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Supersymmetric approach to coherent states for nonlinear oscillator with spatially dependent effective mass

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ABSTRACT

A nonlinear oscillator with variable mass is studied in the approach of Supersymmetric quantum mechanics. Ladder operators in association with the shape invariance condition allowed us to find the coherent states of the system in the sense of Barut and Girardello. The statistical properties of these particular states are studied using the probability of distribution. In addition, the vibrational partition function is calculated. We see that the thermodynamic functions of the system such as mean energy and entropy depend on the nonlinearity parameter, except for the specific heat.

1. Introduction

Scientists have long sought a unifying explanation of all basic interactions in nature, namely strong, electroweak and gravitational interactions [1]. Various efforts have been made in the last forty years, and nowadays it is known that Supersymmetry is the key to any unifying approach. This approach which discusses bosonic and fermionic degrees of freedom, relies on Lie algebra based on commutation and anti-commutation relations. In a crystal, an electron can behave as if it had a mass different from that of a free electron m_0. There are crystals in which the effective mass m^* of charge carriers is larger or smaller than m_0. The effective mass can be anisotropic or even negative [2, 3, 4]. The significant point is that in a periodic potential, an electron is accelerated relative to the lattice in an electric or magnetic field as if its mass were equal to its effective mass. Neutrons propagating in a crystal under the diffraction conditions have a lower effective mass than the rest mass in vacuum and a positive or negative electric charge [5]. This has been verified experimentally by measuring the deflection of neutrons subjected to a magnetic force at the crossing of a silicon crystal. There is a multitude of work on quantum systems of variable mass. We can mention among others: the approach of the displacement operator, the application of the Meijer G-functions for the analytical solutions, the transport and dispersion properties at heterojunctions, etc. [6]. This field of study has emerged in favor of the development of the fabrication of semiconductors with very small dimensions and remarkable quantum effects. Variable mass Schrödinger equations play an important role in the study of the electronic properties of inhomogeneous crystals, quantum dots and quantum liquids [7]. In 1926 Schrödinger introduced a system of non-orthogonal wave functions to describe non-propagated wave packets of quantum oscillators. A few years later, a large subset of these wave functions was used to partition the phase plane of a one-dimensional system into regular cells. This approach was carefully studied by Glauber who qualified these states as coherent states and showed that they are adequate to describe a coherent laser beam in the context of quantum theory in 1953 [8]. The exceptional property of a system of coherent states is that it contains more states than it takes to decompose any state vector. Such a system cannot be treated as it is customary. Compared to the usual orthogonal system, this system has many advantages. It has been successfully applied to quantum optics, radiophysics, superfluidity of a Bose gas, spin waves in the Heisenberg model of ferromagnetism and quantum electrodynamics. Coherent states are the subject of particular attention in the literature [9]. The formalism of the construction of the coherent states of the harmonic oscillator is based on the Heisenberg-Weyl algebra [10]. These states are defined as: eigenstates of the annihilation operator, displaced vacuum states or states of minimum uncertainty. As a consequence of the many applications in mathematics and physics, the notion of coherent states has been generalized using the algebraic structure of the system [9].

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In 1971 Barut and Girardello developed coherent states of noncompact groups [11]. In the context of constant mass systems, contributions have been made by many authors. However the generalization of coherent states for variable mass systems has not yet been studied in a significant way [11, 12]. We expect in this work to determine coherent states for a particular variable mass system in the sense of Barut-Girardello and then compare their properties to those already known. This article is structured as follows. In section 2 we present ladder operators that factorize the quantum Hamiltonian of the variable mass system under consideration. Section 3 is reserved to the construction of coherent states of the system in the sense of Barut-Girardello. Some statistical features of these states are discussed in Section 4. The system's vibrational partition function is calculated in Section 5 where we explore the thermodynamic properties of the system. We end our work with some remarks in section 6.

2. Model

Researchers study nonlinear oscillations because many realistic phenomena present this type of oscillations. In the literature we can find several models of variable mass systems. We choose:

$$m(x) = \lambda \left(1 + \delta^2 x^2\right)^{-1} \tag{1}$$

where λ is a real parameter and δ a constant that measures the force of the nonlinearity of the oscillator. This expression is one of the three effective mass profiles proposed by A.P. Zang and his team when they solved the Schrödinger equation of variable mass for some physical potentials [13]. The classical Hamiltonian of the nonlinear oscillator is:

$$H(x) = \frac{p^2}{2m(x)} + V(x) \tag{2}$$

where $V(x)$ is a deformed quadratic potential given by:

$$V(\omega, x) = \frac{1}{2}\left(\frac{\lambda}{1 + \delta^2 x^2}\right)\omega^2 x^2 \tag{3}$$

and ω the frequency of the oscillations. The quantization of the classical system takes place by replacing position and momentum canonical variables x and p by their corresponding operators satisfying the knowing commutation relation [14]. A particular ordering applied to the kinetic operator [14, 15, 16, 17, 18, 19] in association with Eq. (3) provides the quantum Hamiltonian which allows us to write the Schrödinger equation independent of the time of this system. In the standard system of atomic units (a.u.) [17], $\hbar = 1$. The effective mass profile chosen in Eq. (1) is introduced into the Hamiltonian to yield:

$$H(\omega, x) = \frac{1}{2\lambda}\left[-(1 + \delta^2 x^2)\frac{d^2}{dx^2} - 2\delta^2 x \frac{d}{dx} + \frac{\lambda^2\omega^2 x^2}{1 + \delta^2 x^2}\right] \tag{4}$$

By applying the technique of elaboration of ladder operators in Supersymmetric quantum mechanics for variable mass systems [20, 21], we obtain the pair of operators:

$$\xi^-(\omega) = \frac{1}{\sqrt{2\lambda}}\left[\sqrt{1 + \delta^2 x^2}\frac{d}{dx} + \frac{\lambda\omega x}{\sqrt{1 + \delta^2 x^2}}\right] \quad ; \qquad \xi^+(\omega) = \frac{1}{\sqrt{2\lambda}}\left[-\sqrt{1 + \delta^2 x^2}\frac{d}{dx} + \frac{\left(\lambda\omega - \delta^2\right)x}{\sqrt{1 + \delta^2 x^2}}\right] \tag{5}$$

Previous operators are constructed so that:

$$\xi^-(\omega)\left|\psi_0\right\rangle = 0 \tag{6}$$

where ψ_0 represents the wave function of the ground state of the quantum system studied. Products of operators $\xi^+(\omega)$ and $\xi^-(\omega)$ can be obtained by applying them on any wave function $\psi(x)$. It is clear that the pair of previous operators factorizes the Hamiltonian and provides us with ε_0 the energy of the fundamental level of the system:

$$\xi^+(\omega)\xi^-(\omega) = H(\omega) - \varepsilon_0 \tag{7}$$

$$\varepsilon_0 = \omega/2 \tag{8}$$

This pair of operators intervenes in the condition of shape invariance as follows [22, 23]:

$$\xi^-(\omega_1)\xi^+(\omega_1) = \xi^+(\omega_2)\xi^-(\omega_2) + r(\omega_1) \tag{9}$$

In which, the remainder

$$r(\omega_1) = \omega_2 = \omega_1 - \delta^2/\lambda \tag{10}$$

shows that for the considered system, the parameters ω_1 and ω_2 are connected by means of a translation with step equal to $-\delta^2/\lambda$. We can therefore write:

$$r(\omega_n) = \omega_{n+1} = \omega - n\delta^2/\lambda \tag{11}$$

This implies that parameters ω_{n+1} can be obtained from ω_n by using the following unitary translation operator $\widehat{T}(\omega)$ [22]:

$$\widehat{T}(\omega) = exp(-\frac{\delta^2}{\lambda}\frac{\partial}{\partial\omega}) \quad ; \quad \widehat{T}^{-1}(\omega) = exp(+\frac{\delta^2}{\lambda}\frac{\partial}{\partial\omega}) \tag{12}$$

Moreover, the commutation relation between operators $\xi^-(\omega)$ and $\xi^+(\omega)$ depends on the position x. For this reason these operators are not good ladder operators. We construct the appropriate operators as follows [21]:

$$b^-(\omega) = \widehat{T}^{-1}(\omega)\xi^-(\omega) \quad ; \quad b^+(\omega) = \xi^+(\omega)\widehat{T}(\omega) \tag{13}$$

It is therefore easy to see that:

$$\xi^+(\omega)\xi^-(\omega) = b^+(\omega)b^-(\omega) \tag{14}$$

Thus, the quantity (14) becomes the first supersymmetric partner Hamiltonian, say $H'(\omega)$. From Eq. (7), we write:

$$H(\omega) - \varepsilon_0 = H'(\omega) \tag{15}$$

The eigenvalues of $H'(\omega)$ are derived from Eq. (11) by [9, 21, 23, 24, 25]:

$$\varepsilon'_n(\omega) = \sum_{i=1}^{n} r(\omega_i) = n\omega - \frac{\delta^2}{2\lambda}n(n+1) \tag{16}$$

Eq. (8) combined with Eq. (16) provides us with the general expression of the energy spectrum of the variable mass oscillator:

$$\varepsilon_n(\omega) = \omega\left(n + \frac{1}{2}\right) - \frac{\delta^2}{2\lambda}n(n+1) \tag{17}$$

Let's take a look at the annihilation and creation actions of the newly built ladder operators:

$$b^-(\omega)\left|\psi_n\right\rangle = \sqrt{n\omega - \frac{\delta^2}{2\lambda}n(n+1)}\left|\psi_{n-1}\right\rangle \quad ; \qquad b^+(\omega)\left|\psi_n\right\rangle = \sqrt{(n+1)\omega - \frac{\delta^2}{2\lambda}(n+1)(n+2)}\left|\psi_{n+1}\right\rangle \tag{18}$$

From Eq. (12) and the Campbell-Hausdorff identity [26, 27], we have computed the commutator of the operators newly built and obtained:

$$\left[b^-(\omega), b^+(\omega)\right] = \omega \tag{19}$$

In the standard system of atomic units (a.u.) [17, 26], we have $\omega = 1$, thus the commutator indicated in Eq. (19) turns to be the unity. We see that the algebra hidden under the studied shape invariance potential has a finite dimension. The eigenstates of the nonlinear oscillator with variable mass are thus given by [21]:

$$\left|\psi_n\right\rangle = \frac{\left(b^+\right)^n\left|\psi_0\right\rangle}{\sqrt{[n]!}} \tag{20}$$

where $[n]!$ is the generalized factorial that can be written in the form [9, 21]:

$$[n]! = \varepsilon'_n \varepsilon'_{n-1} ... \varepsilon'_1 = n! \left(-\frac{\delta^2}{2\lambda}\right)^n (2 - \frac{2\lambda}{\delta^2})_n \tag{21}$$

$(u)_n = u(u+1)...(u+n-1)$ represents the Pochhammer symbol.

3. Materials and methods

We have previously seen that the nonlinear oscillator with variable mass has a finite dimensional Lie algebra. However the generators of the Lie group obtained in Eq. (18) have a complex structure. It is for this reason that Barut has proposed new operators simple to handle than the previous ones [26]:

$$a^- = b^- \sqrt{H'} \quad ; \quad a^+ = \sqrt{H'} b^+ \quad ; \quad a^0 = 1/2 + H' \tag{22}$$

From Eq. (22) we get the following commutators, as well as the corresponding Casimir operator:

$$\begin{aligned} &\left[a^0, a^+\right] = +a^+ \quad ; \quad \left[a^0, a^-\right] = -a^- \quad ; \quad \left[a^-, a^+\right] = 2a^0 \quad ; \\ &\hat{c} = a^+ a^- - a^0(a^0 - 1) = 1/4 \end{aligned} \tag{23}$$

Results obtained in Eq. (23) are the commutation relations and the Casimir operator of the generators of the Lie algebra su(1,1) [26, 28]. Thus the dynamic group of the nonlinear oscillator with variable mass is the non-compact group SU(1,1). Our appropriate ladder operators act as follows:

$$\begin{aligned} &a^- \left|\psi_n\right\rangle = [n - kn(n+1)] \left|\psi_{n-1}\right\rangle \quad ; \\ &a^+ \left|\psi_n\right\rangle = [(n+1) - k(n+1)(n+2)] \left|\psi_{n+1}\right\rangle \end{aligned} \tag{24}$$

where we have set $k = \frac{\delta^2}{2\lambda}$. Note that there are four irreducible representations for the Lie algebra su(1,1) [11]. Strong of the preceding generators, we can determine the coherent states for the nonlinear oscillator with variable mass in the sense of Barut-Girardello. These states are defined as [29]:

$$a^- |s\rangle = s |s\rangle \tag{25}$$

where s is any complex number and a^- is the annihilation operator given by Eq. (24). The states $|s\rangle$ can be expanded onto energy eigenstates of the Hamiltonian H [30, 31]:

$$|s\rangle = \sum_{n=0}^{\infty} \Omega_n \left|\psi_n\right\rangle \tag{26}$$

By substituting Eq. (26) in Eq. (25), we obtain a recursive relation which permits to express the solutions:

$$\Omega_n = \frac{\Omega_0 (-1/k)^n}{n! (2 - 1/k)_n} s^n \tag{27}$$

Setting

$$\rho_n = n!(-k)^n (2 - 1/k)_n \tag{28}$$

the desired coherent states are given by:

$$|s\rangle = \frac{1}{K(s)} \sum_{n=0}^{\infty} \frac{s^n}{\rho_n} \left|\psi_n\right\rangle \tag{29}$$

where the normalization factor $\Omega_0 = \frac{1}{K(s)}$ can be obtained through the condition $\langle s \mid s \rangle = 1$:

$$K^2(s) = \sum_{n=0}^{\infty} \left(\frac{(1/k)^n}{n! (2 - 1/k)_n}\right)^2 |s|^{2n} \tag{30}$$

which in terms of hypergeometric function [32], is equal to:

$$K^2(s) = {}_0F_3\left(1, 2 - 1/k, 2 - 1/k; \left(\frac{|s|}{k}\right)^2\right) \tag{31}$$

Barut-Girardello coherent states obtained in Eq. (29) satisfy the minimum Kauder requirements for any coherent state [12]. Using the Eq. (29) we perform the inner product of two coherent states $|s\rangle$ and $|s'\rangle$ for the nonlinear oscillator with variable mass. It comes that:

$$\left\langle s \mid s' \right\rangle = \frac{1}{K(s)K(s')} \sum_{n=0}^{\infty} \left(\frac{(1/k)^n}{n! (2 - 1/k)_n}\right)^2 \left(s' s^*\right)^n \tag{32}$$

Eq. (32) shows that Barut-Girardello coherent states for the nonlinear oscillator with variable mass do not form an orthogonal set. In fact $s' s^*$ is a positive number and therefore does not admit any zero. This is not a paradox because coherent states are eigenstates of the annihilation operator, which is not hermitian. Consequently, any coherent state can be expressed in terms of all other coherent states. Thus the coherent states are not linearly independent. We therefore say that the set $\{|s\rangle\}$ is overcomplete [33]. Moreover, the convergence radius is important because a coherent state exists only if its radius of convergence is nonzero. For the nonlinear oscillator with variable mass considered, the convergence radius ℓ of the coherent states is defined from Eq. (29) as follows [11]:

$$\ell = \lim_{n\to\infty} \sqrt[n]{\rho_n} = \lim_{n\to\infty} \sqrt[n]{n!(-k)^n (2 - 1/k)_n} \tag{33}$$

Computation of ℓ provides us with the following:

$$\ell = \infty \tag{34}$$

The result given in Eq. (34) shows that Barut-Girardello coherent states for the nonlinear oscillator with variable mass are defined on the entire complex plane.

4. Analysis

The statistical properties of the preceding coherent states can be explored through the probability of occupation of the eigenstates in the configuration space. It is a quantity that indicates the way the total probability of one is distributed over the entire population. This occupation probability is evaluated from Eq. (28) by [11]:

$$\wp_n = \left|\Omega_n\right|^2 \tag{35}$$

whose graph is given in Fig. 1. We see that this distribution is narrower than that of the linear oscillator. For this reason, the probability distribution of the nonlinear oscillator with variable mass is sub-poissonian. Specifically, the sub-poissonian statistics refers to a photon number distribution for which the variance is less than the expected value. The first moment represents the expected value $\langle n\rangle$, also called mean of the distribution is:

$$\langle n\rangle = \sum_{n=0}^{\infty} n\wp_n = \frac{1}{K^2(s)} \left(\frac{|s|}{2k-1}\right)^2 \sum_{n=0}^{\infty} \frac{(1/k)^2}{(2)_n (3 - 1/k)_n (3 - 1/k)_n} \frac{|s|^{2n}}{n!} \tag{36}$$

equivalent to the following programmable form:

$$\langle n\rangle = \frac{1}{K^2(s)} \left(\frac{|s|}{2k-1}\right)^2 {}_0F_3\left(2, 3 - 1/k, 3 - 1/k; \left(\frac{|s|}{k}\right)^2\right) \tag{37}$$

The second moment of the distribution is given by:

$$\left\langle n^2\right\rangle = \sum_{n=0}^{\infty} n^2 \wp_n = \frac{1}{K^2(s)} \left(\frac{|s|}{2k-1}\right)^2 \sum_{n=0}^{\infty} \frac{(1/k)^2}{(1)_n (3 - 1/k)_n (3 - 1/k)_n} \frac{|s|^{2n}}{n!} \tag{38}$$

which is equivalent to:

$$\left\langle n^2\right\rangle = \frac{1}{K^2(s)} \left(\frac{|s|}{2k-1}\right)^2 {}_0F_3\left(1, 3 - 1/k, 3 - 1/k; \left(\frac{|s|}{k}\right)^2\right) \tag{39}$$

The variance of this distribution is:

$$(\Delta n)^2 = \left\langle n^2\right\rangle - \langle n\rangle^2 \tag{40}$$

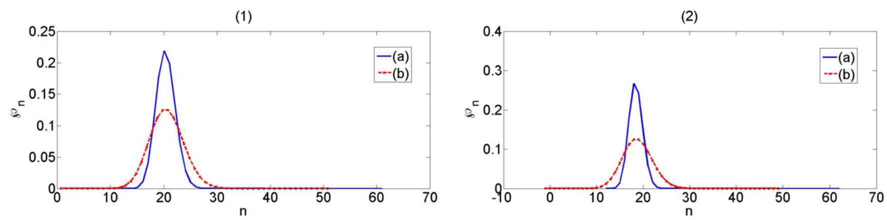

Fig. 1. Plot of the probability distribution for (a) the nonlinear oscillator and (b) the linear oscillator as a function of quantum number n with $|s| = 20$ for (1) $k = -0.10$ and (2) $k = -0.90$.

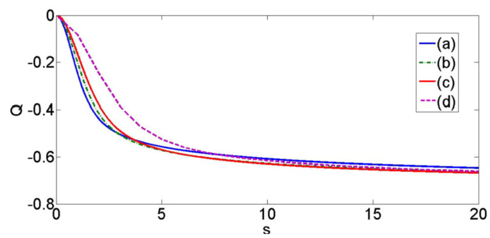

Fig. 2. Plot of the Mandel parameter as function of the coherent states parameter s for (a) $k = -0.10$, (b) $k = -0.25$, (c) $k = -0.39$ and (d) $k = -0.90$.

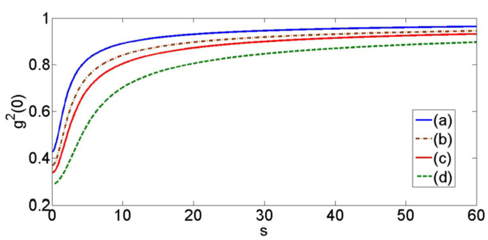

Fig. 3. Plot of the second order correlation function as a function of the coherent state parameter s for (a) $k = -0.10$, (b) $k = -0.25$, (c) $k = -0.39$ and (d) $k = -0.90$.

Generally, the Mandel parameter Q is an indispensable tool for determining the nature of a distribution. A distribution of probability is poissonian if $Q = 0$, sub-poissonian if $Q < 0$ and super-poissonian if $Q > 0$. This parameter is defined as follows [11, 34, 35, 36, 37]:

$$Q = \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle} \tag{41}$$

Fig. 2 presents the variations of the Mandel parameter according to the parameter s of coherent states. The sub-poissonian character of the distribution is confirmed by this graph. Another important quantity that can indicate the nature of the distribution is the second order correlation function $g^2(0)$ defined by [34]:

$$g^2(0) = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} \tag{42}$$

For $g^2(0) < 1$ (resp. $g^2(0) > 1$) the antibunching (resp. bunching) effect appears [11]. The coherent states for the linear oscillator record $g^2(0) = 1$. For the system under consideration, the second order correlation function is plotted in Fig. 3. This graph shows that Barut-Girardello coherent states for the nonlinear oscillator with variable mass exhibit the antibunching behavior which reveals the quantum nature of particles. This phenomenon which has no analogue in classical and semi-classical theories, can be described with a simplified two-level energy diagram in which an atom taken in an excited state requires a lifetime to return back to his ground state by emission of a photon. The duration between adjacent photons is consequently determined by the excite-state lifetime. This effect known as antibunching and observed in quantum dots, carbon nanotubes and in diamond nanocrystals has his applications based on single photons sources [38].

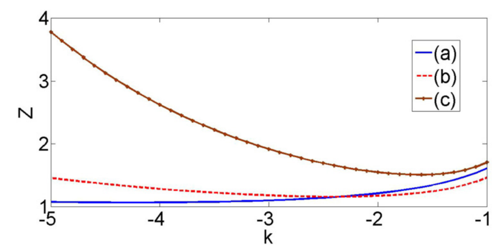

Fig. 4. Plot of the partition function Z as function of k for (a) $\beta = 0.5$, (b) $\beta = 1$ and (c) $\beta = 2$.

5. Results and discussion

We are now exploring the thermodynamic properties of the variable mass system. They are useful to compare our system with others found in the literature. For this, it is necessary to obtain the vibrational partition function of this system [39]:

$$Z = \sum_{n=0}^{\infty} exp(-\beta \varepsilon_n) \tag{43}$$

in which $\beta = \frac{1}{k_B T}$, T is the temperature of the system and k_B the Boltzmann factor. ε_n represents the vibrational energy spectrum of the system. Eq. (17) together with Eq. (43) enables us to write:

$$Z = exp\left(-\frac{\beta}{2}\right) \sum_{n=0}^{\infty} exp\left[\beta k n^2 + \beta (k-1) n\right] \tag{44}$$

Since the energy levels of the system are close to each other, we can replace the discrete summation with an integral without, however, introducing serious mathematical errors [40]. This does not mean that the energy spectrum becomes continuous. No, quantification remains important because of the presence of the Planck constant. Eq. (44) becomes thereby:

$$Z = \frac{1}{2}\sqrt{-\frac{\pi}{\beta k}} exp\left\{-\frac{\beta}{2}\left[1 + \frac{(k-1)^2}{2k}\right]\right\} \tag{45}$$

For the linear oscillator, the partition function is:

$$Z_0 = \frac{exp(-\beta/2)}{1 - exp(-\beta)} \tag{46}$$

Eq. (45) shows that the partition function of the variable mass system depends on the nonlinearity parameter k as well as the temperature T. For a given temperature, the variations of the partition function as a function of the nonlinearity parameter are summarized in Fig. 4. We see that Z increases when β increases. Moreover, for the nonlinearity parameter $k = -0.90$, the partition function of the linear oscillator tends to zero more slowly than that of the nonlinear oscillator when β increases. Fig. 5 shows that if $\beta > 0.6$, Z_0 is less than Z. The vibrational mean energy U can be obtained as follows:

$$U = -\frac{\partial ln Z}{\partial \beta} = \frac{1}{2}\left[1 + \frac{1}{\beta} + \frac{(k-1)^2}{2k}\right] \tag{47}$$

The previous implies that:

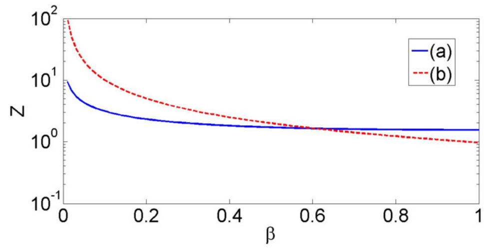

Fig. 5. Comparison of the partition function between (a) Z and (b) Z_0 for the parameter $k = -0.90$.

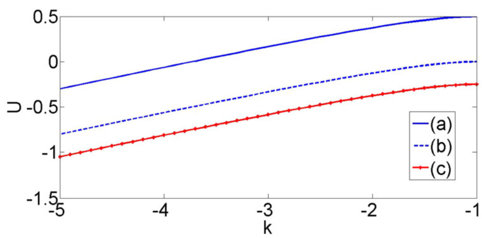

Fig. 6. Plot of the mean energy U as function of k for (a) $\beta = 0.5$, (b) $\beta = 1$ and (c) $\beta = 2$.

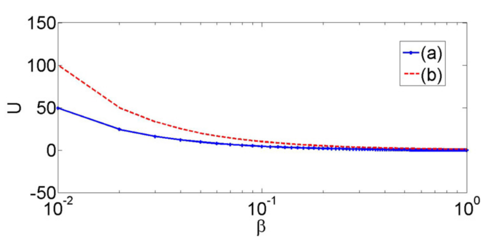

Fig. 7. Plot showing the comparison of the mean energy between (a) U and (b) U_0 for $k = -0.90$.

$$U = \frac{1}{2}\left[1 + \frac{(k-1)^2}{2k}\right] \quad ; \quad \beta \longrightarrow \infty \tag{48}$$

In the case of the linear oscillator, the mean energy is given by:

$$U_0 = \frac{1}{2} + \frac{1}{e^{\beta} - 1} \tag{49}$$

Thus:

$$U_0 = \frac{1}{2} \quad ; \quad \beta \longrightarrow \infty \tag{50}$$

From Eqs. (48) and (50), we say that for low temperatures ($\beta \longrightarrow \infty$), the mean energy of the nonlinear oscillator is different from that of the linear one. The nonlinearity introduces a shift of $\frac{(k-1)^2}{4k}$ in the mean energy of our system. From Fig. 6, one notices a monotonic increase of U with k; but for a fixed value of k, U decreases as β increases. Moreover, Fig. 7 shows that the difference between U and U_0 decreases as β increases. The vibrational specific heat of the nonlinear oscillator with variable mass may be calculated in the following way:

$$C = \frac{\partial U}{\partial T} = \frac{k_B}{2} \tag{51}$$

Eq. (51) suggests the following comment: the nonlinearity parameter has no influence on the specific heat of the system. This result is surprising, knowing that the specific heat is a function of the structure of a substance. Particularly, it depends on the number of degrees of freedom that are available to the particles in the substance. In addition, for low temperatures ($\beta \longrightarrow \infty$), the specific heat of our variable mass system vanishes. This is explained by the third law of thermodynamics and is due to the existence of a non-zero energy between the fundamental level and the first excited state [41, 42]. In general, the heat capacity

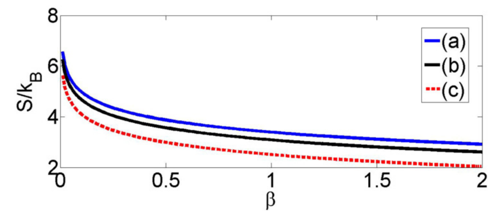

Fig. 8. Plot of the vibrational entropy S as function of β for (a) $k = -0.25$, (b) $k = -0.39$ and (c) $k = -0.90$.

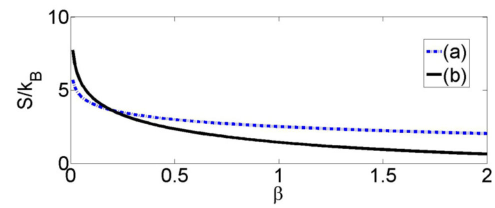

Fig. 9. Comparison of the vibrational entropy between (a) the variable mass system for $k = -0.90$ and (b) the linear oscillator.

of a system approaches zero as the temperature tends to absolute zero because of loss of available degrees of freedom. Then, consider the free energy F given by:

$$F = -\frac{lnZ}{\beta} = -\frac{1}{2}\left[1 + \frac{1}{\beta} ln\left(-\frac{4\pi}{k\beta}\right) + \frac{(k-1)^2}{2k}\right] \tag{52}$$

which provides us with the vibrational entropy S of the variable mass system:

$$S = -\frac{\partial F}{\partial T} = \frac{k_B}{2}\left[1 + ln\left(-\frac{4\pi}{k\beta}\right)\right] \tag{53}$$

Consequently the entropy of the system depends on the nonlinearity parameter k. Fig. 8 shows that the entropy of the considered system decreases as β increases. If $k = -0.90$, the entropy of the nonlinear oscillator is greater than that of the linear oscillator for $\beta > 0.2$ as can be seen in Fig. 9. Increases in entropy correspond to irreversible changes in a system as the temperature increases, because an amount of energy is wasted in the form of heat. On the other hand the entropy S tends towards a limit value when the temperature decreases. This result is consistent with Nernst's theorem which states that any quantum system admits the following property [39]:

$$\lim_{T \to 0} S = e_0 \tag{54}$$

where e_0 is a constant number independent of all parameters of the system.

6. Conclusion

The condition of shape invariance allowed us to study the coherent states of a nonlinear oscillator with variable mass. These states have been shown to fulfill the basic conditions of coherent states. The Mandel parameter for the system and the second order correlation function obtained showed that these coherent states obey the sub-poissonian statistics and consequently exhibit antibunching effect. The induced nonlinearity gives rise to a nonclassical photon phenomenon that is useful for quantum information and communication. In addition, the vibrational partition function of the system permit us to compare the variable mass oscillator with the constant mass one. We were surprised to find that the specific heat of the system does not depend on the nonlinearity parameter. On the other hand, the entropy of the system decreases when the nonlinearity parameter increases. Thermodynamic quantities presented a behavior very similar to the corresponding ones found in standard systems, as the absolute temperature undergoes a variation. Finally, note that the shape invariance potential considered here cannot be reduced to the problem solving procedure of a constant mass oscillator. It would be interesting to study a potential that admits the limit case of the linear oscillator. Some of these problems and quantum systems with positive nonlinearity parameter which introduce imaginary entropies, will be treated elsewhere.

Declarations

Author contribution statement

M. Tchoffo: Conceived and designed the analysis; Contributed analysis tools or data. F.B. Migueu: Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper. M. Vubangsi: Conceived and designed the analysis. L.C. Fai: Contributed analysis tools or data.

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The authors declare no conflict of interest.

Additional information

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