_y^2 + \delta_x^2\delta_y^2\right] z_{i,j} = \frac{h^2}{15}\Big[& -\lambda^2\left(z_{i+1,j+1} + z_{i+1,j-1} + z_{i-1,j+1} + z_{i-1,j-1}\right) \\ & + \frac{\lambda^2}{2}\left(z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1} + 16 z_{i,j}\right) \\ & - 24\lambda^2\left(\bar{z}_{i+\frac{1}{2},j} + \bar{z}_{i-\frac{1}{2},j} + \bar{z}_{i,\ j+\frac{1}{2}} + \bar{z}_{i,\ j-\frac{1}{2}}\right) \\ & + f_{i+1,j+1} + f_{i+1,j-1} + f_{i-1,j+1} + f_{i-1,j-1} \\ & - \frac{1}{2}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 16 f_{i,j}\right) \\ & + 24\left(f_{i+\frac{1}{2},j} + f_{i-\frac{1}{2},j} + f_{i,\ j+\frac{1}{2}} + f_{i,\ j-\frac{1}{2}}\right)\Big],\end{aligned} \tag{4}$$

where, $\delta_x z_{i,j} = z_{i+\frac{1}{2},j} - z_{i-\frac{1}{2},j}$, $\mu_x z_{i,j} = \frac{1}{2}(z_{i+\frac{1}{2},j} + z_{i-\frac{1}{2},j})$, and $\delta_y z_{i,j} = z_{i,j+\frac{1}{2}} - z_{i,j-\frac{1}{2}}$, $\mu_y z_{i,j} = \frac{1}{2}(z_{i,j+\frac{1}{2}} + z_{i,j-\frac{1}{2}})$ are central and average difference operators with respect to *x*- and *y*-directions respectively, and

$$\bar{z}_{xxy_{i,j}} = \frac{1}{2h^3}\left[\left(z_{i+1,j+1} - z_{i+1,j-1} + z_{i-1,j+1} + z_{i-1,j-1}\right) - 2\left(z_{i,j+1} - z_{i,j-1}\right)\right], \tag{5}$$

$$\bar{z}_{xyy_{i,j}} = \frac{1}{2h^3}\left[(z_{i+1,j+1} + z_{i+1,j-1} - z_{i-1,j+1} - z_{i-1,j-1}) - 2\left(z_{i+1,j} - z_{i-1,j}\right)\right], \tag{6}$$

$$\bar{z}_{xxyy_{i,j}} = \frac{1}{h^4}\left[(z_{i+1,j+1,} + z_{i+1,j-1} + z_{i-1,j+1} + z_{i-1,j-1}) - 2\left(z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1}\right) + 4z_{i,j}\right], \tag{7}$$

$$\begin{aligned}\bar{z}_{i+\frac{1}{2},j} = {} & \frac{1}{2}\left(z_{i+1,j} + z_{i,j}\right) - \frac{1}{8}\left(z_{i+1,j} - 2z_{i,j} + z_{i-1,j}\right) - \frac{h^2}{32}\left(f_{i+1,j} - f_{i-1,j}\right) \\ & - \frac{h^2}{128}\left(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}\right) + \frac{h^3}{16}\bar{z}_{xyy_{i,j}} + \frac{h^4}{128}\bar{z}_{xxyy_{i,j}},\end{aligned} \tag{8}$$

$$\begin{aligned}\bar{z}_{i-\frac{1}{2},j} = {} & \frac{1}{2}\left(z_{i-1,j} + z_{i,j}\right) - \frac{1}{8}\left(z_{i+1,j} - 2z_{i,j} + z_{i-1,j}\right) + \frac{h^2}{32}\left(f_{i+1,j} - f_{i-1,j}\right) \\ & - \frac{h^2}{128}\left(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}\right) - \frac{h^3}{16}\bar{z}_{xyy_{i,j}} + \frac{h^4}{128}\bar{z}_{xxyy_{i,j}},\end{aligned} \tag{9}$$

$$\begin{aligned}\bar{z}_{i,j+\frac{1}{2}} = {} & \frac{1}{2}\left(z_{i,j+1} + z_{i,j}\right) - \frac{1}{8}\left(z_{i,j+1} - 2z_{i,j} + z_{i,j-1}\right) - \frac{h^2}{32}\left(f_{i,j+1} - f_{i,j-1}\right) \\ & - \frac{h^2}{128}\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right) + \frac{h^3}{16}\bar{z}_{xxy_{i,j}} + \frac{h^4}{128}\bar{z}_{xxyy_{i,j}},\end{aligned} \tag{10}$$

$$\begin{aligned}\bar{z}_{i,j-\frac{1}{2}} = {} & \frac{1}{2}\left(z_{i,j-1} + z_{i,j}\right) - \frac{1}{8}\left(z_{i,j+1} - 2z_{i,j} + z_{i,j-1}\right) + \frac{h^2}{32}\left(f_{i,j+1} - f_{i,j-1}\right) \\ & - \frac{h^2}{128}\left(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}\right) - \frac{h^3}{16}\bar{z}_{xxy_{i,j}} + \frac{h^4}{128}\bar{z}_{xxyy_{i,j}}.\end{aligned} \tag{11}$$

Simplifying (4) with the aid of (5)–(11), we get

$$\left[6\delta_x^2 + 6\delta_y^2 + \delta_x^2\delta_y^2\right]z_{i,j} + \frac{7R}{30}\delta_x^2\delta_y^2 z_{i,j} + \left(R + \frac{R^2}{10}\right)\left(\delta_x^2 + \delta_y^2\right)z_{i,j} + 12Rz_{i,j} = \sum f, \tag{12}$$

where

$$\begin{aligned}\sum f = {} & \frac{h^2}{15}\Bigg[f_{i+1,j+1} + f_{i+1,j-1} + f_{i-1,j+1} + f_{i-1,j-1} - \frac{1}{2}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 16f_{i,j}\right) \\ & + 24\left(\mathrm{f}_{i+\frac{1}{2},j} + \mathrm{f}_{i-\frac{1}{2},j} + \mathrm{f}_{i,\ j+\frac{1}{2}} + \mathrm{f}_{i,\ j-\frac{1}{2}}\right) + \frac{3R}{4}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j}\right)\Bigg].\end{aligned} \tag{13}$$

We may re-write (12) explicitly as

$$\begin{aligned}& \left(1 + \frac{7R}{30}\right)(z_{i+1,j+1,} + z_{i+1,j-1} + z_{i-1,j+1} + z_{i-1,j-1}) + \left(4 + \frac{8R}{15} + \frac{R^2}{10}\right)\left(z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1}\right) \\ & + \left(-20 + \frac{134R}{15} - \frac{4R^2}{10}\right)z_{i,j} = \sum f;\, i,j = 1,2,\ldots,N.\end{aligned} \tag{14}$$

Note that the method (14) is a 9-point compact sixth order accurate scheme for the solution of the Helmholtz Eq. (3) and free from the derivatives of $f(x,y)$. Hence right side of (14) can be computed directly.

Three-phase block-AGE iteration method

Merging boundary values in Eq. (14) yields the equation in matrix form

$$Az = RH \tag{15}$$

Here

$$A = \begin{bmatrix} \boldsymbol{D} & \boldsymbol{B} & & & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{B} & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{B} \\ \boldsymbol{0} & & & \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}_{N^2\times N^2}$$

with

$$\boldsymbol{B} = \begin{pmatrix} B_0 & B_1 & & & \boldsymbol{0} \\ B_2 & B_0 & B_1 & & \\ & \ddots & \ddots & \ddots & \\ & & B_2 & B_0 & B_1 \\ \boldsymbol{0} & & & B_2 & B_0 \end{pmatrix}_{N\times N} \text{ and } \boldsymbol{D} = \begin{pmatrix} A_0 & A_1 & & & \boldsymbol{0} \\ A_2 & A_0 & A_1 & & \\ & \ddots & \ddots & \ddots & \\ & & A_2 & A_0 & A_1 \\ \boldsymbol{0} & & & A_2 & A_0 \end{pmatrix}_{N\times N}$$

are three-diagonal matrices of order N, where

$B_2 = B_1 = 1 + \frac{7R}{30},\ B_0 = 4 + \frac{8R}{15} + \frac{R^2}{10}, A_2 = A_1 = 4 + \frac{8R}{15} + \frac{R^2}{10}, A_0 = -20 + \frac{134R}{15} - \frac{4R^2}{10}$ and

$$\boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \vdots \\ \vdots \\ \boldsymbol{z}_N \end{bmatrix}_{N^2\times 1} \quad \text{with } \boldsymbol{z_i} = \begin{bmatrix} z_{1,i} \\ z_{2,i} \\ \vdots \\ \vdots \\ z_{N,i} \end{bmatrix}_{N\times 1}, i = 1(1)N$$

is the vector with numerical solutions $z_{i,j}$,

$$RH = \begin{bmatrix} \boldsymbol{RH}_1 \\ \boldsymbol{RH}_2 \\ \vdots \\ \vdots \\ \boldsymbol{RH}_N \end{bmatrix}_{N^2\times 1} \quad with\ RH_{i=} \begin{bmatrix} RH_{1,i} \\ RH_{2,i} \\ \vdots \\ \vdots \\ RH_{N,i} \end{bmatrix}_{N\times 1}, i = 1(1)N$$

is the vector which contains the exact boundary values and of $f_{i,j},\ f_{i\pm\frac{1}{2},j},\ f_{i,j\pm\frac{1}{2}}$.

We use the technique given in [25]. Let

$$A = M_1 + M_2 \tag{16}$$

where

$$\boldsymbol{M}_1 = \begin{bmatrix} \frac{1}{2}\boldsymbol{D} & & & & & \boldsymbol{0} \\ & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} & & & \\ & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} \\ \boldsymbol{0} & & & & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} \end{bmatrix}, \boldsymbol{M}_2 = \begin{bmatrix} \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} & & & & \boldsymbol{0} \\ \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & & & & \\ & & \ddots & & & \\ & & & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} & \\ & & & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & \\ \boldsymbol{0} & & & & & \frac{1}{2}\boldsymbol{D} \end{bmatrix}$$

if N is odd, and

$$\boldsymbol{M}_1 = \begin{bmatrix} \frac{1}{2}\boldsymbol{D} & & & & & \boldsymbol{0} \\ \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} \\ \boldsymbol{0} & & & & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} \end{bmatrix}, \boldsymbol{M}_2 = \begin{bmatrix} \frac{1}{2}\boldsymbol{D} & & & & & & \boldsymbol{0} \\ & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} & & & & \\ & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & & & & \\ & & & \ddots & & & \\ & & & & \frac{1}{2}\boldsymbol{D} & \boldsymbol{B} & \\ & & & & \boldsymbol{B} & \frac{1}{2}\boldsymbol{D} & \\ \boldsymbol{0} & & & & & & \frac{1}{2}\boldsymbol{D} \end{bmatrix}$$

if N is even, where $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ satisfy the conditions $\det(\boldsymbol{M}_1+\rho\boldsymbol{I}) \neq 0$ and $\det(\boldsymbol{M}_2+\rho\boldsymbol{I}) \neq 0$ for any $\rho > 0$.

Substituting (16) into (15), we may re-write the matrix equation

$$\left(M_1 + M_2\right)z = RH \tag{17}$$

Then the 3-phase block-AGE method is given by

$$\boldsymbol{w}^{(k)} = \left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\boldsymbol{z}^{(k)}, k = 0, 1, 2, \dots, \tag{18.1}$$

$$\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)\boldsymbol{z}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{RH} - \boldsymbol{w}^{(k)}, k = 0, 1, 2, \dots, \tag{18.2}$$

$$\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)\boldsymbol{z}^{(k+1)} = 2\rho\boldsymbol{z}^{\left(k+\frac{1}{2}\right)} + \boldsymbol{w}^{(k)}, k = 0, 1, 2, \dots, \tag{18.3}$$

where

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \vdots \\ \boldsymbol{w}_N \end{bmatrix}_{\text{N}^2\times 1} \quad \text{with } \boldsymbol{w}_i = \begin{bmatrix} w_{1,i} \\ w_{2,i} \\ \vdots \\ \vdots \\ w_{N,i} \end{bmatrix}_{\text{N}\times 1}, \text{i} = 1, 2, \dots, \text{N},$$

$z^{(k+\frac{1}{2})}$ is an intermediate vector at k^{th}-iteration, and $\rho>0$ is the acceleration parameters of the 3-phase block-AGE method.

Now we discuss the algorithm for 3-phase block-AGE, when N is even.

We denote:

$$\boldsymbol{D}_1 = \frac{1}{2}\boldsymbol{D} + \rho\boldsymbol{I} = \left[\frac{1}{2}A_2, \frac{1}{2}A_0 + \rho, \frac{1}{2}A_1\right],$$

$$\boldsymbol{D}_2 = \frac{1}{2}\boldsymbol{D} - \rho\boldsymbol{I} = \left[\frac{1}{2}A_2, \frac{1}{2}A_0 - \rho, \frac{1}{2}A_1\right].$$

Then the 3-phase block-AGE iteration algorithm (18.1)–(18.3) takes the matrix form

$$\begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \vdots \\ \vdots \\ \boldsymbol{w}_{N-1} \\ \boldsymbol{w}_N \end{bmatrix}^{(k)} = \begin{bmatrix} \boldsymbol{D}_2 & \boldsymbol{B} & & & & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D}_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \boldsymbol{D}_2 & \boldsymbol{B} \\ \boldsymbol{0} & & & & \boldsymbol{B} & \boldsymbol{D}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \vdots \\ \vdots \\ \vdots \\ \boldsymbol{z}_{N-1} \\ \boldsymbol{z}_N \end{bmatrix}^{(k)}, k = 0, 1, 2, \dots, \tag{19.1}$$

$$\begin{bmatrix} \boldsymbol{D}_1 & & & & & & 0 \\ & \boldsymbol{D}_1 & \boldsymbol{B} & & & & \\ & \boldsymbol{B} & \boldsymbol{D}_1 & & & & \\ & & & \ddots & & & \\ & & & & \boldsymbol{D}_1 & \boldsymbol{B} & \\ & & & & \boldsymbol{B} & \boldsymbol{D}_1 & \\ 0 & & & & & & \boldsymbol{D}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \\ \vdots \\ \vdots \\ \boldsymbol{z}_{N-2} \\ \boldsymbol{z}_{N-1} \\ \boldsymbol{z}_N \end{bmatrix}^{\left(k+\frac{1}{2}\right)} = \begin{bmatrix} \boldsymbol{RH}_1 - \boldsymbol{w}_1 \\ \boldsymbol{RH}_2 - \boldsymbol{w}_2 \\ \boldsymbol{RH}_3 - \boldsymbol{w}_3 \\ \vdots \\ \vdots \\ \boldsymbol{RH}_{N-2} - \boldsymbol{w}_{N-2} \\ \boldsymbol{RH}_{N-1} - \boldsymbol{w}_{N-1} \\ \boldsymbol{RH}_N - \boldsymbol{w}_N \end{bmatrix}^{(k)}, k = 0, 1, 2, \dots, \tag{19.2}$$

$$\begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{B} & & & & & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D}_1 & & & & & \\ & & \boldsymbol{D}_1 & \boldsymbol{B} & & & \\ & & \boldsymbol{B} & \boldsymbol{D}_1 & & & \\ & & & & \ddots & & \\ & & & & & \boldsymbol{D}_1 & \boldsymbol{B} \\ \boldsymbol{0} & & & & & \boldsymbol{B} & \boldsymbol{D}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \\ \vdots \\ \vdots \\ \boldsymbol{z}_{N-2} \\ \boldsymbol{z}_{N-1} \\ \boldsymbol{z}_N \end{bmatrix}^{(k+1)} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \vdots \\ \boldsymbol{w}_{N-1} \\ \boldsymbol{w}_N \end{bmatrix}^{(k)} + 2\rho \begin{bmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \\ \boldsymbol{z}_3 \\ \vdots \\ \vdots \\ \boldsymbol{z}_{N-2} \\ \boldsymbol{z}_{N-1} \\ \boldsymbol{z}_N \end{bmatrix}^{\left(k+\frac{1}{2}\right)}, k = 0, 1, 2, \dots . \tag{19.3}$$

By carrying out the necessary calculation, (19.1)-(19.3) can be written in explicit form and the following 3-phase block-AGE algorithm:

Phase-1: For $m = 1(2)N\text{-}1$, $l = 1(1)N$, we have

$$\boldsymbol{w}_m^{(k)} = \boldsymbol{D}_2\boldsymbol{z}_m^{(k)} + \boldsymbol{B}\,\boldsymbol{z}_{m+1}^{(k)}, k = 0, 1, 2, \dots . \tag{20.1}$$

$$\boldsymbol{w}_{m+1}^{(k)} = \boldsymbol{B}\,\boldsymbol{z}_m^{(k)} + \boldsymbol{D}_2\,\boldsymbol{z}_{m+1}^{(k)}, k = 0, 1, 2, \dots . \tag{20.2}$$

with $z_{0,m}^{(k)} = z_{N+1,m}^{(k)} = 0$.

P*hase-II*: For $m = 1$ and $l = 1(1)N$, we set

$$\boldsymbol{D}_1\boldsymbol{z}_1^{\left(k+\frac{1}{2}\right)} = \boldsymbol{RH}_1 - \boldsymbol{w}_1^{(k)} \equiv \boldsymbol{Y}_0^{(k)}. \tag{21}$$

The linear system (21) is three-diagonal, hence $\boldsymbol{z}_1^{\left(k+\frac{1}{2}\right)}$can be computed using a tri-diagonal solver.

For $m = 2(2)N\text{-}2$ and $l = 1(1)N$, we set

$$\boldsymbol{D}_1\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} + \boldsymbol{B}\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{RH}_m - \boldsymbol{w}_m^{(k)} \equiv \boldsymbol{Y}_1^{(k)}, \tag{22.1}$$

$$\boldsymbol{B}\,\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} + \boldsymbol{D}_1\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{RH}_{m+1} - \boldsymbol{w}_{m+1}^{(k)} \equiv \boldsymbol{Y}_2^{(k)}. \tag{22.2}$$

Re-writing (22.1)-(22.2)

$$(\boldsymbol{B}^{-1}\boldsymbol{D}_1)\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} + \boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{B}^{-1}\boldsymbol{Y}_1^{(k)}, \tag{23.1}$$

$$\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} + (\boldsymbol{B}^{-1}\boldsymbol{D}_1)\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{B}^{-1}\boldsymbol{Y}_2^{(k)}. \tag{23.2}$$

Multiplying (23.1) by $(\boldsymbol{B}^{-1}\boldsymbol{D}_1)$ and then subtracting from (23.2), we have

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}=\boldsymbol{D}_1\boldsymbol{B}^{-1}\boldsymbol{Y}_1^{(k)}-\boldsymbol{Y}_2^{(k)}. \tag{24}$$

In order to solve (24), let
$\boldsymbol{B}^{-1}\boldsymbol{Y}_1^{(k)}=\boldsymbol{Y}_3^{(k)}$, this implies, $\boldsymbol{B}\boldsymbol{Y}_3^{(k)}=\boldsymbol{Y}_1^{(k)}$, so that $\boldsymbol{Y}_3^{(k)}$ can be evaluated with the aid of a tri-diagonal solver.
The Eq. (24) takes the form

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}=\boldsymbol{D}_1\boldsymbol{Y}_3^{(k)}-\boldsymbol{Y}_2^{(k)}\equiv\boldsymbol{Y}_4^{(k)}. \tag{25}$$

Let

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{Y}_5^{(k)}=\boldsymbol{Y}_4^{(k)}. \tag{26}$$

The left-hand side of (26) is a tri-diagonal matrix, thus (26) can be solved for $\boldsymbol{Y}_5^{(k)}$ using a tri-diagonal solver, where

$$\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}=\boldsymbol{Y}_5^{(k)},$$

or,

$$(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}=\boldsymbol{B}\boldsymbol{Y}_5^{(k)}\equiv\ \boldsymbol{Y}_6^{(k)}, \tag{27}$$

which is a tri-diagonal linear system and can be solved for $\boldsymbol{z}_m^{(k+\frac{1}{2})}$.
From (22.1), we have

$$\boldsymbol{B}\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)}=\boldsymbol{Y}_1{}^{(k)}-\ \boldsymbol{D}_1\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}\equiv\boldsymbol{Y}_7^{(k)}, \tag{28}$$

which can be solved for the intermediate vector $\boldsymbol{z}_{m+1}^{(k+\frac{1}{2})}$.
Finally, for $m=N$ and $l=1(1)N$, we have

$$\boldsymbol{D}_1\boldsymbol{z}_N^{\left(k+\frac{1}{2}\right)}=\boldsymbol{RH}_{\mathrm{N}}-\boldsymbol{w}_N^{(k)}\equiv\boldsymbol{Y}_8^{(k)}. \tag{29}$$

Above system is a three-diagonal linear system, can be solved for $\boldsymbol{z}_N^{(k+\frac{1}{2})}$.
Phase-III: For $m=1(2)N$-1 and $l=1(1)N$, we have

$$\boldsymbol{D}_1\boldsymbol{z}_m^{(k+1)}+\boldsymbol{B}\boldsymbol{z}_{m+1}^{(k+1)}=\boldsymbol{w}_m^{(k)}+2\rho\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)}\equiv\boldsymbol{Y}_9^{(k)}, \tag{30.1}$$

$$\boldsymbol{B}\boldsymbol{z}_m^{(k+1)}+\boldsymbol{D}_1\boldsymbol{z}_{m+1}^{(k+1)}=\boldsymbol{w}_{m+1}^{(k)}+2\rho\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)}\equiv\boldsymbol{Y}_{10}^{(k)}, \tag{30.2}$$

Eqs. (30.1), (30.2) can be re-written as

$$(\boldsymbol{B}^{-1}\boldsymbol{D}_1)\boldsymbol{z}_m^{(k+1)}+\boldsymbol{z}_{m+1}^{(k+1)}=\boldsymbol{B}^{-1}\boldsymbol{Y}_9^{(k)}, \tag{31.1}$$

$$\boldsymbol{z}_m^{(k+1)}+(\boldsymbol{B}^{-1}\boldsymbol{D}_1)\boldsymbol{z}_{m+1}^{(k+1)}=\boldsymbol{B}^{-1}\boldsymbol{Y}_{10}^{(k)}. \tag{31.2}$$

Multiplying (31.1) by $(\boldsymbol{B}^{-1}\boldsymbol{D}_1)$, subtracting from (31.2) and rearranging, we get

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{(k+1)}=\boldsymbol{D}_1\boldsymbol{B}^{-1}\boldsymbol{Y}_9^{(k)}-\boldsymbol{Y}_{10}^{(k)}. \tag{32}$$

In order to solve (32), let
$\boldsymbol{B}^{-1}\boldsymbol{Y}_9^{(k)}=\boldsymbol{Y}_{11}^{(k)}$, this implies, $\boldsymbol{B}\boldsymbol{Y}_{11}^{(k)}=\boldsymbol{Y}_9^{(k)}$, so that $\boldsymbol{Y}_{11}^{(k)}$ can be evaluated with the aid of a tri-diagonal solver.
Then Eq. (32) simplified to

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{(k+1)}=\boldsymbol{D}_1\boldsymbol{Y}_{11}^{(k)}-\boldsymbol{Y}_{10}^{(k)}\equiv\boldsymbol{Y}_{12}^{(k)}. \tag{33}$$

Let

$$(\boldsymbol{D}_1-\boldsymbol{B})\boldsymbol{Y}_{13}^{(k)}=\boldsymbol{Y}_{12}^{(k)}, \tag{34}$$

which can be solved for $\boldsymbol{Y}_{13}^{(k)}$ using a tri-diagonal solver, where

$$\boldsymbol{B}^{-1}(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{(k+1)}=\boldsymbol{Y}_{13}^{(k)},$$

or,

$$(\boldsymbol{D}_1+\boldsymbol{B})\boldsymbol{z}_m^{(k+1)}=\boldsymbol{B}\boldsymbol{Y}_{13}^{(k)}\equiv\ \boldsymbol{Y}_{14}^{(k)}, \tag{35}$$

which is a linear tri-diagonal system and can be computed for $\boldsymbol{z}_m^{(k+1)}$.

Then from (30.1), we have

$$\boldsymbol{B}\boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{Y}_9^{(k)} - \boldsymbol{D}_1\boldsymbol{z}_m^{(k+1)} \equiv \boldsymbol{Y}_{15}^{(k)}, \tag{36}$$

which can be solved for $\boldsymbol{z}_{m+1}^{(k+1)}$ using a tri-diagonal solver.

In a similar, we can write 3-phase block-AGE algorithm, when N is odd.

Error analysis

Now we discuss the convergence of the 3-step block-AGE iteration algorithm (18.1)–(18.3).

Combining the Eqs. (18.1)–(18.3), we get

$$\boldsymbol{z}^{(k+1)} = \boldsymbol{M}\boldsymbol{z}^{(k)} + \boldsymbol{H} \tag{37}$$

where

$$\boldsymbol{M} = \left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1}\left(\boldsymbol{M}_1 - \rho\boldsymbol{I}\right)\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)^{-1}\left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right) \tag{38}$$

is called the 3-phase block-AGE iteration matrix and

$$\boldsymbol{H} = \left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1}\left[\boldsymbol{I} - \left(\boldsymbol{M}_1 - \rho\boldsymbol{I}\right)\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)^{-1}\right]\boldsymbol{RH}.$$

The exact solution value $\boldsymbol{Z}$ satisfies

$$\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)\boldsymbol{Z} = \boldsymbol{RH} - \left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\boldsymbol{Z}, \tag{39.1}$$

$$\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)\boldsymbol{Z} = 2\rho\boldsymbol{Z} + \left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\boldsymbol{Z}. \tag{39.2}$$

Let $\boldsymbol{\varepsilon}^{(k)} = \boldsymbol{z}^{(k)} - \boldsymbol{Z}$ be the error vector at kth iteration. Subtracting (39.1) from (18.2) and (39.2) from (18.3), we get

$$\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)\varepsilon^{\left(k+\frac{1}{2}\right)} = -\left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\varepsilon^{(k)}, k = 0, 1, 2, \dots, \tag{40.1}$$

$$\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)\varepsilon^{(k+1)} = 2\rho\varepsilon^{\left(k+\frac{1}{2}\right)} + \left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\varepsilon^{(k)}, k = 0, 1, 2, \dots, \tag{40.2}$$

and with the aid of (40.1), from (40.2), the error equation is given by

$$\boldsymbol{\varepsilon}^{(k+1)} = \boldsymbol{M}\ \boldsymbol{\varepsilon}^{(\mathbf{k})}, \quad k = 0, 1, 2, \dots. \tag{41}$$

For convergence it is required to prove that the spectral radius $S(\boldsymbol{M}) < 1$, for $\rho>0$.

Let

$$\boldsymbol{M}^* = \left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)\boldsymbol{M}\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1} = \left[\boldsymbol{I} - 2\rho\left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)^{-1}\right]\left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1}, \tag{42}$$

then $\boldsymbol{M}^*$ is similar to $\boldsymbol{M}$, and hence $S(\boldsymbol{M}) = S(\boldsymbol{M}^*)$.

$$\text{Now } ||\mathbf{M}^*||_{\mathbf{2}} \le \left|\left|\boldsymbol{I} - 2\rho\ \left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)^{-1}\right|\right|_2 \cdot \left|\left|\left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1}\right|\right|_2. \tag{43}$$

If $\boldsymbol{M}_1$ has eigen values η_{ij}, $i, j = 1(1)N,$ then

$$\left|\left|\boldsymbol{I} - 2\rho\ \left(\boldsymbol{M}_1 + \rho\boldsymbol{I}\right)^{-\mathbf{1}}\right|\right|_2 = \max\left|1 - \frac{2\rho}{\eta_{ij} + \rho}\right| = \max\left|\frac{\eta_{ij} - \rho}{\eta_{ij} + \rho}\right| < 1, \tag{44}$$

where $R_e(\eta_{ij})>0$; $i, j = 1(1)N$.

In a similar manner,

$$\left|\left|\left(\boldsymbol{M}_2 - \rho\boldsymbol{I}\right)\left(\boldsymbol{M}_2 + \rho\boldsymbol{I}\right)^{-1}\right|\right|_2 < 1. \tag{45}$$

Thus from (43), we obtain

$$S(\boldsymbol{M}) = S(\boldsymbol{M}*) \le ||\mathbf{M}^*||_{\mathbf{2}} < 1. \tag{46}$$

Hence the convergence follows.

Two-phase block-AGE iteration method

Referring to the matrix Eq. (17), we may write 2-phase block-AGE iteration method as

$$(\boldsymbol{M}_1+\rho\boldsymbol{I})\boldsymbol{z}^{\left(k+\frac{1}{2}\right)}=\boldsymbol{RH}-(\boldsymbol{M}_2-\rho\boldsymbol{I})\boldsymbol{z}^{(k)}, k=0,1,2,\dots, \tag{47.1}$$

$$(\boldsymbol{M}_2+\rho\boldsymbol{I})\boldsymbol{z}^{(k+1)}=\boldsymbol{RH}-(\boldsymbol{M}_1-\rho\boldsymbol{I})\boldsymbol{z}^{\left(k+\frac{1}{2}\right)}, k=0,1,2,\dots, \tag{47.2}$$

where $\rho>0$ is the acceleration parameters associated with the 2-phase block-AGE method and $\boldsymbol{z}^{(k+\frac{1}{2})}$ is an intermediate vector.

Combining (47.1)-(47.2), we acquire

$$(\boldsymbol{M}_2+\rho\boldsymbol{I})\boldsymbol{z}^{(k+1)}=\boldsymbol{RH}-(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}\left[\boldsymbol{RH}\ -\ (\boldsymbol{M}_2-\rho\boldsymbol{I})\ \boldsymbol{z}^{(k)}\right]. \tag{48}$$

Simplifying further, we establish the general iteration method

$$\boldsymbol{z}^{(k+1)}=\boldsymbol{P}\boldsymbol{z}^{(k)}+\boldsymbol{Q} \tag{49}$$

where

$$\boldsymbol{P}=(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}(\boldsymbol{M}_2-\rho\boldsymbol{I}) \tag{50}$$

is the two-phase block-AGE iteration matrix and

$$\boldsymbol{Q}=(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}\left[\boldsymbol{I}-(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}\right]\boldsymbol{RH}. \tag{51}$$

Let the error vector at kth iterate is defined by $\boldsymbol{err}^{(k)}=\boldsymbol{z}^{(k)}-\boldsymbol{z}$.

As discussed in previous section, the corresponding error equation is found to be

$$\boldsymbol{err}^{(k+1)}=\ \boldsymbol{P}.\ \boldsymbol{err}^{(k)}k=0,1,2,\dots. \tag{52}$$

In order to validate the convergence, it is obligatory to reveal that the spectral radius S($\boldsymbol{P}$) < 1, for $\rho>0$. Let

$$\boldsymbol{P}^*=(\boldsymbol{M}_2+\rho\boldsymbol{I})\boldsymbol{P}(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}=(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}(\boldsymbol{M}_2-\rho\boldsymbol{I})(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}, \tag{53}$$

then $\boldsymbol{P}^*$ is similar to $\boldsymbol{P}$, and hence $S(\boldsymbol{P})=S(\boldsymbol{P}^*)$.

With the aid of spectral norm from (53), we set

$$||\mathbf{P}^*||_2\ \le\left\|(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}\right\|_2.\left\|(\boldsymbol{M}_2-\rho\boldsymbol{I})(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}\right\|_2. \tag{54}$$

It has been verified that $R_e(\xi_{ij})>0$, where ξ_{ij}; $i,j=1(1)N$ are the eigenvalues of $\boldsymbol{M}_1$ and

$$\left\|(\boldsymbol{M}_1-\rho\boldsymbol{I})(\boldsymbol{M}_1+\rho\boldsymbol{I})^{-1}\right\|_2<1. \tag{55}$$

Equally,

$$\left\|(\boldsymbol{M}_2-\rho\boldsymbol{I})(\boldsymbol{M}_2+\rho\boldsymbol{I})^{-1}\right\|_2<1. \tag{56}$$

From Eq. (54), it is convenient to write

$$S(\boldsymbol{P})=S(\boldsymbol{P}*)\le||\mathbf{P}^*||_2\ <1. \tag{57}$$

Hence the method (47.1)–(47.2) convergences.

Now we discuss the 2-phase block-AGE, when N is odd.

As usual, let

$$\boldsymbol{D}_1=\frac{1}{2}\boldsymbol{D}+\rho\boldsymbol{I}=\left[\frac{1}{2}A_2,\frac{1}{2}A_0+\rho,\frac{1}{2}A_1\right],$$

$$\boldsymbol{D}_2=\frac{1}{2}\boldsymbol{D}-\rho\boldsymbol{I}=\left[\frac{1}{2}A_2,\frac{1}{2}A_0-\rho,\frac{1}{2}A_1\right].$$

Then the two-phase block-AGE iteration algorithm (47.1), (47.2) takes the matrix form

$$\begin{bmatrix}\boldsymbol{D}_1 & & & & & 0\\ & \boldsymbol{D}_1 & \boldsymbol{B} & & & \\ & \boldsymbol{B} & \boldsymbol{D}_1 & & & \\ & & & \ddots & & \\ & & & & \boldsymbol{D}_1 & \boldsymbol{B}\\ 0 & & & & \boldsymbol{B} & \boldsymbol{D}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{z}_1\\ \boldsymbol{z}_2\\ \boldsymbol{z}_3\\ \vdots\\ \vdots\\ \boldsymbol{z}_{N-1}\\ \boldsymbol{z}_N\end{bmatrix}^{\left(k+\frac{1}{2}\right)}=\begin{bmatrix}\boldsymbol{RH}_1-\boldsymbol{D}_2\boldsymbol{z}_1-\boldsymbol{B}\boldsymbol{z}_2\\ \boldsymbol{RH}_2-\boldsymbol{B}\boldsymbol{z}_1-\boldsymbol{D}_2\boldsymbol{z}_2\\ \boldsymbol{RH}_3-\boldsymbol{D}_2\boldsymbol{z}_3-\boldsymbol{B}\boldsymbol{z}_4\\ \vdots\\ \boldsymbol{RH}_{N-1}-\boldsymbol{B}\boldsymbol{z}_{N-2}-\boldsymbol{D}_2\boldsymbol{z}_{N-1}\\ \boldsymbol{RH}_N-\boldsymbol{D}_2\boldsymbol{z}_N\end{bmatrix}^{(k)},\quad k=0,1,2,3,\dots \tag{58.1}$$

$$\begin{bmatrix} D_1 & B & & & & 0 \\ B & D_1 & & & & \\ & & \ddots & & & \\ & & D_1 & B & & \\ & & & B & D_1 & \\ 0 & & & & & D_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ \vdots \\ z_{N-1} \\ z_N \end{bmatrix}^{(k+1)} = \begin{bmatrix} RH_1 - D_2 z_1 \\ RH_2 - D_2 z_2 - Bz_3 \\ \vdots \\ RH_{N-2} - Bz_{N-3} - D_2 z_{N-2} \\ RH_{N-1} - D_2 z_{N-1} - Bz_N \\ RH_N - Bz_{N-1} - D_2 z_N \end{bmatrix}^{\left(k+\frac{1}{2}\right)}, k = 0, 1, 2, \dots \tag{58.2}$$

Simplifying (58.1)-(58.2), we get the following 2-phase block-AGE algorithms.

Phase-1 algorithm:

With $m = 1$ and $l = 1(1)N$, we set

$$D_1 z_1^{\left(k+\frac{1}{2}\right)} = RH_1 - D_2 z_1^{(k)} - Bz_2^{(k)} \equiv X_0^{(k)}, \tag{59.1}$$

The linear system (59) is tri-diagonal, hence very easy to solve for $z_1^{(k+\frac{1}{2})}$ using a tri-diagonal solver (Gaussian-elimination method).

For $m = 2(2)N$-1 and $l = 1(1)N$, we have

$$D_1 z_m^{\left(k+\frac{1}{2}\right)} + Bz_{m+1}^{\left(k+\frac{1}{2}\right)} = RH_m - Bz_{m-1}^{(k)} - D_2 z_m^{(k)} \equiv X_1^{(k)}, \tag{60.1}$$

$$Bz_m^{\left(k+\frac{1}{2}\right)} + D_1 z_{m+1}^{\left(k+\frac{1}{2}\right)} = RH_{m+1} - D_2 z_{m+1}^{(k)} - Bz_{m+2}^{(k)} \equiv X_2^{(k)}, \tag{60.2}$$

with $z_{N+1}^{(k)} = 0$.

We can re-write (60.1)-(60.2) as

$$z_m^{\left(k+\frac{1}{2}\right)} + \left(D_1^{-1} B\right) z_{m+1}^{\left(k+\frac{1}{2}\right)} = D_1^{-1} X_1^{(k)}, \tag{61.1}$$

$$\left(D_1^{-1} B\right) z_m^{\left(k+\frac{1}{2}\right)} + z_{m+1}^{\left(k+\frac{1}{2}\right)} = D_1^{-1} X_2^{(k)}. \tag{61.2}$$

Multiplying (61.2) by $(D_1^{-1} B)$ and subtracting from (61.1) and simplifying, we get

$$\left(B - D_1\right) D_1^{-1} \left(B + D_1\right) z_m^{\left(k+\frac{1}{2}\right)} = BD_1^{-1} X_2^{(k)} - X_1^{(k)}. \tag{62}$$

To solve (62), let

$$D_1^{-1} X_2^{(k)} = X_3^{(k)}. \tag{63}$$

This implies

$$D_1 X_3^{(k)} = X_2^{(k)}, \tag{64}$$

which can be computed with the aid of a tri-diagonal solver.

Therefore (62) shortens to

$$\left(B - D_1\right) D_1^{-1} \left(B + D_1\right) z_m^{\left(k+\frac{1}{2}\right)} = BX_3^{(k)} - X_1^{(k)} \equiv X_4^{(k)}. \tag{65}$$

Let

$$D_1^{-1} \left(B + D_1\right) z_m^{\left(k+\frac{1}{2}\right)} = X_5^{(k)}. \tag{66}$$

Then (65) moderates to a linear tri-diagonal system

$$\left(B - D_1\right) X_5^{(k)} = X_4^{(k)}, \tag{67}$$

which can be easily solved for $X_5^{(k)}$.

Thus from (66), we obtain

$$\left(B + D_1\right) z_m^{\left(k+\frac{1}{2}\right)} = D_1 X_5^{(k)} \equiv X_6^{(k)}, \tag{68}$$

which is again a linear tri-diagonal system and can be computed for $z_m^{(k+\frac{1}{2})}$.

Lastly, from (60.2), we set the linear tri-diagonal structure

$$\boldsymbol{D}_1\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} = \boldsymbol{X}_2^{(k)} - \boldsymbol{B}\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} \equiv \boldsymbol{X}_7^{(k)}, \tag{69}$$

which can be solved for $\boldsymbol{z}_{m+1}^{(k+\frac{1}{2})}$.

Phase-2 *algorithm*:

With $m = 1(2)N$-2 and $l = 1(1)N$, we set

$$\boldsymbol{D}_1\boldsymbol{z}_m^{(k+1)} + \boldsymbol{B}\boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{R}\boldsymbol{H}_m - \boldsymbol{B}\boldsymbol{z}_{m-1}^{\left(k+\frac{1}{2}\right)} - \boldsymbol{D}_2\boldsymbol{z}_m^{\left(k+\frac{1}{2}\right)} \equiv \boldsymbol{X}_8^{(k)}, \tag{70.1}$$

$$\boldsymbol{B}\boldsymbol{z}_m^{(k+1)} + \boldsymbol{D}_1\boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{R}\boldsymbol{H}_{m+1} - \boldsymbol{D}_2\boldsymbol{z}_{m+1}^{\left(k+\frac{1}{2}\right)} - \boldsymbol{B}\boldsymbol{z}_{m+2}^{\left(k+\frac{1}{2}\right)} \equiv \boldsymbol{X}_9^{(k)}, \tag{70.2}$$

with $\boldsymbol{z}_0^{(k+\frac{1}{2})} = 0$.

We may re-write (70.1)-(70.2) as:

$$\boldsymbol{z}_m^{(k+1)} + \left(\boldsymbol{D}_1^{-1}\boldsymbol{B}\right)\boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{D}_1^{-1}\boldsymbol{X}_8^{(k)}, \tag{71.1}$$

$$\left(\boldsymbol{D}_1^{-1}\boldsymbol{B}\right)\boldsymbol{z}_m^{(k+1)} + \boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{D}_1^{-1}\boldsymbol{X}_9^{(k)}. \tag{71.2}$$

Multiplying (71.2) by $(\boldsymbol{D}_1^{-1}\boldsymbol{B})$ and subtracting from (71.1) and simplifying, we get

$$\left(\boldsymbol{B} - \boldsymbol{D}_1\right)\boldsymbol{D}_1^{-1}\left(\boldsymbol{B} + \boldsymbol{D}_1\right)\boldsymbol{z}_m^{(k+1)} = BD_1^{-1}\boldsymbol{X}_9^{(k)} - \boldsymbol{X}_8^{(k)}. \tag{72}$$

To determine $\boldsymbol{z}_m^{(k+1)}$, we first solve

$$\boldsymbol{D}_1\boldsymbol{X}_{10}^{(k)} = \boldsymbol{X}_9^{(k)}, \tag{73}$$

which is a linear tri-diagonal structure can be computed for

$$\boldsymbol{X}_{10}^{(k)} = \boldsymbol{D}_1^{-1}\boldsymbol{X}_9^{(k)}. \tag{74}$$

Then (72) reduces to

$$\left(\boldsymbol{B} - \boldsymbol{D}_1\right)\boldsymbol{D}_1^{-1}\left(\boldsymbol{B} + \boldsymbol{D}_1\right)\boldsymbol{z}_m^{(k+1)} = \boldsymbol{B}\boldsymbol{X}_{10}^{(k)} - \boldsymbol{X}_8^{(k)} \equiv \boldsymbol{X}_{11}^{(k)}. \tag{75}$$

Let

$$\boldsymbol{D}_1^{-1}\left(\boldsymbol{B} + \boldsymbol{D}_1\right)\boldsymbol{z}_m^{(k+1)} = \boldsymbol{X}_{12}^{(k)}. \tag{76}$$

Then (75) moderates to a tri-diagonal matrix form

$$\left(\boldsymbol{B} - \boldsymbol{D}_1\right)\boldsymbol{X}_{12}^{(k)} = \boldsymbol{X}_{11}^{(k)}, \tag{77}$$

which can be solved for $\boldsymbol{X}_{12}^{(k)}$.

From (76), we have

$$\left(\boldsymbol{B} + \boldsymbol{D}_1\right)\boldsymbol{z}_m^{(k+1)} = \boldsymbol{D}_1\boldsymbol{X}_{12}^{(k)} \equiv \boldsymbol{X}_{13}^{(k)}, \tag{78}$$

which is a linear tri-diagonal structure and can be computed for $\boldsymbol{z}_m^{(k+1)}$.

Therefore, from (70.2), we have the system

$$\boldsymbol{D}_1\boldsymbol{z}_{m+1}^{(k+1)} = \boldsymbol{X}_9^{(k)} - \boldsymbol{B}\boldsymbol{z}_m^{(k+1)} \equiv \boldsymbol{X}_{14}^{(k)}, \tag{79}$$

which is again a linear tri-diagonal system and can be computed for $\boldsymbol{z}_{m+1}^{(k+1)}$.

Lastly, for $m = N$ and $l = 1(1)N$, we set

$$\boldsymbol{D}_1\boldsymbol{z}_N^{(k+1)} = \boldsymbol{R}\boldsymbol{H}_N - \boldsymbol{B}\boldsymbol{z}_{N-1}^{\left(k+\frac{1}{2}\right)} - \boldsymbol{D}_2\boldsymbol{z}_N^{\left(k+\frac{1}{2}\right)} \equiv \boldsymbol{X}_{15}^{(k)}, \tag{80}$$

which is also a linear tri-diagonal form, can be computed for $\boldsymbol{z}_N^{(k+1)}$.

In an alike approach, we can write the procedure for N is even.

Table 1
Test Example 1.

h	Block-SOR Method			2-phase block-AGE Method			3-phase block-AGE Method			MAEs
	ρ_{opt}	Itera-tions	CPU Time	ρ_{opt}	Itera- tions	CPU Time	ρ_{opt}	Itera-tions	CPU Time	
1/10	1.446	32	0.0028	0.464	28	0.0024	0.639	26	0.0024	4.5557e-08
1/20	1.677	64	0.0165	0.341	46	0.0155	0.416	42	0.0144	6.7248e-10
1/30	1.773	96	0.0258	0.245	65	0.0223	0.272	58	0.0205	5.7991e-11
1/40	1.824	127	0.0514	0.162	98	0.0486	0.208	87	0.0402	1.0228e-11
1/60	1.886	189	0.1158	0.116	126	0.1067	0.128	105	0.0911	9.0327e-13
1/80	1.934	255	0.1988	0.090	196	0.1784	0.103	157	0.1566	1.5864e-13

Table 2
Test Example 2.

h	Block-SOR Method			2-phase block-AGE Method			3-phase block-AGE Method			MAEs
	ρ_{opt}	Itera-tions	CPU Time	ρ_{opt}	Itera- tions	CPU Time	ρ_{opt}	Itera-tions	CPU Time	
1/10	1.428	30	0.0026	0.449	24	0.0023	0.611	23	0.0023	4.7924e-07
1/20	1.658	59	0.0153	0.320	42	0.0138	0.408	34	0.0121	7.0629e-09
1/30	1.756	84	0.0237	0.238	57	0.0208	0.266	44	0.0182	6.0887e-10
1/40	1.818	117	0.0488	0.151	86	0.0375	0.208	65	0.0257	1.0743e-10
1/60	1.880	169	0.1113	0.109	114	0.0972	0.121	93	0.0863	9.3424e-12
1/80	1.918	233	0.1807	0.088	148	0.1616	0.101	115	0.1343	1.6539e-12

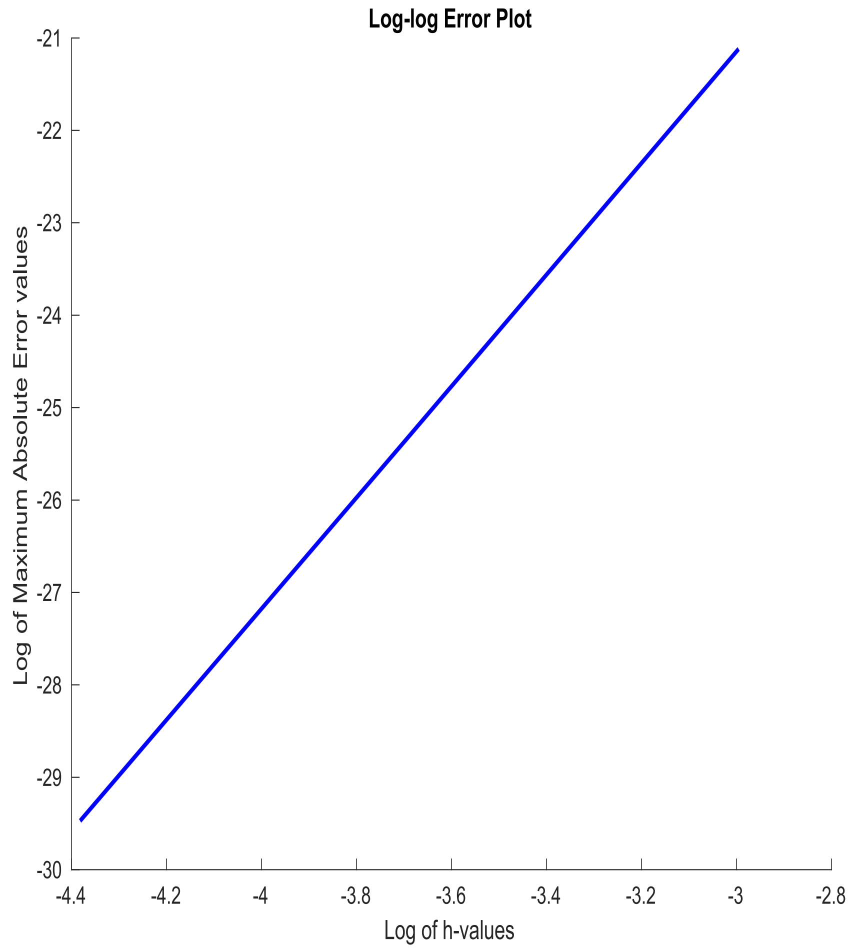

Fig. 1. Test. Example 1: Log-Log Error Plot.

Table 3
Test Example 3.

h	Block-SOR Method			2-phase block-AGE Method			3-phase block-AGE Method			MAEs
	ρ_{opt}	Itera-tions	CPU Time	ρ_{opt}	Itera- tions	CPU Time	ρ_{opt}	Itera-tions	CPU Time	
1/10	1.412	28	0.0024	0.486	22	0.0022	0.642	20	0.0021	1.2839 e-07
1/20	1.654	55	0.0144	0.362	36	0.0138	0.425	32	0.0124	1.8959e-09
1/30	1.755	76	0.0218	0.267	45	0.0204	0.276	40	0.0196	1.6349e-10
1/40	1.812	109	0.0410	0.188	65	0.0394	0.211	57	0.0323	2.8838e-11
1/60	1.876	155	0.1082	0.127	91	0.0984	0.136	82	0.0812	2.4733e-12
1/80	1.921	221	0.1728	0.094	112	0.1518	0.104	101	0.1222	4.4317e-13

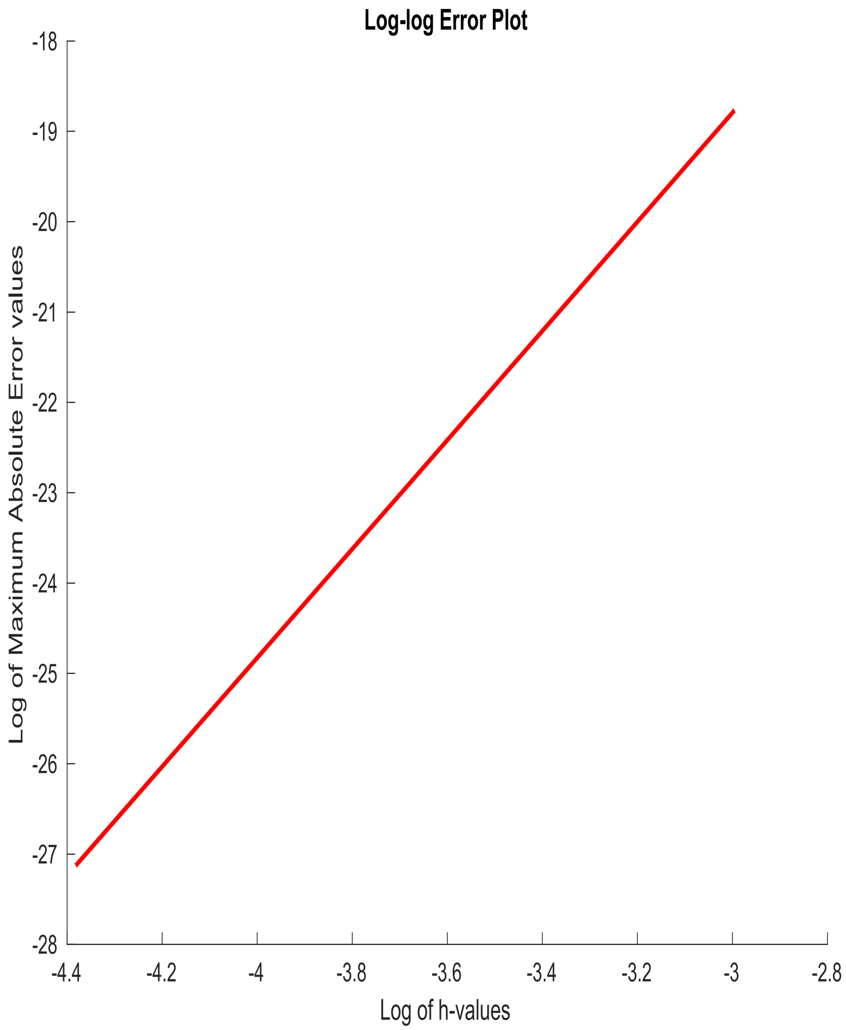

Fig. 2. Test. Example 2: Log-Log Error Plot.

Validation of the proposed iteration methods

The Eq. (15) can be written as

$$\left(\boldsymbol{A}_M - \boldsymbol{A}_L - \boldsymbol{A}_U\right)\boldsymbol{z} = \boldsymbol{RH}, \tag{81}$$

where $\boldsymbol{A} = \boldsymbol{A}_M - \boldsymbol{A}_L - \boldsymbol{A}_U$ represents a tri-block-diagonal matrix with $\boldsymbol{A}_M$, $\boldsymbol{A}_L$ and $\boldsymbol{A}_U$ as main-, lower- and upper- tri-diagonal matrices of order N.

The block-SOR iteration method [37,38] for Eq. (81) may be written as:

$$\boldsymbol{A}_M \boldsymbol{z}^{(k+1)} = \rho\left[\boldsymbol{A}_L \boldsymbol{z}^{(k+1)} + \boldsymbol{A}_U \boldsymbol{z}^{(k)} + \boldsymbol{RH}\right] + (1-\rho)\boldsymbol{A}_M \boldsymbol{z}^{(k)}, \tag{82}$$

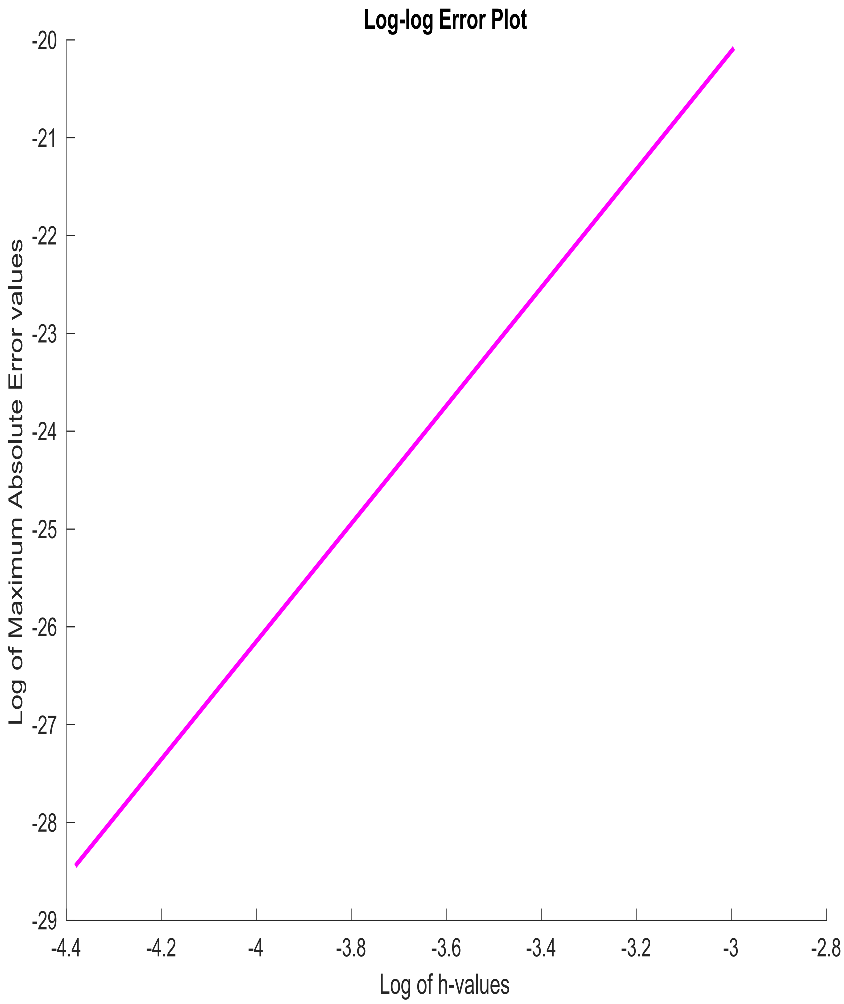

Fig. 3. Test. Example 3: Log-Log Error Plot.

where $\rho \in (0,2)$ represents as a relaxation parameter.

We solve the following benchmark problems using the proposed multi-phase block-AGE iterative methods and compared the results with the corresponding block-SoR method [37,38]. The right-side homogeneous functions and boundary conditions are set up by the closed form solution as an experiment. In all cases, zero vectors are chosen as initial guess and CPU times are reported for all computations. Theoretically, it is difficult to obtain the optimal relaxation parameters (ρ_{opt}) for the proposed block-AGE iteration methods. However, we have obtained values of ρ_{opt} through experiment. Tri-diagonal solver is used in all cases. The computation of proposed methods is possible in the absence of fictitious points. Fictitious points arise due to the discretization of derivative boundary conditions and use of high accuracy non-compact schemes. Since we are using sixth-order compact scheme and the solutions are known exactly on the boundary, the fictitious points do not arise for the computation. The iterations were terminated when the absolute error acceptance $\leq 10^{-15}$ stood accomplished. MATLAB codes were used for executing the computational part.

Test Example 1: The Eq. (3) is solved in the region $0 < x,y < \pi$. The closed form of the solution is given by $z(x,y) = \sin(x)\sin(\lambda\, y)$. The maximum absolute errors (MAEs), CPU time and ρ_{opt} are reported in Table 1 for λ =0.5. The log-log error plot is portrayed in Fig. 1.

Test Example 2: The Eq. (3) is solved in the region $0 < x,y < 1$ with the exact solution $z(x,y) = \sin(\pi x)\sin(\pi y)$. The MAEs, CPU time and ρ_{opt} are reported in Table 2 for λ =0.5. The log-log error graph is plotted in Fig. 2.

Test Example 3: The Eq. (3) is solved for λ =0 in the region $0 < x,y < 1$ with the exact solution $z(x,y) = \exp(2x) sin(\pi y)$. The MAEs, CPU time and ρ_{opt} are reported in Table 3. The log-log error graph is presented in Fig. 3.

Conclusions and final remarks

We have employed compact 9-grid points sixth order approximation for the solution of 2D Helmholtz and Poisson equations. The resulting block liner system that arises due to discretization is solved using multi-phase block-AGE iteration algorithms. In numerical test, the MAEs, CPU time and ρ_{opt} are discussed in terms of the number of iterations required for convergence. The proposed multi-phase block-AGE algorithms require less number of iterations and CPU time for convergence in comparison with block-SoR method for the known values of ρ_{opt}. In 2-phase method the common term is not evaluated separately, whereas in 3-phase method the common term is evaluated first and then used in subsequent computations, thus take less CPU time in comparison with 2-phase method. Though the cost for multi-phase block-AGE methods is more than the block-SoR method, the block-AGE methods have an in-built parallelism; hence more computational time can be saved on parallel designs. Error tabulation, confirms, the projected technique has sixth order convergence, assuredly. The proposed methods have some limitations: (i) not applicable to nonlinear EBVPs, and (ii) not applicable to EBVPs with Neumann boundary conditions. However, it is expected that the proposed block-AGE iteration technique can be successfully implemented on other linear cases.

Ethics statements

This research does not involve human participation, animal experiments and data collection from social media platforms.

Related research article

R.K. Mohanty and Niranjan, Nine-point compact sixth-order approximation for two-dimensional nonlinear elliptic partial differential equations: Application to bi- and tri-harmonic boundary value problems, *Comput. Math. Applics.,* 152 (2023) 239–249, 10.1016/j.camwa.2023.10.030.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

R.K. Mohanty: Conceptualization, Investigation, Methodology, Project administration, Formal analysis, Writing – original draft, Writing – review & editing. **Niranjan:** Conceptualization, Investigation, Data curation, Methodology, Validation, Visualization.

Data availability

No data was used for the research described in the article.

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