Heliyon 8 (2022) e12274

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Optimal control strategy analysis for an human-animal brucellosis infection model with multiple delays

Man Wu, Xamxinur Abdurahman *, Zhidong Teng

College of Mathematics and Systems Science, Xinjiang University, China

ARTICLE INFO

Keywords:
Brucellosis
Delay
Stability analysis
Optimal control

ABSTRACT

Firstly, we consider an animal-human infection model of brucellosis with three distributed delays, representing the latent period of brucellosis in infected animal and human population and the survival time of brucella in the environment, respectively. The equilibrium points and basic reproduction number R_0 are calculated. By building appropriate Lyapunov functionals and applying LaSalle's invariance principle, the sufficient conditions for global asymptotic stability of two equilibria are given. Secondly, by introducing four control variables, we set the corresponding optimal control model and drive the first order necessary conditions for the existence of optimal control solution. Finally, we perform several numerical simulations to validate our theoretical results and show effects of different control strategies.

1. Introduction

Brucellosis is a zoonotic infectious disease that spreads in three main ways. One is the transmission through contact with the mucous membrane of the skin and is believed as the most important way of transmission. Human or animals infection caused through direct contact with infected livestock, or contact with their excreta, vaginal secretions and so on. Humans also can be infected by careless feeding, milking, shearing, slaughtering and processing of the skin, hair and meat of infected animals through minor cuts in the skin, or the conjunctiva of the eye. The second route of transmission is through the digestive tract, which is spread by eating contaminated food and water, drinking raw dairy products, eating undercooked meat and internal organs. The third is transmission through the respiratory tract. Brucella discharged by infected animals can pollute the environment and after the formation of polluted aerosols floating in the air, taking into the respiratory tract by breathing also causes the infection [1]. Although human-to-human transmission is rare, but there is still some reported cases [2].

In animals, main symptoms of brucellosis are abortion and orchitis, which have adverse effects on reproduction and fertility, newborn survival and milk yield, thus leads to huge economic losses [3, 4]. Human brucellosis usually presents as an acute febrile illness with weakness, drowsiness and fever. According to the national notifiable diseases report released by the Chinese center for disease control, human brucellosis also has a very low mortality rate, but may further develop into a chronic and disabling disease with serious complications such as bone and joint complications, gastrointestinal complications and expiratory tract complications [5, 6]. Based on the statistical data of China, there were 44,036 new human brucellosis cases in 2019 and 47,245 in 2020. Since 2021, a total of 44,134 cases of brucellosis have been reported nationwide until October. According to the data of newly released cases of brucellosis every month since 2021, we can reasonably predict that the number of newly released cases of brucellosis this year will exceed that of 2020, forming a trend of annual increase of newly released cases of brucellosis for three consecutive years [7].

Mathematical modeling is always recognized as one of the effective tools for better understanding and controlling the transmission of epidemic diseases. Since the latent period of the disease is not negligible in the process of transmission, many scholars have established and discussed the epidemic models with latency delay [8, 9, 10, 11]. As for the brucellosis model, the latency period, immune period and the time required to detect and eliminate infected animals are mainly studied. Hou et al. have done a series of works on brucellosis modeling. For example, the literature [12] established a general SEIB dynamic model for the transmission of brucellosis in animals. The authors considered the general incidence of brucellosis in animals and the discrete delay of latency τ and analyzed the dynamic behavior of the model equilibrium. A general dynamic model

* Corresponding author.
E-mail address: xamxinur@sina.com (X. Abdurahman).

https://doi.org/10.1016/j.heliyon.2022.e12274
Received 21 March 2022; Received in revised form 7 September 2022; Accepted 2 December 2022

and its corresponding discrete time-delay model are established in [13]. It is found that the time delay has no effect on the equilibrium stability of the discrete time-delay model.

Based on the transmission mechanism of brucellosis, many scholars have established dynamic models of brucellosis and analyzed dynamical behaviors. As we mentioned earlier, brucellosis is a zoonosis, so some scholars have analyzed the human-animal infection models of brucellosis. Hao [14] et al. considered a dynamical model of human-animal brucellosis infection, indicated that Brucella in the environment is also a major means of brucellosis transmission. In practice, the disease can be effectively controlled by controlling the amount of introduction, timely and effective sterilization and slaughtering of infected animals. Li [15] et al. evaluated the brucellosis control strategy in China based on their model. The scale of human brucellosis in mainland China can be quantified by selecting 11 provinces (and the entire country) and estimating the basic reproduction number. These preliminary estimates allow for better implementation of control strategies. Zhou [16] et al. studied the transmission dynamics of Brucellosis in Inner Mongolia by establishing a multi-population epidemiological model, and discussed the control strategies of brucellosis. Their theoretical and numerical results suggest that brucellosis will increase gradually over the next few decades and peak around 2030.

It is known that the people infected with brucellosis are mainly animal husbandry employees, veterinary students and residents in the epidemic areas, and the infection rate is at a relatively high level. Therefore, the study of human-animal brucellosis infection model is still of great interest. As the brucellosis infection, both in human and in animals, has the latent period in the process of infection. In addition, Brucella can live in the environment for a long time, which can create a "cumulative" infection. Based on these considerations, we firstly established a animal-human brucellosis infection model with distributed delays and analyzed stability of the model. Then considering different control measures, we set an optimal control model and studied the existence of optimal control solution.

2. The model and equilibria

The animal population is classified into the susceptible compartment $S_a(t)$, the vaccinated compartment $V_a(t)$ and the infectious compartment $I_a(t)$. The number of brucellosis pathogens in the environment is denoted by $B_a(t)$. The human population is also classified into the susceptible $S_h(t)$, the acute human brucellosis $I_h(t)$ and the chronic human brucellosis $R_h(t)$. The latent period of animal-to-animal transmission and the latent period of animal-to-human transmission are represented by τ_1 and τ_3 respectively. As mentioned earlier, brucellosis pathogens can survive in the environment for a long time, so we need to consider a persistence of pathogens when we think about pathogens in the environment. τ_2 is the survival time of pathogens in the environment, and $\int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\mu_a+\delta_a)\sigma_2} h(I_a(t-\sigma_2))\, d\sigma_2$ is the accumulated number of pathogens from 0 to τ_2. Therefore, we have the following model:

$$\begin{cases} \dfrac{dS_a}{dt} = A_a + \epsilon V_a - (\mu_a+\theta)S_a - S_a\left(f\left(I_a\right)+g\left(B_a\right)\right), \\ \dfrac{dV_a}{dt} = \theta S_a - \epsilon V_a - \mu_a V_a - \eta V_a\left(f\left(I_a\right)+g\left(B_a\right)\right), \\ \dfrac{dI_a}{dt} = \displaystyle\int_0^{\tau_1} \varphi_1\left(\sigma_1\right) e^{-(\mu_a+\delta_a)\sigma_1}\left(S_a\left(t-\sigma_1\right)+\eta V_a\left(t-\sigma_1\right)\right) \\ \qquad \cdot\left(f\left(I_a\left(t-\sigma_1\right)\right)+g\left(B_a\left(t-\sigma_1\right)\right)\right) d\sigma_1 - \left(\mu_a+\delta_a\right) I_a, \\ \dfrac{dB_a}{dt} = \displaystyle\int_0^{\tau_2} \varphi_2\left(\sigma_2\right) e^{-(\gamma+\alpha)\sigma_2} h\left(I_a\left(t-\sigma_2\right)\right) d\sigma_2 - (\gamma+\alpha)B_a, \\ \dfrac{dS_h}{dt} = A_h + (1-n)\beta I_h - \mu_h S_h - S_h\left(p\left(I_a\right)+q\left(B_a\right)\right), \\ \dfrac{dI_h}{dt} = \displaystyle\int_0^{\tau_3} \varphi_3\left(\sigma_3\right) e^{-(\mu_h+\beta)\sigma_3} S_h\left(t-\sigma_3\right)\left(p\left(I_a\left(t-\sigma_3\right)\right)+q\left(B_a\left(t-\sigma_3\right)\right)\right) d\sigma_3, \\ \qquad -\left(\mu_h+\beta\right) I_h, \\ \dfrac{dR_h}{dt} = n\beta I_h - \mu_h R_h. \end{cases} \tag{2.1}$$

Here, A_a and A_h represent the recruitment rates of human and sheep respectively, μ_a and μ_h represent the natural elimination rates of human and sheep population respectively. The slaughtering rate of individuals in compartment $I_a(t)$ due to disease is the constant δ_a, θ denotes the vaccination rate of individuals in compartment $S_a(t)$, ϵ denotes the rate of immunity loss in compartment $V_a(t)$, η is the invalid vaccination rate, α represents the decaying rate of brucella, γ is the product of the number of disinfection times and the effective disinfection rate, $\frac{1}{\beta}$ denotes the acute onset period of human brucellosis, n is the fraction of acute human brucellosis turned into chronic cases, $\varphi_i(\sigma_i)(i=1,2,3)$ are non-negative continuous functions.

The initial conditions of system (2.1) are given as

$$\begin{cases} S_a(\xi) = \phi_1(\xi), V_a(\xi) = \phi_2(\xi), I_a(\xi) = \phi_3(\xi), B_a(\xi) = \phi_4(\xi), \\ S_h(\xi) = \phi_5(\xi), I_h(\xi) = \phi_6(\xi), R_h(\xi) = \phi_7(\xi), \\ \phi_i(\xi) \in C([-\tau, 0], R_+), \xi \in [-\tau, 0], i = 1, 2, \dots, 7, \end{cases} \tag{2.2}$$

where $\tau = max\{\tau_1, \tau_2, \tau_3\}$.

Assume that f, g, h, p and q are second-order continuous differentiable functions and satisfy assumptions (H_1)–(H_5) in order to make the system (2.1) have epidemiological significance.

(H_1) $f(0) = g(0) = h(0) = 0$ and $f(I_a), g(B_a), h(I_a) > 0$ for $I_a, B_a > 0$;
(H_2) $f'(I_a), g'(B_a) > 0$ and $f''(I_a), g''(B_a) \le 0$ for $B_a, I_a \ge 0$;

(H_3) $h'(I_a) > 0$ and $h''(I_a) \le 0$ for $I_a \ge 0$;
(H_4) $p(0) = q(0) = 0$ and $p(I_a), q(B_a)$ for $I_a, B_a > 0$;
(H_5) $p'(I_a), q'(B_a) > 0$ and $p''(I_a), q''(B_a) \le 0$ for $B_a, I_a \ge 0$.

2.1. Positivity and boundedness of solutions

For the positivity and boundedness of solutions for system (2.1), we have the following result.

Theorem 2.1. *Let $(S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t))$ be arbitrary solution of system (2.1) with initial conditions (2.2), then $(S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t))$ are non-negative on $[0, +\infty)$ and ultimately bounded.*

Proof. Assume that $S_a(t)$ is the first to reach the t-axis in the maximum existence interval of the solution, then $\exists t_1 \in [0, T)$, such that $S_a(t_1) = 0$ and $S_a(t) > 0$ for all $t \in [0, t_1)$. Therefore, we can easily see that $\dot{S}_a(t_1) \le 0$. On the other hand, from the second equation of system (2.1), we can see that $\dot{S}_a(t_1) = A_a + \epsilon V_a(t_1) > 0$. This leads to a contradiction.

Similarly, we can prove that $(V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t))$ are not reaching the t-axis for all $t \in [0, T)$. Therefore, $(S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t))$ is non-negative for all $t \in [0, T)$.

Next, we prove that solutions to the system (2.1) are bounded. Define

$$F_1(t) = \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(S_a(t-\sigma_1) + V_a(t-\sigma_1)\right) d\sigma_1 + I_a(t).$$

We have

$$\begin{aligned}
\frac{dF_1(t)}{dt} &= \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(A_a - \mu_a\left(S_a(t-\sigma_1) + V_a(t-\sigma_1)\right)\right) d\sigma_1 - (\mu_a + \delta_a) I_a(t) \\
&\le -\mu_a\left(\int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(S_a(t-\sigma_1) + V_a(t-\sigma_1)\right) d\sigma_1 + I_a(t)\right) \\
&\quad + \int_0^{\tau_1} A_a \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} d\sigma_1 \\
&= -\mu_a F_1(t) + A_a m_1,
\end{aligned}$$

where $m_1 = \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} d\sigma_1$.

This implies that

$$\lim_{t\to\infty} sup\, F_1(t) \le \frac{m_1 A_a}{\mu_a}.$$

We know that $h(I_a)$ is convex, and convex functions are upper bounded on bounded closed intervals, so $h(I_a)$ is upper bounded.

$$\begin{aligned}
\frac{dB_a}{dt} &= \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} h(I_a(t-\sigma_2)) d\sigma_2 - (\gamma+\alpha) B_a \\
&\le \int_0^{\tau_2} M_1 \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} d\sigma_2 - (\gamma+\alpha) B_a,
\end{aligned}$$

where M_1 is the upper bounded of $h(I_a)$. This implies that

$$\lim_{t\to\infty} sup\, B_a(t) \le \frac{m_2 M_1}{\gamma+\alpha},$$

where $m_2 = \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} d\sigma_2$.

This implies that $B_a(t)$ is bounded on $t \in [0, T)$.

Similarly, define

$$F_2(t) = \int_0^{\tau_3} \varphi_3(\sigma_3) e^{-(\mu_h+\beta)\sigma_3} S_h(t-\sigma_3) d\sigma_3 + I_h(t).$$

Then

$$\begin{aligned}
\frac{dF_2(t)}{dt} &= \int_0^{\tau_3} \varphi_3(\sigma_3) e^{-(\mu_h+\beta)\sigma_3} \left(A_h + (1-n)\beta I_h - \mu_h S_h(t-\sigma_3)\right) d\sigma_3 - (\mu_h + \beta) I_h(t) \\
&\le -\mu_h\left(\int_0^{\tau_3} \varphi_3(\sigma_3) e^{-(\mu_h+\beta)\sigma_3} S_h(t-\sigma_3) d\sigma_3 + I_h(t)\right)
\end{aligned}$$

$$+\int_0^{\tau_3} A_h\varphi_3(\sigma_3)e^{-(\mu_h+\beta)\sigma_3}d\sigma_3$$
$$=-\mu_h F_2(t)+A_h m_3,$$

where $m_3=\int_0^{\tau_3}\varphi_3(\sigma_3)e^{-(\mu_h+\beta_a)\sigma_3}d\sigma_3$.

This implies that $F_2(t)$ is bounded on $t\in[0,T)$, then S_h, I_h is bounded on $t\in[0,T)$ too. As we have

$$\lim_{t\to\infty} sup\, R_h \le \frac{\gamma I_h}{\mu_h}.$$

So R_h is bounded on $t\in[0,T)$. This ends the proof. □

2.2. The basic reproduction number and the endemic equilibrium

For system (2.1) there is always a disease-free equilibrium $P^0=(S_a^0,V_a^0,0,0,S_h^0,0,0)$, where $S_a^0=\frac{A_a(\epsilon+\mu_a)}{\mu_a(\mu_a+\epsilon+\theta)}$, $V_a^0=\frac{\theta A_a}{\mu_a(\mu_a+\epsilon+\theta)}$, $S_h^0=\frac{A_h}{\mu_h}$. The basic reproduction number R_0 is calculated using the method described in [17] and given as follows:

$$R_0=\frac{m_1(S_a^0+\eta V_a^0)f_{I_a}(0)}{\mu_a+\delta_a}+\frac{m_1m_2(S_a^0+\eta V_a^0)h_{I_a}(0)g_{B_a}(0)}{(\gamma+\alpha)(\mu_a+\delta_a)}.$$

Lemma 2.1. *Assume that conditions (H_1)–(H_3) are hold. Then functions $\frac{f(I_a)}{I_a}$, $\frac{h(I_a)}{I_a}$, $\frac{g(B_a)}{B_a}$ and $\frac{g(h(I_a))}{I_a}$ are monotonic decreasing for $I_a, B_a>0$.*

Proof. Since $f''(I_a)\le 0$, it shows that $f'(I_a)$ is monotonic decreasing, it follows that

$$\frac{f(I_a)}{I_a}=\frac{f(I_a)-f(0)}{I_a-0}=f'(\xi_1)\ge f'(I_a),\quad \xi_1\in(0,I_a),$$

and

$$\left(\frac{f(I_a)}{I_a}\right)'=\frac{f'(I_a)I_a-f(I_a)}{I^2}=\frac{f'(I_a)-\frac{f(I_a)}{I_a}}{I_a}=\frac{f'(I_a)-f'(\xi_1)}{I_a}\le 0.$$

That is to say, $\frac{f(I_a)}{I_a}$ is a monotonic decreasing function. Using the similar steps, we can show that $\frac{h(I_a)}{I_a}$ and $\frac{g(B_a)}{B_a}$ are also monotonic decreasing.

Noting that

$$\begin{aligned}\frac{g(h(I_a))}{I_a}&=\frac{g(h(I_a))-g(h(0))}{h(I_a)-h(0)}\frac{h(I_a)}{I_a}\\&=g'(h(\xi_2))h'(\xi_3)\\&\ge g'(h(I_a))h'(I_a),\quad \xi_2,\xi_3\in(0,I_a).\end{aligned}$$

It can deduce that

$$\left(\frac{g(h(I_a))}{I_a}\right)'=\frac{g'(h(I_a))h'(I_a)I_a-g(h(I_a))}{I_a^2}\le 0.$$

Therefore, $\frac{g(h(I_a))}{I_a}$ is also monotonic decreasing. □

Furthermore, for all I_a, I_a^*, B_a, B_a^*, we have

$$\left(\frac{f(I_a)}{f\left(I_a^*\right)}-1\right)\left(1-\frac{f\left(I^*\right)I_a}{f(I_a)I_a^*}\right)\le 0,$$
$$\left(\frac{g(B_a)}{g\left(B_a^*\right)}-1\right)\left(1-\frac{g\left(B^*\right)I_a}{g(B_a)I_a^*}\right)\le 0,$$

and

$$\left(\frac{h(I_a)}{h\left(I_a^*\right)}-1\right)\left(1-\frac{h\left(I_a^*\right)B_a}{h(I_a)B_a^*}\right)\le 0.$$

Next, we calculate the endemic point $(S_a^*,V_a^*,I_a^*,B_a^*,S_h^*,I_h^*,R_h^*)$. The value of S_a^*,V_a^*,I_a^*,B_a^* does not depend on the fifth to seventh equations of the system (2.1), so we use equations (2.3) to solve the endemic equilibrium S_a^*,V_a^*,I_a^*,B_a^*.

$$\begin{cases}A_a-(\mu_a+\theta)S_a^*+\epsilon V_a^*-S_a^*(f(I_a^*)+g(B_a^*))=0,\\ \theta S_a^*-(\mu_a+\epsilon)V_a^*-\eta V_a^*(f(I_a^*)+g(B_a^*))=0,\\ m_1(S_a^*+\eta V_a^*)(f(I_a^*)+g(B_a^*))-(\mu_a+\delta_a)I_a^*=0,\\ m_2h(I_a^*)-(\gamma+\alpha)B_a^*=0.\end{cases}\tag{2.3}$$

From the last equation in (2.3), we can obtain

$$B_a^* = \frac{m_2 h(I_a^*)}{\gamma+\alpha} \triangleq H(I_a^*). \tag{2.4}$$

Substituting (2.4) into the first two equations in (2.3), we get

$$A_a + \frac{\epsilon\theta S_a^*}{\epsilon+\mu_a+\eta\left(f(I_a^*)+g(H(I_a^*))\right)} - (\mu_a+\theta)S_a^* - S_a^*\left(f(I_a^*)+g(H(I_a^*))\right) = 0$$

$$m_1\left(S_a^* + \eta\frac{\theta S_a^*}{\epsilon+\mu_a+\eta(f(I_a^*)+g(H(I_a^*)))}\right)\left(f(I_a^*)+g(H(I_a^*))\right) - (\mu_a+\delta_a)I_a^* = 0$$

Let us define

$$G_1(S_a,I_a) \triangleq A_a + \frac{\epsilon\theta S_a}{\epsilon+\mu_a+\eta(f(I_a)+g(H(I_a)))} - (\mu_a+\theta)S_a - S_a(f(I_a)+g(H(I_a))),$$

$$G_2(S_a,I_a) \triangleq m_1\left(S_a + \eta\frac{\theta S_a}{\epsilon+\mu_a+\eta(f(I_a)+g(H(I_a)))}\right)(f(I_a)+g(H(I_a))) - (\mu_a+\delta_a)I_a.$$

Since $G_1(S_a,I_a)$ is monotonic decreasing for $S_a>0$ and

$$\begin{aligned}
&G_1(0,I_a)G_1(S_a^0,I_a)\\
&= A_a\left(A_a + \frac{\epsilon\theta S_a^0}{\epsilon+\mu_a+\eta(f(I_a)+g(H(I_a)))} - (\mu_a+\theta)S_a^0 - S_a^0(f(I_a)+g(H(I_a)))\right)\\
&\le A_a\left(A_a + \frac{\epsilon\theta S_a^0}{\epsilon+\mu_a} - (\mu_a+\theta)S_a^0\right) = 0,
\end{aligned}$$

for $I_a>0$, the equation $G_1(S_a,I_a)=0$ can be uniquely solved with S_a as a function of I_a for all I_a. That is to say, there is a function $S_a=\xi_1(I_a)$ which satisfies

$$\frac{A_a}{\xi_1(I_a)} - \frac{\mu_a(\mu_a+\epsilon+\theta)+\eta(\mu_a+\theta)(f(I_a)+g(H(I_a)))}{\epsilon+\mu_a+\eta(f(I_a)+g(H(I_a)))} = f(I_a)+g(H(I_a)).$$

Since $f(I_a)+g(H(I_a))$ is monotonic increasing, it follows that ξ_1 is monotonic decreasing. From (2.3) and (2.4), we have $\lim_{I_a\to\frac{m_1A_a}{\mu_a}}\xi_1(I_a)=0$.

Noticing that $G_2(S_a,I_a)$ is monotonic increasing for $S_a>0$ and $G_2(0,I_a)<0$ for all I_a.

$$\begin{aligned}
\left.\frac{dG_2(S_a^0,I_a)}{dI_a}\right|_{I_a=0} &= m_1(S_a^0+\eta V_a^0)\left(f_{I_a}(0)+h_{I_a}(0)g_{B_a}(0)\frac{m_2}{\gamma+\alpha}\right) - (\mu_a+\delta_a)\\
&= \frac{1}{\mu_a+\delta_a}(R_0-1),\\
G_2(S_a^0,0) &= 0.
\end{aligned}$$

So $G_2(S_a^0,I_a)$ is positive for I_a close to 0 if $R_0>1$. Since we are searching for a unique endemic equilibrium and for a uniquely corresponding I_a^*, it needs the local solvability of the equation $G_2(S_a,I_a)=0$ on a certain condition.

When $R_0>1$, the equation $G_2(S_a,I_a)=0$ can be uniquely solved with S_a as a function of I_a (locally for I_a). That is, there is a function $S_a=\xi_2(I_a)$ which satisfies

$$m_1\left(\xi_2(I_a) + \frac{\theta\eta}{\epsilon+\mu_a+\eta(f(I(a))+g(H(I(a))))}\xi_2(I_a)\right) = \frac{(\mu_a+\delta_a)}{\frac{f(I(a))+g(B(a))}{I_a}}.$$

From Lemma 2.1 we know that $\frac{f(I(a))+g(H(I(a)))}{I_a}$ is strictly decreasing for $I_a>0$, then ξ_2 is monotonic increasing.

Since ξ_1 is monotonic decreasing, ξ_2 is monotonic increasing and $\lim_{I_a\to\frac{m_1A_a}{\mu_a}}\xi_1(I_a)=0$, the curves defined by $S_a=\xi_1(I_a)$ and $S_a=\xi_2(I_a)$ have a common point (S_a^*,I_a^*) with $S_a^*>0, I_a^*>0$ on condition that if and only if $\xi_1(0)>\xi_2(0)$.

$$\begin{aligned}
\xi_1(0)-\xi_2(0) &= S_a^0 - \lim_{I_a\to0}\frac{\mu_a+\delta_a}{m_1}\frac{I_a\left(\epsilon+\mu_a+\eta\left(f\left(I_a\right)+g\left(H\left(I_a\right)\right)\right)\right)}{\left(f\left(I_a\right)+g\left(H\left(I_a\right)\right)\right)\left(\epsilon+\mu_a+\theta\eta+\eta\left(f\left(I_a\right)+g\left(H\left(I_a\right)\right)\right)\right)}\\
&= S_a^0 - \frac{\mu_a+\delta_a}{m_1}\frac{\epsilon+\mu_a}{(\epsilon+\mu_a+\theta\eta)\left(f_{I_a}(0)+h_{I_a}(0)g_{B_a}(0)\frac{m_2}{\gamma+\alpha}\right)}\\
&= S_a^0 - \frac{(\epsilon+\mu_a)\left(S_a^0+\eta V_a^0\right)}{R_0\left(\epsilon+\mu_a+\theta\eta\right)}\\
&= \frac{S_a^0\left(R_0-1\right)}{R_0} > 0.
\end{aligned}$$

So, when $R_0>1$, the equations (2.3) have a unique positive solutions $(S_a^*,V_a^*,I_a^*,B_a^*)$. The system (2.1) also has a unique positive equilibrium $P^*=(S_a^*,V_a^*,I_a^*,B_a^*,S_h^*,I_h^*,R_h^*)$.

3. Local stability

3.1. Local stability of disease-free equilibrium P^0

The following conclusions can be drawn about the stability of P^0 of the system (2.1).

Theorem 3.1. *The disease-free equilibrium P^0 of system (2.1) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. The characteristic equation can be obtained by linearizing the system (2.1) at P^0.

$$\begin{aligned}&\left(\lambda+\mu_a\right)\left(\lambda+\mu_h+\beta\right)\left(\lambda+\mu_h\right)^2\left(\lambda+\mu_a+\theta+\epsilon\right)\\&\Big[(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a-m_1\left(S_a^0+\eta V_a^0\right)f_{I_a}(0)e^{-\lambda\tau_1}\right)\\&-m_1m_2\left(S_a^0+\eta V_a^0\right)g_{B_a}(0)h_{I_a}(0)e^{-\lambda(\tau_1+\tau_2)}\Big]=0.\end{aligned}$$

Obviously $\lambda_{1,2}=-\mu_a, \lambda_3=-\mu_h, \lambda_4=-(\mu_h+\beta), \lambda_5=-(\mu_a+\theta+\epsilon)$ are negative, and the other two eigenvalues are determined by the following equation

$$\begin{aligned}Q(\lambda)=&(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a-m_1\left(S_a^0+\eta V_a^0\right)f_{I_a}(0)e^{-\lambda\tau_1}\right)\\&-m_1m_2\left(S_a^0+\eta V_a^0\right)g_{B_a}(0)h_{I_a}(0)e^{-\lambda(\tau_1+\tau_2)}=0.\end{aligned}\tag{3.1}$$

When $R_0 < 1$, rewrite the equation (3.1) as

$$\frac{m_1\left(S_a^0+\eta V_a^0\right)\left((\lambda+\gamma+\alpha)f_{I_a}(0)e^{-\lambda\tau_1}+m_2g_{B_a}(0)h_{I_a}(0)e^{-\lambda(\tau_1+\tau_2)}\right)}{(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)}=1.$$

Let $\lambda=x+iy(x,y\in R)$ be the root of equation (3.1), and $x\ge 0$, then we have

$$\begin{aligned}|1|=&\left|\frac{m_1\left(S_a^0+\eta V_a^0\right)\left((\lambda+\gamma+\alpha)f_{I_a}(0)e^{-\lambda\tau_1}+m_2g_{B_a}(0)h_{I_a}(0)e^{-\lambda(\tau_1+\tau_2)}\right)}{(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)}\right|\\&\le m_1\left(S_a^0+\eta V_a^0\right)\left(\left|\frac{f_{I_a}(0)e^{-\lambda\tau_1}}{\lambda+\mu_a+\delta_a}\right|+\left|\frac{m_2g_{B_a}(0)h_{I_a}(0)e^{-\lambda(\tau_1+\tau_2)}}{(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)}\right|\right)\\&\le m_1\left(S_a^0+\eta V_a^0\right)\left(\frac{f_{I_a}(0)}{\mu_a+\delta_a}+\frac{m_2g_{B_a}(0)h_{I_a}(0)}{(\gamma+\alpha)\left(\mu_a+\delta_a\right)}\right)\\&=R_0<1.\end{aligned}$$

This is contradictory. So if $R_0 < 1$, then the roots of equation (3.1) have negative real parts, that is to say P^0 is locally asymptotically stable.

When $R_0 > 1$, we have

$$Q(0)=(\gamma+\alpha)\left(\mu_a+\delta_a-m_1\left(S_a^0+\eta V_a^0\right)f_{I_a}(0)\right)-m_1m_2\left(S_a^0+\eta V_a^0\right)g_{B_a}(0)h_{I_a}(0)<0$$

and $\lim_{\lambda\to+\infty}Q(\lambda)=+\infty$. So $Q(\lambda)=0$ has at least one positive root. Therefore, P^0 is unstable. □

3.2. Local stability of endemic equilibrium P^*

Theorem 3.2. *The endemic equilibrium P^* of system (2.1) is locally asymptotically stable if $R_0 > 1$.*

Proof. The system (2.1) is linearized at P^* and the characteristic equation is written

$$\begin{aligned}&\left(\lambda+\mu_h+\beta\right)\left(\lambda+\mu_h\right)^2\Big[(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\\&+\eta\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\\&+\eta\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a+\theta\right)\\&+\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a+\epsilon\right)\\&-m_1f_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)e^{-\lambda\tau_1}\\&-m_1m_2g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)e^{-\lambda(\tau_1+\tau_2)}\\&-m_1\eta f_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)e^{-\lambda\tau_1}\\&-m_1m_2\eta g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)\left(\lambda+\mu_a\right)e^{-\lambda(\tau_1+\tau_2)}\Big]=0.\end{aligned}$$

Obviously $\lambda_{1,2}=-\mu_h, \lambda_3=-(\mu_h+\beta)$ are the negative eigenvalues, and the other four eigenvalues are given by the following equation

$$Q(\lambda)=U_1+U_2-U_3-U_4=0,\tag{3.2}$$

where

$$
\begin{aligned}
U_1 =& (\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right),\\
U_2 =& \eta\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\\
&+\eta\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a+\theta\right)\\
&+\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a+\epsilon\right),\\
U_3 =& m_1 f_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)e^{-\lambda\tau_1}\\
&+m_1 m_2 g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)\left(S_a^*+\eta V_a^*\right)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)e^{-\lambda\left(\tau_1+\tau_2\right)},\\
U_4 =& m_1\eta f_{I_a}\left(I_a^*\right)\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)e^{-\lambda\tau_1}\\
&+m_1 m_2\eta g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)\left(\lambda+\mu_a\right)e^{-\lambda\left(\tau_1+\tau_2\right)}.
\end{aligned}
$$

The equation (3.2) is sorted as

$$
U_1+U_2=U_3+U_4. \tag{3.3}
$$

Set $\lambda=x+iy (x, y\in R)$ is root of equation (3.2), and $x\geq 0$, then we have

$$
\begin{aligned}
\left|U_3\right| \leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\right|\\
&\cdot\left|\left(m_1(S_a^*+\eta V_a^*)\right)\left(f_{I_a}(I_a^*)e^{-\lambda\tau_1}+\frac{m_2 g_{B_a}(B_a^*)h_{I_a}(I_a^*)e^{-\lambda(\tau_1+\tau_2)}}{\lambda+\gamma+\alpha}\right)\right|\\
\leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\right|\\
&\cdot\left|\left(m_1(S_a^*+\eta V_a^*)\right)\left(f_{I_a}(I_a^*)+\frac{m_2 g_{B_a}(B_a^*)h_{I_a}(I_a^*)}{\gamma+\alpha}\right)\right|\\
\leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\right|\left|\frac{m_1(S_a^*+\eta V_a^*)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)}{I_a^*}\right|\\
\leq & \left|(\mu_a+\delta_a)(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\right|\\
\leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a+\delta_a\right)\left(\lambda+\mu_a\right)\left(\lambda+\mu_a+\theta+\epsilon\right)\right|\\
= & \left|U_1\right|,\\
\left|U_2\right| > & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\left|\left(\eta\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2+(\eta+1)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)\left(\mu_a+\delta_a\right)\right)\right|\\
> & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\frac{m_1(\eta+1)\left(S_a^*+\eta V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2}{I_a^*}\\
\geq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\frac{m_1\eta\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2}{I_a^*},\\
\left|U_4\right| = & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\\
&\cdot\left|m_1\eta\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)\left(f_{I_a}\left(I_a^*\right)e^{-\lambda\tau_1}+\frac{m_2 g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)e^{-\lambda\left(\tau_1+\tau_2\right)}}{\lambda+\gamma+\alpha}\right)\right|\\
\leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\\
&\cdot\left|m_1\eta\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)\left(f_{I_a}\left(I_a^*\right)+\frac{m_2 g_{B_a}\left(B_a^*\right)h_{I_a}\left(I_a^*\right)}{\gamma+\alpha}\right)\right|\\
\leq & \left|(\lambda+\gamma+\alpha)\left(\lambda+\mu_a\right)\right|\frac{m_1\eta\left(S_a^*+V_a^*\right)\left(f\left(I_a^*\right)+g\left(B_a^*\right)\right)^2}{I_a^*}.
\end{aligned}
$$

Thus, $|U_1+U_2|>|U_3+U_4|$. This is contradiction with equation (3.3). Therefore, if $R_0>1$, then the roots of equation (3.2) have negative real parts, that is to say P^* is locally asymptotically stable. □

4. Global stability

Let us first consider subsystem with variables $S_a V_a I_a B_a$:

$$\begin{cases} \frac{dS_a}{dt} = A_a + \epsilon V_a - \mu_a S_a - \theta S_a - S_a(f(I_a) + g(B_a)), \\ \frac{dV_a}{dt} = \theta S_a - \epsilon V_a - \mu_a V_a - \eta V_a(f(I_a) + g(B_a)), \\ \frac{dI_a}{dt} = \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1)) \\ \qquad \cdot (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1)))d\sigma_1 - (\mu_a + \delta_a) I_a, \\ \frac{dB_a}{dt} = \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} h(I_a(t-\sigma_2))d\sigma_2 - (\gamma+\alpha)B_a. \end{cases} \tag{4.1}$$

Note that system (4.1) can be considered independently of system (2.1). Similarly, system (4.1) also has a unique disease-free equilibrium $(S_a^0, V_a^0, 0, 0)$ and a unique endemic equilibrium $(S_a^*, V_a^*, I_a^*, B_a^*)$ if $R_0 > 1$.

4.1. Global stability of the disease-free equilibrium

Theorem 4.1. *The disease-free equilibrium $(S_a^0, V_a^0, 0, 0)$ of the system (4.1) is globally asymptotically stable when $R_0 < 1$.*

Proof. We define a Lyapunov functional L as follows:

$$L(t) = L_1(t) + L_2(t),$$

where

$$L_1 = S_a - S_a^0 - S_a^0 \ln \frac{S_a}{S_a^0} + V_a - V_a^0 - V_a^0 \ln \frac{V_a}{V_a^0} + \frac{1}{m_1} I_a + \frac{(S_a^0 + \eta V_a^0) g_{B_a}(0)}{\gamma + \alpha} B_a$$

$$\begin{aligned} L_2 =& \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \int_{t-\sigma_1}^{t} (S_a(\theta) + \eta V_a(\theta))(f(I_a(\theta)) + g(B_a(\theta)))d\theta d\sigma_1 \\ &+ \frac{(S_a^0 + \eta V_a^0) g_{B_a}(0)}{\gamma + \alpha} \int_0^{\tau_2} \rho_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \int_{t-\sigma_2}^{t} h(I_a(\theta))d\theta d\sigma_2. \end{aligned}$$

Then the derivative of L_1 and L_2 along with the solution of system (4.1) is

$$\begin{aligned} \frac{dL_1}{dt} =& \left(1 - \frac{S_a^0}{S_a}\right)\left(A_a - (\mu_a + \theta)S_a + \epsilon V_a - S_a(f(I_a) + g(B_a))\right) + \frac{1}{m_1}\frac{dI}{dt} \\ &+ \left(1 - \frac{V_a^0}{V_a}\right)\left(\theta S_a - (\mu_a + \epsilon)V_a - \eta V_a(f(I_a) + g(B_a))\right) + \frac{(S_a^0 + \eta V_a^0) g_{B_a}(0)}{\gamma + \alpha}\frac{dB}{dt} \\ =& \mu_a S_a^0 \left(2 - \frac{S_a^0}{S_a} - \frac{S_a}{S_a^0}\right) + \theta S_a^0 \left(3 - \frac{S_a^0}{S_a} - \frac{V_a}{V_a^0} - \frac{S_a V_a^0}{S_a^0 V_a}\right) \\ &+ \epsilon V_a \left(\frac{V_a}{V_a^0} + \frac{S_a^0}{S_a} - \frac{S_a^0 V_a}{S_a V_a^0} - 1\right) + (S_a^0 + \eta V_a^0 - (S_a + \eta V_a))(f(I_a) + g(B_a)) \\ &+ \frac{1}{m_1}\left(\int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1}(S_a(t-\sigma_1) + \eta V_a(t-\sigma_1))\right. \\ &\left. \cdot (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1)))d\sigma_1 - (\mu_a + \delta_a)I_a\right) \\ &+ \frac{(S_a^0 + \eta V_a^0) g_{B_a}(0)}{\gamma + \alpha}\left(\int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} h(I_a(t-\sigma_2))d\sigma_2 - (\gamma + \alpha)B_a\right), \end{aligned}$$

$$\begin{aligned} \frac{dL_2}{dt} =& \frac{1}{m_1}\left(m_1\left((S_a + \eta V_a)\right)\left(f(I_a) + g(B_a)\right) - \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1}\right. \\ &\left. \cdot (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1))(f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1)))d\sigma_1\right) \\ &+ \frac{(S_a^0 + \eta V_a^0) g_{B_a}(0)}{\gamma + \alpha}\left(m_2 h(I_a) - \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} h(I_a(t-\sigma_2))d\sigma_2\right). \end{aligned}$$

So there are

$$\begin{aligned}\frac{dL_1}{dt}+\frac{dL_2}{dt}=&\theta S_a^0\left(3-\frac{S_a^0}{S_a}-\frac{V_a}{V_a^0}-\frac{S_aV_a^0}{S_a^0V_a}\right)+\epsilon V_a\left(\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1\right)\\&+(S_a^0+\eta V_a^0)(f(I_a)+g(B_a))+\frac{m_2(S_a^0+\eta V_a^0)g_{B_a}(0)h(I_a)}{\gamma+\alpha}\\&-\frac{1}{m_1}(\mu_a+\delta_a)I_a-(S_a^0+\eta V_a^0)g_{B_a}(0)B_a\\ \le&\theta S_a^0\left(3-\frac{S_a^0}{S_a}-\frac{V_a}{V_a^0}-\frac{S_aV_a^0}{S_a^0V_a}\right)+\epsilon V_a\left(\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1\right)\\&+(S_a^0+\eta V_a^0)(I_af_{I_a}(0)+B_ag_{B_a}(0))+\frac{m_2(S_a^0+\eta V_a^0)g_{B_a}(0)h_{I_a}(0)I_a}{\gamma+\alpha}\\&-\frac{1}{m_1}(\mu_a+\delta_a)I_a-(S_a^0+\eta V_a^0)g_{B_a}(0)B_a\\=&\theta S_a^0\left(3-\frac{S_a^0}{S_a}-\frac{V_a}{V_a^0}-\frac{S_aV_a^0}{S_a^0V_a}\right)+\epsilon V_a\left(\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1\right)\\&+\frac{I_a}{m_1}(\mu_a+\delta_a)(R_0-1).\end{aligned}$$

Case 1. $\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1\le 0$, we have

$$\frac{dL_1}{dt}+\frac{dL_2}{dt}\le 0$$

Case 2. $\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1>0$.

Since $\epsilon V_a=\theta S_a^0-\mu_aV_a^0<\theta S_a^0$, we have

$$\begin{aligned}&\theta S_a^0(3-\frac{S_a^0}{S_a}-\frac{V_a}{V_a^0}-\frac{S_aV_a^0}{S_a^0V_a})+\epsilon V_a(\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1)\\<&\theta S_a^0(3-\frac{S_a^0}{S_a}-\frac{V_a}{V_a^0}-\frac{S_aV_a^0}{S_a^0V_a})+\theta S_a^0(\frac{V_a}{V_a^0}+\frac{S_a^0}{S_a}-\frac{S_a^0V_a}{S_aV_a^0}-1)\\=&\theta S_a^0(2-\frac{S_aV_a^0}{S_a^0V_a}-\frac{S_a^0V_a}{S_aV_a^0}).\end{aligned}$$

So, $\frac{dL_1}{dt}+\frac{dL_2}{dt}\le 0$. The equality $\frac{dL}{dt}=0$ holds if and only if $S_a=S_a^0, V_a=V_a^0, I_a=0, B_a=0$ and $R_0=1$. Since $(S_a^0,V_a^0,0,0)$ is the only invariant set of system (4.1) in $\{(S_a,V_a,I_a,B_a|\frac{dL}{dt}=0)\}$, the disease-free equilibrium $(S_a^0,V_a^0,0,0)$ is globally asymptotically stable by LaSalle's Invariance Principle [18]. □

Theorem 4.2. *The disease-free equilibrium P^0 of the system (2.1) is globally asymptotically stable when $R_0<1$.*

Proof. According to theorem 4.1, the disease-free equilibrium of system (4.1) is global asymptotically stable if $R_0<1$. To prove the globally stability of the equilibrium $(S_a^0,V_a^0,0,0,S_h^0,0,0)$ of system (2.1) with the animals components already at the disease-free steady state given by

$$\begin{cases}\frac{dS_h}{dt}=A_h+(1-n)\beta I_h-\mu_hS_h,\\ \frac{dI_h}{dt}=-(\mu_h+\beta)I_h,\\ \frac{dR_h}{dt}=n\beta I_h-mu_hR_h.\end{cases}$$

Solving:

$$S_h=\frac{A_h+(1-n)\beta I_h}{\mu_h}+D_1e^{-\mu_ht}, I_h=D_2e^{-(\mu_h+\beta)t}, R_h=\frac{n\beta I_h}{\mu_h}+D_3e^{-\mu_ht},$$

where D_1,D_2,D_3 are integrating constants.

It is clear that $S_h\to S_h^0, I_h\to 0$ and $R_h\to 0$ when $t\to\infty$ and so P^0 of system (2.1) is globally asymptotically stable. □

4.2. Global stability of the endemic equilibrium

Theorem 4.3. *The endemic equilibrium $(S_a^*,V_a^*,I_a^*,B_a^*)$ of the system (4.1) is globally asymptotically stable when $R_0>1$.*

Proof. Define a Lyapunov functional L as follows:

$$L(t) = L_1(t) + L_2(t),$$

where

$$\begin{aligned}
L_1 =& S_a - S_a^* - S_a^* \ln \frac{S_a}{S_a^*} + V_a - V_a^* - V_a^* \ln \frac{V_a}{V_a^*} + \frac{1}{m_1}(I_a - I_a^* - I_a^* \ln \frac{I_a}{I_a^*}) \\
&+ \frac{(S_a^* + \eta V_a^*)g(B_a^*)}{m_2 h(I_a^*)}(B_a - B_a^* - B_a^* \ln \frac{B_a}{B_a^*}), \\
L_2 =& \frac{S_a^* f(I_a^*)}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \int_{t-\sigma_1}^{t} \left(\frac{S_a(\theta) f(I_a(\theta))}{S_a^* f(I_a^*)} - 1 - \ln \frac{S_a(\theta) f(I_a(\theta))}{S_a^* f(I_a^*)} \right) d\theta d\sigma_1 \\
&+ \frac{S_a^* g(B_a^*)}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \int_{t-\sigma_1}^{t} \left(\frac{S_a(\theta) g(B_a(\theta))}{S_a^* g(B_a^*)} - 1 - \ln \frac{S_a(\theta) g(B_a(\theta))}{S_a^* g(B_a^*)} \right) d\theta d\sigma_1 \\
&+ \frac{\eta V_a^* f(I_a^*)}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \int_{t-\sigma_1}^{t} \left(\frac{V_a(\theta) f(I_a(\theta))}{V_a^* f(I_a^*)} - 1 - \ln \frac{V_a(\theta) f(I_a(\theta))}{V_a^* f(I_a^*)} \right) d\theta d\sigma_1 \\
&+ \frac{\eta V_a^* g(B_a^*)}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \int_{t-\sigma_1}^{t} \left(\frac{V_a(\theta) g(B_a(\theta))}{V_a^* g(B_a^*)} - 1 - \ln \frac{V_a(\theta) g(B_a(\theta))}{V_a^* g(B_a^*)} \right) d\theta d\sigma_1 \\
&+ \frac{(S_a^* + \eta V_a^*)g(B_a^*)}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \int_{t-\sigma_2}^{t} \left(\frac{h(I_a(\theta))}{h(I_a^*)} - 1 - \ln \frac{h(I_a(\theta))}{h(I_a^*)} \right) d\theta d\sigma_2.
\end{aligned}$$

Then the derivative of L_1 along the solutions of system (4.1) is

$$\begin{aligned}
\frac{dL_1}{dt} =& \left(1 - \frac{S_a^*}{S_a}\right) \left(A_a + \epsilon V_a - S_a(f(I_a) + g(B_a)) - \frac{S_a}{S_a^*} \left(A_a + \epsilon V_a^* - S_a^*(f(I_a^*) + g(B_a^*)) \right) \right) \\
&+ \left(1 - \frac{V_a^*}{V_a}\right) \left(\theta S_a - \eta V_a(f(I_a) + g(B_a)) - \frac{V_a}{V_a^*} \left(\theta S_a^* - \eta V_a^*(f(I_a^*) + g(B_a^*)) \right) \right) \\
&+ \frac{1}{m_1} \left(1 - \frac{I_a^*}{I_a}\right) \left(\int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1)) \right. \\
&\left. \cdot (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1))) d\sigma_1 - \frac{m_1 I_a}{I_a^*}(S_a^* + \eta V_a^*)(f(I_a^*) + g(B_a^*)) \right) \\
&+ \frac{(S_a^* + \eta V_a^*)g(B_a^*)}{m_2 h(I_a^*)} \left(1 - \frac{B_a^*}{B_a}\right) \left(\int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} h(I_a(t-\sigma_2)) d\sigma_2 - m_2 h(I_a^*) \frac{B_a}{B_a^*} \right) \\
=& A_a(2 - \frac{S_a^*}{S_a} - \frac{S_a}{S_a^*}) + \epsilon V_a^*(1 + \frac{V_a}{V_a^*} - \frac{S_a}{S_a^*} - \frac{S_a^* V_a}{S_a V_a^*}) + \theta S_a^*(1 + \frac{S_a}{S_a^*} - \frac{V_a}{V_a^*} - \frac{S_a V_a^*}{S_a^* V_a}) \\
&- (S_a - S_a^*)(f(I_a) + g(B_a)) - \eta(V_a - V_a^*)(f(I_a) + g(B_a)) + (S_a + \eta V_a)(f(I_a^*) + g(B_a^*)) \\
&+ \frac{1}{m_1} \left(1 - \frac{I_a^*}{I_a}\right) \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1)) \\
&\cdot (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1))) d\sigma_1 - \frac{I_a}{I_a^*}(S_a^* + \eta V_a^*)(f(I_a^*) + g(B_a^*)) \\
&+ (S_a^* + \eta V_a^*) g(B_a^*) \left(\frac{1}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \left(1 - \frac{B_a^*}{B_a}\right) \frac{h(I_a(t-\sigma_2))}{h(I_a^*)} d\sigma_2 + 1 - \frac{B_a}{B_a^*} \right) \\
=& A_a(2 - \frac{S_a^*}{S_a}) + \epsilon V_a^*(1 - \frac{S_a^* V_a}{S_a V_a^*}) + \theta S_a^*(1 - \frac{S_a V_a^*}{S_a^* V_a}) - \mu_a S_a - \mu_a V_a \\
&- (S_a - S_a^*)(f(I_a) + g(B_a)) - \eta(V_a - V_a^*)(f(I_a) + g(B_a)) \\
&+ \frac{1}{m_1} \left(1 - \frac{I_a^*}{I_a}\right) \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1)) \\
&\cdot (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1))) d\sigma_1 - \frac{I_a}{I_a^*}(S_a^* + \eta V_a^*)(f(I_a^*) + g(B_a^*))
\end{aligned}$$

$$
\begin{aligned}
&+ (S_a^* + \eta V_a^*) g(B_a^*) \left(\frac{1}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \left(1 - \frac{B_a^*}{B_a}\right) \frac{h(I_a(t-\sigma_2))}{h(I_a^*)} d\sigma_2 + 1 - \frac{B_a}{B_a^*} \right) \\
=& \mu_a S_a (2 - \frac{S_a^*}{S_a} - \frac{S_a}{S_a^*}) + \mu_a V_a (3 - \frac{S_a^*}{S_a} - \frac{V_a}{V_a^*} - \frac{S_a V_a^*}{S_a^* V_a}) + \epsilon V_a^* (2 - \frac{S_a V_a^*}{S_a^* V_a} - \frac{S_a^* V_a}{S_a V_a^*}) \\
&+ S_a^* f(I_a^*) \left(2 + \frac{f(I_a)}{f(I_a^*)} - \frac{S_a f(I_a)}{S_a^* f(I_a^*)} - \frac{S_a^*}{S_a} - \frac{I_a}{I_a^*} \right. \\
&\left. + \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(1 - \frac{I_a^*}{I_a}\right) \frac{S_a(t-\sigma_1) f(I_a(t-\sigma_1))}{S_a^* f(I_a^*)} d\sigma_1 \right) \\
&+ S_a^* g(I_a^*) \left(3 + \frac{g(B_a)}{g(B_a^*)} - \frac{S_a g(B_a)}{S_a^* g(B_a^*)} - \frac{S_a^*}{S_a} - \frac{I_a}{I_a^*} - \frac{B_a}{B_a^*} \right. \\
&+ \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(1 - \frac{I_a^*}{I_a}\right) \frac{S_a(t-\sigma_1) g(B_a(t-\sigma_1))}{S_a^* g(B_a^*)} d\sigma_1 \\
&\left. + \frac{1}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \left(1 - \frac{B_a}{B_a^*}\right) \frac{h(I_a(t-\sigma_2))}{h(I_a^*)} d\sigma_2 \right) \\
&+ \eta V_a^* f_a(I_a^*) \left(3 + \frac{f(I_a)}{f(I_a^*)} - \frac{V_a f(I_a)}{V_a^* f(I_a^*)} - \frac{S_a^*}{S_a} - \frac{S_a V_a^*}{S_a^* V_a} - \frac{I_a}{I_a^*} \right. \\
&\left. + \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(1 - \frac{I_a^*}{I_a}\right) \frac{V_a(t-\sigma_1) f(I_a(t-\sigma_1))}{V_a^* f(I_a^*)} d\sigma_1 \right) \\
&+ \eta V_a^* g(I_a^*) \left(4 + \frac{g(B_a)}{g(B_a^*)} - \frac{V_a g(B_a)}{V_a^* g(B_a^*)} - \frac{S_a^*}{S_a} - \frac{S_a V_a^*}{S_a^* V_a} - \frac{I_a}{I_a^*} - \frac{B_a}{B_a^*} \right. \\
&+ \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} \left(1 - \frac{I_a^*}{I_a}\right) \frac{V_a(t-\sigma_1) g(B_a(t-\sigma_1))}{V_a^* g(B_a^*)} d\sigma_1 \\
&\left. + \frac{1}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} \left(1 - \frac{B_a^*}{B_a}\right) \frac{h(I_a(t-\sigma_2))}{h(I_a^*)} d\sigma_2 \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{dL_1}{dt} + \frac{dL_2}{dt} \\
\le& S_a^* f(I_a^*) \left(2 + \frac{f(I_a)}{f(I_a^*)} - \frac{S_a^*}{S_a} - \frac{I_a}{I_a^*} - \frac{S_a f(I_a) I_a^*}{S_a^* f(I_a^*) I_a} - M\left(\frac{S_a f(I_a) I_a^*}{S_a^* f(I_a^*) I_a} \right) \right. \\
&\left. + \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} M\left(\frac{S_a(t-\sigma_1) f(I_a(t-\sigma_1)) I_a^*}{S_a^* f(I_a^*) I_a} \right) d\sigma_1 \right) \\
&+ S_a^* g(B_a^*) \left(3 + \frac{g(B_a)}{g(B_a^*)} + \frac{h(I_a)}{h(I_a^*)} - \frac{S_a^*}{S_a} - \frac{I_a}{I_a^*} - \frac{B_a}{B_a^*} - \frac{S_a g(B_a) I_a^*}{S_a^* g(B_a^*) I_a} - \frac{h(I_a) B_a^*}{h(I_a^*) B_a} \right. \\
&+ \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} M\left(\frac{S_a(t-\sigma_1) g(B_a(t-\sigma_1)) I_a^*}{S_a^* g(B_a^*) I_a} \right) d\sigma_1 - M\left(\frac{S_a g(B_a) I_a^*}{S_a^* g(B_a^*) I_a} \right) \\
&\left. + \frac{1}{m_2} \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma+\alpha)\sigma_2} M\left(\frac{h(I_a(t-\sigma_2)) B_a^*}{h(I_a^*) B_a} \right) d\sigma_2 - M\left(\frac{h(I_a) B_a^*}{h(I_a^*) B_a} \right) \right) \\
&+ \eta V_a^* f(I_a^*) \left(3 + \frac{f(I_a)}{f(I_a^*)} - \frac{S_a^*}{S_a} - \frac{S_a V_a^*}{S_a^* V_a} - \frac{I_a}{I_a^*} - \frac{V_a f(I_a) I_a^*}{V_a^* f(I_a^*) I_a} - M\left(\frac{V_a f(I_a) I_a^*}{V_a^* f(I_a^*) I_a} \right) \right. \\
&\left. + \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} M\left(\frac{V_a(t-\sigma_1) f(I_a(t-\sigma_1)) I_a^*}{V_a^* f(I_a^*) I_a} \right) d\sigma_1 \right) \\
&+ \eta V_a^* g(I_a^*) \left(4 + \frac{g(B_a)}{g(B_a^*)} - \frac{h(I_a)}{h(I_a^*)} - \frac{S_a^*}{S_a} - \frac{S_a V_a^*}{S_a^* V_a} - \frac{I_a}{I_a^*} - \frac{B_a}{B_a^*} - \frac{V_a g(B_a) I_a^*}{V_a^* g(B_a^*) I_a} - \frac{h(I_a) B_a^*}{h(I_a^*) B_a} \right. \\
&+ \frac{1}{m_1} \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a+\delta_a)\sigma_1} M\left(\frac{V_a(t-\sigma_1) g(B_a(t-\sigma_1)) I_a^*}{V_a^* g(B_a^*) I_a} \right) d\sigma_1 - M\left(\frac{V_a g(B_a) I_a^*}{V_a^* g(B_a^*) I_a} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{m_2}\int_0^{\tau_2}\varphi_2(\sigma_2)e^{-(\gamma+\alpha)\sigma_2}M\left(\frac{h(I_a(t-\sigma_2))B_a^*}{h(I_a^*)B_a}\right)d\sigma_2 - M\left(\frac{h(I_a)B_a^*}{h(I_a^*)B_a}\right)\Bigg)\\
&\le S_a^* f(I_a^*)\left(\left(\frac{f(I_a)}{f(I_a^*)}-1\right)\left(1-\frac{f(I_a^*)I_a}{f(I_a)I_a^*}\right)+M\left(\frac{S_a^*}{S_a}\right)+M\left(\frac{f(I_a)I_a^*}{f(I_a^*)I_a}\right)\right.\\
&\left.+M\left(\frac{S_a f(I_a)I_a^*}{S_a^* f(I_a^*)I_a}\right)-M\left(\frac{S_a f(I_a)I_a^*}{S_a^* f(I_a^*)I_a}\right)\right)\\
&+S_a^* g(B_a^*)\left(\left(\frac{g(B_a)}{g(B_a^*)}-1\right)\left(1-\frac{g(B_a^*)I_a}{g(B_a)I_a^*}\right)+\left(\frac{h(I_a)}{h(I_a^*)}-1\right)\left(1-\frac{h(I_a^*)B_a^*}{h(I_a)B_a}\right)+M\left(\frac{S_a^*}{S_a}\right)\right.\\
&+M\left(\frac{g(B_a^*)I_a}{g(B_a)I_a^*}\right)+M\left(\frac{h(I_a^*)B_a}{h(I_a)B_a^*}\right)+M\left(\frac{S_a g(B_a)I_a^*}{S_a^* g(B_a^*)I_a}\right)+M\left(\frac{h(I_a)B_a^*}{h(I_a^*)B_a}\right)\\
&\left.-M\left(\frac{S_a g(B_a)I_a^*}{S_a^* g(B_a^*)I_a}\right)-M\left(\frac{h(I_a)B_a^*}{h(I_a^*)B_a}\right)\right)\\
&+\eta V_a^* f(I_a^*)\left(\left(\frac{f(I_a)}{f(I_a^*)}-1\right)\left(1-\frac{f(I_a^*)I_a}{f(I_a)I_a^*}\right)+M\left(\frac{S_a^*}{S_a}\right)+M\left(\frac{S_aV_a^*}{S_a^*V_a}\right)+M\left(\frac{f(I_a)I_a^*}{f(I_a^*)I_a}\right)\right.\\
&\left.+M\left(\frac{V_a f(I_a)I_a^*}{V_a^* f(I_a^*)I_a}\right)-M\left(\frac{V_a f(I_a)I_a^*}{V_a^* f(I_a^*)I_a}\right)\right)\\
&+\eta V_a^* g(I_a^*)\left(\left(\frac{g(B_a)}{g(B_a^*)}-1\right)\left(1-\frac{g(B_a^*)I_a}{g(B_a)I_a^*}\right)+\left(\frac{h(I_a)}{h(I_a^*)}-1\right)\left(1-\frac{h(I_a^*)B_a^*}{h(I_a)B_a}\right)+M\left(\frac{S_a^*}{S_a}\right)\right.\\
&+M\left(\frac{S_aV_a^*}{S_a^*V_a}\right)+M\left(\frac{g(B_a^*)I_a}{g(B_a)I_a^*}\right)+M\left(\frac{h(I_a^*)B_a}{h(I_a)B_a^*}\right)+M\left(\frac{V_a g(B_a)I_a^*}{V_a^* g(B_a^*)I_a}\right)+M\left(\frac{h(I_a)B_a^*}{h(I_a^*)B_a}\right)\\
&\left.-M\left(\frac{S_a g(B_a)I_a^*}{S_a^* g(B_a^*)I_a}\right)-M\left(\frac{h(I_a)B_a^*}{h(I_a^*)B_a}\right)\right)\\
&\le 0.
\end{aligned}
$$

The equality $\frac{dL}{dt}=0$ holds if and only if $S_a=S_a^*, V_a=V_a^*, I_a=I_a^*, B_a=B_a^*$. Since $(S_a^*,V_a^*,I_a^*,B_a^*)$ is the only invariant set of system (3.1) in $\{(S_a,V_a,I_a,B_a|\frac{dL}{dt}=0)\}$, the endemic equilibrium $(S_a^*,V_a^*,I_a^*,B_a^*)$ is globally asymptotically stable by LaSalle's Invariance Principle. □

Theorem 4.4. *The endemic equilibrium P^* of the system (2.1) is globally asymptotically stable when $R_0>1$.*

Proof. According to theorem 3.3 the endemic equilibrium of system (4.1) is global asymptotically stable if $R_0>1$. To prove the global stability of the equilibrium $(S_a^*,V_a^*,I_a^*,B_a^*,S_h^*,I_h^*,R_h^*)$ of system (2.1) with the animals components already at the endemic steady state given by

$$
\begin{cases}
\dfrac{dS_h}{dt}=A_h+(1-n)\beta I_h-\mu_h S_h-S_h\left(p(I_a^*)+q(B_a^*)\right),\\
\dfrac{dI_h}{dt}=\displaystyle\int_0^{\tau_3}\varphi_3(\sigma_3)e^{-(\mu_h+\beta)\sigma_3}S_h(t-\sigma_3)(p(I_a^*)+q(B_a^*))d\sigma_3-(\mu_h+\beta)I_h,\\
\dfrac{dR_h}{dt}=n\beta I_h-\mu_h R_h.
\end{cases}
$$

Then it follows

$$
\begin{aligned}
S_h&=\frac{A_h+(1-n)\beta I_h}{\mu_h+p(I_a^*)+q(B_a^*)}+D_4e^{-(\mu_h+p(I_a^*)+q(B_a^*))t},\\
I_h&=\frac{\int_0^{\tau_3}\varphi_3(\sigma_3)e^{-(\mu_h+\beta)\sigma_3}S_h(t-\sigma_3)(p(I_a^*)+q(B_a^*))d\sigma_3}{\mu_h+\beta}+D_5e^{-(\mu_h+\beta)t},\\
R_h&=\frac{n\beta I_h}{\mu_h}+D_6e^{-\mu_h t},
\end{aligned}
$$

where D_4, D_5, D_6 are integrating constants.

It is clear that $S_h\to S_h^*$ $I_h\to I_h^*$ and $R_h\to R_h^*$ when $t\to\infty$ and so P^* of system (2.1) is globally asymptotically stable. □

5. The optimal control

In reality, financial support is limited and cannot meet all the actual needs of brucellosis control. With a limited financial budget, it is important to achieve the best control effect. In this section, we consider optimal control measures for distributed delayed brucellosis model (2.1) and introduce four control variables. The control $v_1(t)$ represents the vaccination rate for susceptible compartments S_a. The control $v_2(t)$ represents the slaughtering rate of individuals in compartment $I_a(t)$ due to disease. The control $v_3(t)$ represents disinfection rate of environment. The control $v_4(t)$ represents educational campaign to the compartment S_h.

$$\begin{cases} \frac{dS_a}{dt} = A_a + \epsilon V_a - \mu_a S_a - \upsilon_1(t) S_a - S_a(f(I_a) + g(B_a)), \\ \frac{dV_a}{dt} = \upsilon_1(t) S_a - \epsilon V_a - \mu_a V_a - \eta V_a(f(I_a) + g(B_a)), \\ \frac{dI_a}{dt} = \int_0^{\tau_1} \varphi_1(\sigma_1) e^{-(\mu_a + \upsilon_2(\sigma_1))\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1)) \\ \qquad (f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1)))d\sigma_1 - (\mu_a + \upsilon_2(t)) I_a, \\ \frac{dB_a}{dt} = \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma + \upsilon_3(\sigma_2))\sigma_2} h(I_a(t-\sigma_2)) d\sigma_2 - (\gamma + \upsilon_3(t)) B_a, \\ \frac{dS_h}{dt} = A_h + (1-n)\beta I_h - \mu_h S_h - (1 - \upsilon_4(t)) S_h(p(I_a) + q(B_a)), \\ \frac{dI_h}{dt} = \int_0^{\tau_3} \varphi_3(\sigma_3) e^{-(\mu_h + \beta)\sigma_3} (1 - \upsilon_4(t-\sigma_3)) S_h(t-\sigma_3) \\ \qquad \cdot (p(I_a(t-\sigma_3)) + q(B_a(t-\sigma_3)))d\sigma_3 - (\mu_h + \beta) I_h, \\ \frac{dR_h}{dt} = n\beta I_h - \mu_h R_h. \end{cases} \tag{5.1}$$

We construct the following objective function of system (5.1):

$$J(\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t)) = \int_0^T L(S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t)) dt,$$

with

$$\begin{aligned} L = & B_1 \upsilon_1(t) S_a(t) + B_2 \upsilon_2(t) I_a(t) + B_3 \upsilon_3(t) B_a(t) + B_4 \upsilon_4(t) S_h(t) \\ & + \frac{1}{2}(C_1(\upsilon_1(t))^2 + C_2 \upsilon_2(t))^2 + C_3 \upsilon_3(t))^2 + +C_4 \upsilon_4(t))^2), \end{aligned}$$

where $B_i, C_i (i = 1, 2, 3, 4)$ are positive weights that balance the size of the terms.

Our aim is to find out an optimal pair $(\upsilon_1^{**}(t), \upsilon_2^{**}(t), \upsilon_3^{**}(t), \upsilon_4^{**}(t))$ such that

$$\begin{aligned} & J(\upsilon_1^{**}(t), \upsilon_2^{**}(t), \upsilon_3^{**}(t), \upsilon_4^{**}(t)) \\ & = min\{ J(\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t)) | (\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t)) \in U \}, \end{aligned}$$

where the admissible control set U is given as

$$\begin{aligned} U = & \{ (\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t)) | \upsilon_i(t) (i = 1, 2, 3, 4) is \;\; Lebesgue\ measurable, \\ & \upsilon_i(t) \in [0, 1], t \in [0, T] \}. \end{aligned}$$

5.1. Existence of optimal control

We can rewrite system (5.1) in the following form:

$$\frac{dW(t)}{dt} = AW(t) + F\left(W(t), W_{\tau_1}(t), W_{\tau_2}(t), W_{\tau_3}(t) \right) + C(\upsilon, \upsilon_{\tau_3}, W(t), W_{\tau_3}(t)),$$

where $W(t) = \left(S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t) \right)^T, W_{\tau_i} = W(t - \tau_i)$, $\upsilon = (\upsilon_1(t), \upsilon_2(t), \upsilon_3(t), \upsilon_4(t))^T, \upsilon_{\tau_3} = \upsilon(t - \tau_3)$,

$$A = \begin{pmatrix} -\mu_a & \epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -\left(\mu_a + \epsilon\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & & -\mu_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu_h & (1-n)\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\mu_h + \beta) & 0 \\ 0 & 0 & 0 & 0 & 0 & n\beta & -\mu_h \end{pmatrix},$$

$$\begin{aligned} & F\left(W(t), W_{\tau_1}(t), W_{\tau_2}(t), W_{\tau_3}(t) \right) \\ & = \begin{pmatrix} A_a - S_a(f(I_a) + g(B_a)) \\ -\eta V_a(f(I_a) + g(B_a)) \\ \int_0^{\tau_1} \varphi 1(\sigma_1) e^{-(\mu_a + \upsilon_2(\sigma_1))\sigma_1} (S_a(t-\sigma_1) + \eta V_a(t-\sigma_1))(f(I_a(t-\sigma_1)) + g(B_a(t-\sigma_1)))d\sigma_1 \\ \int_0^{\tau_2} \varphi_2(\sigma_2) e^{-(\gamma + \upsilon_3(\sigma_2))\sigma_2} h(I_a(t-\sigma_2)) d\sigma_2 \\ A_h - S_h(p(I_a) + q(B_a)) \\ \int_0^{\tau_3} \varphi_3(\sigma_3) e^{-(\mu_h + \beta)\sigma_3} S_h(t-\sigma_3)(p(I_a(t-\sigma_3)) + q(B_a(t-\sigma_3)))d\sigma_3 \\ 0 \end{pmatrix}, \end{aligned}$$

$$C(v, v_{\tau_3}, W(t), W_{\tau_3}(t)) = \begin{pmatrix} -v_1(t)S_a \\ v_1(t)S_a \\ v_2(t)I_a \\ v_3(t)B_a \\ v_4(t)S_h(p(I_a)+q(B_a)) \\ -\int_0^{\tau_3}\varphi_3(\sigma_3)e^{-(\mu_h+\beta)\sigma_3}v_4(t-\sigma_3)S_h(t-\sigma_3)(p(I_a(t-\sigma_3))+q(B_a(t-\sigma_3)))d\sigma_3 \\ 0 \end{pmatrix}.$$

System (5.1) is a nonlinear system with bounded coefficients. Let

$$G\left(W(t), W_{\tau_1}(t), W_{\tau_2}(t), W_{\tau_3}(t)\right) = AW(t) + F\left(W(t), W_{\tau_1}(t), W_{\tau_2}(t), W_{\tau_3}(t)\right).$$

The function $F\left(W(t), W_{\tau_1}(t), W_{\tau_2}(t), W_{\tau_3}(t)\right)$ satisfies

$$\begin{aligned} &\left|F\left(W_1(t), \left(W_{\tau_1}\right)_1(t), \left(W_{\tau_2}\right)_1(t), \left(W_{\tau_3}\right)_1(t)\right) - F\left(W_2(t), \left(W_{\tau_1}\right)_2(t), \left(W_{\tau_2}\right)_2(t), \left(W_{\tau_3}\right)_2(t)\right)\right| \\ &\le M_1\left|W_1(t)-W_2(t)\right| + M_2\left|\left(W_{\tau_1}\right)_1(t)-\left(W_{\tau_1}\right)_2(t)\right| \\ &+ M_3\left|\left(W_{\tau_2}\right)_1(t)-\left(W_{\tau_2}\right)_2(t)\right| + M_4\left|\left(W_{\tau_3}\right)_1(t)-\left(W_{\tau_3}\right)_2(t)\right|, \end{aligned}$$

where M_1, M_2, M_3 and M_4 are all positive constants that don't depend on the state variables $S_a(t), V_a(t), I_a(t), B_a(t), S_h(t), I_h(t), R_h(t)$ and

$$\begin{aligned} |W_1(t)-W_2(t)| = &\left|(S_a)_1(t)-(S_a)_2(t)\right| + \left|(V_a)_1(t)-(V_a)_2(t)\right| + \left|(I_a)_1(t)-(I_a)_2(t)\right| \\ &+ \left|(B_a)_1(t)-(B_a)_2(t)\right| + \left|(S_h)_1(t)-(S_h)_2(t)\right| + \left|(I_h)_1(t)-(I_h)_2(t)\right| \\ &+ \left|(R_h)_1(t)-(R_h)_2(t)\right|, \\ \left|\left(W_{\tau_i}\right)_1(t)-\left(W_{\tau_i}\right)_2(t)\right| = &\left|(S_a)_1(t-\tau_i)-(S_a)_2(t-\tau_i)\right| + \left|(V_a)_1(t-\tau_i)-(V_a)_2(t-\tau_i)\right| \\ &+ \left|(I_a)_1(t-\tau_i)-(I_a)_2(t-\tau_i)\right| + \left|(B_a)_1(t-\tau_i)-(B_a)_2(t-\tau_i)\right| \\ &+ \left|(S_h)_1(t-\tau_i)-(S_h)_2(t-\tau_i)\right| + \left|(I_h)_1(t-\tau_i)-(I_h)_2(t-\tau_i)\right| \\ &+ \left|(R_h)_1(t-\tau_i)-(R_h)_2(t-\tau_i)\right|, \end{aligned}$$

where $i = 1, 2, 3$.

Moreover, we get

$$\begin{aligned} &\left|G\left(W_1(t), (W_{\tau_1})_1(t), (W_{\tau_2})_1(t), (W_{\tau_3})_1(t)\right) - G\left(W_2(t), (W_{\tau_1})_2(t), (W_{\tau_2})_2(t), (W_{\tau_3})_2(t)\right)\right| \\ &\le L\Big(|W_1(t)-W_2(t)| + \left|(W_{\tau_1})_1(t)-(W_{\tau_1})_2(t)\right| \\ &+ \left|(W_{\tau_2})_1(t)-(W_{\tau_2})_2(t)\right| + \left|(W_{\tau_3})_1(t)-(W_{\tau_3})_2(t)\right|\Big), \end{aligned}$$

where $L = \max\left\{M_1, M_2, M_3, M_4, \|A\|\right\} < \infty$. It implies that function G is uniformly Lipschitz continuous and system (5.1) admits a solution.

Theorem 5.1. *There exists an optimal control pair $v^{**} = (v_1^{**}, v_2^{**}, v_3^{**}, v_4^{**}) \in U$ and a corresponding optimal state $(S_a^{**}, V_a^{**}, I_a^{**}, B_a^{**}, S_h^{**}, I_h^{**}, R_h^{**})$ such that*

$$\begin{aligned} &J(v_1^{**}(t), v_2^{**}(t), v_3^{**}(t), v_4^{**}(t)) \\ &= \min\{J(v_1(t), v_2(t), v_3(t), v_4(t)) | (v_1(t), v_2(t), v_3(t), v_4(t)) \in U\}, \end{aligned}$$

subject to the control system (5.1) with the initial conditions (2.2).

Proof. We will use the results of Fleming and Rishel [19, 20] to prove the existence of optimal control. Firstly, the solution set of system (5.1) is non-empty with the control variables in U. Because according to theorem 2.1 we know that the solutions of the system (5.1) are bounded for each bounded control variable $v_i \in U (i = 1, 2, 3, 4)$, and that the right-hand side functions of the system (5.1) satisfy the Lipschitz condition about state variables. Secondly, the admissible control set U is closed convex, and the system (5.1) can be rewritten as a linear function of the control variables whose coefficients depend on the state variables. Thirdly, the Hessian matrix $H(L) = \text{diag}\left\{C_1, C_2, C_3, C_4\right\}$ is positive definite with $C_i > 0 (i = 1, 2, 3, 4)$, so L is convex. Besides, there exists constants $\eta_1, \eta_2 > 0$, $\rho > 1$ such that

$$\begin{aligned} &L\left(S_a, V_a, I_a, S_h, I_h, R_h, v_1, v_2, v_3, v_4\right) \\ =&B_1 v_1(t)S_a(t) + B_2 v_2(t)I_a(t) + B_3 v_3(t)B_a(t) + B_4 v_4(t)S_h(t) \\ &+ \frac{1}{2}(C_1(v_1(t))^2 + C_2 v_2(t))^2 + C_3 v_3(t))^2 + C_4 v_4(t))^2) \\ \ge&\eta_1(|v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2)^{\frac{\rho}{2}} - \eta_2. \end{aligned}$$

This ends the proof. □

5.2. Characterization of optimal control

In this section, based on Pontryagin's maximum principle [21], the optimal control is characterized by deducing the necessary conditions for optimal control. For simplicity, let's first define a characteristic function $\chi_{[a,b]}(t)$ as

$$\chi_{[a,b]}(t)=\begin{cases}1, & \text{if } t\in[a,b],\\ 0, & \text{otherwise.}\end{cases}$$

The Hamiltonian is defined as follows:

$$\begin{aligned}
&H(\mathbf{x},\mathbf{x}_{\tau_1},\mathbf{x}_{\tau_2},\mathbf{x}_{\tau_3},\upsilon,\upsilon_{\tau_3},\lambda)\\
&=B_1\upsilon_1(t)S_a(t)+B_2\upsilon_2(t)I_a(t)+B_3\upsilon_3(t)B_a(t)+B_4\upsilon_4(t)S_h(t)\\
&\quad+\frac{1}{2}(C_1(\upsilon_1(t))^2+C_2\upsilon_2(t))^2+C_3\upsilon_3(t))^2+C_4\upsilon_4(t))^2)\\
&\quad+\lambda_1(t)\left[A_a+\epsilon V_a-\mu_a S_a-\upsilon_1(t)S_a-S_a(f(I_a)+g(B_a))\right]\\
&\quad+\lambda_2(t)\left[\upsilon_1(t)S_a-\epsilon V_a-u_a V_a-\eta V_a(f(I_a)+g(B_a))\right]\\
&\quad+\lambda_3(t)\left[\int_0^{\tau_1}\varphi_1(\sigma_1)e^{-(\mu_a+\upsilon_2(\sigma_1))\sigma_1}(S_a(t-\sigma_1)+\eta V_a(t-\sigma_1))\right.\\
&\quad\left.\cdot(f(I_a(t-\sigma_1))+g(B_a(t-\sigma_1)))d\sigma_1-(\mu_a+\upsilon_2(t))I_a\right]\\
&\quad+\lambda_4(t)\left[\int_0^{\tau_2}\varphi_2(\sigma_2)e^{-(\gamma+\upsilon_3(\sigma_2))\sigma_2}h(I_a(t-\sigma_2))d\sigma_2-(\gamma+\upsilon_3(t))B_a\right]\\
&\quad+\lambda_5(t)\left[A_h+(1-n)\beta I_h-\mu_h S_h-(1-\upsilon_4(t))S_h(p(I_a)+q(B_a))\right]\\
&\quad+\lambda_6(t)\left[\int_0^{\tau_3}\varphi_3(\sigma_3)e^{-(\mu_h+\beta)\sigma_3}(1-\upsilon_4(t-\sigma_3))S_h(t-\sigma_3)\right.\\
&\quad\left.\cdot(p(I_a(t-\sigma_3))+q(B_a(t-\sigma_3)))d\sigma_3-(\mu_h+\beta)I_h\right]\\
&\quad+\lambda_7(t)\left[n\beta I_h-\mu_h R_h\right],
\end{aligned}\tag{5.2}$$

where $\mathbf{x}(t)=\left(S_a(t),V_a(t),I_a(t),B_a(t),S_h(t),I_h(t),R_h(t)\right),\mathbf{x}_{\tau_i}(t)=\mathbf{x}(t-\tau_i)$.

Using the necessary conditions for the optimality problem [22, 23], there exists a continuous function $\lambda(t)$ on $[0,T]$ satisfies the following equations:

the state equations

$$\frac{d\mathbf{x}}{dt}=\frac{\partial H}{\partial\lambda},$$

the optimal condition

$$0=\left[\frac{\partial H}{\partial\upsilon}(t)+\chi_{[0,T-\tau_3]}(t)\left(\frac{\partial H}{\partial\upsilon_{\tau_3}}(t)\right)_{t=t+\tau_3}\right]_{\upsilon=\upsilon^{**}},$$

and the adjoint equation

$$\begin{aligned}
-\frac{d\lambda}{dt}=&\left[\frac{\partial H}{\partial\mathbf{x}}(t)+\chi_{[0,T-\tau_1]}(t)\left(\frac{\partial H}{\partial\mathbf{x}_{\tau_1}}(t)\right)_{t=t+\tau_1}+\chi_{[0,T-\tau_2]}(t)\left(\frac{\partial H}{\partial\mathbf{x}_{\tau_2}}(t)\right)_{t=t+\tau_2}\right.\\
&\left.+\chi_{[0,T-\tau_3]}(t)\left(\frac{\partial H}{\partial\mathbf{x}_{\tau_3}}(t)\right)_{t=t+\tau_3}\right]_{\mathbf{x}=\mathbf{x}^{**}}.
\end{aligned}$$

Applying this necessary conditions, we can draw the following conclusion.

Theorem 5.2. *Let $\mathbf{x}^{**}=\left(S_a^{**},V_a^{**},I_a^{**},B_a^{**},S_h^{**},I_h^{**},R_h^{**}\right)$ be an optimal state solution associated with the optimal control variable $\upsilon^{**}=\left(\upsilon_1^{**},\upsilon_2^{**},\upsilon_3^{**},\upsilon_4^{**}\right)$ for the optimal control problem. Then there exist four adjoint variables $\lambda_i(t)(i=1,2,\cdots,7)$ satisfying the adjoint equations as follows*

$$\begin{aligned}
-\dot\lambda_1(t)=&B_1\upsilon_1^{**}-(\mu_a+\upsilon_1^{**}+f(I_a^{**})+g(B_a^{**}))\lambda_1(t)+\upsilon_1^{**}\lambda_2(t)\\
&+\chi_{[0,T-\tau_1]}(t)m_1(f(I_a^{**})+g(B_a^{**}))\lambda_3(t+\tau_1),\\
-\dot\lambda_2(t)=&\epsilon\lambda_1(t)-(\mu_a+\epsilon+\eta(f(I_a^{**})+g(B_a^{*})))\lambda_2(t)\\
&+\chi_{[0,T-\tau_1]}(t)m_1\eta(f(I_a^{**})+g(B_a^{**}))\lambda_3(t+\tau_1),
\end{aligned}$$

$$
\begin{aligned}
-\dot{\lambda}_3(t) =& B_2 \upsilon_2^{**} - f_{I_a}\left(I_a^{**}\right)(S_a^{**}\lambda_1(t) + \eta V_a^{**}\lambda_2(t) + (1-\upsilon_4^{**})S_h^{**}\lambda_5(t)) \\
&+ \chi_{[0,T-\tau_1]}(t)(m_1(S_a^{**}+\eta V_a^{**})f_{I_a}(I_a^{**}) - (\mu_a+\upsilon_2^{**}))\lambda_3(t+\tau_1) \\
&+ \chi_{[0,T-\tau_2]}(t) m_2 h_{I_a}(I_a^{**})\lambda_4(t+\tau_2) \\
&+ \chi_{[0,T-\tau_3]}(t) m_3(1-\upsilon_4^{**})S_h^{**}\cdot p_{I_a}(I_a^{**})\lambda_6(t+\tau_3), \\
-\dot{\lambda}_4(t) =& B_3 \upsilon_3^{**} - g_{B_a}\left(B_a^{**}\right)(S_a^{**}\lambda_1(t) + \eta V_a^{**}\lambda_2(t) + (1-\upsilon_4^{**})S_h^{**}\lambda_5(t)) \\
&+ \chi_{[0,T-\tau_1]}(t) m_1(S_a^{**}+\eta V_a^{**}) g_{B_a}(B_a^{**})\lambda_3(t+\tau_1) - (\gamma+\upsilon_3^{**})\lambda_4(t) \\
&+ \chi_{[0,T-\tau_3]}(t) m_3(1-\upsilon_4^{**})S_h^{**}\cdot q_{B_a}(B_a^{**})\lambda_6(t+\tau_3), \\
-\dot{\lambda}_5(t) =& -(\mu_h + (1-\upsilon_4^{**})(p_{I_a}(I_a^{**}) + q_{B_a}(B_a^{**})))\lambda_5(t), \\
&+ \chi_{[0,T-\tau_3]}(t) m_3(1-\upsilon_4^{**})S_h^{**}(p(I_a^{**}) + q(B_a^{**}))\lambda_6(t+\tau_3), \\
-\dot{\lambda}_6(t) =& (1-n)\beta\lambda_5(t) - (\mu_h+\beta)\lambda_6(t) + \beta\lambda_7(t), \\
-\dot{\lambda}_7(t) =& -\mu_h\lambda_7(t)
\end{aligned}
\tag{5.3}
$$

with transversality conditions $\lambda_i(T)=0, \quad i=1,2,\cdots,7$. *Moreover, the optimal controls* $\upsilon_1^{**}, \upsilon_2^{**}, \upsilon_3^{**}$ *and* υ_4^{**} *are characterized in the following*

$$
\begin{aligned}
\upsilon_1^{**} &= \max\left\{\min\left\{\frac{(\lambda_1(t)-\lambda_2(t))S_a^{**} - B_1 S_a^{**}}{C_1}, 1\right\}, 0\right\}, \\
\upsilon_2^{**} &= \max\left\{\min\left\{\frac{\lambda_3(t) I_a^{**} - B_2 I_a^{**}}{C_2}, 1\right\}, 0\right\}, \\
\upsilon_3^{**} &= \max\left\{\min\left\{\frac{\lambda_4(t) B_a^{**} - B_3 B_a^{**}}{C_3}, 1\right\}, 0\right\}, \\
\upsilon_4^{**} &= \max\left\{\min\left\{\frac{S_h(p(I_a)+q(B_a))(\chi_{[0,T-\tau_3]}(t) m_3\lambda_6(t+\tau_3) - \lambda_5(t)) - B_4 S_h}{C_4}, 1\right\}, 0\right\}.
\end{aligned}
$$

Proof. By differentiating the Hamiltonian (5.2) with respect to $\mathbf{x}(t)=\mathbf{x}^{**}(t)$, we obtain

$$
\begin{aligned}
-\frac{d\lambda_1}{dt} &= \left[\frac{\partial H}{\partial S_a} + \chi_{[0,T-\tau_1]}(t)\frac{\partial H}{\partial S_a(t-\tau_1)}\,|_{(t+\tau_1)}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_2}{dt} &= \left[\frac{\partial H}{\partial V_a} + \chi_{[0,T-\tau_1]}(t)\frac{\partial H}{\partial V_a(t-\tau_1)}\,|_{(t+\tau_1)}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_3}{dt} &= \left[\frac{\partial H}{\partial I_a} + \chi_{[0,T-\tau_1]}(t)\frac{\partial H}{\partial I_a(t-\tau_1)}\,|_{(t+\tau_1)} + \chi_{[0,T-\tau_2]}(t)\frac{\partial H}{\partial I_a(t-\tau_2)}\,|_{(t+\tau_2)}\right. \\
&\quad \left. + \chi_{[0,T-\tau_3]}(t)\frac{\partial H}{\partial I_a(t-\tau_3)}\,|_{(t+\tau_3)}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_4}{dt} &= \left[\frac{\partial H}{\partial B_a} + \chi_{[0,T-\tau_1]}(t)\frac{\partial H}{\partial B_a(t-\tau_1)}\,|_{(t+\tau_1)} + \chi_{[0,T-\tau_3]}(t)\frac{\partial H}{\partial B_a(t-\tau_3)}\,|_{(t+\tau_3)}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_5}{dt} &= \left[\frac{\partial H}{\partial S_h} + \chi_{[0,T-\tau_3]}(t)\frac{\partial H}{\partial S_h(t-\tau_3)}\,|_{(t+\tau_3)}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_6}{dt} &= \left[\frac{\partial H}{\partial I_h}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}, \\
-\frac{d\lambda_7}{dt} &= \left[\frac{\partial H}{\partial R_h}\right]_{(\mathbf{x},\upsilon)=(\mathbf{x}^{**},\upsilon^{**})}.
\end{aligned}
\tag{5.4}
$$

The adjoint equation (5.3) is obtained by substituting the corresponding derivatives $\frac{\partial H}{\partial \mathbf{x}}, \frac{\partial H}{\partial \mathbf{x}_{\tau_1}}, \frac{\partial H}{\partial \mathbf{x}_{\tau_2}}$ and $\frac{\partial H}{\partial \mathbf{x}_{\tau_3}}$ into (5.4).

Further, by the optimal conditions, we have

$$
\begin{aligned}
\frac{\partial H}{\partial \upsilon_1} =& B_1 S_a + C_1\upsilon_1(t) - \lambda_1(t)S_a + \lambda_2(t)S_a = 0, \\
\frac{\partial H}{\partial \upsilon_2} =& B_2 I_a + C_2\upsilon_2(t) - \lambda_3(t) I_a = 0, \\
\frac{\partial H}{\partial \upsilon_3} =& B_3 B_a + C_3\upsilon_3(t) - \lambda_4(t) B_a = 0, \\
\frac{\partial H}{\partial \upsilon_4} =& B_4 S_h + C_4\upsilon_4(t) + \lambda_5(t) S_h(p(I_a)+q(B_a)) \\
& - \lambda_6(t+\tau_3)\chi_{[0,T-\tau_3]}(t) m_3 S_h(p(I_a)+q(B_a)) = 0,
\end{aligned}
$$

which indicates

Table 1. Parameters and their values (unit: year^{-1}).

Parameter	Value	Comments	Source
A_a	13146980	The recruitment rate of the sheep population	[A]
μ_a	0.22	The natural elimination rate of sheep	[A]
A_h	1437640	The recruitment rate of the human population	[B]
μ_h	0.0566	The natural elimination rate of human	[B]
δ_a	0.15	The elimination rate caused by brucellosis	[24]
θ	0.316	The sheep vaccination rate	[24]
ϵ	0.4	The sheep loss of vaccination rate	[24]
η	0.18	The invalid vaccination rate	[24]
α	3.6	The decaying rate of brucella in the environment	[24]
γ	0	The decaying rate of brucella caused by disinfection in the environment	[24]
$n\beta$	0.6	Transfer rate from acute infections to chronic infections	[24]
$(1-n)\beta$	0.4	Transfer rate from acute infections to susceptible	[24]
β_{11}	1.1×10^{-9}	The sheep-to-sheep transmission rate	Assumption
β_{12}	5×10^{-10}	The brucella-to-sheep transmission rate	Assumption
β_{21}	1×10^{-11}	The sheep-to-human transmission rate	Assumption
β_{22}	6×10^{-12}	The brucella-to-susceptible human transmission rate	Assumption
k	20	The brucella shedding rate by infected sheep	Assumption

[A] sheep life span is about $4-5$ years [24]. According to statistics published in the China Statistical Yearbook [25], the annual total number of sheep in China in recent years is about 5.9759×10^7. We estimate that the average annual elimination rate of sheep μ_a is $\frac{1}{4.5}\approx 0.22$. So the recruitment rate A_a is taken as 13146980.

[B] According to statistics published in the China Statistical Yearbook [25], the annual total number of human in China's Inner Mongolia in recent years is about 2.54×10^7 and the average annual elimination rate of human μ_a is 0.0566. So the recruitment rate A_h is taken as 1437640.

$$
\begin{aligned}
v_1 &= \frac{(\lambda_1(t)-\lambda_2(t))S_a - B_1 S_a}{C_1},\\
v_2 &= \frac{\lambda_3(t)I_a - B_2 I_a}{C_2},\\
v_3 &= \frac{\lambda_4(t)B_a - B_3 B_a}{C_3},\\
v_4 &= \frac{S_h(p(I_a)+q(B_a))(\lambda_6(t+\tau_3)\chi_{[0,T-\tau_3]}(t)m_3 - \lambda_5(t)) - B_4 S_h}{C_4}.
\end{aligned}
$$

Considering the restriction conditions in the admissible control set U, we obtain

$$
\begin{aligned}
v_1^{**} &= \max\left\{\min\left\{\frac{(\lambda_1(t)-\lambda_2(t))S_a^{**} - B_1 S_a^{**}}{C_1},1\right\},0\right\},\\
v_2^{**} &= \max\left\{\min\left\{\frac{\lambda_3(t)I_a^{**} - B_2 I_a^{**}}{C_2},1\right\},0\right\},\\
v_3^{**} &= \max\left\{\min\left\{\frac{\lambda_4(tB_a^{**} - B_3 B_a^{**}}{C_3},1\right\},0\right\},\\
v_4^{**} &= \max\left\{\min\left\{\frac{S_h(p(I_a)+q(B_a))(\chi_{[0,T-\tau_3]}(t)m_3\lambda_6(t+\tau_3) - \lambda_5(t)) - B_4 S_h}{C_4},1\right\},0\right\}.
\end{aligned}
$$

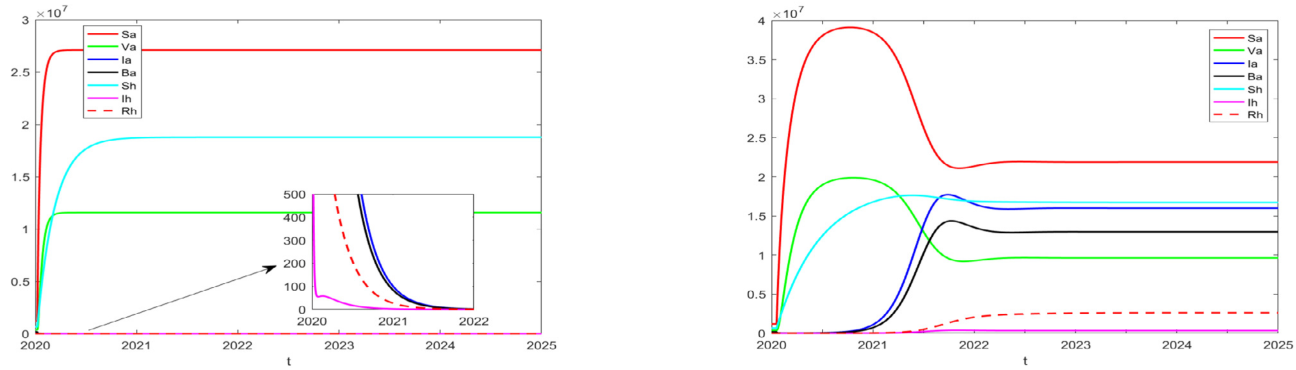
This completes our proof. □

6. Numerical simulations

In this part, several examples are given for numerical simulation. From [19], we indicate the similar kernel functions $\varphi_i(\sigma_i)=e^{-0.1\sigma_i}$ for all $\sigma_i\in[-\tau_i,0]$. It is easy to verify that assumptions (H_1)–(H_5) are satisfied if $f(I)=\beta_{11}I, g(B)=\beta_{12}B, p(I)=\beta_{21}I, q(B)=\beta_{22}B$ and $H(I)=kI$.

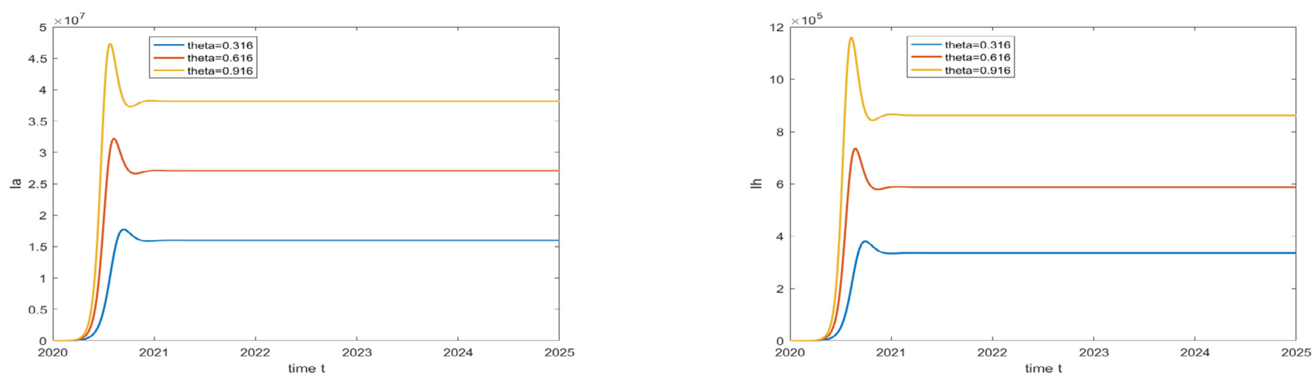
We use the parameter values shown in Table 1 and initial values: $S_a=1189310, V_a=375820, I_a=29730, B_a=164380, S_h=653150, I_h=1000, R_h=2000$. And let $\tau_1=0.04, \tau_2=0.1, \tau_3=0.02$. Fig. 1(a) shows that the disease-free equilibrium point is globally asymptotically stable when the basic reproduction number is less than 1. Similarly, Fig. 1(b) shows that positive equilibrium exists and is globally asymptotically stable when the basic reproduction number is greater than 1.

In order to better understand the control strategy of brucellosis, we simulated the influence of some key parameters or factors through sensitivity analysis. From Fig. 2(a) and Fig. 2(b), we know that the vaccination of sheep can effectively control the brucellosis in some extent but can not eradicate it from either the sheep or human population. From Fig. 3(a) and Fig. 3(b), we know that increased culling rate of infected sheep can reduce the incidence of brucellosis in humans and animals. When the culling rate reaches a certain level, brucellosis can even be eliminated. As shown in Fig. 4(a) and Fig. 4(b), increasing the frequency of disinfecting the environment can control the spread of the disease to some extent (assuming that the effective disinfection rate of the environment is 0.82. If disinfection could be placed every six days, that means disinfection about 61 times in one year, then $G\approx 50$). People and animals could be infected from exposure to brucella in the environment, so theoretically this way of transmission could be cut off by adequately disinfecting the environment. From Fig. 5, we know that human brucellosis can be controlled to a certain extent when people fully receive the education of brucellosis prevention and control and put it into practice. From Fig. 6(a) and Fig. 6(b),



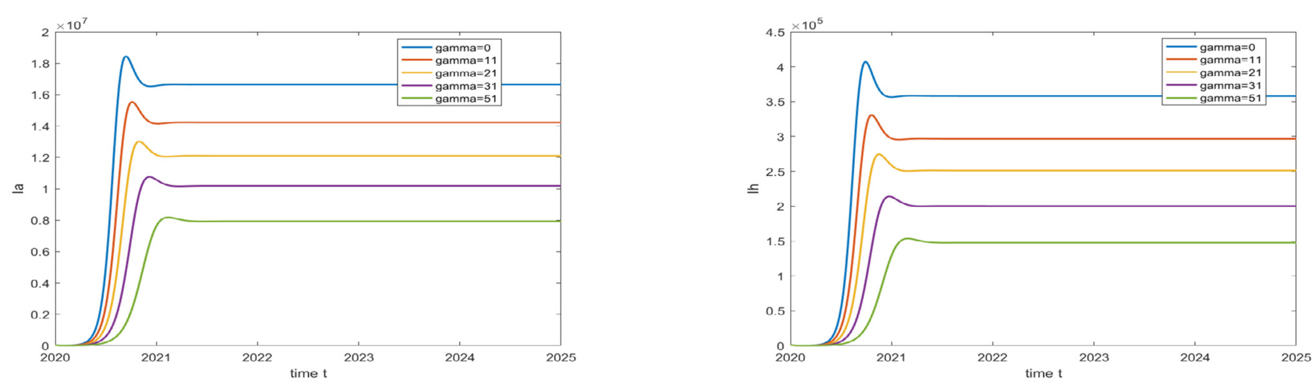
(a) (b)

Fig. 1. (a) When $\mu_a = 0.3$, the basic reproduction number $R_0 = 0.6132 < 1$, the disease goes to extinction. (b) When $\mu_a = 0.022$, the basic reproduction number $R_0 = 3.5030 > 1$, the disease persists.



(a) (b)

Fig. 2. Influence of vaccination (θ) on animal and human brucellosis.



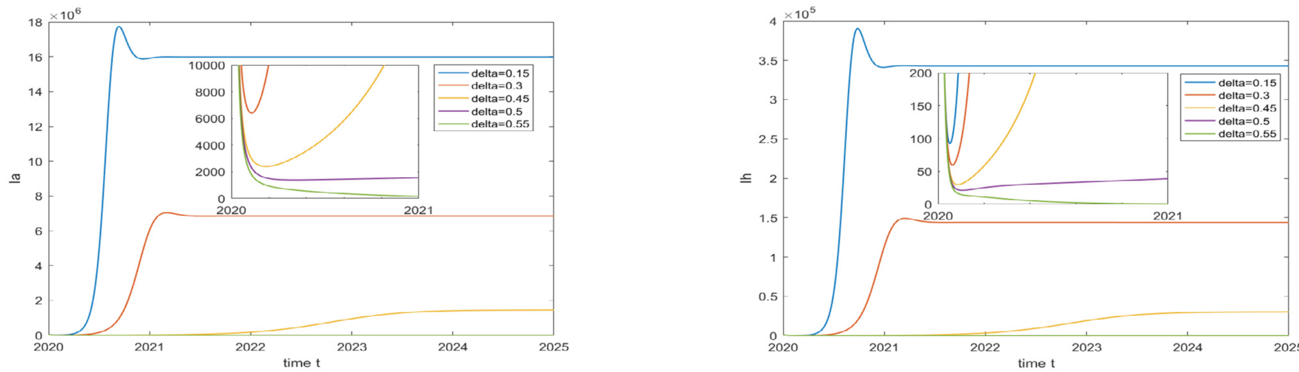
(a) (b)

Fig. 3. Influence of disinfection (γ) on animal and human brucellosis.

we can see that different control strategies have different control effects. It is clear that the best results are achieved when all available control strategies are implemented.

7. Conclusions and discussions

Based on the transmission mechanism of brucellosis, a dynamic model of human-animal brucellosis with time delay was established. The results show that the disease-free equilibrium is globally asymptotically stable when $R_0 < 1$. When $R_0 > 1$, the endemic equilibrium is also globally asymptotically stable. In other words, delay does not change the dynamic properties of the system. In order to study the influence of different control strategies, an optimal control problem described by a delay differential equation with multiple delays is considered, and the necessary conditions for the existence of optimal control are obtained. From the numerical simulation, we can see that elimination of infected animals, disinfection of



(a) (b)

Fig. 4. Influence of culling of the infected sheep (δ_a) on animal and human brucellosis.

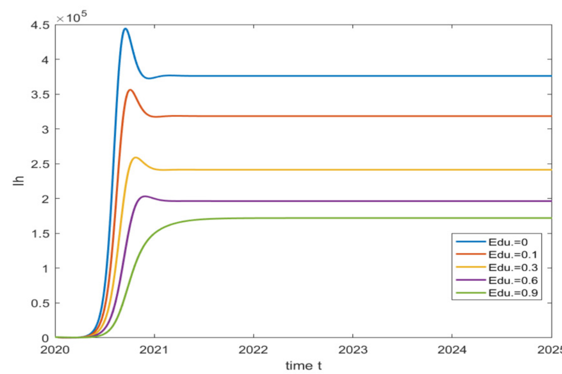
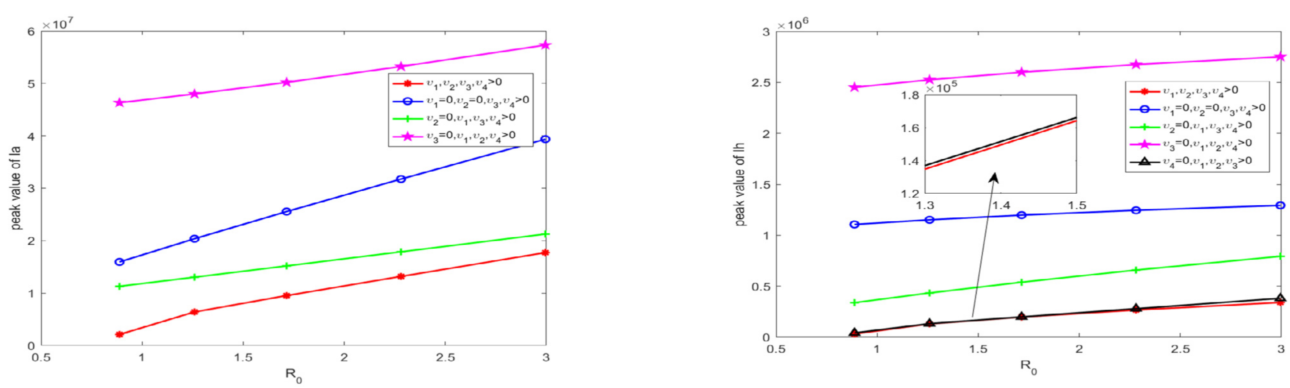


Fig. 5. Influence of education campaign on human brucellosis.



(a) (b)

Fig. 6. The relationship between the basic reproduction number and the peak values of animal and human infections under different control strategies.

the environment, vaccination and education of people are the effective prevention and control strategies. Fig. 6 shows that killing sick animals may be the most effective means.

In the numerical simulation part, we simulated the prevention and control effects of different control strategies. Unfortunately, due to the limitations of statistical data, we can't achieve the values of the corresponding optimal control variables really and effectively, which concluded to some differences between our simulation results and the actual situation. We will focus more on real world application of theoretical results in our future works.

Declarations

Author contribution statement

Man Wu: Performed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Xamxinur Abdurahman: Conceived and designed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

Zhidong Teng: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

Funding statement

Prof. Xamxinur Abdurahman was supported by National Natural Science Foundation of China [11861063].

Data availability statement

Data included in article/supp. material/referenced in article.

Declaration of interests statement

The authors declare no competing interests.

Additional information

No additional information is available for this paper.

References

[1] M.T. Li, G.Q. Sun, Y.F. Wu, et al., Transmission dynamics of a multi-group brucellosis model with mixed cross infection in public farm, Appl. Math. Comput. 237 (2014) 582–594.
[2] Francesca F. Norman, Begoña Monge-Maillo, et al., Imported brucellosis: a case series and literature review, Trav. Med. Infect. Dis. 14 (3) (2016) 182–199.
[3] E. Moreno, Retrospective and prospective perspectives on zoonotic brucellosis, Front. Microbiol. 5 (2014) 1–19.
[4] F. Roth, J. Zinsstag, D. Orkhon, et al., Human health benefits from livestock vaccination for brucellosis: case study, Bull. World Health Organ. 81 (12) (2003) 867–876.
[5] V. Punda-Polić, Z. Cvetnić, Human brucellosis in Croatia, Lancet Infect. Dis. 6 (9) (2006) 540–541.
[6] P. Bossi, A. Tegnell, A. Baka, et al., Bichat guidelines for the clinical management of brucellosis and bioterrorism-related brucellosis, Euro Surveill. 9 (12) (2004) 1–5.
[7] Bureau for Disease Control and Prevention, National Epidemic Situation of Notifiable Infectious Diseases, Chinese Center for Disease Control and Prevention, 2021, http://www.nhc.gov.cn/jkj/s2907/new_list.shtml?tdsourcetag=s_pcqq_aiomsg.
[8] K.L. Cooke, P. van den Driessche, Analysis of an SEIRS epidemic model with two delays, J. Math. Biol. 35 (1996) 240–260.
[9] T.L. Zhang, J.L. Liu, Z.D. Teng, Dynamic behavior for a nonautonomous SIRS epidemic model with distributed delays, Appl. Math. Comput. 214 (2009) 624–631.
[10] M.Y. Li, Z.H. Shuai, C.C. Wang, Global stability of multi-group epidemic models with distributed delays, J. Math. Anal. Appl. 361 (1) (2010) 38–47.
[11] H.W. Hethcote, P. van den Driessche, Two SIS epidemiologic models with delays, J. Math. Biol. 40 (1) (2000) 3–26.
[12] Q. Hou, F. Zhang, Global dynamics of a general brucellosis model with discrete delay, J. Appl. Anal. Comput. 6 (1) (2016) 227–241.
[13] Q. Hou, T. Wang, Global stability and a comparison of SVEIP and delayed SVIP epidemic models with indirect transmission, Commun. Nonlinear Sci. Numer. Simul. 43 (2017) 271–281.
[14] B. Hao, R.M. Zhang, Dynamics of animal-human epidemic model of brucellosis and analysis of its stability, Hubei Agric. Sci. 54 (24) (2015) 6324–6327.
[15] M.T. Li, G.Q. Sun, W.Y. Zhang, et al., Model-based evaluation of strategies to control brucellosis in China, Int. J. Environ. Res. Public Health 14 (3) (2017) 295–310.
[16] L.H. Zhou, M. Fan, Q. Hou, et al., Transmission dynamics and optimal control of brucellosis in Inner Mongolia of China, Math. Biosci. Eng. 2 (15) (2018) 543–567.
[17] P. van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29–48.
[18] J.P. Lasalle, The Stability of Dynamical Systems, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1976.
[19] W.H. Fleming, R.W. Rishel, Deterministic and Stochastic Optimal Control, Springer Verlag, New York, 1975.
[20] W.K. Hackbush, A numerical method for solving parabolic equations with opposite orientations, Computing 20 (1978) 229–240.
[21] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, et al., The Mathematical Theory of Optimal Process, IEEE, New York, 1962.
[22] B.M.B. Eihab, C.P. Kailash, Optimal control of an epidemiological model with multiple time delays, Appl. Math. Comput. 292 (2017) 47–56.
[23] L. Göllmann, D. Kern, H. Maurer, Optimal control problems with delays in state and control variables subject to mixed control-state constraints, Optim. Control Appl. Methods 30 (2009) 341–365.
[24] Q. Hou, X.D. Sun, J. Zhang, et al., Modeling the transmission dynamics of sheep brucellosis in Inner Mongolia Autonomous Region, China, Math. Biosci. 242 (2013) 51–58.
[25] Committee of China National Bureau of Statistics, China Statistical Yearbook, China Statistics Press, Beijing, 2021.