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Research article

Cosine similarity and distance measures for $p,q-$ quasirung orthopair fuzzy sets: Applications in investment decision-making

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ABSTRACT

Similarity measures and distance measures are used in a variety of domains, such as data clustering, image processing, retrieval of information, and recognizing patterns, in order to measure the degree of similarity or divergence between elements or datasets. $p,q-$ quasirung orthopair fuzzy ($p,q-$ QOF) sets are a novel improvement in fuzzy set theory that aims to properly manage data uncertainties. Unfortunately, there is a lack of research on similarity and distance measure between $p,q-$ QOF sets. In this paper, we investigate different cosine similarity and distance measures between to $p,q-$ quasirung orthopair fuzzy sets ($p,q-$ ROFSs). Firstly, the cosine similarity measure and the Euclidean distance measure for $p,q-$ QOFSs are defined, followed by an exploration of their respective properties. Given that the cosine measure does not satisfy the similarity measure axiom, a method is presented for constructing alternative similarity measures for $p,q-$ QOFSs. The structure is based on the suggested cosine similarity and Euclidean distance measures, which ensure adherence to the similarity measure axiom. Furthermore, we develop a cosine distance measure for $p,q-$ QOFSs that connects similarity and distance measurements. We then apply this technique to decision-making, taking into account both geometric and algebraic perspectives. Finally, we present a practical example that demonstrates the proposed justification and efficacy of the proposed method, and we conclude with a comparison to existing approaches.

1. Introduction

Fuzzy sets [1], introduced by Zadeh in 1965, are an extension of classical set theory that allows for the representation of uncertainty and vagueness in data. In classical set theory, an element either belongs to a set or does not. However, in fuzzy set theory, elements can belong to a set to a degree between 0 and 1, reflecting the degree of membership or possibility. Applications of fuzzy sets span various fields, including Control Systems [2], Pattern Recognition [3], Decision Making and Optimization [4,5], Information Retrieval [6],

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Medicine and Healthcare [7] and Natural Language Processing [8]. However, the FS is not suitable for capturing comprehensive decision information; it just describes the membership degree (MD) of connected information. For example, a ten-person expert panel assesses applicants when a university considers hiring teaching staff members. In this case, two experts declare opposition to recruiting, seven experts are in agreement with it, and one expert is undecided. By adding non-membership degrees, Atanassov [9] extended the FS to an intuitionistic FS (IFS) in order to communicate the earlier indicated information. The decision information for recruiting may be written as $\mathscr{I} = \{0.50, 0.20\}$, where 0.50 and 0.20 represent MD and non-MD (NMD), respectively. The hesitancy MD is determined as 0.30 ($1 - 0.50 - 0.20 = 0.30$). For an IFS defined as $\mathscr{I} = (t, u_{\mathscr{I}}(t_i), v_{\mathscr{I}}(t_i))$, where $t_i \in T$ and T is a set $\{t_1, t_2, \ldots, t_n\}$ with n elements, the sum of the membership degree ($u_{\mathscr{I}}(t_i)$) and non-membership degree ($v_{\mathscr{I}}(t_i)$) for any element xi satisfies the condition $u_{\mathscr{I}}(t_i) + v_{\mathscr{I}}(t_i) \leqslant 1$, ensuring $u_{\mathscr{I}}(t_i) + v_{\mathscr{I}}(t_i) + \Pi \leqslant 1$, where Π represents the hesitancy MD.

Because of the complexities of the decision-making environment, IFS have constraints in effectively capturing decision data. For example, suppose a decision maker assigns membership and non-membership degrees of 0.80 and 0.60, respectively, resulting in a total greater than one. In that case, the IFS is inadequate for communicating the relevant information. To handle these types of information, Yager [10] expanded the notion of IFS by introducing the Pythagorean Fuzzy Set (PFS), expressed as: $\mathscr{P} = \{t, u_{\mathscr{P}}(t_i), v_{\mathscr{P}}(t_i)\}$, where $(u_{\mathscr{P}}(t_i))^2 + v_{\mathscr{P}}(t_i)^2 \leqslant 1$. As a result, the data provided can be represented as $\mathscr{P} = (0.80, 0.60)$, indicating a Pythagorean membership grade rather than an intuitionistic membership grade, because the sum of 0.80 and 0.60 is greater than one. When compared to IFS, PFS have a higher spatial membership degree, indicating that PFS may have a wider range of applications. PFS has been the subject of much research and application in many different fields since its introduction. For example, Can et al. [11] introduced the notion of circular Pythagorean fuzzy sets as a new extension of PFSs. Pan et al. [12] introduced a quaternion model of Pythagorean fuzzy set (QPFS), where membership, non-membership, and hesitation functions are represented using quaternions. QPFS offers a significant advantage over PFS in that its representation space for fuzzy information extends from the real plane to the hypercomplex plane. Jamal et al. [13] suggested a solution for linear correlated fuzzy differential equations within the linear correlated fuzzy spaces.

The PFS is unable to effectively represent a decision maker with MD and NMD of 0.80 and 0.75, respectively, as the sum of these values exceeds the allowed range, i.e., $0.80^2 + 0.75^2 \leqslant 1$. The deal with such information, Yager [14] introduced $q-$ rung orthopair fuzzy sets ($q-$ ROFSs) which is the generalization of IFSs and PFSs. The $q-$ ROFS can be expressed as $\mathscr{Q} = \{u_{\mathscr{Q}}(t_i), v_{\mathscr{Q}}(t_i)\}$ with the condition $(u_{\mathscr{Q}}(t_i))^q + (v_{\mathscr{Q}}(t_i))^q \leqslant 1$ $(q \geqslant 1)$. IFS and PFS are instances of $q-$ ROFS, as indicated by $q = 1$ and $q = 2$, respectively. For $q \geqslant 3$, the data presented in the scenario above can be reflected as a q-ROFS $\mathscr{Q} = (0.8, 0.75)_q$ such that $0.80^q + 0.75^q \leqslant 1$. The $q-$ ROFS offers decision-makers greater flexibility in expressing membership grades. Decision-makers can choose the parameter q to influence the range of information expression in the $q-$ ROFS. $q-$ ROFS has seen broad research and application in a variety of fields. For example, Ali [15] devised a q-ROF distance measure by utilizing a matrix norm and a strictly increasing (or decreasing) function. Chen et al. [16] introduced a multi-attribute decision-making approach utilizing the q-rung orthopair probabilistic hesitant fuzzy Schweizer-Sklar power weighted Hamy mean operator. Vimala et al. [17] explored two novel concepts, namely q-rung orthopair multi-fuzzy set and q-rung orthopair multi-fuzzy soft set, merging the advantages of both q-rung orthopair fuzzy set and multi-fuzzy soft set. Shahzad et al. [18] established fuzzy fixed-point results for sequences of locally fuzzy mappings that adhere to rational-type almost contractions within complete dislocated metric spaces. Subsequently, they extended these findings to derive results applicable to both set-valued and single-valued mappings. Seikh and Mandal [19] introduced operational laws for q-rung orthopair fuzzy sets

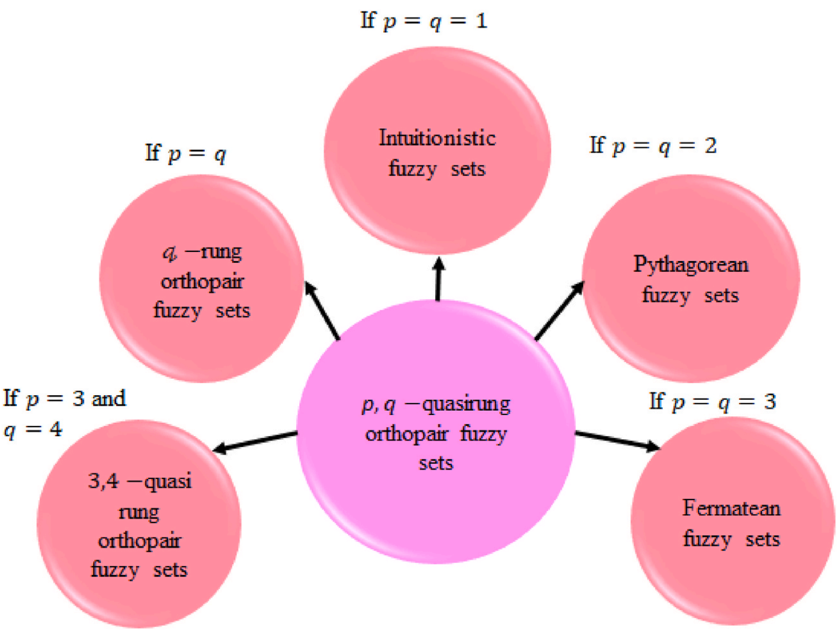

Fig. 1. $p, q-$ QOFSs and their specific instances.

utilizing Archimedean t-conorms and t-norms, subsequently deriving a set of aggregation operators and a model for addressing multiple attribute decision-making (MADM) problems.

In $q-$ ROFSs, decision-makers are required to use an equivalent q value for both MD and NDM when managing provided data. However, in some real-world scenarios, decision-makers may need to use different powers for MD and NDM. Seikh and Mandal [20] addressed this need by introducing p,q-quasirung orthopair fuzzy sets ($p,q-$ QOFSs), which are a generalization of $q-$ ROFS. $p,q-$ QOFSs can be expressed as $\mathscr{V} = \{t, \mathscr{u}_{\mathscr{V}}(t_i), \mathscr{v}_{\mathscr{V}}(t_i)\}$ such that $(\mathscr{u}_{\mathscr{V}}(t_i))^p + (\mathscr{v}_{\mathscr{V}}(t_i))^p \preccurlyeq 1$. In the $p,q-$ QOF context, the parameters p and q are positive integers ($p,\ q \succcurlyeq 1$) such that $p \prec q$, $p \succ q$, or $p = q$. The given information can be represented as a $p,q-$ QOFSs $\mathscr{V} = (0.80, 0.75)_{p,q}$ with the condition $0.80^p + 0.75^q \preccurlyeq 1$ ($p = 4$ and $q = 3$). After the $p,q-$ QOFSs was introduced, several scholars have conducted studies on it. As an illustration, Rahim et al. [21] introduced $p,q-$ QOF weighted and geometric operators employing the sine function to address intricate decision-making challenges. Ali and Naeem [22] used Aczel-Alsina operations and proposed Aczel-Alsina operators to aggregate $p,q-$ QOF information. Rahim et al. [23] proposed confidence levels-based aggregation operators for $p,q-$ QOF numbers ($p,q-$ QOFNs. Fig. 1 depicts the distinct scenarios of $p,q-$ QOFSs according to different conditions for the values of p and q.

On the other hand, similarity measures (SMs) constitute a significant aspect within FS theory, finding extensive application in pattern recognition, medical diagnosis, and various other domains. For example, Kirişci [24] introduced distance metrics and cosine similarity measures for Fermatean fuzzy sets (FFSs), and their respective properties were investigated. Verma and Mittal [25] formulated the generalized Pythagorean fuzzy probabilistic ordered weighted cosine similarity operator, which incorporates probabilistic information. Liu [26] formulated several cosine similarity measures and Euclidean distance measures for complex q-rung orthopair fuzzy sets, and subsequently explored their properties. Ejegwa [27] introduced similarity measures and distance measures for comparing two IFSs. Gohain et al. [28] introduced a nonlinear distance formula for IFSs and demonstrated its validity as a distance measure through explicit proof of its properties.

The literature review highlights a gap in research regarding similarity measures specifically designed for $p,q-$ QOFSs. While similarity measures have been extensively studied for other types of fuzzy sets, such as IFS, PFS, and $q-$ ROFSs, there is a notable absence of research addressing similarity measures for $p,q-$ QOFSs. Therefore, the paper aims to fill this gap by investigating various cosine similarity and distance measures tailored for $p,q-$ QOFSs, which are crucial for decision-making applications. The contributions of the paper are as follows:

1. The paper defines cosine similarity and Euclidean distance measures for $p,q-$ QOFSs and explores their properties. This establishes a foundational understanding of similarity and distance metrics in the context of $p,q-$ QOFSs.
2. Recognizing that the cosine similarity measure does not satisfy the similarity measure axiom, the paper presents a method for constructing alternative similarity measures for $p,q-$ QOFSs. These measures are designed to adhere to the similarity measure axiom while being based on the suggested cosine similarity and Euclidean distance measures.
3. The paper extends the TOPSIS method by incorporating the newly proposed cosine distance measure for $p,q-$ QOFSs. This enhancement allows for more robust decision-making in scenarios with multiple criteria, considering both geometric and algebraic perspectives.
4. A practical example is provided to demonstrate the effectiveness and applicability of the proposed methodology. This real-world illustration showcases how the suggested strategy can be implemented in decision-making contexts.

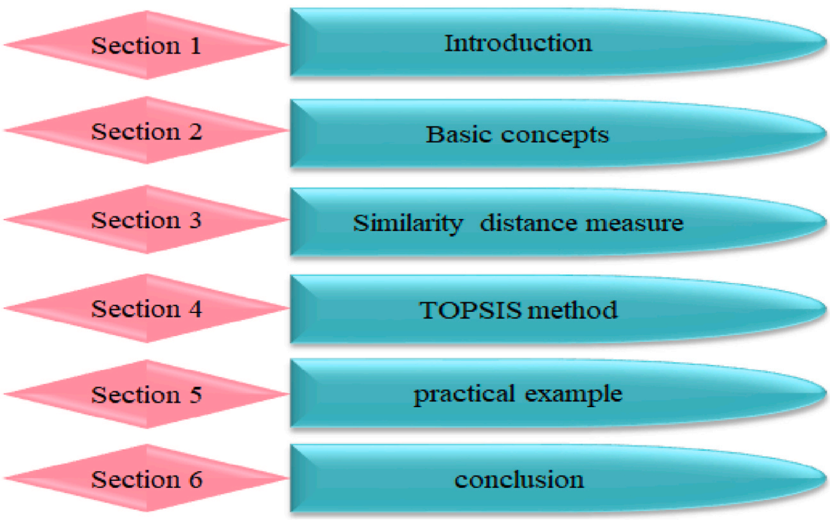

Fig. 2. Paper layout.

5. The proposed approach is compared with other well-known approaches, providing insights into its advantages and distinguishing features. This comparative analysis offers valuable context for understanding the strengths and limitations of the proposed methodology.

The proposed cosine similarity measures offer several advantages, including insensitivity to vector length, robustness to outliers, and efficient computation, making them well-suited for comparing data samples or feature vectors of varying lengths, handling noisy data, and processing high-dimensional datasets efficiently. On the other hand, the proposed distance measure for $p,q-$ QOFSs addresses the specific characteristics and complexities of these sets, ensuring its applicability in decision-making and uncertainty management. Through explicit proofs of its properties, the proposed distance measure satisfies the necessary criteria to be considered a valid distance metric, guaranteeing its reliability and utility in analytical and computational contexts. Moreover, by integrating cosine similarity and distance measures, the proposed distance measure provides a comprehensive framework for analyzing and comparing $p,q-$ QOFSs, enhancing its interpretability and effectiveness in decision-making processes.

The rest of the paper is presented in the following sections. A summary of the basic ideas behind IFS, PFS, $q-$ ROFS, $p,q-$ QOFS and their operational principles is given in Section 2. Assuring the axiom of similarity, we suggest several new similarity measures for $p,q-$ QOFSs in Section 3 that are based on the cosine similarity and Euclidean distance (ED) measures. Furthermore, we use the relationship between similarity and distance measures to derive a cosine distance measure for $p,q-$ QOFSs. We extend the TOPSIS method by incorporating the newly proposed cosine distance measure for q-ROFSs in Section 4. A real-world example is provided in Section 5 to show the applicability and effectiveness of the suggested strategy. For context, this approach is compared with other well-known approaches. The conclusion is given in Section 6. The structure of the proposed work is presented in Fig. 2.

2. Preliminaries

This section provides essential background knowledge, definitions of key terms, and relevant theoretical frameworks.

Definition 1. [9] Consider $T=\{t_1,t_2,\ldots,t_n\}$ as a fixed set; in this context, IFS $\mathscr{I}$ on T can be defined as:

$$\mathscr{I}=\{t,\langle u_{\mathscr{I}}(t_i),v_{\mathscr{I}}(t_i)\rangle|t_i\in T\} \tag{1}$$

In the context, $u_{\mathscr{I}}(t_i)\in[0,1]$ represent the membership grade and $v_{\mathscr{I}}(t_i)\in[0,1]$ represent the non-membership grade of an element $t_i\in T$ where $i=1,2,\ldots,n$. IFS satisfy the condition $u_{\mathscr{I}}(t_i)+v_{\mathscr{I}}(t_i)\preccurlyeq 1$. The degree of hesitancy for IFS as presented in Equation (1) can be expressed as $\Pi_{\mathscr{I}}=1-u_{\mathscr{I}}(t_i)-v_{\mathscr{I}}(t_i)$ where $0\preccurlyeq\Pi_{\mathscr{I}}\preccurlyeq 1$.

Definition 2. [10] Assume $T=\{t_1,t_2,\ldots,t_n\}$ as a fixed set; then PFS $\mathscr{P}$ on T can be defined as:

$$\mathscr{P}=\{t,\langle u_{\mathscr{P}}(t_i),v_{\mathscr{P}}(t_i)\rangle|t_i\in T\} \tag{2}$$

where, $u_{\mathscr{I}}(t_i)\in[0,1]$ represent the membership grade and $v_{\mathscr{I}}(t_i)\in[0,1]$ represent the non-membership grade of an element $t_i\in T$ where $i=1,2,\ldots,n$. IFS satisfy the condition $(u_{\mathscr{P}}(t_i))^2+(v_{\mathscr{P}}(t_i))^2\preccurlyeq 1$ for $t_i\in T$. The degree of hesitancy for PFSs as presented in Equation (2) can be expressed as $\Pi_{\mathscr{P}}=\sqrt{1-(u_{\mathscr{P}}(t_i))^2-(v_{\mathscr{P}}(t_i))^2}$ where $0\preccurlyeq\Pi_{\mathscr{P}}\preccurlyeq 1$.

Definition 3. [14] For any finite set $T=\{t_1,t_2,\ldots,t_n\}$, a $q-$ ROFS $\mathscr{Q}$ over an element $t_i\in T$ can be defined as follows:

$$\mathscr{Q}=\{t,\langle u_{\mathscr{Q}}(t_i),v_{\mathscr{Q}}(t_i)\rangle|t_i\in T\} \tag{3}$$

In this context, the membership and non-membership grades are represented by $u_{\mathscr{Q}}(t_i)$ and $v_{\mathscr{Q}}(t_i)$. Where $0\preccurlyeq v_{\mathscr{Q}}(t_i)$, $v_{\mathscr{Q}}(t_i)\preccurlyeq 1$ such that $(v_{\mathscr{Q}}(t_i))^q+(v_{\mathscr{Q}}(t_i))^q\preccurlyeq 1$ for all $i=1,2,\ldots,n$ and $q\geqslant 1$. The degree of hesitancy for $q-$ ROFSs as presented in Equation (3) can be calculated as $\Pi_{\mathscr{Q}}=\sqrt[q]{1-(u_{\mathscr{Q}}(t_i))^q-(v_{\mathscr{Q}}(t_i))^q}$.

Definition 4. [20] For any finite set $T=\{t_1,t_2,\ldots,t_n\}$, a $p,q-$ QOFS $\mathscr{V}$ over an element $t_i\in T$ can be defined as follows:

$$\mathscr{V}=\{t,\langle u_{\mathscr{V}}(t_i),v_{\mathscr{V}}(t_i)\rangle|t_i\in T\} \tag{4}$$

In Equation (4), the membership and non-membership grades are represented by $u_{\mathscr{Q}}(t_i)$ and $v_{\mathscr{Q}}(t_i)$ of $p,q-$ QOFS. Where $0\preccurlyeq v_{\mathscr{V}}(t_i)$, $v_{\mathscr{V}}(t_i)\preccurlyeq 1$ such that $(v_{\mathscr{V}}(t_i))^p+(v_{\mathscr{V}}(t_i))^q\preccurlyeq 1$ for all $i=1,2,\ldots,n$ and $p,q\geqslant 1$. The degree of hesitancy can be calculated as $\Pi_{\mathscr{V}}=\sqrt[l]{1-(u_{\mathscr{V}}(t_i))^p-(v_{\mathscr{V}}(t_i))^q}$. For the sake of simplicity, we designate $p,q-$ QOFS as $\mathscr{V}=(u_{\mathscr{V}},v_{\mathscr{V}})_{p,q}$ such that $(v_{\mathscr{V}})^p+(v_{\mathscr{V}})^q\preccurlyeq 1$ and called $p,q-$ QOFN.

Remake 1. [20] The parameters p and q are two positive integers such that

(a) $p\succ q$, $p\prec q$, or $p=q$,
(b) l is the least common multiple of p and q and can be expressed as $l=LCM(p,q)$.

Definition 5. [20] Let $\mathscr{V}_1=(u_{\mathscr{V}_1},v_{\mathscr{V}_1})_{p,q}$ and $\mathscr{V}_2=(u_{\mathscr{V}_2},v_{\mathscr{V}_2})_{p,q}$ are any two $p,q-$ QOFNs then

(a) $\mathscr{V}_1 \cup \mathscr{V}_2 = (\max(u_{\mathscr{V}_1}, u_{\mathscr{V}_2}), \min(v_{\mathscr{V}_1}, v_{\mathscr{V}_2}))_{p,q}$,
(b) $\mathscr{V}_1 \cap \mathscr{V}_2 = (\min(u_{\mathscr{V}_1}, u_{\mathscr{V}_2}), \max(v_{\mathscr{V}_2}, v_{\mathscr{V}_2}))_{p,q}$,
(c) $\mathscr{V}_1^C = (v_{\mathscr{V}_1}, u_{\mathscr{V}_1})_{p,q}$, where $\mathscr{V}_1^C$ represent the complement of $p,q-$ QOFN $\mathscr{V}_1$.
(d) $\mathscr{V}_1 \preccurlyeq \mathscr{V}_2$ if and only if $u_{\mathscr{V}_1} \preccurlyeq u_{\mathscr{V}_2}$ and $v_{\mathscr{V}_1} \succcurlyeq v_{\mathscr{V}_2}$.

The standard approach for comparing two $p,q-$ QOFNs is to use score and accuracy functions. When the score value of $p,q-$ QOFN $\mathscr{V}_1$ is greater than $\mathscr{V}_2$, it means that $\mathscr{V}_1$ is sparser than $\mathscr{V}_2$. On the other hand, accuracy values must be determined using an accuracy function when the score values of the two $p,q-$ QOFNs are equal.

Definition 6. [20] Let $\mathscr{V} = (u_{\mathscr{V}}, v_{\mathscr{V}})_{p,q}$ be a $p,q-$ QOFN, the score function (*SF*) and accuracy function (*AF*) of $\mathscr{V}$ can be defined as follows:

$$SF(\mathscr{V}) = \frac{1}{2}(1 + u_{\mathscr{V}}{}^p - v_{\mathscr{V}}{}^q) \tag{5}$$

$$AF(\mathscr{V}) = u_{\mathscr{V}}{}^p + v_{\mathscr{V}}{}^q \tag{6}$$

where $0 \preccurlyeq AF(\mathscr{V}), SF(\mathscr{V}) \preccurlyeq 1$ $(p,q \succcurlyeq 1)$.

Definition 7. [20] Let $\mathscr{V}_1 = (u_{\mathscr{V}_1}, v_{\mathscr{V}_1})_{p,q}$ and $\mathscr{V}_2 = (u_{\mathscr{V}_2}, v_{\mathscr{V}_2})_{p,q}$ are any two $p,q-$ QOFNs then

(a) If $SF(\mathscr{V}_1) \succ SF(\mathscr{V}_2)$ then $\mathscr{V}_1 \succ \mathscr{V}_2$,
(b) If $SF(\mathscr{V}_1) \prec SF(\mathscr{V}_2)$ then $\mathscr{V}_1 \prec \mathscr{V}_2$,
(c) If $SF(\mathscr{V}_1) = SF(\mathscr{V}_2)$ then
- If $AF(\mathscr{V}_1) \succ AF(\mathscr{V}_2)$ then $\mathscr{V}_1 \succ \mathscr{V}_2$,
- If $AF(\mathscr{V}_1) \prec AF(\mathscr{V}_2)$ then $\mathscr{V}_1 \prec \mathscr{V}_2$,
- If $AF(\mathscr{V}_1) = AF(\mathscr{V}_2)$ and $SF(\mathscr{V}_1) = SF(\mathscr{V}_2)$ then $\mathscr{V}_1 = \mathscr{V}_2$.

3. Cosine similarity and distance measures between $p,q-$ QOFSs

The cosine similarity measure (CSM) is a metric for calculating the similarity of two vectors in a multidimensional space. It is calculated as the cosine of the angle between two vectors. CSM is widely used in many fields, including information retrieval, natural language processing, and data mining, to determine the similarity of materials, texts, or data points. In this section, we introduce a CSM designed for $p,q-$ QOFSs.

Definition 8. Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non-empty finite set. We suppose two $p,q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$. The CSM between $\mathscr{V}_1$ and $\mathscr{V}_2$ can be expressed as follows:

$$CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{(u_{\mathscr{V}_1}(t_i))^p(u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q(v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} - (v_{\mathscr{V}_1}(t_i))^{2q}}\sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} - (v_{\mathscr{V}_2}(t_i))^{2q}}}\right) \tag{7}$$

In Equation (7), $p \succ q$, $p \prec q$, or $p = q$.

Theorem 1. Let $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ be any two $p,q-$ QOFSs. The CSM $CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)$ satisfy the following conditions.

(a) $0 \preccurlyeq CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \preccurlyeq 1$ $(p,q \succcurlyeq 1)$,
(b) $CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = CSM_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$,
(c) $CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 1$ if $\mathscr{V}_1 = \mathscr{V}_2$ that is $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$.

Proof. Assertions (a) and (b) are self-evident given the criteria stated in Definition 8.
In the case of $\mathscr{V}_1 = \mathscr{V}_2$, where $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$, the result is $CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 1$.

Definition 9. Let $T = \{t_1, t_2, \ldots, t_n\}$ be a non-empty finite set. We suppose two $p,q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$. The weighted CSM (WCSM) between $\mathscr{V}_1$ and $\mathscr{V}_2$ can be expressed as follows:

$$WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \sum_{i=1}^{n}\eta_i\left(\frac{(u_{\mathscr{V}_1}(t_i))^p(u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q(v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} + (v_{\mathscr{V}_1}(t_i))^{2q}}\sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} + (v_{\mathscr{V}_2}(t_i))^{2q}}}\right) \tag{8}$$

where $\eta_i \in [0,1]$, $\sum_{i=1}^{n}\eta_i = 1$ and $p \succ q$, $p \prec q$, or $p = q$.

Remark 2. If the weights $\eta_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$ i.e., $(\eta_1, \eta_2, \ldots, \eta_n) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ the WCSM reduce to CSM.

Theorem 2. Let $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ be any two p, q− QOFSs. The WCSM $WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)$ satisfy the following conditions.

(a) $0 \leqslant WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$ $(p, q \geqslant 1)$,
(b) $WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = WCSM_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$,
(c) $WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 1$ if $\mathscr{V}_1 = \mathscr{V}_2$ that is $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$. where $\eta_i \in [0, 1]$, $\sum_{i=1}^{n} \eta_i = 1$.

Proof. The proof is like that of Theorem 1; therefore, we decide to omit it in this context.

Example 1 illustrates the computational procedure of the Weighted Cosine Similarity Measure (WCSM).

Example 1. Let $T = \{t_1, t_2, t_3, t_4, t_5\}$ be a finite set, suppose that for two p, q− QOFSs $\mathscr{V}_1 = \Big\{\langle t_1, (0.30, 0.40)_{p,q}\rangle, \langle t_2, (0.60, 0.50)_{p,q}\rangle, \langle t_3, (0.50, 0.70)_{p,q}\rangle, \langle t_4, (0.60, 0.40)_{p,q}\rangle, \langle t_5, (0.35, 0.50)_{p,q}\rangle\Big\}$, $\mathscr{V}_2 = \Big\{\langle t_1, (0.50, 0.30)_{p,q}\rangle, \langle t_2, (0.30, 0.40)_{p,q}\rangle, \langle t_3, (0.40, 0.50)_{p,q}\rangle, \langle t_4, (0.30, 0.45)_{p,q}\rangle, \langle t_5, (0.55, 0.65)_{p,q}\rangle\Big\}$, if $p = q = 4$ and the weights are $(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = (0.20, 0.25, 0.15, 0.10, 0.30)$, the WCSM defined in Equation (8) can be calculated as:

$$0.20 \times \left(\frac{(0.30)^4(0.50)^4 - (0.40)^4(0.30)^4}{\sqrt{(0.30)^8 + (0.40)^8}\sqrt{(0.50)^8 + (0.30)^8}}\right) + 0.25 \times \left(\frac{(0.60)^4(0.30)^4 - (0.50)^4(0.40)^4}{\sqrt{(0.60)^8 + (0.50)^8}\sqrt{(0.30)^8 + (0.40)^8}}\right)$$

$$0.15 \times \left(\frac{(0.50)^4(0.40)^4 - (0.70)^4(0.50)^4}{\sqrt{(0.50)^8 + (0.70)^8}\sqrt{(0.40)^8 + (0.50)^8}}\right) + 0.10 \times \left(\frac{(0.60)^4(0.30)^4 - (0.40)^4(0.45)^4}{\sqrt{(0.60)^8 + (0.40)^8}\sqrt{(0.30)^8 + (0.45)^8}}\right)$$

$$0.30 \times \left(\frac{(0.35)^4(0.55)^4 - (0.50)^4(0.65)^4}{\sqrt{(0.35)^8 + (0.50)^8}\sqrt{(0.55)^8 + (0.65)^8}}\right) = 0.7340. \text{ Thus, } WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 0.7340 \in [0, 1].$$

A similarity measure is considered authentic if it follows the axiom of similarity measures.

Proposition 1. Consider two fuzzy sets: $\mathscr{V}_1$ and $\mathscr{V}_2$. If a similarity measure (SM), denoted as $SM(\mathscr{V}_1, \mathscr{V}_2)$, has the following properties:

(a) $0 \leqslant SM(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$,
(b) $SM(\mathscr{V}_1, \mathscr{V}_2) = 1$ if and only if $\mathscr{V}_1 = \mathscr{V}_2$,
(c) $SM(\mathscr{V}_1, \mathscr{V}_2) = SM(\mathscr{V}_2, \mathscr{V}_1)$.

In this case, $SM(\mathscr{V}_1, \mathscr{V}_2)$ is considered an authentic SM.

If a SM $SM(\mathscr{V}_1, \mathscr{V}_2)$, follows Proposition 2, the distance measure $Dis(\mathscr{V}_1, \mathscr{V}_2)$, can be expressed as $Dis(\mathscr{V}_1, \mathscr{V}_2) = 1 - SM(\mathscr{V}_1, \mathscr{V}_2)$. The CSMs outlined in Definitions 8 and 9 do not meet the criteria for authentic SM because they fail to satisfy property (b) of Proposition 2 in specific scenarios, as illustrated by Example 2.

Example 2. Assume $T = \{t_1, t_2\}$ be a non-empty finite set. $\mathscr{V}_1 = \Big\{\langle t_1, (0.40, 0.40)_{p,q}\rangle, \langle t_2, (0.30, 0.30)_{p,q}\rangle\Big\}$, $\mathscr{V}_2 = \Big\{\langle t_1, (0.50, 0.50)_{p,q}\rangle, \langle t_2, (0.40, 0.40)_{p,q}\rangle\Big\}$, if $p = q = 1$ and the weights are $(\eta_1, \eta_2) = (0.45, 0.55)$ then the WCSM $WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 1$, however, $\mathscr{V}_1$ and $\mathscr{V}_2$ are not equal i.e., $\mathscr{V}_1 \neq \mathscr{V}_1$.

Considering that the CSM outlined in Definitions 6 and 7 fail to satisfy the axiom of SM, we now introduce a novel SM, denoted as $SM_{p,q-QOFS}$, for any two p, q− QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$. This measure is created by combining the proposed CSM and the Euclidean distance measure (EDM) $ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)$.

Definition 10. Let $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ be any two p, q− QOFSs. The EDM $ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)$ can be expressed as follows:

$$ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2n}\sum_{t_i \in T}\left(\left|(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p\right|^2 + \left|(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q\right|^2\right)\right)^{\frac{1}{2}} \tag{9}$$

Using the assumption that η_i represents the weight of $t_i \in T$ and $\eta_i \in [0, 1]$ for all $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \eta_i = 1$, we can calculate the weighted EDM $WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)$ between two p, q− QOFSs $\mathscr{V}_1$ and $\mathscr{V}_2$ as follows:

$$WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2}\sum_{t_i \in T}\eta_i\left(\left|(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p\right|^2 + \left|(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q\right|^2\right)\right)^{\frac{1}{2}} \tag{10}$$

In Equations (9) and (10), the parameters $p, q \geqslant 1$ such that $p \succ q$, $p \prec q$, or $p = q$.

Theorem 3. For any two $p,q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$, then weighted EDM $WED_{p,q-QOF}$ satisfies the following properties:

(a) $0 \leqslant WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$ $(p, q \geqslant 1)$,
(b) $WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = WED_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$,
(c) $WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 0$ if $\mathscr{V}_1 = \mathscr{V}_2$ that is $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$. where $\eta_i \in [0, 1]$, $\sum_{i=1}^{n} \eta_i = 1$.

Proof. Since $0 \leqslant u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i), u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$ then for $p, q \geqslant 1$ $0 \leqslant |(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p|^2 \leqslant 1$ and $0 \leqslant |(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q|^2 \leqslant 1$. Thus, $0 \leqslant WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant \left(\frac{1}{2}\right)^{\frac{1}{2}} \times \left(2 \sum_{i=1}^{n} \eta_i\right)^{\frac{1}{2}} = 1$. And hence $0 \leqslant WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$.

Definition 10states that (b) is self-evident.

If $WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 0$ then it implies that $|(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p|^2 = 0$ and $|(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q|^2 = 0$. And hence $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$.

Definition 11. For any two $p,q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$, then, a new SM $\widetilde{SM}_{p,q-QOFS}$ between $\mathscr{V}_1$ and $\mathscr{V}_2$ can be expressed as:

$$\widetilde{SM}_{p,q-QOFS}(\mathscr{V}_1, \mathscr{V}_2) = \frac{CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) + 1 - ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)}{2} \tag{11}$$

In Equation (11),

$$CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{(u_{\mathscr{V}_1}(t_i))^p (u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q (v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} - (v_{\mathscr{V}_1}(t_i))^{2q}} \sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} - (v_{\mathscr{V}_2}(t_i))^{2q}}} \right)$$

and

$$ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2n} \sum_{t_i \in T} \left(|(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p|^2 + |(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q|^2 \right) \right)^{\frac{1}{2}}.$$

Definition 12. For any two $p,q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$, then, a new weighted SM $\widetilde{WSM}_{p,q-QOFS}$ between $\mathscr{V}_1$ and $\mathscr{V}_2$ can be expressed as:

$$\widetilde{WSM}_{p,q-QOFS}(\mathscr{V}_1, \mathscr{V}_2) = \frac{WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) + 1 - WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)}{2} \tag{12}$$

In Equation (12)

$$WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \sum_{i=1}^{n} \eta_i \left(\frac{(u_{\mathscr{V}_1}(t_i))^p (u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q (v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} + (v_{\mathscr{V}_1}(t_i))^{2q}} \sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} + (v_{\mathscr{V}_2}(t_i))^{2q}}} \right)$$

and

$$WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2} \sum_{t_i \in T} \eta_i \left(|(u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p|^2 + |(v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q|^2 \right) \right)^{\frac{1}{2}}.$$

Remark 3. If the weights $\eta_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$ i.e., $(\eta_1, \eta_2, \ldots, \eta_n) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ then $\widetilde{WSM}$ reduced to $\widetilde{SM}$.

Example 3. Consider a fixed set $T = \{t_1, t_2\}$, and let's assume two $p,q-$ ROFSs: $\mathscr{V}_1 = \left\{ \langle t_1, (0.40, 0.40)_{p,q} \rangle, \langle t_2, (0.20, 0.20)_{p,q} \rangle \right\}$ and $\mathscr{V}_2 = \left\{ \langle t_1, (0.40, 0.40)_{p,q} \rangle, \langle t_2, (0.30, 0.30)_{p,q} \rangle \right\}$. If $p = q = 1$ and $\eta = (0.45, 0.55)$ be the weights of $\mathscr{V}_i$ $(1, 2)$, then the WSM $\widetilde{WSM}_{p,q-QOFS}(\mathscr{V}_1, \mathscr{V}_2) = 0.95$. When $\mathscr{V}_1 = \mathscr{V}_2$, the SM $\widetilde{WSM}_{p,q-QOFS}$ does not equal 1. This feature distinguishes it from the weighted cosine similarity measure $WCSM_{p,q-QOF}$, avoiding the drawback of the latter. Importantly, $\widetilde{WSM}_{p,q-QOF}$ follows the axiom of similarity measure.

Theorem 4. For any two $p, q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$, then weighted EDM $WED_{p,q-QOF}$ satisfies the following properties:

(a) $0 \leqslant \widetilde{WSM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$ $(p, q \geqslant 1)$,

(b) $\widetilde{WSM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \widetilde{WSM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$,

(c) $\widetilde{WSM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 0$ if $\mathscr{V}_1 = \mathscr{V}_2$ that is $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$. Where $\eta_i \in [0, 1]$, $\sum_{i=1}^{n} \eta_i = 1$.

Proof. Easy proof.

Remark 4. The SM $\widetilde{SM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$ for any two $p, q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ in T also conforms to the axiom of SM.

If the SM follows to the axiom of distance measure (DM), a corresponding DM can be derived based on the relationship between DMs and SMs. Given that the proposed SM $\widetilde{SM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$ is indeed an authentic SM, we can establish a corresponding DM $\widetilde{WDM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$ between any two $p, q-$ QOFSs $\mathscr{V}_1$ and $\mathscr{V}_1$ using the following approach.

Definition 13. Let $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ be any two $p, q-$ QOFSs. The EDM $\widetilde{WDM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$ can be expressed as follows:

$$\widetilde{WDM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1) = 1 - WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \frac{1 - WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) + WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)}{2} \tag{13}$$

In Equation (13),

$$WCSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \sum_{i=1}^{n} \eta_i \left(\frac{(u_{\mathscr{V}_1}(t_i))^p (u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q (v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} + (v_{\mathscr{V}_1}(t_i))^{2q}} \sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} + (v_{\mathscr{V}_2}(t_i))^{2q}}} \right)$$

and

$$WED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2} \sum_{t_i \in T} \eta_i \left(\left| (u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p \right|^2 + \left| (v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q \right|^2 \right) \right)^{\frac{1}{2}}.$$

Remark 5. If the weights $\eta_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$ i.e., $(\eta_1, \eta_2, \ldots, \eta_n) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ then $\widetilde{WDM}$ reduced to $\widetilde{DM}$.

Definition 14. Let $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$ be any two $p, q-$ QOFSs. The DM $\widetilde{DM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$ can be expressed as follows:

$$\widetilde{DM} = 1 - \frac{1 - CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) + ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2)}{2} \tag{14}$$

In Equation (14),

$$CSM_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{(u_{\mathscr{V}_1}(t_i))^p (u_{\mathscr{V}_2}(t_i))^p - (v_{\mathscr{V}_1}(t_i))^q (v_{\mathscr{V}_2}(t_i))^q}{\sqrt{(u_{\mathscr{V}_1}(t_i))^{2p} - (v_{\mathscr{V}_1}(t_i))^{2q}} \sqrt{(u_{\mathscr{V}_2}(t_i))^{2p} - (v_{\mathscr{V}_2}(t_i))^{2q}}} \right)$$

and

$$ED_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \left(\frac{1}{2n} \sum_{t_i \in T} \left(\left| (u_{\mathscr{V}_1}(t_i))^p - (u_{\mathscr{V}_2}(t_i))^p \right|^2 + \left| (v_{\mathscr{V}_1}(t_i))^q - (v_{\mathscr{V}_2}(t_i))^q \right|^2 \right) \right)^{\frac{1}{2}}.$$

Theorem 4For any two $p, q-$ QOFSs $\mathscr{V}_1 = \{t, \langle u_{\mathscr{V}_1}(t_i), v_{\mathscr{V}_1}(t_i)\rangle | t_i \in T\}$ and $\mathscr{V}_2 = \{t, \langle u_{\mathscr{V}_2}(t_i), v_{\mathscr{V}_2}(t_i)\rangle | t_i \in T\}$, then weighted DM $ED_{p,q-QOF}$ satisfies the following properties:

(a) $0 \leqslant \widetilde{DM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) \leqslant 1$ $(p, q \geqslant 1)$,

(b) $\widetilde{DM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = \widetilde{DM}_{p,q-QOF}(\mathscr{V}_2, \mathscr{V}_1)$,

(c) $\widetilde{DM}_{p,q-QOF}(\mathscr{V}_1, \mathscr{V}_2) = 0$ if $\mathscr{V}_1 = \mathscr{V}_2$ that is $u_{\mathscr{V}_1}(t_i) = u_{\mathscr{V}_2}(t_i)$ and $v_{\mathscr{V}_1}(t_i) = v_{\mathscr{V}_2}(t_i)$ for all $i = 1, 2, \ldots, n$. Where $\eta_i \in [0, 1]$, $\sum_{i=1}^{n} \eta_i = 1$.

Proof. Easy proof.

4. MCDM TOPSIS approach with $p, q-$ QOFNs

In the following section, we construct a TOPSIS method to manage MCDM for $p, q-$ QOF information.

Assume decision-makers are assessing alternatives $\mathscr{Z} = \{\mathscr{Z}_1, \mathscr{Z}_2, \ldots, \mathscr{Z}_m\}$ with criteria $\mathscr{E} = \{\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n\}$, each denoted by $p, q-$ QOFNs $\mathscr{V}_{ij} = \langle u_{\mathscr{V}_{ij}}, v_{\mathscr{V}_{ij}} \rangle_{p,q}$, where $u_{\mathscr{V}_{ij}}, v_{\mathscr{V}_{ij}} \in [0,1]$ and $(u_{\mathscr{V}_{ij}})^p + (v_{\mathscr{V}_{ij}})^q \leqslant 1$. Assume that the criteria weight vector is $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ such that $\eta_i \in [0,1]$ and $\sum_{j=1}^{n} \eta_j = 1$. Then, the $p, q-$ QOF decision matrix $\mathscr{G} = (\mathscr{Z}_{ij})_{m \times n} = \left(\langle u_{\mathscr{V}_{ij}}, v_{\mathscr{V}_{ij}} \rangle_{p,q} \right)_{m \times n}$ can be expressed as follows:

$$\mathscr{G} = \begin{pmatrix} \mathscr{Z}_{11} & \mathscr{Z}_{12} & \cdots & \mathscr{Z}_{1n} \\ \mathscr{Z}_{21} & \mathscr{Z}_{22} & \cdots & \mathscr{Z}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{Z}_{m1} & \mathscr{Z}_{m2} & \cdots & \mathscr{Z}_{mn} \end{pmatrix} \tag{15}$$

In Equation (15), $\mathscr{Z}_{ij}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ represents $p, q-$ QOFNs in the $p, q-$ QOF decision matrix.

Listed below are the phases in the process of decision-making that use the proposed CSM:

Phase 1. Cost and benefit criteria are essential to evaluate alternatives in MCDM scenarios. Cost criteria assess each option's economic viability by considering financial considerations, time implications, and resource utilization. On the other hand, benefit criteria evaluate each alternative's overall performance and beneficial implications while emphasizing effectiveness, quality, and sustainability. For example, when choosing a waste management system for a city, decision-makers may compare the advantages of effectiveness, quality of results, and sustainability with the costs of time, resources, and financial investment. It is customary to harmonies cost and benefit criteria in MCDM problems by utilizing a negation operator as presented in Equation (16), to change cost-type criteria ($\mathscr{C}$) into benefit-type criteria ($\mathscr{B}$).

$$\widetilde{\mathscr{Z}}_{ij} = \langle \widetilde{u}_{\mathscr{V}_{ij}}, \widetilde{v}_{\mathscr{V}_{ij}} \rangle_{p,q} = \begin{cases} \langle u_{\mathscr{V}_{ij}}, v_{\mathscr{V}_{ij}} \rangle_{p,q} \text{ for } \mathscr{B}_j \\ \langle v_{\mathscr{V}_{ij}}, u_{\mathscr{V}_{ij}} \rangle_{p,q} \text{ for } \mathscr{C}_j \end{cases} \tag{16}$$

Phase 4. For all alternatives determined under the same criteria, we may obtain the positive ideal solution (PIS) $\mathscr{Z}^+$ and the negative ideal solution (NIS) $\mathscr{Z}^-$. These solutions are determined by the SF and AF described in Equations (5) and (6).

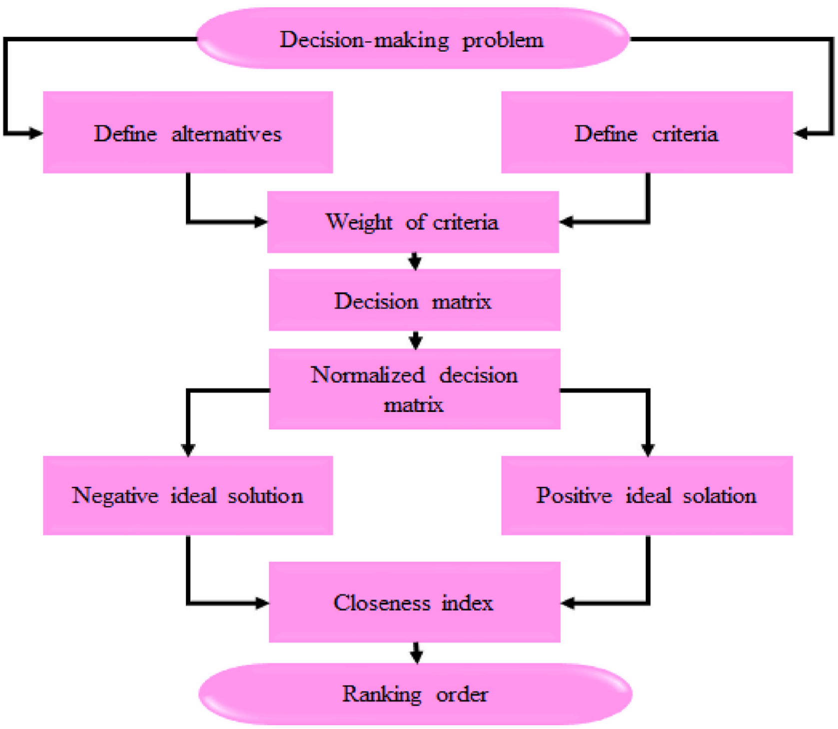

Fig. 3. Schematic depiction of the proposed MCDM.

$$\begin{cases} \mathscr{Z}_j^+ = \max\{SF(\mathscr{Z}_{1j}), SF(\mathscr{Z}_{2j}), \dots, SF(\mathscr{Z}_{mj})\} \\ \mathscr{Z}_j^- = \min\{SF(\mathscr{Z}_{1j}), SF(\mathscr{Z}_{2j}), \dots, SF(\mathscr{Z}_{mj})\} \end{cases} \text{for } j = 1, 2, \dots, n \tag{17}$$

In a $p, q-$ QOFN, where $SF(*)$ signifies the SF as per Definition 6, if the SF value is expressed as $SF(\mathscr{Z}_{1j}) = SF(\mathscr{Z}_{2j}) = \dots = SF(\mathscr{Z}_{mj})$, then the values of the AF values may be compared.

Phase 3. Use the proposed distance measure $\widetilde{WDM}_{p,q-QOF}$ to calculate the distance between each of the alternatives and the PIS $\mathscr{Z}^+$ and NIS $\mathscr{Z}^-$. The separation measure for alternative $\mathscr{Z}_i$ (where $i = 1, 2, \dots, n$) from $\mathscr{Z}^+$ is stated as $\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+) = \sum_{j=1}^{n} \eta_j \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_{ij}, \mathscr{Z}^+)$, while the separation measure from $\mathscr{Z}^-$ is given by $\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-) = \sum_{j=1}^{n} \eta_j \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_{ij}, \mathscr{Z}^-)$. Next, the closeness index γ_i for choice $\mathscr{Z}_i$ is calculated, reflecting its near to the ideal solutions.

$$\gamma_i = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+) + \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-)} \ (i = 1, 2, \dots, m) \tag{18}$$

Phase 4. Arrange alternatives based on proximity coefficient values γ_i, with a lower value indicating a more satisfactory ranking for alternative $\mathscr{Z}_i$ (where $i = 1, 2, \dots, m$). The step-by-step pathway of the proposed MCDM approach is presented in Fig. 3.

5. Implementation of the presented model in investment decision-making

In this section, the presented approach is applied to address obstacles encountered in Investment Decision-making. To demonstrate the practicality of the proposed technique, it is compared to numerous existing methods.

5.1. Problem background

To demonstrate the practicality of the proposed approach in MCDM, we utilize a specific example that was presented by Szmidt and Kacprzyk with a few modifications. Imagine an investment company preparing to allocate its funds smartly to optimize profits. Five possible organizations that are available for investment in the market are $\mathscr{Z}_1$, $\mathscr{Z}_2$, $\mathscr{Z}_3$, $\mathscr{Z}_4$ and $\mathscr{Z}_5$. To thoroughly assess these firms, the investment firm invested in specialists in the areas of $\mathscr{E}_1$ (risk analysis), $\mathscr{E}_2$ (growth condition), $\mathscr{E}_3$ (social effect), $\mathscr{E}_4$ (environmental impact), and $\mathscr{E}_5$ (development of society). For these qualities, the corresponding weight vector is $\eta = (0.21, 0.24, 0.18, 0.20, 0.17)$. As shown in Table 1, the assessment values are expressed using $p, q-$ QOFNs. Fig. 4 illustrates the proposed model's schematic depiction.

Since we realize that criterion $\mathscr{E}_1$ is of the cost type and the other criteria, $\mathscr{E}_2$, $\mathscr{E}_3$, $\mathscr{E}_4$, and $\mathscr{E}_5$, are of the benefit type, we can use Equation (16) to normalize the $p, q-$ QOF decision matrix.

The PIS and NIS can be obtained using Equation (5). In this scenario we set $p = q = 3$.

PIS and NIS for criteria $\mathscr{E}_1$.

$$SF(\mathscr{Z}_{11}) = \frac{1}{2}\left(1 + 0.20^3 - 0.35^3\right) = 0.4825, SF(\mathscr{Z}_{21}) = \frac{1}{2}\left(1 + 0.30^3 - 0.40^3\right) = 0.4815,$$

$$SF(\mathscr{Z}_{31}) = \frac{1}{2}\left(1 + 0.15^3 - 0.25^3\right) = 0.4939, SF(\mathscr{Z}_{41}) = \frac{1}{2}\left(1 + 0.30^3 - 0.45^3\right) = 0.4679,$$

$$SF(\mathscr{Z}_{51}) = \frac{1}{2}\left(1 + 0.15^3 - 0.30^3\right) = 0.4882.$$

From the above calculation we can observe that $SF(\mathscr{Z}_{31}) \succ SF(\mathscr{Z}_{51}) \succ SF(\mathscr{Z}_{11}) \succ SF(\mathscr{Z}_{21}) \succ SF(\mathscr{Z}_{41})$, and hence the PIS for criteria $\mathscr{E}_1$ is $\mathscr{Z}_{31} = (0.15, 0.25)_{p,q}$ and NIF is $\mathscr{Z}_{41} = (0.30, 0.45)_{p,q}$. Similarly, we can compute the PIS and NIS for the remaining criteria using the same approach and listed below.

$$\mathscr{Z}^+ = \left\{(0.15, 0.25)_{p,q}, (0.55, 0.35)_{p,q}, (0.65, 0.40)_{p,q}, (0.60, 0.50)_{p,q}, (0.70, 0.40)_{p,q}\right\},$$

Table 1
Information about alternatives $\mathscr{Z}_i$ with respect to criteria $\mathscr{E}_j$ provided by the decision-makers.

$\mathscr{Z}_i$	$\mathscr{E}_1$	$\mathscr{E}_2$	$\mathscr{E}_3$	$\mathscr{E}_4$	$\mathscr{E}_5$
$\mathscr{Z}_1$	$(0.35, 0.20)_{p,q}$	$(0.50, 0.60)_{p,q}$	$(0.65, 0.40)_{p,q}$	$(0.20, 0.50)_{p,q}$	$(0.70, 0.40)_{p,q}$
$\mathscr{Z}_2$	$(0.40, 0.30)_{p,q}$	$(0.55, 0.35)_{p,q}$	$(0.25, 0.30)_{p,q}$	$(0.45, 0.30)_{p,q}$	$(0.50, 0.35)_{p,q}$
$\mathscr{Z}_3$	$(0.25, 0.15)_{p,q}$	$(0.45, 0.40)_{p,q}$	$(0.40, 0.25)_{p,q}$	$(0.50, 0.35)_{p,q}$	$(0.55, 0.60)_{p,q}$
$\mathscr{Z}_4$	$(0.45, 0.30)_{p,q}$	$(0.30, 0.20)_{p,q}$	$(0.65, 0.70)_{p,q}$	$(0.60, 0.50)_{p,q}$	$(0.45, 0.55)_{p,q}$
$\mathscr{Z}_5$	$(0.30, 0.15)_{p,q}$	$(0.35, 0.30)_{p,q}$	$(0.60, 0.30)_{p,q}$	$(0.40, 0.25)_{p,q}$	$(0.60, 0.45)_{p,q}$

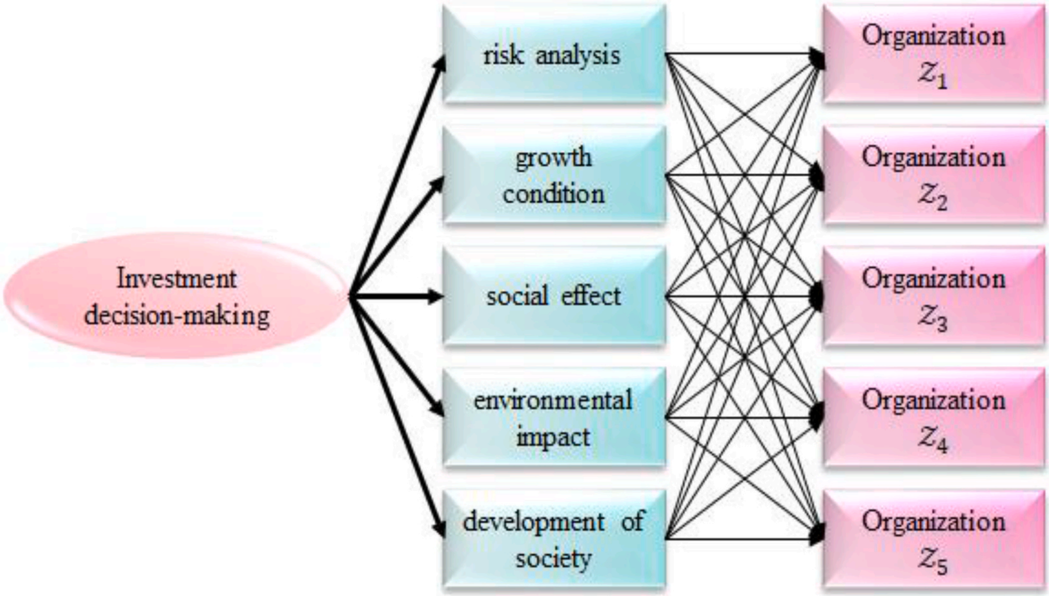

Fig. 4. Flowchart of the proposed model.

$$\mathscr{Z}^- = \left\{(0.30, 0.45)_{p,q}, (0.50, 0.60)_{p,q}, (0.65, 0.70)_{p,q}, (0.20, 0.50)_{p,q}, (0.50, 0.35)_{p,q}\right\}.$$

The vector of weights for criteria $\mathscr{E}_j$ $(j = 1, 2, 3, 4, 5)$ is given by $\eta = (0.21, 0.24, 0.18, 0.20, 0.17)$. In general, assuming $p = q = 3$, the proposed DM $\widetilde{WDM}_{p,q-QOF}$ is used to calculate the distance between each alternative relative to $\mathscr{Z}^+$ and $\mathscr{Z}^-$.

$$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+) = \frac{1 - WCSM_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+) + WED_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+)}{2} \tag{19}$$

$$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-) = \frac{1 - WCSM_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-) + WED_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-)}{2} \tag{20}$$

Using Equations (19) and (20) to calculate $\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^+)$ and $\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_i, \mathscr{Z}^-)$. The results are listed in Table 3.
Using Equation (18) to find the closeness index γ_i for alternative $\mathscr{Z}_i$ in the following manner.

$$\gamma_1 = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_1, \mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_1, \mathscr{Z}^+) + \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_1, \mathscr{Z}^-)} = \frac{0.2390}{0.2390 + 0.2038} = 0.5362,$$

$$\gamma_2 = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_2, \mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_2, \mathscr{Z}^+) + \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_2, \mathscr{Z}^-)} = \frac{0.1984}{0.1984 + 0.2174} = 0.4771,$$

$$\gamma_3 = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_3, \mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_3, \mathscr{Z}^+) + \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_3, \mathscr{Z}^-)} = \frac{0.2021}{0.2021 + 0.1820} = 0.5261,$$

$$\gamma_4 = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_4, \mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_4, \mathscr{Z}^+) + \widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_4, \mathscr{Z}^-)} = \frac{0.1793}{0.1793 + 0.2285} = 0.4396,$$

Table 2
Normalized decision-matrix.

$\mathscr{Z}_i$	$\mathscr{E}_1$	$\mathscr{E}_2$	$\mathscr{E}_3$	$\mathscr{E}_4$	$\mathscr{E}_5$
$\mathscr{Z}_1$	$(0.20, 0.35)_{p,q}$	$(0.50, 0.60)_{p,q}$	$(0.65, 0.40)_{p,q}$	$(0.20, 0.50)_{p,q}$	$(0.70, 0.40)_{p,q}$
$\mathscr{Z}_2$	$(0.30, 0.40)_{p,q}$	$(0.55, 0.35)_{p,q}$	$(0.25, 0.30)_{p,q}$	$(0.45, 0.30)_{p,q}$	$(0.50, 0.35)_{p,q}$
$\mathscr{Z}_3$	$(0.15, 0.25)_{p,q}$	$(0.45, 0.40)_{p,q}$	$(0.40, 0.25)_{p,q}$	$(0.50, 0.35)_{p,q}$	$(0.55, 0.60)_{p,q}$
$\mathscr{Z}_4$	$(0.30, 0.45)_{p,q}$	$(0.30, 0.20)_{p,q}$	$(0.65, 0.70)_{p,q}$	$(0.60, 0.50)_{p,q}$	$(0.45, 0.55)_{p,q}$
$\mathscr{Z}_5$	$(0.15, 0.30)_{p,q}$	$(0.35, 0.30)_{p,q}$	$(0.60, 0.30)_{p,q}$	$(0.40, 0.25)_{p,q}$	$(0.60, 0.45)_{p,q}$

Table 3
Distance between each alternative relative to $\mathscr{Z}^+$ and $\mathscr{Z}^-$.

$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_1,\mathscr{Z}^+)$	**0.2390**	$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_1,\mathscr{Z}^-)$	**0.2038**
$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_2,\mathscr{Z}^+)$	0.1984	$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_2,\mathscr{Z}^-)$	0.2174
$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_3,\mathscr{Z}^+)$	0.2021	$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_3,\mathscr{Z}^-)$	0.1820
$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_4,\mathscr{Z}^+)$	0.1793	$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_4,\mathscr{Z}^-)$	0.2285
$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_5,\mathscr{Z}^+)$	0.1634	$\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_5,\mathscr{Z}^-)$	0.2066

$$\gamma_5 = \frac{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_5,\mathscr{Z}^+)}{\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_5,\mathscr{Z}^+)+\widetilde{WDM}_{p,q-QOF}(\mathscr{Z}_5,\mathscr{Z}^-)} = \frac{0.1634}{0.1634+0.2066} = 0.4416.$$

Ranking order of alternatives based on γ_i $(j = 1,2,3,4,5)$. $\mathscr{Z}_4$ is the top-ranked option, followed by $\mathscr{Z}_5$, $\mathscr{Z}_2$, $\mathscr{Z}_3$ and $\mathscr{Z}_1$. The relative closeness of the alternatives is graphically presented in Fig. 5.

5.2. The impact of parameters *p* and *q*

To determine the impact of the parameters p and q on the decision ranking results, several pairs of values are used in the distance measure $\widetilde{WDM}_{p,q-QOF}$, and the results are shown in Table 4.

The data presented in Table 2 cannot be managed using pairs of parameters $(1,1)$, $(1,2)$ and $(2,1)$ since these pairs are unable to satisfy the condition $(u_{\mathscr{V}}(t_i))^p + (v_{\mathscr{V}}(t_i))^p \preccurlyeq 1$ for $p,q-$ QOFSs. For instance, the information $(0.65,0.70)_{p,q}$ does not meet the specified condition for $p,q-$ QOFSs, i.e., $0.65^1+0.70^1 = 1.35 \succ 1$, $0.65^1+0.70^2 = 0.65+0.49 = 1.14 \succ 1$ and $0.65^2+0.70^1 = 1.1225 \succ 1$. Therefore, this information cannot be accommodated with the pairs $(1,1)$, $(1,2)$ and $(2,1)$. Table 4 shows differences in the Closeness index of alternatives for different sets of parameters p and q. Regardless of these variances, the overall ranking order is similar across all parameter pairings. The graphical view of closeness index of alternatives is presented in Fig. 7.

5.3. Comparative analysis

The suggested approach's efficacy and feasibility are demonstrated by comparing it to existing approaches [29–33]. The closeness index of alternatives and ranking order of alternatives obtained by different existing approach is presented in Table 5.

The methodologies presented in Table 5 are constrained by specific limitations. For instance, approaches [29,31,32], and [33] operate within the Pythagorean fuzzy framework, where the information provided by decision-makers is confined to the condition $(u_{\mathscr{P}}(t_i))^2 + v_{\mathscr{P}}(t_i)^2 \preccurlyeq 1$. In contrast, approach [30] is formulated within the q-rung orthopair fuzzy environment, where information is bound by the condition $(u_{\mathscr{Q}}(t_i))^q + v_{\mathscr{Q}}(t_i)^q \preccurlyeq 1$ $(q \succcurlyeq 1)$. The graphical representation of closeness index obtained by existing approaches is presented in Fig. 6.

The proposed method overcomes these limits, providing decision-makers with a more adaptive environment. Both the variables p and q play an important role in adapting membership degrees to specific decision needs. Unlike previous systems, our suggested method allows for the change of membership degrees based on the defined circumstances. This characteristic makes the suggested

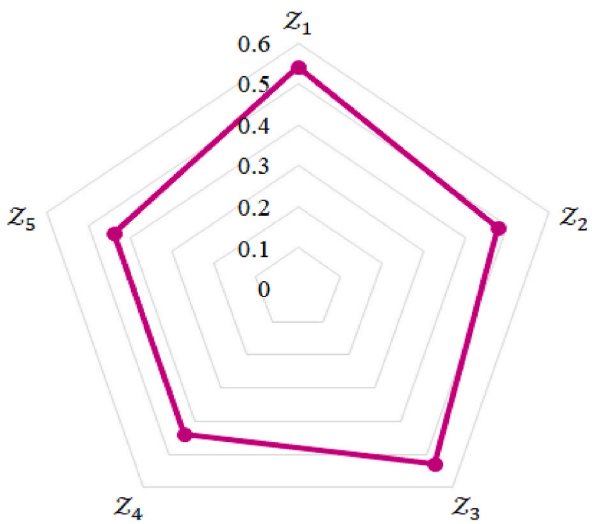

Fig. 5. Relative closeness of the alternatives.

Table 4
Influence of parameters p and q.

(p,q)	Closeness index γ_i				
	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$
$(1,1)$	–	–	–	–	–
$(1,2)$	–	–	–	–	–
$(2,1)$	–	–	–	–	–
$(2,2)$	0.5246	0.4601	0.5112	0.4233	0.4317
$(2,3)$	0.5287	0.4645	0.5174	0.4269	0.4359
$(3,2)$	0.5319	0.4662	0.5227	0.4380	0.4395
$(3,3)$	0.5362	0.4771	0.5261	0.4396	0.4416
$(3,4)$	0.5383	0.4808	0.5277	0.4405	0.4427
$(4,3)$	0.5395	0.4829	0.5291	0.4421	0.4441
$(4,4)$	0.5406	0.4843	0.5304	0.4436	0.4474
$(4,5)$	0.5414	0.4860	0.5318	0.4451	0.4498
$(5,4)$	0.5421	0.4878	0.5326	0.4468	0.4514
$(5,5)$	0.5431	0.4893	0.5331	0.4479	0.4521

Table 5
Comparative analysis.

Approaches	γ_i					Ranking order	Best option
	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{Z}_4$	$\mathcal{Z}_5$		
Zhang et al. [29]	0.2680	0.2197	0.2511	0.1673	0.1846	$\mathcal{Z}_4 \prec \mathcal{Z}_5 \prec \mathcal{Z}_2 \prec \mathcal{Z}_3 \prec \mathcal{Z}_1$	$\mathcal{Z}_4$
Liu et al. [30]	0.3064	0.2766	0.2814	0.2412	0.2582	$\mathcal{Z}_4 \prec \mathcal{Z}_5 \prec \mathcal{Z}_2 \prec \mathcal{Z}_3 \prec \mathcal{Z}_1$	$\mathcal{Z}_4$
Ejegwa [31]	0.3526	0.3249	0.3307	0.2975	0.3189	$\mathcal{Z}_4 \prec \mathcal{Z}_5 \prec \mathcal{Z}_2 \prec \mathcal{Z}_3 \prec \mathcal{Z}_1$	$\mathcal{Z}_4$
Li and Lu [32]	0.3898	0.3406	0.3562	0.3104	0.3341	$\mathcal{Z}_4 \prec \mathcal{Z}_5 \prec \mathcal{Z}_2 \prec \mathcal{Z}_3 \prec \mathcal{Z}_1$	$\mathcal{Z}_4$
Firozja et al. [33]	0.5173	0.4670	0.4861	0.4158	0.4522	$\mathcal{Z}_4 \prec \mathcal{Z}_5 \prec \mathcal{Z}_2 \prec \mathcal{Z}_3 \prec \mathcal{Z}_1$	$\mathcal{Z}_4$

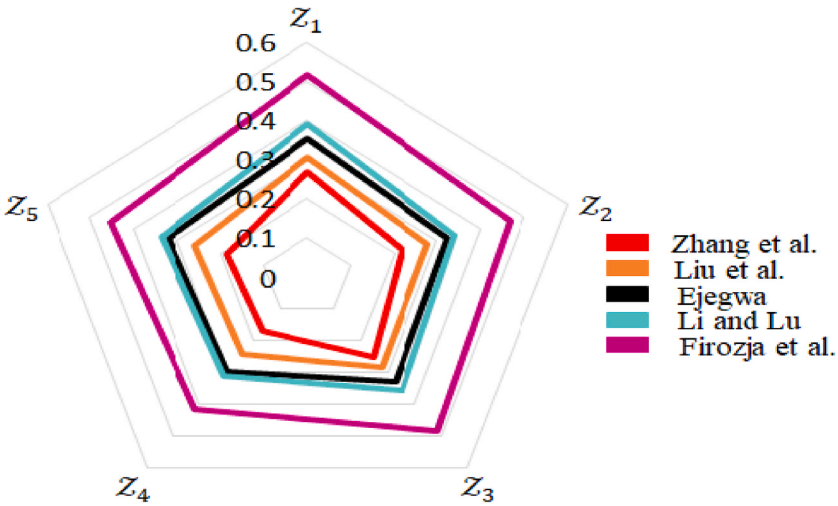

Fig. 6. Closeness index obtained by existing approaches.

technique more realistic and suitable for a broader range of decision-making settings.

Some advantages of the presented work are listed below:

1. The $p,q-$ QOFS provides more comprehensive decision-related information than the IFS, PFS and $q-$ ROFS. It gives decision-makers more freedom in expressing their views on membership degrees.
2. The distance metrics for $p,q-$ QOFSs use both cosine similarity and Euclidean distance. In other words, the suggested distance measure tackles decision-making issues from both geometric and algebraic perspectives.

Remark 6. The suggested Cosine Similarity Measures (CSMs) and Distance Measures (DMs) outlined in Equations (8)–(14)simplify to

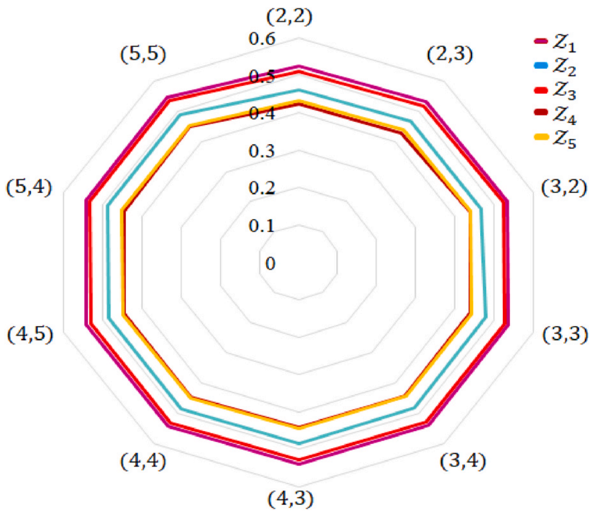

Fig. 7. The graphical view of Closeness index for different pairs of parameters p and q.

those applicable in Intuitionistic Fuzzy Sets when the parameters p and q are both set to 1 [34,35].

Remark 7. The suggested CSMs and DMs outlined in Equations (8)–(14)simplify to those applicable in Pythagorean Fuzzy Sets when the parameters p and q are both set to 2 [32].

Remark 8. The suggested CSMs and DMs outlined in Equations (8)–(14)simplify to those applicable in Fermatean Fuzzy Sets when the parameters p and q are both set to 3 [24].

Remark 9. The suggested CSMs and DMs outlined in Equations (8)–(14)simplify to those applicable in $q-$ rung orthopair Fuzzy Sets when the parameters p and q are equal [30].

Based on the preceding discourse, it's evident that the proposed methodology is adept at managing various types of fuzzy information, including intuitionistic fuzzy, Pythagorean fuzzy, Fermatean fuzzy, and $q-$ rung orthopair fuzzy sets. However, it is noteworthy that existing approaches may not be equipped to effectively handle the intricacies associated with $p, q-$ quasirung information, underscoring the significance of the proposed approach in extending the scope of fuzzy information management.

6. Conclusion

This study delves into the application of cosine similarity (CS) and distance measures (DM) in the context of $p, q-$ quasirung orthopair fuzzy sets ($p, q-$ QOFSs). We introduce two distinct measures, namely cosine similarity and Euclidean distance, tailored specifically for $p, q-$ QOFSs, and conduct a comprehensive analysis of their characteristics. Additionally, we develop alternative similarity measures for $p, q-$ QOFSs based on the proposed cosine similarity and Euclidean distance measures, ensuring adherence to the similarity measure axiom while addressing decision-making concerns from both geometric and algebraic perspectives. By incorporating the proposed cosine distance measure, we modify the classical TOPSIS method to provide a more robust decision-making framework. A numerical example is presented to illustrate the feasibility and utility of our proposed method. The main finding of our study is the efficacy of the proposed measures in quantifying similarity and dissimilarity between $p, q-$ QOFSs, thereby enhancing the decision-making process. However, it is important to acknowledge some limitations. The proposed measures may have constraints in capturing the full complexity of decision-making scenarios, and further empirical validation across diverse application domains is warranted to assess their generalizability and robustness. Despite these limitations, our study contributes to advancing the understanding and application of fuzzy set theory in decision-making contexts.

For future work, several avenues can be explored to extend the findings of this study. Firstly, further investigation into the application of cosine similarity and distance measures in more complex decision-making scenarios [36] could provide valuable insights. This could involve exploring their effectiveness in handling larger datasets or in addressing multi-criteria decision-making problems with additional constraints. Additionally, research could focus on refining the proposed measures to enhance their applicability in specific domains, such as healthcare [37], finance [38], or engineering. Furthermore, comparative studies with other existing similarity and distance measures could be conducted to evaluate the relative performance and advantages of the proposed approach.

Data availability

The accompanying paper does not contain any associated data. The paper only presents the written text and does not have any additional data that supports the claims and conclusions presented in the paper.

CRediT authorship contribution statement

Muhammad Rahim: Writing – original draft, Validation, Methodology, Investigation, Conceptualization. **Shougi Suliman Abosuliman:** Writing – review & editing, Formal analysis, Conceptualization. **Roobaea Alroobaea:** Validation, Software, Formal analysis, Data curation. **Kamal Shah:** Methodology, Formal analysis, Data curation. **Thabet Abdeljawad:** Validation, Software, Formal analysis, Data curation.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT to improve language and readability. After using this tool, the authors reviewed and edited the content as needed and takes full responsibility for the content of the publication.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965/06/01/1965) 338–353.
[2] N.D. Phu, N.N. Hung, A. Ahmadian, N. Senu, A new fuzzy PID control system based on fuzzy PID controller and fuzzy control process, Int. J. Fuzzy Syst. 22 (7) (2020) 2163–2187.
[3] G.A. Papakostas, A.G. Hatzimichailidis, V.G. Kaburlasos, Distance and similarity measures between intuitionistic fuzzy sets: a comparative analysis from a pattern recognition point of view, Pattern Recogn. Lett. 34 (14) (2013) 1609–1622.
[4] P.Y. Ekel, Fuzzy sets and models of decision making, Comput. Math. Appl. 44 (7) (2002) 863–875.
[5] S. Ali, A. Ali, A.B. Azim, A. Aloqaily, N. Mlaiki, Averaging aggregation operators under the environment of q-rung orthopair picture fuzzy soft sets and their applications in MADM problems, AIMS Mathematics 8 (4) (2023) 9027–9053.
[6] S. Miyamoto, Information retrieval based on fuzzy associations, Fuzzy Set Syst. 38 (2) (1990) 191–205.
[7] A. Mardani, et al., Application of decision making and fuzzy sets theory to evaluate the healthcare and medical problems: a review of three decades of research with recent developments, Expert Syst. Appl. 137 (2019) 202–231.
[8] V. Novák, Fuzzy logic in natural language processing, in: 2017 IEEE International Conference on Fuzzy Systems, FUZZ-IEEE, IEEE, 2017, pp. 1–6.
[9] K.T. Atanassov, "Intuitionistic fuzzy sets, Fuzzy Set Syst. 20 (1986).
[10] R.R. Yager, Pythagorean fuzzy subsets, in: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), IEEE, 2013, pp. 57–61.
[11] M. Olgun, M. Ünver, Circular Pythagorean fuzzy sets and applications to multi-criteria decision making, Informatica 34 (4) (2023) 713–742.
[12] L. Pan, Y. Deng, K.H. Cheong, Quaternion model of Pythagorean fuzzy sets and its distance measure, Expert Syst. Appl. 213 (2023) 119222.
[13] N. Jamal, M. Sarwar, N. Mlaiki, A. Aloqaily, Solution of linear correlated fuzzy differential equations in the linear correlated fuzzy spaces, AIMS Mathematics 9 (2) (2024) 2695–2721.
[14] R.R. Yager, Generalized orthopair fuzzy sets, IEEE Trans. Fuzzy Syst. 25 (5) (2016) 1222–1230.
[15] J. Ali, Norm-based distance measure of q-rung orthopair fuzzy sets and its application in decision-making, Comput. Appl. Math. 42 (4) (2023) 184.
[16] Z. Chen, D. Shen, F. Yu, X. Tang, Z. Zhang, Multi-attribute decision-making method based on q-rung orthopair probabilistic hesitant fuzzy schweizer-sklar power weighted hamy mean operator, PLoS One 18 (2) (2023) e0266779.
[17] J. Vimala, P. Mahalakshmi, A.U. Rahman, M. Saeed, A customized TOPSIS method to rank the best airlines to fly during COVID-19 pandemic with q-rung orthopair multi-fuzzy soft information, Soft Comput. 27 (20) (2023) 14571–14584.
[18] A. Shahzad, A. Shoaib, N. Mlaiki, S. Subhi Aiadi, Results for fuzzy mappings and stability of fuzzy sets with applications, Fractal and Fractional 6 (10) (2022) 556.
[19] M.R. Seikh, U. Mandal, q-Rung orthopair fuzzy Archimedean aggregation operators: application in the site selection for software operating units, Symmetry 15 (9) (2023) 1680.
[20] M.R. Seikh, U. Mandal, Multiple attribute group decision making based on quasirung orthopair fuzzy sets: application to electric vehicle charging station site selection problem, Eng. Appl. Artif. Intell. 115 (2022) 105299.
[21] M. Rahim, H. Garg, S. Khan, H. Alqahtani, H.A.E.-W. Khalifa, Group decision-making algorithm with sine trigonometric p, q-quasirung orthopair aggregation operators and their applications, Alex. Eng. J. 78 (2023) 530–542.
[22] J. Ali, M. Naeem, Analysis and application of p, q-quasirung orthopair fuzzy Aczel–Alsina aggregation operators in multiple criteria decision-making, IEEE Access (2023) 11 (2023) 49081–49101.
[23] M. Rahim, K. Shah, T. Abdeljawad, M. Aphane, A. Alburaikan, H.A.E.-W. Khalifa, Confidence levels-based p, q-quasirung orthopair fuzzy operators and its applications to criteria group decision making problems, IEEE Access (2023) 11 (2023) 109983–109996.
[24] M. Kirişci, New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach, Knowl. Inf. Syst. 65 (2) (2023) 855–868.
[25] R. Verma, A. Mittal, Multiple attribute group decision-making based on novel probabilistic ordered weighted cosine similarity operators with Pythagorean fuzzy information, Granular Computing 8 (1) (2023) 111–129.
[26] P. Liu, Z. Ali, T. Mahmood, Some cosine similarity measures and distance measures between complex q-rung orthopair fuzzy sets and their applications, Int. J. Comput. Intell. Syst. 14 (1) (2021) 1653–1671.
[27] P.A. Ejegwa, J.M. Agbetayo, Similarity-distance decision-making technique and its applications via intuitionistic fuzzy pairs, Journal of Computational and Cognitive Engineering 2 (1) (2023) 68–74.
[28] B. Gohain, P. Dutta, S. Gogoi, R. Chutia, Construction and generation of distance and similarity measures for intuitionistic fuzzy sets and various applications, Int. J. Intell. Syst. 36 (12) (2021) 7805–7838.

[29] Q. Zhang, J. Hu, J. Feng, A. Liu, Y. Li, New similarity measures of Pythagorean fuzzy sets and their applications, IEEE Access 7 (2019) 138192–138202.
[30] D. Liu, X. Chen, D. Peng, Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets, Int. J. Intell. Syst. 34 (7) (2019) 1572–1587.
[31] P.A. Ejegwa, Personnel Appointments: a Pythagorean fuzzy sets approach using similarity measure, J. Inf. Comput. Sci. 14 (2) (2019) 94–102.
[32] Z. Li, M. Lu, Some novel similarity and distance measures of pythagorean fuzzy sets and their applications, J. Intell. Fuzzy Syst. 37 (2) (2019) 1781–1799.
[33] M.A. Firozja, B. Agheli, E.B. Jamkhaneh, A new similarity measure for Pythagorean fuzzy sets, Complex & Intelligent Systems 6 (1) (2020) 67–74.
[34] J. Ye, Cosine similarity measures for intuitionistic fuzzy sets and their applications, Math. Comput. Model. 53 (1–2) (2011) 91–97.
[35] F. Xiao, A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems, IEEE Transactions on Systems, Man, and Cybernetics: Systems 51 (6) (2019) 3980–3992.
[36] U. Mandal, M.R. Seikh, Interval-valued spherical fuzzy MABAC method based on Dombi aggregation operators with unknown attribute weights to select plastic waste management process, Appl. Soft Comput. 145 (2023) 110516.
[37] M. Rahim, F. Amin, K. Shah, T. Abdeljawad, S. Ahmad, Some distance measures for pythagorean cubic fuzzy sets: application selection in optimal treatment for depression and anxiety, MethodsX (2024) 102678.
[38] M.R. Seikh, P. Chatterjee, Determination of best renewable energy sources in India using SWARA-ARAS in confidence level based interval-valued Fermatean fuzzy environment, Appl. Soft Comput. (2024) 111495.