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Fractional-order model on vaccination and severity of COVID-19

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Abstract
Coronavirus disease 2019 (COVID-19), an infection that is highly contagious. It has a regrettable effect on the world and has resulted in more than 4.6 million deaths to date (July 2021). For this contagious disease, numerous nations implemented control measures. Every country has vaccination programs in place to achieve the best results. This research is done in two stages, including partial and complete vaccination, to enhance the efficiency and effectiveness of the vaccination. Our study found that receiving this vaccination lowers the risk of contracting a disease and its side effects, such as severity, hospitalization, need for oxygen, admission to the intensive care unit, and infection-related death. Taking into account, the system is built using fractional-order Caputo sense nonlinear differential equations. A basic reproduction number is calculated to determine the transmission rate. The bifurcation analysis predicts chaotic behavior of a system for this threshold value. The suggested system's recovery rate is optimized using fractional optimum controls. For the fractional-order differential equation, numerical results are simulated using MATLAB software using real-validated data (July 2021).

Keywords Fractional-order · Caputo derivative · Bifurcation · Optimal control · Vaccination · Severity

1 Introduction

The extremely contagious COVID-19 severe acute respiratory syndrome (SARS) disease, which was initially categorized as a novel coronavirus, marked in the starting period of the decade 2020. With 221 million cases and more than 5 million fatalities worldwide, COVID-19 is spreading in an unexpected way in almost every nation (https://www.worldometers.info/coronavirus/) [1]. The globe needs a potent COVID-19 vaccine in order to reduce or stop the disease's irregular spread. The first COVID-19 vaccine was registered for human clinical testing on March 16, 2020. By the end of 2020, there will be more than 200 COVID-19 vaccine candidates in development, 52 of which are now being tested on humans, according to the World Health Organization (WHO) [2].

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India has the second-highest number of infected cases due to the disease's highly contagious nature and populated locations. The Indian government implemented many protective measures and began one of the biggest lockdowns ever on March 25, 2020. Additionally, it restricts travel between states and enforces social segregation in all workplaces [3]. However, despite taking all necessary precautions, the nation was unable to stop the spread of the disease, and in September 2020, a sizable number of infected cases (97,570) were reported [4]. During February 2021, the transmission was under control for a few days. And it was urgently hastened following the second wave of COVID-19 infection in March 2021. In the first week of May 2021, more than 4 lakhs cases per day were reported in the nation.

India began administering mass vaccinations on January 16, 2021, using two vaccine kinds, Covishield and Covaxin, supplied by Serum Institute of India Ltd. and Bharat Biotech International Ltd. [5]. 37 percent of the population in India had received the first dose as of September 7, 2021, and 11 percent had received all three doses. In the current research, a mathematical model is created to examine how vaccination affects COVID-19 transmission in India. In which we calculated the severity of the illness in the demographic groups that received vaccinations and those who did not. We have calculated the impact of vaccination on the spread of COVID-19 by comparing the simulated results produced from actual data, and we have noticed the largest relative reduction in mortality due to COVID-19.

In order to simulate the COVID-19 outbreak, Higazy [6] created a fractional SIDARTHE model with continuous function, state variables, and controls. To stop the spread of COVID-19, Shah et al. [7] used optimal control, and Shah et al. [8, 9] used fractional optimal control. A dynamics and control on COVID-19 using fractional derivative for India was developed by Shaikh et al. [10]. Through mathematical modelling, Shah et al. [6] applied a qualitative technique to analyze the COVID-19. FODE was utilized by Ahmad et al. [11] to create an epidemiological compartmental model.

Evaluating regional and global stability an SIR model with a fractional-order derivative was created by Mouaouine et al. [12]. A predator–prey and rabies model using a fractional-order derivative was developed by Ahmed et al. [13]. A fractional SIR model was created by Shah et al. [8, 9]. For the analysis of nonlinear fractional-order models, Shah et al. [14] contrasted the Euler method (MEM) with nonstandard finite difference (NSFD). Shah et al. [12, 15] employed the Caputo derivative to simulate the outcomes using the Haar wavelet collocation technique.

The fractional-order derivative of Caputo was used in this study. A popular tool for modelling combining the conventional technique with long-term memory and long-term spatial interface is fractional order. Accurate result can be found for this epidemic model through fractional-order derivative.

Definition 1 The Caputo fractional-order derivative of a function y in the interval $[0, T]$ is defined by

$$
{}^{C}D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}y^{(n)}(s)\,\mathrm{d}s
$$

where C represents Caputo derivative, D^{α} denotes Caputo fractional derivative of order $n = [\alpha] + 1$, and $[\alpha]$ represents the integer part of α.

Definition 2 Laplace transform of Caputo derivative is defined as,

$$
L\left\{D^{\alpha}y(t)\right\} = s^{\alpha}y(s) - \sum_{k=0}^{n-1}s^{\alpha-k-1}y^{(k)}(0),\quad n-1<\alpha<n,\ n\in N.
$$

In Sect. 2, the mathematical transmission model has been prepared. Stability analysis has been worked out in Sect. 3. Then fractional optimal control theory is applied to the model with specified three controls in Sect. 4. Section 5 includes the numerical simulation with the graphical presentation for real data. Section 6 concludes the proposed model.

2 Mathematical modeling

The proposed model has eleven compartments: susceptible populations (S) who are either vaccinated (V) or non-vaccinated (N_V). The infected classes are divided into three sub-classes depending on the intensity of the infection, class of mild (M_I), moderate (M_O), and severely infected individuals (S_E). There are some cases where mild infected cases got recovered without hospitalization or by home-quarantine $\left(H_Q\right)$ while moderated and severely infected cases may need oxygen support (O_S) or may be admitted in ICU (A_{ICU}) to survive in critical cases. This scenario leads to construct the system of nonlinear differential equations using the compartmental model as given in Fig. 1.

July 2nd, 2021, the data for infected cases and the vaccinated population in the country were collected from multiple sources, including websites from the Ministry of Health and Family Welfare, the Government of India, and a website for crowd-sourced information related to COVID-19. Using these data calculated parametric values are given in Table 1 [16, 17].

Total population = 136.65 crore.

Total Vaccination (till July 2nd, 2021) = 33,13,07,026.

Vaccination of 1st dose = 27,30,08,676.

Vaccination of 2.nd dose = 5,82,98,350

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= B - \alpha_1 SV - \alpha_2 SN_V - \mu S\\
\frac{dV}{\mathrm{d}t} &= \alpha_1 SV - \beta_1 V - \beta_2 V + \alpha_3 N_V - \mu V\\
\frac{dN_V}{\mathrm{d}t} &= \alpha_2 SN_V - \gamma_1 N_V - \gamma_2 N_V - \gamma_3 N_V\\
&\quad - \alpha_3 N_V - \mu N_V\\
\frac{\mathrm{d}M_I}{\mathrm{d}t} &= \beta_1 V + \gamma_1 N_V - \varepsilon_1 M_I - \eta_1 M_I - \mu M_I\\
\frac{\mathrm{d}M_O}{\mathrm{d}t} &= \beta_2 V + \gamma_2 N_V + \varepsilon_1 M_I - \varepsilon_2 M_O + \delta_1 H_Q\\
&\quad - \eta_1 M_O - \mu M_O\\
\frac{\mathrm{d}S_E}{\mathrm{d}t} &= \gamma_3 N_V + \varepsilon_2 M_O + \delta_2 H_Q - \eta_3 S_E - \mu S_E\\
\frac{dH_Q}{\mathrm{d}t} &= \eta_1 M_I - \delta_1 H_Q - \delta_2 H_Q - \theta H_Q - \mu H_Q\\
\frac{dO_S}{\mathrm{d}t} &= \eta_2 M_O + \eta_3 S_E - \xi O_S - \mu O_S\\
\frac{dA_{\text{ICU}}}{\mathrm{d}t} &= \xi O_S - \rho_1 A_{\text{ICU}} - \rho_2 A_{\text{ICU}} - \mu A_{\text{ICU}}\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \theta H_Q + \rho_1 A_{\text{ICU}} - \mu R\\
\frac{dF}{\mathrm{d}t} &= \rho_2 A_{\text{ICU}} - \mu F
\end{aligned}
\tag{1}
$$

Fig. 1 Schematic diagram

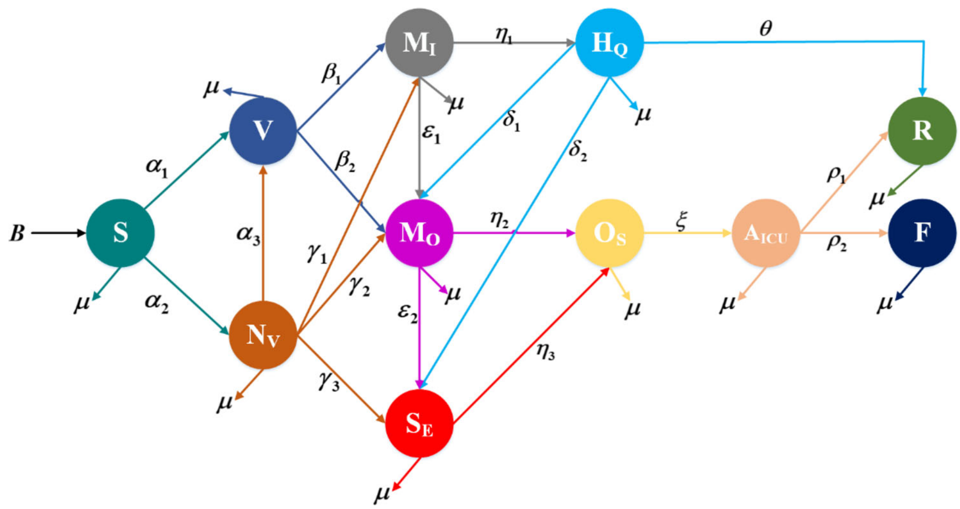

Table 1 Description of parameters

Parameter	Description	References
B	Birth rate	[6]
α_1/α_2	The rate as a result of contacting among susceptible individuals and vaccinated/non-vaccinated individuals	[18]/Assumed
α_3	The rate at which non-vaccinated individuals get a vaccine	[18]
β_1/β_2	The rate of vaccinated individuals is getting mild/moderate COVID-19 infection	Assumed
$\gamma_1/\gamma_2\gamma_2/\gamma_3$ Assumed	The rate of non-vaccinated individuals is getting mild/moderate/severe COVID-19 infection	Assumed/ [19]/
ε_1	The rate at which mildly infected individuals moves to a class of moderately infected individuals	Assumed
ε_2	The rate at which moderately infected individuals moves to a class of severely infected individuals	[19]
η_1	The rate at which mildly infected individuals goes for treatment with home-quarantine	Assumed
η_2/η_3	The rate at which moderate/severe infected individuals needs oxygen support	[20]/Assumed
ρ_1/ρ_2	The rate at which individuals admitted to ICU moves to recover/fatal class	[20]/ [18]
δ_1/δ_2	The rate of home-quarantined individuals gets moderate/severe infection	Assumed
ξ	The rate of oxygen supported individuals admitted to ICU	[20]
θ	The rate of home-quarantined individuals shifted to the recovered class	[19]
μ	Natural death rate	Assumed

Summing all equations, the feasible region of the model is obtained as,

$$\Lambda = \left\{ \begin{array}{l} (S,\ V,\ N_V,\ M_I,\ M_O,\ S_E,\ H_Q,\ O_S,\ A_{\text{ICU}},\ R,\ F) \\ \in R_+^{11} : S + V + N_V + M_I + M_O + S_E + H_Q + O_S \\ + A_{\text{ICU}} + R + F \leq \dfrac{B}{\mu} \end{array} \right\}.$$

where

$$R_+^{11} = \left\{ \begin{array}{l} (S,\ V,\ N_V,\ M_I,\ M_O,\ S_E,\ H_Q,\ O_S,\ A_{\text{ICU}},\ R,\ F) \\ \in R_+^{11} : S > 0,\ V,\ N_V,\ M_I,\ M_O,\ S_E,\ H_Q,\ O_S, \\ A_{\text{ICU}},\ R,\ F \geq 0 \end{array} \right\}.$$

(i) Disease-free equilibrium point:

$$E_0\left(\frac{B}{\mu},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right)$$

$$E_0\left(\frac{B}{\mu},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right)$$

(ii) Non-vaccinated-free equilibrium point: $E_1(S^1,\ V^1,\ 0,\ M_I^1,\ M_O^1,\ S_E^1,\ H_Q^1,\ O_S^1,\ A_{\text{ICU}}^1,\ R^1,\ F^1)$ where $S^1 = \frac{\beta_1+\beta_2+\mu}{\alpha_1}$, $V^1 = \frac{B\alpha_1-\mu(\beta_1+\beta_2+\mu)}{\alpha_1(\beta_1+\beta_2+\mu)}$, $N_V^1 = 0$, $M_I^1 = \frac{\beta_1 V^1+\gamma_1 N_V^1}{\varepsilon_1+\eta_1+\mu}$, $M_O^1 = \frac{\beta_2 V^1+\gamma_2 N_V^1+\varepsilon_1 M_I^1+\delta_1 H_Q^1}{\varepsilon_2+\eta_2+\mu}$, $S_E^1 = \frac{\beta_1 V^1+\gamma_1 N_V^1}{\varepsilon_1+\eta_1+\mu}$, $H_Q^1 = \frac{\eta_1 M_I^1}{\delta_1+\delta_2+\theta+\mu}$, $O_S^1 = \frac{\eta_2 M_O^1+\eta_3 S_E^1}{\xi+\mu}$, $A_{\text{ICU}}^1 = \frac{\xi O_S^1}{\rho_1+\rho_2+\mu}$, $R^1 = \frac{\theta H_Q^1+\rho_1 A_{\text{ICU}}^1}{\mu}$, $F^1 = \frac{\rho_2 A_{\text{ICU}}^1}{\mu}$.

(iii) Optimum issue point:

$$E^*\big(S^*, V^*, N_V^*, M_I^*, M_O^*, S_E^*, H_Q^*, O_S^*, \times A_{\mathrm{ICU}}^*, R^*, F^*\big)$$

where $S^* = \frac{\gamma_1+\gamma_2+\gamma_3+\alpha_3+\mu}{\alpha_2}$, $V^* = \frac{\alpha_3 N_V^*}{\beta_1+\beta_2+\mu-\alpha_1 S^*}$, $M_I^* = \frac{\beta_1 V^*+\gamma_1 N_V^*}{\varepsilon_1+\eta_1+\mu}$, $N_V^* = \frac{(B(\gamma_1+\gamma_2+\gamma_3+\alpha_3+\mu)-\alpha_2\mu)((\beta_1+\beta_2+\mu)-\alpha_2(\gamma_1+\gamma_2+\gamma_3+\alpha_3+\mu))}{\beta_1+\beta_2+\mu-\alpha_1 S^*}$, $M_O^* = \frac{\beta_2 V^*+\gamma_2 N_V^*+\varepsilon_1 M_I^*+\delta_1 H_Q^*}{\varepsilon_2+\eta_2+\mu}$, $S_E^* = \frac{\beta_1 V^*+\gamma_1 N_V^*}{\varepsilon_1+\eta_1+\mu}$, $H_Q^* = \frac{\eta_1 M_I^*}{\delta_1+\delta_2+\theta+\mu}$, $O_S^* = \frac{\eta_2 M_O^*+\eta_3 S_E^*}{\xi+\mu}$, $A_{\mathrm{ICU}}^* = \frac{\xi O_S^*}{\rho_1+\rho_2+\mu}$, $R^* = \frac{\theta H_Q^*+\rho_1 A_{\mathrm{ICU}}^*}{\mu}$, $F^* = \frac{\rho_2 A_{\mathrm{ICU}}^*}{\mu}$.

Applying Caputo derivative to the system (1) [21],

$$\begin{aligned}
{}^C D^\alpha S &= B-\alpha_1 SV-\alpha_2 SN_V-\mu S\\
{}^C D^\alpha V &= \alpha_1 SV-\beta_1 V-\beta_2 V\\
&\quad +\alpha_3 N_V-\mu V\\
{}^C D^\alpha N_V &= \alpha_2 SN_V-\gamma_1 N_V-\gamma_2 N_V\\
&\quad -\gamma_3 N_V-\alpha_3 N_V-\mu N_V\\
{}^C D^\alpha M_I &= \beta_1 V+\gamma_1 N_V-\varepsilon_1 M_I-\eta_1 M_I-\mu M_I\\
{}^C D^\alpha M_O &= \beta_2 V+\gamma_2 N_V+\varepsilon_1 M_I-\varepsilon_2 M_O+\delta_1 H_Q\\
&\quad -\eta_1 M_O-\mu M_O\\
{}^C D^\alpha S_E &= \gamma_3 N_V+\varepsilon_2 M_O+\delta_2 H_Q-\eta_3 S_E-\mu S_E\\
{}^C D^\alpha H_Q &= \eta_1 M_I-\delta_1 H_Q-\delta_2 H_Q-\theta H_Q-\mu H_Q\\
{}^C D^\alpha O_S &= \eta_2 M_O+\eta_3 S_E-\xi O_S-\mu O_S\\
{}^C D^\alpha A_{\mathrm{ICU}} &= \xi O_S-\rho_1 A_{\mathrm{ICU}}-\rho_2 A_{\mathrm{ICU}}-\mu A_{\mathrm{ICU}}\\
{}^C D^\alpha R &= \theta H_Q+\rho_1 A_{\mathrm{ICU}}-\mu R\\
{}^C D^\alpha F &= \rho_2 A_{\mathrm{ICU}}-\mu F
\end{aligned} \tag{2}$$

Here, C denotes Caputo derivative having order α with initial conditions $S(0)=S_0$, $V(0)=V_0$, $N_V(0)=N_{V_0}$, $M_I(0)=M_{I_0}$, $M_O(0)=M_{O_0}$, $S_E(0)=S_{E_0}$, $H_Q(0)=H_{Q_0}$, $O_S(0)=O_{S_0}$, $A_{\mathrm{ICU}}(0)=A_{\mathrm{ICU}_0}$, $R(0)=R_0$ and $F(0)=F_0$.

By taking Laplace transform, the generalized system (2) can be written as,

$$\begin{aligned}
S(i+1) &= S(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times(B-\alpha_1 SV-\alpha_2 SN_V-\mu S)\\
V(i+1) &= V(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times(\alpha_1 SV-\beta_1 V-\beta_2 V+\alpha_3 N_V-\mu V)\\
N_V(i+1) &= N_V(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times\begin{pmatrix}\alpha_2 SN_V-\gamma_1 N_V-\gamma_2 N_V-\gamma_3 N_V\\ -\alpha_3 N_V-\mu N_V\end{pmatrix}\\
M_I(i+1) &= M_I(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times(\beta_1 V+\gamma_1 N_V-\varepsilon_1 M_I-\eta_1 M_I-\mu M_I)\\
M_O(i+1) &= M_O(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times\begin{pmatrix}\beta_2 V+\gamma_2 N_V+\varepsilon_1 M_I-\varepsilon_2 M_O\\ +\delta_1 H_Q-\eta_1 M_O-\mu M_O\end{pmatrix}\\
S_E(i+1) &= S_E(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times\big(\gamma_3 N_V+\varepsilon_2 M_O+\delta_2 H_Q-\eta_3 S_E-\mu S_E\big)\\
H_Q(i+1) &= H_Q(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times\big(\eta_1 M_I-\delta_1 H_Q-\delta_2 H_Q-\theta H_Q-\mu H_Q\big)\\
O_S(i+1) &= O_S(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times(\eta_2 M_O+\eta_3 S_E-\xi O_S-\mu O_S)\\
A_{ICU}(i+1) &= A_{ICU}(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\\
&\quad\times(\xi O_S-\rho_1 A_{ICU}-\rho_2 A_{ICU}-\mu A_{ICU})\\
R(i+1) &= R(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}\big(\theta H_Q+\rho_1 A_{ICU}-\mu R\big)\\
F(i+1) &= F(i)+\frac{r^\alpha}{\Gamma(\alpha+1)}(\rho_2 A_{ICU}-\mu F)
\end{aligned} \tag{3}$$

The basic reproduction number is calculated through next generation matrix method as below.

$$F=\begin{bmatrix}\alpha_1 SV\\ \alpha_2 SN_V\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \text{ and}$$

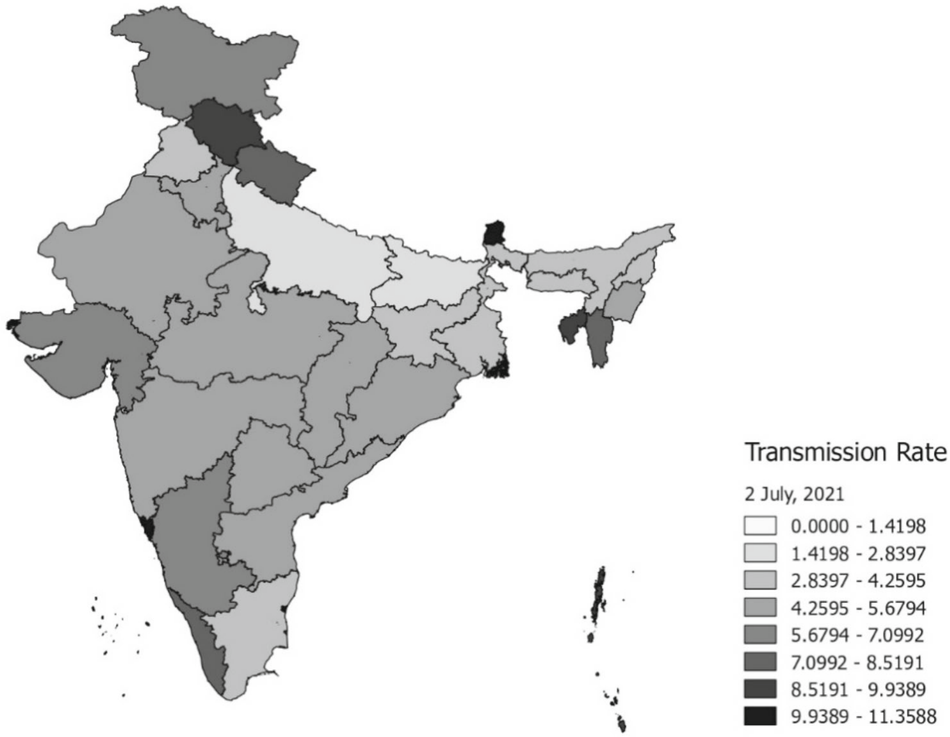

Fig. 2 Transmission rate of COVID-19 in Indian states

$$V = \begin{bmatrix} \beta_1 V + \beta_2 V - \alpha_3 N_V + \mu V \\ \gamma_1 N_V + \gamma_2 N_V + \gamma_3 N_V + \alpha_3 N_V + \mu N_V \\ -\beta_1 V - \gamma_1 N_V + \varepsilon_1 M_I + \eta_1 M_I + \mu M_I \\ -\beta_2 V - \gamma_2 N_V - \varepsilon_1 M_I + \varepsilon_2 M_O \\ -\delta_1 H_Q + \eta_1 M_O + \mu M_O \\ -\gamma_3 N_V - \varepsilon_2 M_O - \delta_2 H_Q + \eta_3 S_E + \mu S_E \\ -\eta_1 M_I + \delta_1 H_Q + \delta_2 H_Q + \theta H_Q + \mu H_Q \\ -\eta_2 M_O - \eta_3 S_E + \xi O_S + \mu O_S \\ -\xi O_S + \rho_1 A_{\mathrm{ICU}} + \rho_2 A_{\mathrm{ICU}} + \mu A_{\mathrm{ICU}} \\ -\theta H_Q - \rho_1 A_{\mathrm{ICU}} + \mu R \\ -\rho_2 A_{\mathrm{ICU}} + \mu F \\ -B + \alpha_1 SV + \alpha_2 S N_V + \mu S \end{bmatrix}$$

Then, the jacobian matrix of the above matrix F and V are computed by matrix $f = \left[\frac{\partial F_i(E_0)}{\partial X_j}\right]$ and $v = \left[\frac{\partial V_i(E_0)}{\partial X_j}\right]$ respectively, where v is a non-singular matrix. Hence, the basic reproduction number at the equilibrium point E_0 is $R_0 = \frac{\alpha_1 B}{\mu(\beta_1+\beta_2+\mu)}$ [22].

The class of vaccinated individuals can be written in terms of R_0 as,

$$V(i+1) = V(i) + \frac{r^{\alpha}(\beta_1 + \beta_2 + \mu)}{\Gamma(\alpha+1)} \times \left(\frac{S^0 V^0 \mu R_0}{B} + \alpha_3 N_V^0 - V^0\right)$$

In Fig. 2, the transmission rate of COVID-19 infection in different parts of India is plotted using QGIS software where the range of reproduction numbers for different states is given. To calculate the value of reproduction number for Indian states data of COVID-19 infection is taken from the Ministry of Health and Family Welfare (the Government of India) on July 2nd, 2021.

3 Stability

After taking the jacobian matrix of system (1), the condition for equilibrium points to be stable is, eigenvalues of the jacobian matrix should be negative.

Therefore, the necessary conditions for E_0 to be locally stable is:

(i) $\frac{B\alpha_2}{\mu} < \alpha_3 + \gamma_1 + \gamma_2 + \gamma_3 + \mu$
(ii) $\frac{B\alpha_1}{\mu} < \beta_1 + \beta_2 + \mu$

E^1 is locally stable without any condition and E^* is locally stable with the following conditions.

(i) $S^*\alpha_2 < \alpha_3 + \gamma_1 + \gamma_2 + \gamma_3 + \mu$
(ii) $S^*\alpha_1 < \beta_1 + \beta_2 + \mu$

Moreover, $|\arg(E_0)| > \frac{\pi}{2}$, $\left|\arg\left(E^1\right)\right| > \frac{\pi}{2}$ and $|\arg(E^*)| > \frac{\pi}{2}$. E_0, E^1 and E^* are locally asymptotically stable.

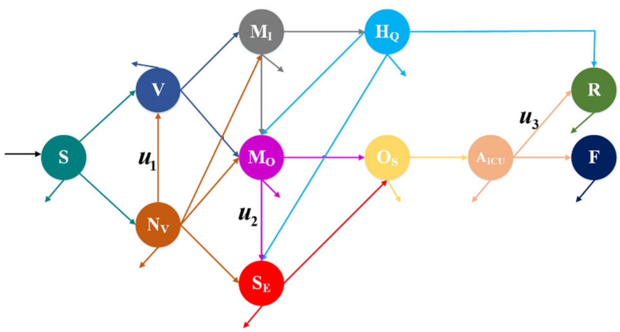

Fig. 3 Diagram with controls

4 Fractional optimal control

In this paper, three controls were applied to the system. The first control (u_1) is to support non-vaccinated people to get the vaccine, the second control (u_2) is to regulate moderately infected individuals to get severity for the infection and the third control (u_3) is to improve recovery of individuals admitted in ICU. After applying optimal control theory, the model is modified as given in Fig. 3,

An objective function can be described as,

$$\begin{aligned} J(c_i, \Lambda) = \int_0^T & (A_1 S^2 + A_2 V^2 + A_3 N_V^2 + A_4 M_I^2 + A_5 M_O^2 \\ & + A_6 S_E^2 + A_7 H_Q^2 + A_8 O_S^2 + A_9 A_{\mathrm{ICU}}^2 + A_{10} R^2 \\ & + A_{11} F^2 + w_1 u_1^2 + w_2 u_2^2 + w_2 u_3^2)\, \mathrm{d}t \end{aligned}$$

where Λ denotes set of all compartmental variables A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11} denote non-negative weight constants for compartments S, V, N_V, M_I, M_O, S_E, H_Q, O_S, A_{ICU}, R and F respectively. w_1, w_2 and w_3 are the weight constants for the controls u_1, u_2 and u_3, respectively.

$$\begin{aligned} J(u_1(t), u_2(t), u_3(t)) = min\{&J(u_1^*, \Lambda), J(u_2^*, \Lambda), \\ &J(u_3^*, \Lambda)/(u_1, u_2, u_3) \in \phi\} \end{aligned}$$

where ϕ is a smooth function on the interval [0, 1].

Derived Langrangian function is,

$$\begin{aligned} L(\Lambda, A_i) = {} & A_1 S^2 + A_2 V^2 + A_3 N_V^2 + A_4 M_I^2 + A_5 M_O^2 + A_6 S_E^2 \\ & + A_7 H_Q^2 + A_8 O_S^2 + A_9 A_{ICU}^2 + A_{10} R^2 + w_1 u_1^2 \\ & + w_2 u_2^2 + w_3 u_3^2 + \lambda_1 (B - \alpha_1 S V - \alpha_2 S N_V - \mu S) \\ & + \lambda_2 (\alpha_1 S V - \beta_1 V - \beta_2 V + \alpha_3 N_V - \mu V + u_1 N_V) \\ & + \lambda_3 (\alpha_2 S N_V - \gamma_1 N_V - \gamma_2 N_V - \gamma_3 N_V \\ & - \alpha_3 N_V - \mu N_V - u_1 N_V) \\ & + \lambda_4 (\beta_1 V + \gamma_1 N_V - \varepsilon_1 M_I - \eta_1 M_I - \mu M_I) \\ & + \lambda_5 (\beta_2 V + \gamma_2 N_V + \varepsilon_1 M_I - \varepsilon_2 M_O + \delta_1 H_Q \\ & - \eta_1 M_O - \mu M_O + u_2 S_E) \\ & + \lambda_6 (\gamma_3 N_V + \varepsilon_2 M_O + \delta_2 H_Q \\ & - \eta_3 S_E - \mu S_E - u_2 S_E) \\ & + \lambda_7 (\eta_1 M_I - \delta_1 H_Q - \delta_2 H_Q - \theta H_Q - \mu H_Q) \\ & + \lambda_8 (\eta_2 M_O + \eta_3 S_E - \xi O_S - \mu O_S) \\ & + \lambda_9 (\xi O_S - \rho_1 A_{ICU} - \rho_2 A_{ICU} \\ & - \mu A_{ICU} - u_3 A_{ICU}) \\ & + \lambda_{10} (\theta H_Q + \rho_1 A_{ICU} - \mu R + u_3 A_{ICU}) \\ & + \lambda_{11} (\rho_2 A_{ICU} - \mu F) \end{aligned}$$

To calculate the adjoint variable $\lambda_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11})$, take partial derivatives of Langrangian function for each state variable (compartment) and we have,

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial L}{\partial S} = -2A_1 S + (\lambda_1 - \lambda_2)\alpha_1 V \\ &\quad + (\lambda_1 - \lambda_3)\alpha_2 N_V + \lambda_1 \mu, \\ \dot{\lambda}_2 &= -\frac{\partial L}{\partial V} = -2A_2 V + (\lambda_1 - \lambda_2)\alpha_1 S \\ &\quad + (\lambda_2 - \lambda_4)\beta_1 + (\lambda_2 - \lambda_6)\beta_2 + \lambda_2 \mu, \\ \dot{\lambda}_3 &= -\frac{\partial L}{\partial N_V} = -2A_3 N_V + (\lambda_1 - \lambda_3)\alpha_2 S + (\lambda_3 - \lambda_2)\alpha_3 \\ &\quad + \gamma_1(\lambda_4 - \lambda_3) + \gamma_2(\lambda_3 - \lambda_5) + \gamma_3(\lambda_3 - \lambda_6) \\ &\quad + \alpha_3(\lambda_3 - \lambda_2) + \lambda_3 \mu + u_1(\lambda_3 - \lambda_2), \\ \dot{\lambda}_4 &= -\frac{\partial L}{\partial M_I} = -2A_4 M_I + (\lambda_4 - \lambda_5)\varepsilon_1 + (\lambda_4 - \lambda_7)\eta_1 + \lambda_4 \mu, \\ \dot{\lambda}_5 &= -\frac{\partial L}{\partial M_O} = -2A_5 M_O + (\lambda_5 - \lambda_6)\varepsilon_2 + (\lambda_5 - \lambda_8)\eta_2 + \lambda_5 \mu \\ \dot{\lambda}_6 &= -\frac{\partial L}{\partial S_E} = -2A_6 S_E + (\lambda_6 - \lambda_5)u_2 + (\lambda_6 - \lambda_8)\eta_3 + \lambda_6 \mu \\ \dot{\lambda}_7 &= -\frac{\partial L}{\partial H_Q} = -2A_7 H_Q + (\lambda_7 - \lambda_5)\delta_1 \\ &\quad + (\lambda_7 - \lambda_6)\delta_2 + \theta(\lambda_7 - \lambda_{10}) + \lambda_7 \mu \\ \dot{\lambda}_8 &= -\frac{\partial L}{\partial O_S} = -2A_8 O_S + (\lambda_8 - \lambda_9)\xi + \lambda_8 \mu \\ \dot{\lambda}_9 &= -\frac{\partial L}{\partial A_{ICU}} = -2A_9 A_{ICU} + (\lambda_9 - \lambda_{10})\rho_1 \\ &\quad + (\lambda_{10} - \lambda_9)u_3 + (\lambda_{11} - \lambda_9)\rho_2 + \lambda_9 \mu \\ \dot{\lambda}_{10} &= -\frac{\partial L}{\partial R} = -2A_{10} R + \lambda_{10} \mu \\ \dot{\lambda}_{11} &= -\frac{\partial L}{\partial F} = -2A_{11} F + \lambda_{11} \mu \end{aligned}$$

The necessary conditions for optimizing Lagrangian function L by taking partial derivatives $-\frac{\partial L}{\partial u_1}$, $-\frac{\partial L}{\partial u_2}$ and $-\frac{\partial L}{\partial u_3}$. Using Pontryagin's [23] principle, the optimized controls are calculated as, $u_1^* = max\left(a_1, min\left(b_1, \frac{N_V(\lambda_3 - \lambda_2)}{2w_1}\right)\right)$, $u_2^* = max\left(a_2, min\left(b_2, \frac{S_E(\lambda_6 - \lambda_5)}{2w_2}\right)\right)$ and $u_3^* = max\left(a_3, min\left(b_3, \frac{A_{\mathrm{ICU}}(\lambda_9 - \lambda_{10})}{2w_3}\right)\right)$.

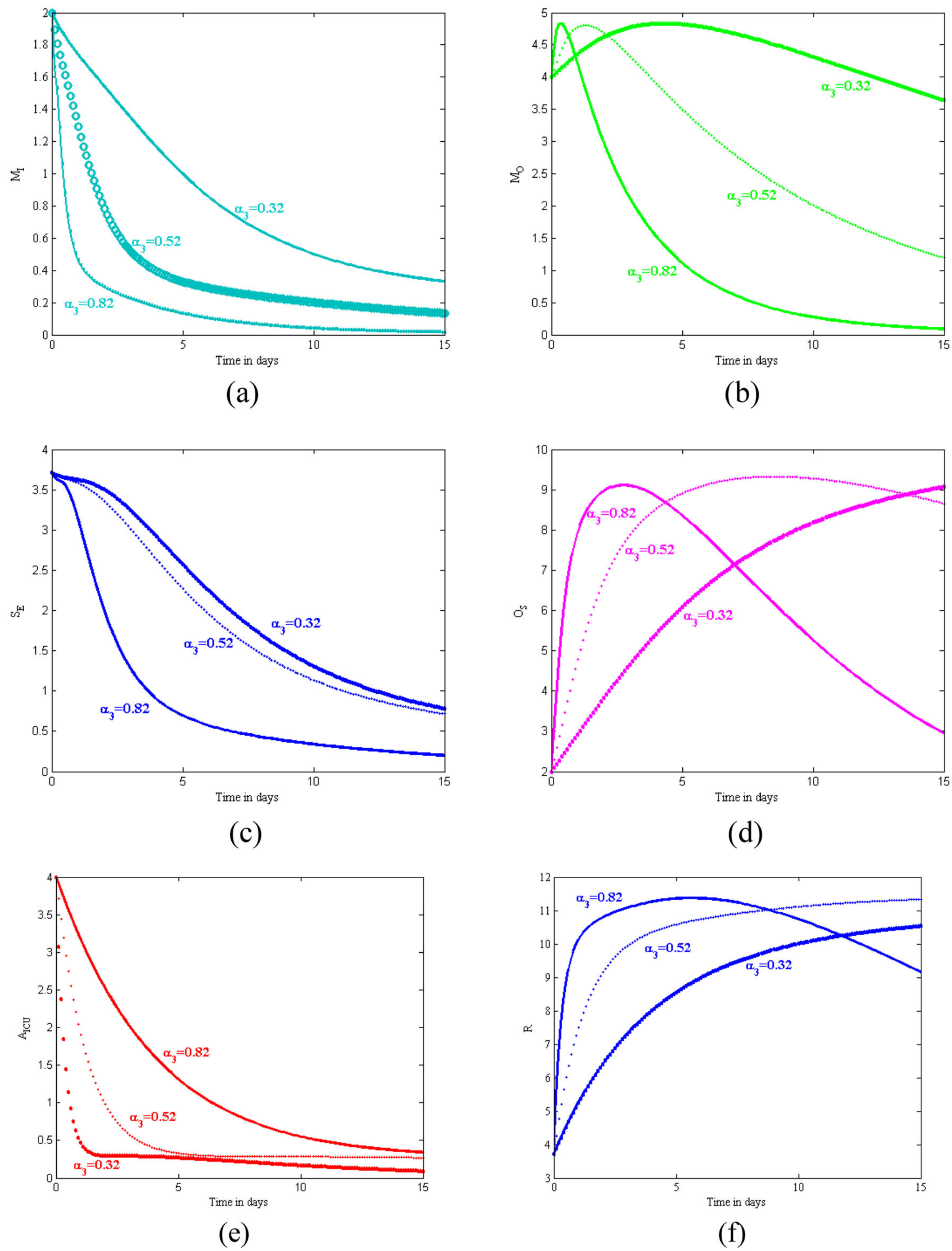

Fig. 4 Change in compartments with change in vaccination rate

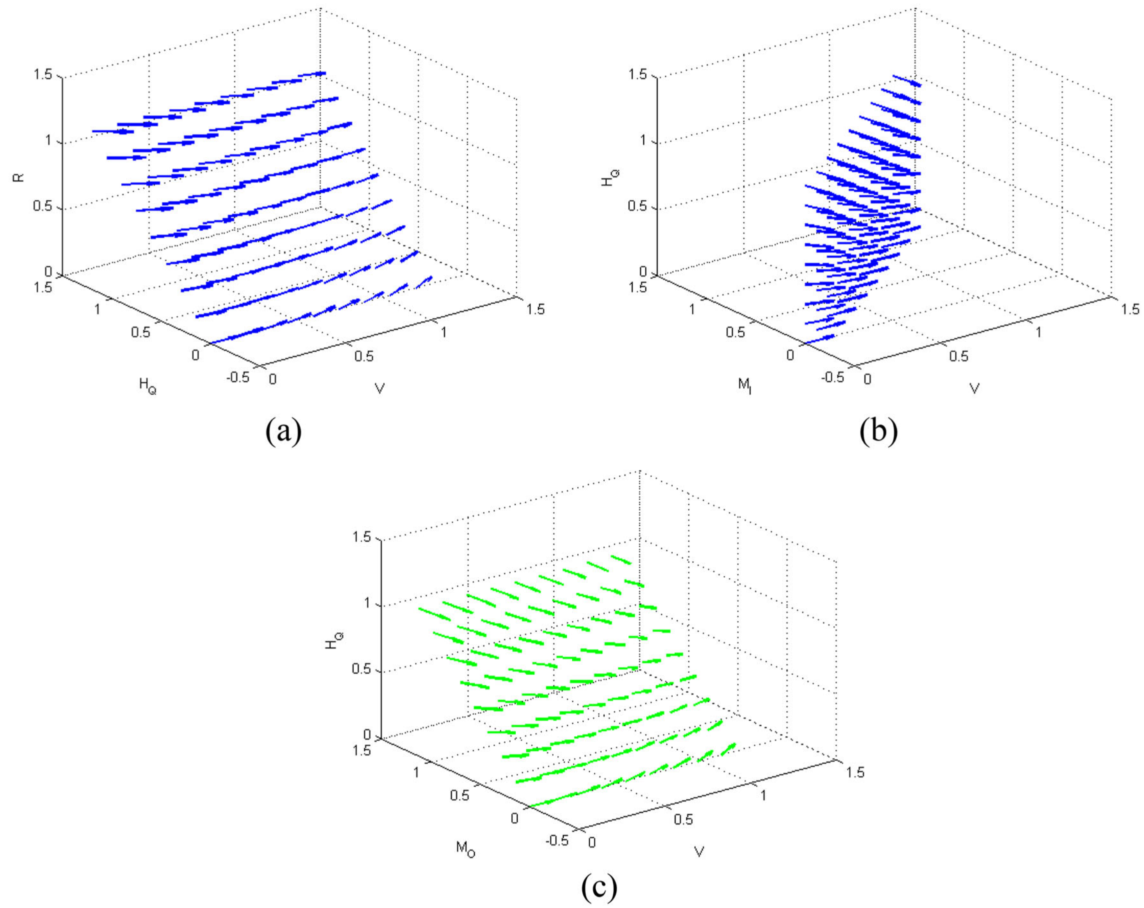

Fig. 5 Directed graphs

5 Numerical simulation

In order to compare the model's output with actual data from reports released by the Ministry of Health and Family Welfare (the Government of India) and worldometer, numerical simulations are carried out.

The vaccination rate plays a vital role to control the transmission of COVID-19 infection. The effect of vaccination on the model is illustrated in Fig. 4 where we have considered variation in the model for three different values of vaccination rate ($\alpha_3 = 0.32, 0.52, 0.82$) with constant parameters $B = 0.017$, $\beta_1 = 0.05$, $\beta_2 = 0.01$, $\alpha_1 = 0.2456$, $\alpha_2 = 0.7819$, $\alpha_3 = 0.32$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\gamma_3 = 0.6$, $\varepsilon_1 = 0.3, \varepsilon_2 = 0.1, \eta_1 = 0.5, \eta_2 = 0.03, \eta_3 = 0.7, \delta_1 = 0.2$, $\delta_2 = 0.1$, $\theta = 0.9697$, $\xi = 0.02$, $\rho_1 = 0.63$, $\rho_2 = 0.01$ and $\mu = 0.018$. From Fig. 4a, b, c, a notable change is observed in mild (M_I), moderate (M_O), and severe (S_E) infected classes when values of α_3 change from 32 to 82%. Also, as Fig. 4d positive impact is observed in recovered class as we increase the rate of vaccination.

Figure 5 directed graphs demonstrate the need for vaccination throughout the COVID-19 outbreak. Graphs show that classes that have recovered, been placed in their homes under quarantine, or are only mildly to moderately infected need vaccinations to provide them with the immunity against the infection. Investigations into the illness are still insufficient to determine the precise period of time that someone is protected after recovering from COVID-19. However, people who are receiving COVID-19 treatment should wait 90 days before becoming vaccinated.

Simultaneously varying the controls, the behavior of different compartments is observed in Fig. 6. Figure 6a results that, control u_1 is more effective while Fig. 6b, c, d,e suggests that, control u_3 is more effective. It concludes that, when u_1 control is applied i.e., if people had been vaccinated, then oxygen support needed individuals is decreasing. And when control u_3 (treatment on people admitted in ICU) is applied then the number of ICU individuals and fatality is more decreasing in nature and recovered people increasing compared to the other two controls.

In this Fig. 7, decreasing the value of an order (α) of differential equation by 20% then mild cases (Fig. 7a) decreases by 16.09%, severe cases (Fig. 7b) are decreases by 22.62%, individuals admitted in ICU (Fig. 7c) decreases by 44.46%,

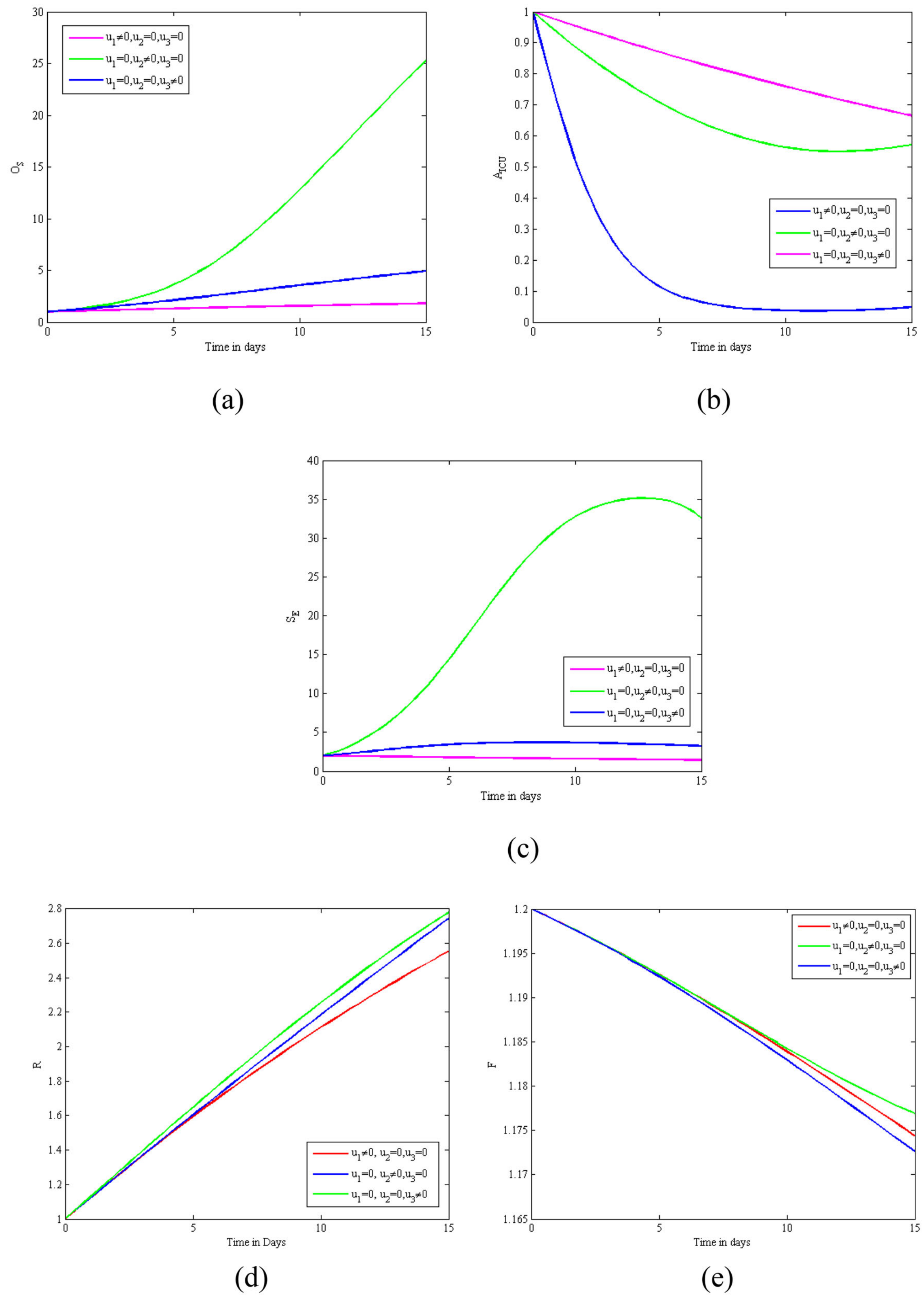

Fig. 6 Variation in compartments under the impact of optimal controls

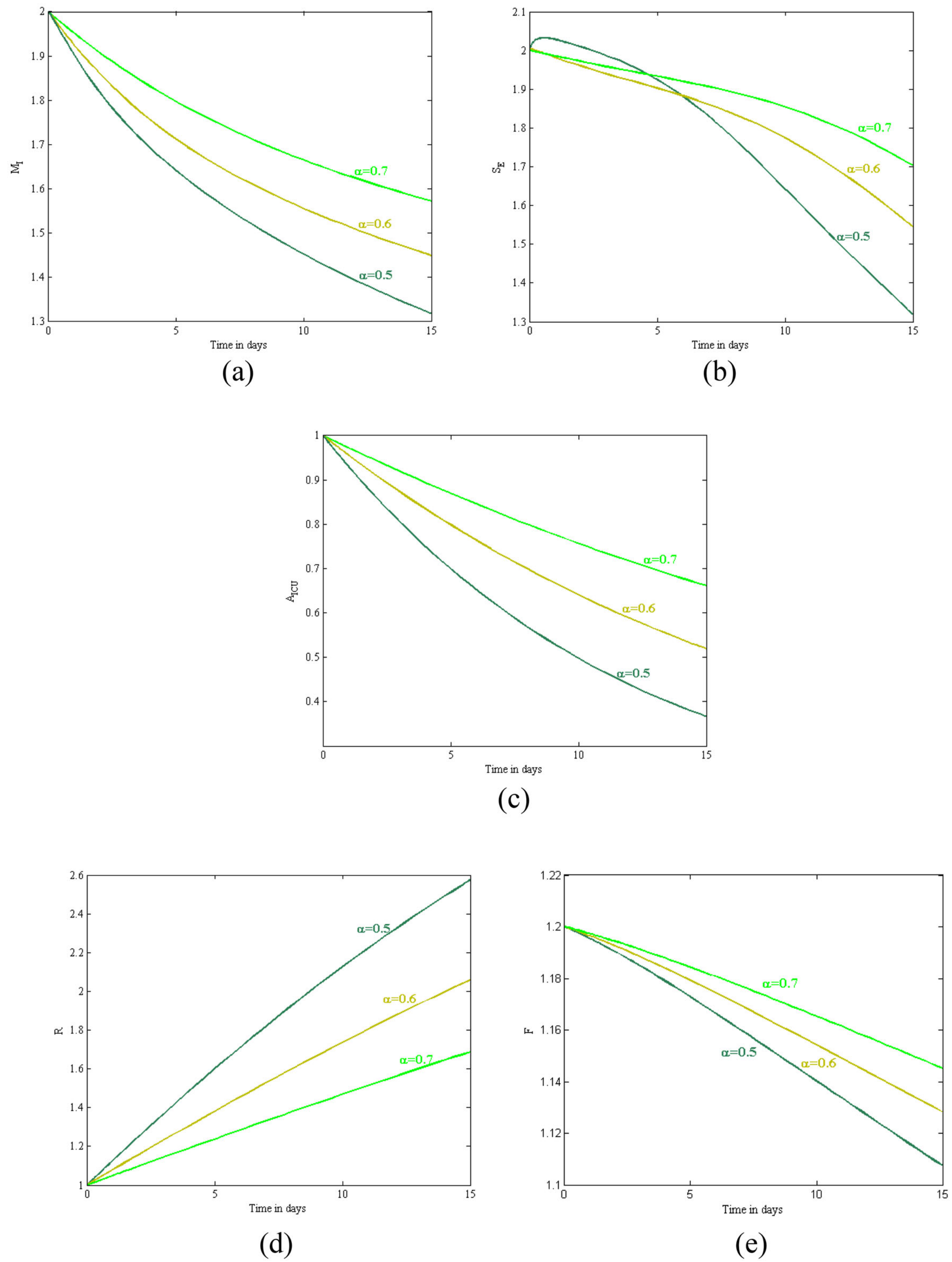

Fig. 7 Variation in compartments after varying order under the impact of controls

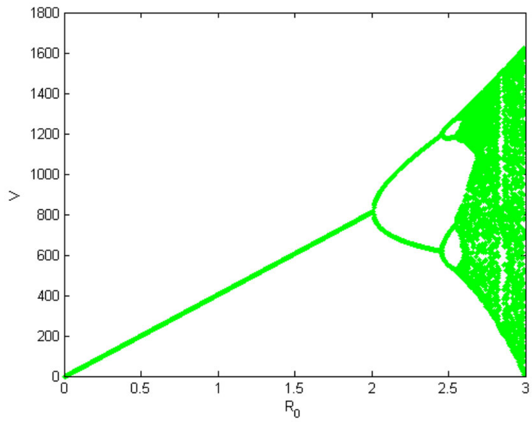

Fig. 8 Period-doubling to chaos

fatality (Fig. 7e) also decreases by 3.32% and recovery of the individual (Fig. 7d) increases by 34.45%.

The system parameters are fixed as mentioned and then the bifurcation diagram is depicted in Fig. 8. The period-doubling situation is observed when $\alpha = 0.85$, $B = 1$ and $\mu = 1$. It connotes that the bifurcation construction of system varying qualitatively that is two periodic then four periodic and so on, with the change in the value of the order α. The route leads to chaos. The area of the chaotic motion increases as the value of R_0 increases.

6 Conclusion

This research has been carried out with eleven epidemic models to resolve the problem of COVID-19 transmission. A nonlinear fractional-ordered mathematical model has been constructed using Caputo derivative operator. The simulation results obtained from the model are valid for $0 < \alpha \leq 1$. The stability with the asymptotic behavior of equilibrium points has been obtained with necessary conditions. The model utilized three controls u_1, u_2 and u_3 in the model to construct strategies to control the transmission of COVID-19. Out of three controls, the most effective control is u_2 which suggests taking extra care of moderately infected individuals and stop them to move into the class of severely infected individuals. Simulation leads to the fact that the severity due to COVID-19 will decrease as the vaccination rate increases. Additionally, the bifurcation analysis of the basic reproduction numbers, which indicate the periodic nature of the infection, is performed on the vaccinated class. It implies that even among those who have had vaccinations, the muted virus may still influence them and cause some minor spread.

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Declarations

Conflict of interest Authors do not have any conflicts of interest.

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