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Monogamy relation of multi-qubit systems for squared Tsallis-*q* entanglement

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Tsallis-*q* entanglement is a bipartite entanglement measure which is the generalization of entanglement of formation for *q* tending to 1. We first expand the range of *q* for the analytic formula of Tsallis-*q* entanglement. For $\frac{5-\sqrt{13}}{2} \le q \le \frac{5+\sqrt{13}}{2}$, we prove the monogamy relation in terms of the squared Tsallis-*q* entanglement for an arbitrary multi-qubit systems. It is shown that the multipartite entanglement indicator based on squared Tsallis-*q* entanglement still works well even when the indicator based on the squared concurrence loses its efficacy. We also show that the μ-th power of Tsallis-*q* entanglement satisfies the monogamy or polygamy inequalities for any three-qubit state.

Quantum entanglement as a physics resource for quantum communication and quantum information processing has been the subject of many recent studies in recent years[1–7]. The study of quantum entanglement from various view points has been a very active area and has led to many interesting results. Monogamy of entanglement(MOE)[8] is an interesting property discovered recently in the context of multi-qubit entanglement, which means that quantum entanglement cannot be shared freely in multi-qubit quantum systems. The bipartite monogamy inequality was first proposed and proved by Coffman, Kundu and Wootters(CKW) in a three-qubit system[9], and it is also named as CKW inequality:

$$C^2(\rho_{A|BC}) \ge C^2(\rho_{AB}) + C^2(\rho_{BC}), \tag{1}$$

where C_{ij}^2 is the squared of concurrence between the pair *i* and *j*[10]. Later, the monogamy inequality was generalized into various entanglement measures such as continuous-variable entanglement[11–13], squashed entanglement[14–16], entanglement negativity[17–21], Tsallis-*q* entanglement[22,23], and Rényi-α entanglement[24–26]. The applications of monogamy relation include many fields of physics such as characterizing the entanglement structure in multipartite quantum systems[27–41], the security proof in quantum cryptography[42], the frustration effects observed in condensed matter physics[43], and even black hole physics[43–48]. Originally, MOE was established in terms of the squared concurrence(SC). Analogously, Bai *et al.*[49,50] have proved that the squared entanglement of formation(SEF) obeys the monogamy relation in arbitrary *N*-qubit mixed state. It should be noted that the entanglement of formation(EOF) itself does not satisfy the monogamy relation even for three-qubit pure states. The new monogamy relation in terms of SEF overcomes some flaws of the SC and can be used to detect all genuine multipartite entanglement for *N*-qubit systems.

On the other hand, Tsallis-*q* entanglement is also a well-defined entanglement measure which is the generalization of EOF. For *q* tending to 1, the Tsallis-*q* entanglement converges to the EOF. A natural question is whether the monogamy relation can be generalized to Tsallis-*q* entanglement. In fact, Kim has derived a monogamy relation in terms of Tsallis-*q* entanglement[22]. However, the result in ref. 22 fails in including EOF as a special case and only holds for $2 \le q \le 3$. In this paper we further consider the monogamy relation in terms of the squared Tsallis-*q* entanglement(ST*q*E). Firstly we expand the range of *q* for the analytic formula of Tsallis-*q* entanglement. Then we prove a monogamy inequality of multi-qubit systems in terms of ST*q*E in an arbitrary *N*-qubit mixed state for $\frac{5-\sqrt{13}}{2} \le q \le \frac{5+\sqrt{13}}{2}$, which covers the case of EOF as a special case. Finally, we show that the μ-th power of the Tsallis-*q* entanglement satisfies the monogamy inequalities for three-qubit state.

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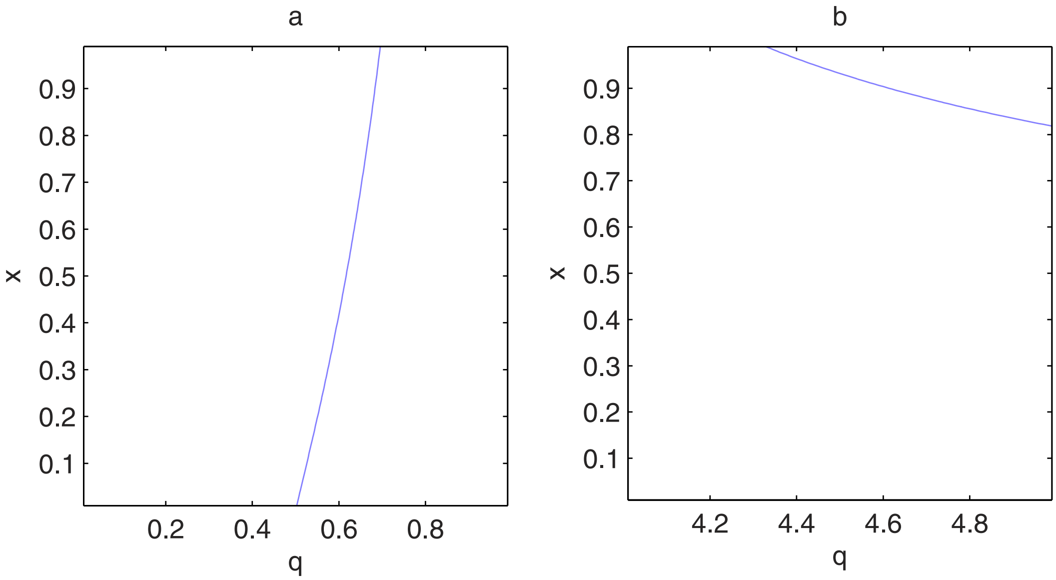

Figure 1. The plot of the dependence of x with q which satisfies the equation $\frac{\partial^2 T_q}{\partial x^2} = 0$ for (**a**) $q \in (0, 1)$ and (**b**) $q \in (4, 5)$ respectively.

Results

Analytic formula of Tsallis-*q* entanglement. Firstly we recall the definition of Tsallis-q entanglement introduced in ref. 22. For a bipartite pure state $|\psi\rangle_{AB}$, the Tsallis-q entanglement is defined as

$$T_q(|\psi\rangle_{AB}) := S_q(\rho_A) = \frac{1}{q-1}(1 - tr\rho_A^q), \tag{2}$$

for any $q > 0$ and $q \neq 1$, where $\rho_A = tr_B|\psi\rangle_{AB}\langle\psi|$ is the reduced density matrix by tracing over the subsystem B. For the case when q tends to 1, $T_q(\rho)$ converges to the von Neumann entropy, that is

$$\lim_{q\to 1} T_q(\rho) = -tr\rho\log\rho = S(\rho). \tag{3}$$

For a bipartite mixed state ρ_{AB}, Tsallis-q entanglement is defined via the convex-roof extension

$$T_q(\rho_{AB}) := \min \sum_i p_i T_q(|\psi_i\rangle_{AB}), \tag{4}$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$.

In ref. 22, Kim has proved an analytic relationship between Tsallis-q entanglement and concurrence for $1 \leq q \leq 4$ as follows

$$T_q(|\psi\rangle_{AB}) = g_q(C(|\psi\rangle_{AB})), \tag{5}$$

where the function $g_q(x)$ is defined as

$$g_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x^2}}{2}\right)^q - \left(\frac{1-\sqrt{1-x^2}}{2}\right)^q\right], \tag{6}$$

According to the results in ref. 22, the analytic formula in Eq. (5) holds for any q such that $g_q(x)$ in Eq. (6) is monotonically increasing and convex. Next we shall generalize the range of q when the function $g_q(x)$ is convex and monotonically increasing with respect to x. The monotonicity and convexity of $g_q(x)$ follow from the nonnegativity of its first and second derivatives. After a direct calculation, we find that the first derivative of $g_q(x)$ with respect to x is always nonnegative for $q \geq 0$[22]. Kim has also proved the nonnegative of the second-order derivative $g_q(x)$ for $1 \leq q \leq 4$. We can further consider the second-order derivative of $g_q(x)$ beyond the region $1 \leq q \leq 4$. We first analyze the nonnegative region for the second-order derivative $g_q(x)$ for $q \in (0, 1)$. Numerical calculation shows that under the condition $\partial^2 T_q(C)/\partial x^2 = 0$, the critical value of x increases monotonically with the parameter q. In Fig. 1(a), we plot the solution (x, q) to this critical condition, where for each fixed x there exists a value of q such that the second-order derivative of $T_q(C)$ is zero. Because x varying monotonically with q, we should only consider the condition $\partial^2 T_q(C)/\partial x^2 = 0$ in the limit $x \to 1$. When $x = 1$, we have

$$\lim_{x\to 1}\frac{\partial^2 T_q}{\partial x^2} = -\frac{2^{1-q}(3-5q+q^2)}{3} \geq 0, \tag{7}$$

which gives the critical point $q_{c1} = \frac{5-\sqrt{13}}{2} \approx 0.7$. When $q > q_{c1}$, the second-order $\partial^2 T_q/\partial x^2$ is always nonnegative. For $q \in (4, 5)$, we find that the value of x decreases monotonically with respect to q as shown in Fig. 1(b). In order to determine the critical point we should only consider the condition $\partial^2 T_q/\partial x^2 = 0$ in the limit $x \to 1$. After direct calculation, we can obtain that the critical point $q_{c2} = \frac{5+\sqrt{13}}{2} \approx 4.3$. When $q < q_{c2}$, the second-order $\partial^2 T_q/\partial x^2$ is always nonnegative. Combining with the previous results in ref. 22, we get that the second derivative of $g_q(x)$ is always a nonnegative function for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$. Thus we have shown that the analytic formula of Tsallis-q entanglement in Eq. (5) holds for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$.

Monogamy inequalities for ST*q*E in *N*-qubit systems. In the following we consider the monogamy properties of STqE. Using the results presented in Methods, we can prove the main result of this paper.

For an arbitrary N-qubit mixed state $\rho_{A_1A_2\cdots A_n}$, the squared Tsallis-q entanglement satisfies the monogamy relation

$$T_q^2(\rho_{A_1|A_2\cdots A_n}) \geq \sum_{i=2}^{n} T_q^2(\rho_{A_1A_i}), \tag{8}$$

where $T_q(\rho_{A_1|A_2\cdots A_n})$ quantifies the Tsallis-q entanglement in the partition $A_1|A_2\cdots A_n$ and $T_q(\rho_{A_1A_i})$ quantifies the one in two-qubit subsystem A_1A_i with the parameter $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5-\sqrt{13}}{2}$.

For proving the above inequality, we first analyze an N-qubit pure state $|\psi\rangle_{A_1A_2\cdots A_n}$. Under the partition $A_1|A_2\cdots A_n$, we have

$$T_q^2(|\psi\rangle_{A_1|A_2\cdots A_n}) = T_q^2\left[C^2_{A_1|A_2\cdots A_n}(|\psi\rangle)\right] \geq T_q^2\left(\sum_{i=2}^{n} C^2_{A_1A_i}\right) \geq \sum_{i=2}^{n} T_q^2(\rho_{A_1A_i}), \tag{9}$$

where in the first inequality we have used the monogamy relation of squared concurrence $C^2_{A_1|A_2\cdots A_n} \geq \sum_{i=2}^{n} C^2_{A_1A_i}$ and the monotonically increasing property of $T_q^2(C^2)$ which has been proved in Methods, and the second inequality is due to the convex property of $T_q^2(C^2)$ (The details for proving the convexity property can be seen from Methods).

Next, we prove the monogamy relation for an N-qubit mixed state $\rho_{A_1A_2\cdots A_n}$. In this case, the formula of Tsallis-q entanglement cannot be applied to $T_q(\rho_{A_1|A_2\cdots A_n})$ since the subsystem $A_2\cdots A_n$ is not a logic qubit in general. But we can still use the definition of Tsallis-q entanglement in Eq. (4). Thus, we have

$$T_q(\rho_{A_1|A_2\cdots A_n}) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i T_q(|\psi_i\rangle_{A_1|A_2\cdots A_n}), \tag{10}$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of the mixed state $\rho_{A_1|A_2\cdots A_n}$. Under the optimal decomposition $\{p_j, |\psi_j\rangle_{A_1|A_2\cdots A_n}\}$, we have

$$\begin{aligned} T_q^2(\rho_{A_1|A_2\cdots A_n}) &= \left[\sum_j p_j T_q(|\psi_j\rangle_{A_1|A_2\cdots A_n})\right]^2 = \left\{\sum_j p_j T_q\left[C_{A_1|A_2\cdots A_n}(|\psi_j\rangle)\right]\right\}^2 \\ &\geq \left\{T_q\left[\sum_j p_j C_{A_1|A_2\cdots A_n}(|\psi_j\rangle)\right]\right\}^2 \geq \left\{T_q\left[C_{A_1|A_2\cdots A_n}(\rho)\right]\right\}^2 \\ &= T_q^2\left[C^2_{A_1|A_2\cdots A_n}(\rho)\right] \geq T_q^2\left[\sum_{i=2}^{n} C^2(\rho_{A_1A_i})\right] \geq \sum_{i=2}^{n} T_q^2\left[C^2(\rho_{A_1A_i})\right] \\ &= \sum_{i=2}^{n} T_q^2(\rho_{A_1A_i}), \end{aligned} \tag{11}$$

where in the second equality we have used the pure state formula of the Tsallis-q entanglement and taken the $T_q(C)$ as a function of the concurrence C for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$; the third inequality is due to that T_q is a monotonically increasing and convex function of the concurrence for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$; the forth inequality is due to the convex property of concurrence for mixed state; and in the sixth and seventh inequalities we used the monotonically increasing and convex properties of $T_q^2(C^2)$ as a function of the squared concurrence for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$ (The details for illustrating the property of STqE can be seen from Methods). Thus we have completed the proof of the monogamy inequalities for STqE in N-qubit systems.

As an application of the established monogamy relation in Eq. (8), we can construct the multipartite entanglement indicator $\tau_q(\rho) = T_q^2(\rho_{A_1|A_2\cdots A_n}) - \sum_{i=2}^{n} T_q^2(\rho_{A_1A_i})$ to detect the genuine multipartite entanglement. We

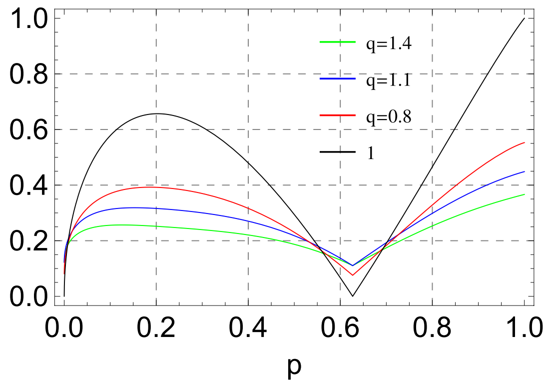

Figure 2. The indicator τ_q for the superposition state $|\psi(p)\rangle$ with $q=0.8$ (red line), $q=1.1$ (blue line), and $q=1.4$ (green line). We also plot the three-tangle of $|\psi(p)\rangle$ with a black line.

consider a three-qubit pure state $|\psi(p)\rangle = \sqrt{p}|GHZ_3\rangle - \sqrt{1-p}|W_3\rangle$, which is the superposition of a GHZ state and a W state with $|GHZ_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|W_3\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. The three-tangle τ introduced in ref. 9 is defined as $\tau(|\psi(p)\rangle) = C_{A|BC}^2 - C_{AB}^2 - C_{AC}^2$. For the quantum state $|\psi(p)\rangle$, its three-tangle is $\tau(|\psi(p)\rangle) = p^2 - 8\sqrt{6}\sqrt{p(1-p)^3}/9$ which has two zero points at $p_1=0$ and $p_2\approx 0.627$. On the other hand, we can directly calculate the value of $\tau_q(|\psi(p)\rangle)$ since the Tsallis-q entanglement has an analytical formula for two-qubit quantum states. In Fig. 2 we plot the three-tangle and the indicator τ_q for the order $q=0.8, 1.1, 1.4$. It is shown that the indicator τ_q is always positive for the different order q in contrast to the three-tangle τ having two zero points. Thus we have shown that the indicator in terms of Tsallis-q entanglement could detect the genuine entanglement in $|\psi(p)\rangle$ better than SC.

Monogamy relation of the μ-th power of Tsallis-q entanglement. Finally, besides the squared Tsallis-q entanglement, we can further consider the monogamy relation of the μ-th power of Tsallis-q entanglement.

For any three-qubit state $\rho_{A_1A_2A_3}$, we can obtain

$$T_q^{\mu}(\rho_{A_1|A_2A_3}) \geq T_q^{\mu}(\rho_{A_1A_2}) + T_q^{\mu}(\rho_{A_1A_3}), \tag{12}$$

for all $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, $\mu \geq 2$.

For proving Eq. (12), we consider the three-qubit case, according to the monogamy relation (8), we have

$$T_q^{2}(\rho_{A_1|A_2A_3}) \geq T_q^{2}(\rho_{A_1A_2}) + T_q^{2}(\rho_{A_1A_3}), \tag{13}$$

for any three-qubit state $\rho_{A_1A_2A_3}$ with $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$. Without loss of generality, assuming $T_q(\rho_{A_1A_2}) > T_q(\rho_{A_1A_3})$, we can obtain

$$\begin{aligned} T_q^{\mu}(\rho_{A_1|A_2A_3}) &\geq (T_q^2(\rho_{A_1A_2}) + T_q^2(\rho_{A_1A_3}))^{\frac{\mu}{2}} = T_q^{\mu}(\rho_{A_1A_2})\left(1 + \frac{T_q^2(\rho_{A_1A_3})}{T_q^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}} \\ &\geq T_q^{\mu}(\rho_{A_1A_2})\left(1 + \left(\frac{T_q^2(\rho_{A_1A_3})}{T_q^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}}\right) = T_q^{\mu}(\rho_{A_1A_2}) + T_q^{\mu}(\rho_{A_1A_3}), \end{aligned} \tag{14}$$

where the second inequality comes from the property $(1+x)^t \geq 1+x^t$ for $x \leq 1$, $t \geq 1$. If $T_q(\rho_{A_1A_2}) = 0$ or $T_q(\rho_{A_1A_3}) = 0$, the inequality obviously holds.

Similarly, we have the following polygamy inequalities. For any three-qubit $\rho_{A_1A_2A_3}$, we have

$$T_q^{\mu}(\rho_{A_1|A_2A_3}) \leq T_q^{\mu}(\rho_{A_1A_2}) + T_q^{\mu}(\rho_{A_1A_3}), \tag{15}$$

for all $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, $\mu \leq 0$.

For any three-qubit state $\rho_{A_1A_2A_3}$ with $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, we have

$$T_q^{\mu}(\rho_{A_1|A_2A_3}) \le (T_q^2(\rho_{A_1A_2}) + T_q^2(\rho_{A_1A_3}))^{\frac{\mu}{2}} = T_q^{\mu}(\rho_{A_1A_2})\left(1 + \frac{T_q^2(\rho_{A_1A_3})}{T_q^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}}$$
$$< T_q^{\mu}(\rho_{A_1A_2})\left(1 + \left(\frac{T_q^2(\rho_{A_1A_3})}{T_q^2(\rho_{A_1A_2})}\right)^{\frac{\mu}{2}}\right) = T_q^{\mu}(\rho_{A_1A_2}) + T_q^{\mu}(\rho_{A_1A_3}), \tag{16}$$

where in the second inequality we have used the inequality $(1+x)^t < 1+x^t$ for $x>0, t\le 0$.

Discussion

In this paper we have generalized the analytic formula of Tsallis-q entanglement to the region $\frac{5-\sqrt{13}}{2} \le q \le \frac{5+\sqrt{13}}{2}$. Then we proved the monogamy relation in terms of STqE for an arbitrary multi-qubit systems, which include previous result in terms of EOF as a special case. Based on the monogamy properties of Tsallis-q entanglement, we have shown that the corresponding indicator can work well even when the indicator based on the squared concurrence loses its efficacy. In addition, we considered the monogamy or polygamy relation of the μ-th power of Tsallis-q entanglement. One distinct advantage of our result is that infinitely many inequalities parameterized by q provides greater flexibility than previous monogamy relation in terms of EOF.

Methods

$T_q^2(C^2)$ is a monotonically-increasing function of the squared concurrence C^2 for all $q\ge 0$. Notice that Eq. (5) can also be written as

$$T_q(|\psi\rangle_{AB}) = f_q(C^2(|\psi\rangle_{AB})), \tag{17}$$

where the function $f_q(x)$ is defined as

$$f_q(x) = \frac{1}{q-1}\left[1 - \left(\frac{1+\sqrt{1-x}}{2}\right)^q - \left(\frac{1-\sqrt{1-x}}{2}\right)^q\right]. \tag{18}$$

The squared Tsallis-q entanglement is a monotonically increasing function of C^2 if the first-order derivative $\partial T_q^2(C^2)/\partial x > 0$ with $x = C^2$. By direct calculation, we have,

$$\frac{\partial T_q^2(C^2)}{\partial x} = 2L(1 - 2^{-q}M^q - 2^{-q}N^q)\left[\frac{2^{-1-q}q(M^{q-1} - N^{q-1})}{\sqrt{1-x}}\right], \tag{19}$$

which is always nonnegative on $0\le x\le 1$ for all $q\ge 0$, where $L = 1/(q-1)^2$, $M = 1 + \sqrt{1-x}$, $N = 1 - \sqrt{1-x}$, and the equality holds only at the boundary. Thus we get that T_q^2 is a monotonically increasing function of x with $x = C^2$.

$T_q^2(C^2)$ is a convex function of the squared concurrence C^2 for $\frac{5-\sqrt{13}}{2} \le q \le \frac{5+\sqrt{13}}{2}$. The convex property of the squared concurrence is satisfied if the second-order derivative $\partial^2 T_q^2(C^2)/\partial x^2 = \partial^2 f_q^2(C^2)/\partial x^2 > 0$ with $x = C^2$. We first define a function $F_q := \partial^2\left[(q-1)^2 T_q^2(C^2)\right]/\partial x^2$ on the domain $D = \{(x, q)|0\le x\le 1, 1\le q\le 4\}$, then the nonnegativity of the second-order derivative T_q^2 can be guaranteed by the nonnegativity of F_q since it varies with $\partial^2 T_q^2(C^2)/\partial x^2$ by a positive constant. After some deduction, we have

$$F_q = \left\{2(1 - 2^{-q}M^q - 2^{-q}N^q)\left[\frac{2^{-2-q}q(M^{q-1} - N^{q-1})}{(1-x)^{3/2}} - \frac{2^{-2-q}(q-1)q(M^{q-2} + N^{q-2})}{1-x}\right] + 2\left[\frac{2^{-1-q}q(M^{q-1} - N^{q-1})}{\sqrt{1-x}}\right]^2\right\}. \tag{20}$$

In order to prove the nonnegativity of F_q, it is suffice to consider its maximum or minimum values on the domain D. The critical points of F_q satisfy the condition

$$\nabla F_q = \left(\frac{\partial F_q}{\partial x}, \frac{\partial F_q}{\partial q}\right) = 0. \tag{21}$$

In Fig. 3(a,b), we have plotted the value of x and q which satisfies the equation $\partial F_q/\partial q = 0$ and $\partial F_q/\partial x = 0$ respectively. Combining the results in Fig. 3(a,b), we find that the solution of the above equation is $q = 1$ which is one of the boundary of domain D. To ensure the nonnegative of F_q, we should only consider the other two cases on the boundary of F_q, i.e., $x = 0$ and $x = 1$.

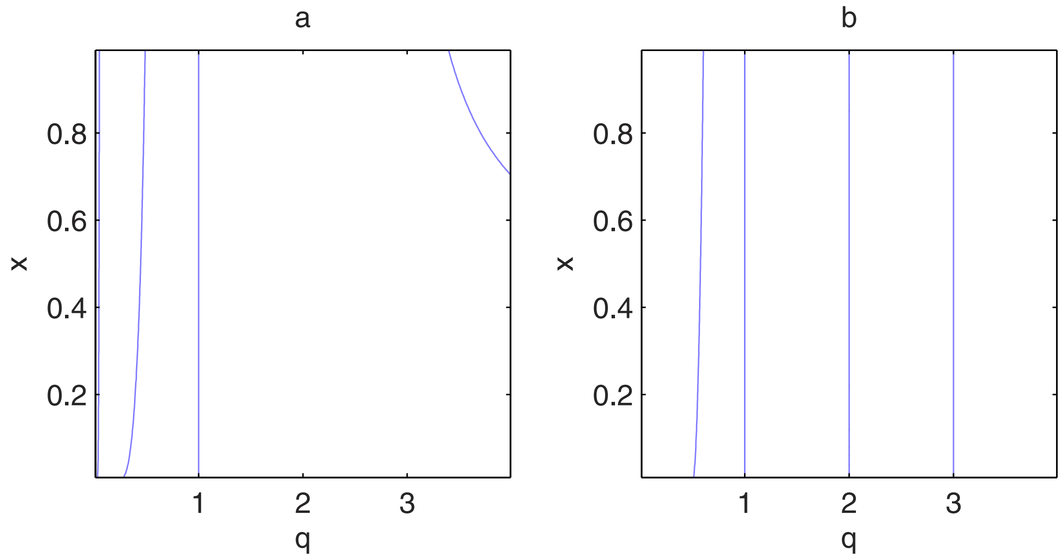

Figure 3. The plot of the dependence of x with q which satisfies the equation (**a**) $\frac{\partial F_q}{\partial q} = 0$ and (**b**) $\frac{\partial F_q}{\partial x} = 0$ respectively.

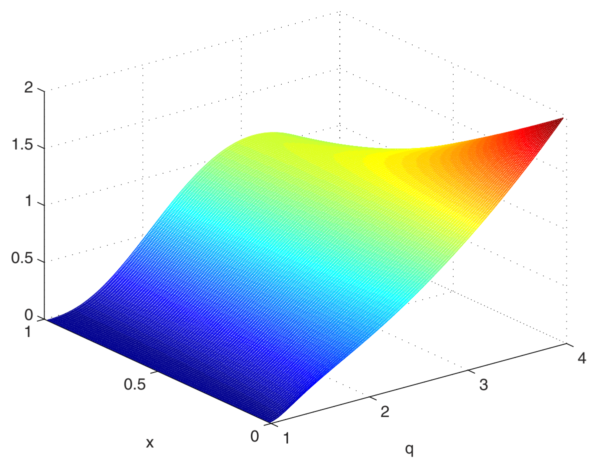

Figure 4. **F_q is plotted as a function of x and q for $0 \leq x \leq 1$, $1 \leq q \leq 4$.**

For the case $x = 0$,

$$\lim_{x \to 0} F_q = 2^{-1-2q} q(2^q - 2)(q - 1), \tag{22}$$

which is always nonnegative in the region $q \in (1, 4)$.

For the case when $x = 1$,

$$\lim_{x \to 1} F_q = \frac{4^{-q}(1 - q)q[6(2^q - 2) + (16 - 5 \times 2^q)q + (2^q - 8)q^2]}{3}, \tag{23}$$

where Eq. (23) is always nonnegative for $q = 1$ and $q = 4$, and the first-order derivative of Eq. (23) increases first and then decreases for $1 \leq q \leq 4$. Thus we prove that Eq. (23) is nonnegative in the region $1 \leq q \leq 4$. Notice that F_q has no critical points in the interior of D, we conclude that F_q is always nonnegative for $1 \leq q \leq 4$. The nonnegative of the F_q is also plotted in Fig. 4.

Furthermore, we can consider the nonnegative region for the second-order derivative $\partial^2 T_q^2 / \partial x^2$ when q ranges in (0, 1). Under the condition $\partial^2 T_q^2 / \partial x^2 = 0$, we find that the critical value of x increases monotonically with the parameter $q \in (0, 1)$. In Fig. 5(a), we plot the solution (x, q) to the critical condition $\partial^2 T_q^2 / \partial x^2 = 0$ where for each fixed x there exists a value of q such that the second-order derivative of T_q^2 is zero. We should only consider the condition $\partial^2 T_q^2 / \partial x^2 \geq 0$ in the limit $x \to 1$. In this case, we have

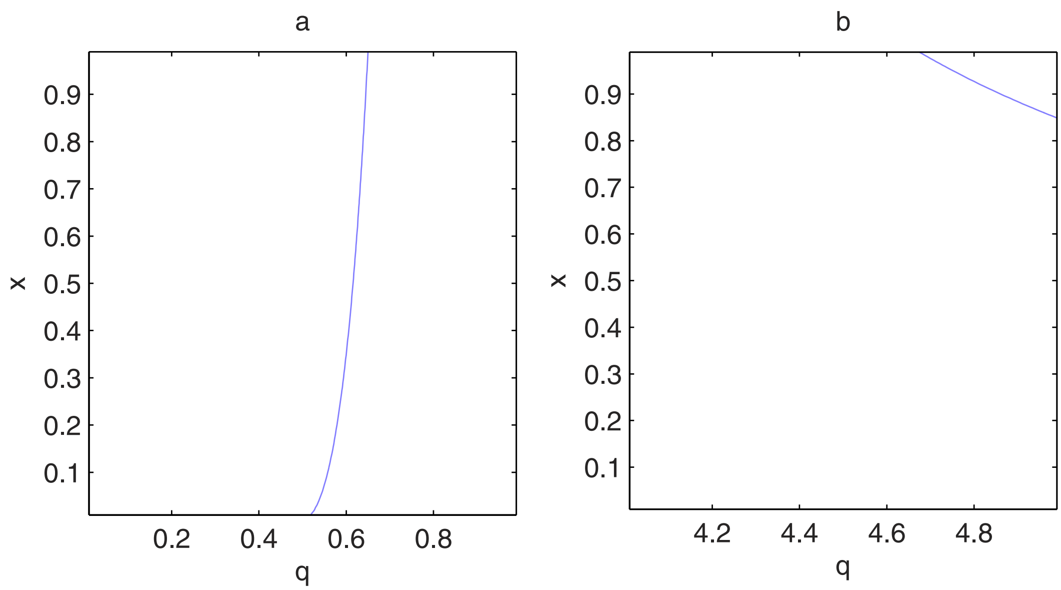

Figure 5. The plot of the dependence of x with q using the equation $\frac{\partial^2 T_q^2}{\partial x^2} = 0$ for (**a**) $q \in (0, 1)$ and (**b**) $q \in (4, 5)$ respectively.

$$\lim_{x\to 1} \frac{\partial^2 T_q^2}{\partial x^2} = -\frac{4^{-q} q[6(2^q - 2) + (16 - 5 \times 2^q)q + (2^q - 8)q^2]}{3(q - 1)} \geq 0, \tag{24}$$

which gives the critical point $q_{c3} \approx 0.65$. When $q \geq q_{c3}$, the second-order $\partial^2 T_q^2/\partial x^2$ is always positive. Similarly, we can also analyze the nonnegative region for the second-order derivative $\partial^2 T_q^2/\partial x^2$ when q ranges in (4, 5). In Fig. 5(b), it is shown that the critical value of x decreases monotonically along with the parameter $q \in (4, 5)$, and the critical point $q_{c4} \approx 4.65$. When $q \leq q_{c4}$, the second-order $\partial^2 T_q^2/\partial x^2$ is always positive. Notice that the analytical formula of T_q is established only for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$, we conclude that the second-order derivative $\partial^2 T_q^2/\partial x^2$ is positive for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$ which completes the proof of the convexity property of $T_q^2(C^2)$ with the squared concurrence C^2 for $\frac{5-\sqrt{13}}{2} \leq q \leq \frac{5+\sqrt{13}}{2}$.

References

1. Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81,** 865 (2009).
2. Mintert, F., Kus, M. & Buchleiyner, A. Concurrence of mixed bipartite quantum states in arbitrary dimensions. *Phys. Rev. Lett.* **92,** 167902 (2004).
3. Chen, K., Albeverio, S. & Fei, S. M. Concurrence of arbitrary dimensional bipartite quantum states. *Phys. Rev. Lett.* **95,** 040504 (2005).
4. Breuer, H. P. Separability criteria and bounds for entanglement measures. *J. Phys. A: Math. Gen.* **39,** 11847 (2006).
5. Breuer, H. P. Optimal entanglement crition for mixed quantum states. *Phys. Rev. Lett.* **97,** 080501 (2006).
6. de Vicente, J. I. Lower bounds on concurrence and separability conditions. *Phys. Rev. A* **75,** 052320 (2007).
7. Zhang, C. J., Zhang, Y. S., Zhang, S. & Guo, G. C. Optimal entanglement witnesses based on local orthogonal observables. *Phy. Rev. A* **76,** 012334 (2007).
8. Terhal, B. M. Is entanglement monogamous? *IBM J. Research and Development* **48,** 71 (2004).
9. Coffman, V., Kundu, J. & Wootters, W. K. Distributed entanglement. *Phys. Rev. A* **61,** 052306 (2000).
10. Wootters, W. K. General monogamy inequality for bipartite qubit entanglement. *Phys. Rev. Lett.* **96,** 220503 (2006).
11. Adesso, G. & Illuminati, F. Continuous variable tangle, monogamy inequality, and entanglement sharing in Gaussian states of continuous variable systems. *New J. Phys.* **8,** 15 (2006).
12. Hiroshima, T., Adesso, G. & Illuminati, F. Monogamy inequality for distributed Gaussian entanglement. *Phys. Rev. Lett.* **98,** 050503 (2007).
13. Adesso, G. & Illuminati, F. Strong monogamy of bipartite and genuine multiparitie entanglement:the Guussian case. *Phys. Rev. Lett.* **99,** 150501 (2007).
14. Koashi, M. & Winter, A. Monogamy of quantum entanglement and other correlations. *Phys. Rev. A* **69,** 022309 (2004).
15. Christandl, M. & Winter, A. Squashed entanglement:an additive entanglement measure. *J. Math. Phys. (NY)* **45,** 829 (2004).
16. Yang, D. *et al.* Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof. *IEEE Trans. Inf. Theory* **55,** 3375 (2009).
17. Ou, Y. C. & Fan, H. Monogamy inequality in terms of negativity for three-qubit states. *Phys. Rev. A* **75,** 062308 (2007).
18. Kim, J. S., Das, A. & Sanders, B. C. Entanglement monogamy of multipartite higher-dimensional quantum systems using convex-roof extend negativity. *Phys. Rev. A* **79,** 012329 (2009).
19. He, H. & Vidal, G. Disentangling theorem and monogamy for entanglement negativity. *Phys. Rev. A* **91,** 012339 (2015).
20. Choi, J. H. & Kim, J. S. Negativity and strong monogamy of multiparty quantum entanglement beyond qubits. *Phys. Rev. A* **92,** 042307 (2015).
21. Luo, Y. & Li, Y. Monogamy of α th power entanglement measurement in qubit system. *Ann. Phys.* **362,** 511 (2015).
22. Kim, J. S. Tsallis entropy and entanglement constraints in multiqubit systems. *Phys. Rev. A* **81,** 062328 (2010).

23. Kim, J. S. Generalized entanglement constraints in multi-qubit systems in terms of Tsallis entropy. arXiv:1603. 02760.
24. Kim, J. S. & Sanders, B. C. Monogamy of multi-qubit entanglement using Rényi entropy. *J. Phys. A: Math. Theor*. **43,** 445305 (2010).
25. Cornelio, M. F. & de Oliveira, M. C. Strong superadditivity and monogamy of the Rényi measure of entanglement. *Phys. Rev. A* **81,** 032332 (2010).
26. Wang, Y. X., Mu, L. Z., Vedral, V. & Fan, H. Entanglement Rényi α entropy. *Phys. Rev. A* **93,** 022324 (2016).
27. Bai, Y. K., Yang, D. & Wang, Z. D. Multipartite quantum correlation and entanglement in four-qubit pure states. *Phys. Rev. A* **76,** 022336 (2007).
28. Bai, Y. K. & Wang, Z. D. Multipartite entanglement in four-qubit Cluster-class states. *Phys. Rev. A* **77,** 032313 (2008).
29. Ou, Y. C., Fan, H. & Fei, S. M. Proper monogamy inequality for arbitrary pure quantum states. *Phys. Rev. A* **78,** 012311 (2008).
30. Jung, E., Park, D. K. & Son, J. W. Three-tangle does not properly quantify tripartite entanglement for Greenberger-Horne-Zeilinger-type states. *Phys. Rev. A* **80,** 010301 (2009).
31. Eltschka, C., Osterloh, A. & Siewert, J. Possibility of generalized monogamy relations for multipartite entanglement beyond three qubits. *Phys. Rev. A* **80,** 032313 (2009).
32. Ren, X. J. & Jiang, W. Entanglement monogamy inequality in a $2\otimes2\otimes4$ system. *Phys. Rev. A* **81,** 024305 (2010).
33. Cornelio, M. F. Multipartite monogamy of the concurrence. *Phys. Rev. A* **87,** 032330 (2013).
34. Regula, B., DiMartino, S., Lee, S. & Adesso, G. Strong monogamy conjecture for multiqubit entanglement: the four-qubit case. *Phys. Rev. Lett.* **113,** 110501 (2014).
35. Kim, J. S. Strong monogamy of quantum entanglement for multiqubit W-class states. *Phys. Rev. A* **90,** 062306 (2014).
36. Bai, Y. K., Ye, M. Y. & Wang, Z. D. Entanglement monogamy and entanglement evolution in multipartite systems. *Phys. Rev. A* **80,** 044301 (2009).
37. Zhu, X. N. & Fei, S. M. Entanglement monogamy relations of qubit systems. *Phys. Rev. A* **90,** 024304 (2014).
38. Zhu, X. N. & Fei, S. M. Generalized monogamy relations of concurrence for N-qubit systems. *Phys. Rev. A* **92,** 062345 (2015).
39. Bai, Y. K., Zhang, N., Ye, M. Y. & Wang, Z. D. Exploring multipartite quantum correlations with the square of quantum discord. *Phys. Rev. A* **88,** 012123 (2013).
40. Aguilar, G. H., Valdes-Hernandez, A., Davidovich, L., Walborn, S. P. & Souto Ribeiro, P. H. Experimental entanglement redistribution under decoherence channels. *Phys. Rev. Lett.* **113,** 240501 (2014).
41. Liu, F., Gao, F. & Wen, Q. Y. Linear monogamy of entanglement in three-qubit systems. *Sci. Rep.* **5,** 16745 (2015).
42. Masanes, L. Universally composable privacy amplification from causality constraints. *Phys. Rev. Lett.* **102,** 140501 (2009).
43. Ma, X. S., Dakic, B., Naylor, W., Zeilinger, A. & Walther, P. Quantum simulation of the wavefunction to probe frustrated Heisenberg spin systems. *Nat. Phys.* **7,** 399 (2011).
44. Toner, B. Monogamy of non-local quantum correlations. *Proc. R. Soc. A* **465,** 59 (2009).
45. Seevinck, M. P. Monogamy of correlations versus monogamy of entanglement. *Quantum Inf. Process.* **9,** 273 (2010).
46. Garca-Saez, A. & Latorre, J. I. Renormalization group contraction of tensor networks in three dimensions. *Phys. Rev. B* **87,** 085130 (2013).
47. Susskind, L. Black hole complementarity and the Harlow-Hayden conjecture. arXiv:1301. 4505.
48. Lloyd, S. & Preskill, J. Unitarity of black hole evaporation in final-state projection models. *J High Energy Phys.* **08,** 126 (2014).
49. Bai, Y. K., Xu, Y. F. & Wang, Z. D. General monogamy relation for the entanglement of formation in multiqubit systems. *Phys. Rev. Lett.* **113,** 100503 (2014).
50. Bai, Y. K., Xu, Y. F. & Wang, Z. D. Hierarchical monogamy relations for the squared entanglement of formation in multipartite systems. *Phys. Rev. A* **90,** 062343 (2014).

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Author Contributions

G.-M.Y. and W.S. carried out the calculations. W.S., M.Y. and D.-C.L. conceived the idea. All authors contributed to the interpretation of the results and the writing of the manuscript. All authors reviewed the manuscript.

Additional Information

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