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Some anomalous exact solutions for the four-component coupled nonlinear Schrödinger equations on complex wave backgrounds

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The coupled nonlinear Schrödinger (CNLS) equations are the important models for the study of the multicomponent Bose-Einstein condensates (BECs). In this paper, we study the four-components CNLS equations via Darboux transformation and obtain the N -soliton solutions with zero seed and non-zero seed solutions ($q_i = 0$ or $q_i = e^{-2it}$). The 1-soliton solution and 2-soliton solution are calculated on complex wave backgrounds, the dark-bright-bright-bright soliton solutions and dark-dark-bright-bright soliton solutions are constructed. We can obtain a new class of dark-bright-bright-bright soliton solutions, which admit one-valley dark soliton in component q_1 and triple-hump bright solitons in the other three components. The collision properties between dark-dark-bright-bright solitons are considered, and the vector solitons are expected to be much more abundant than those of previously reported vector soliton collisions.

As early as 1920, the scientists Bose and Einstein believed that all quantum states of atoms would be clustered in a single quantum state based on Bose's statistical mechanics of photons, which attracted the attention of physicists. In the following decades, the abundant scientists began to explore Bose-Einstein condensation (BEC) through some experiments. For example, London suggested that the tidal phenomenon of liquid nitrogen could be the BEC of helium atoms at low temperatures¹. Bogolyubov described the theoretical model of weakly interacting Bose gases². Hulin suggested that BEC experiments could be performed with exciton in cuprous oxide³. It was not until 1995 that the first BEC was obtained in experiments of the gaseous rubidium atoms by Keightley, Cornell and Wiman⁴. This was an important breakthrough in Bose-Einstein condensation experiments. Initially, these experiments were performed with single atom, meaning that most of the Bose gases occupied the same quantum state, called single-component BEC. With the development of science, the quantum states of BEC experiments were no longer limited to one quantum state, so multicomponent BECs were further studied. The simplest multicomponent BECs are two-component BECs. In this paper, we focus on the model of the four-component BECs and their exact solutions.

The nonlinear partial differential equation is an important tool in modern mathematics, such as KdV equation⁵:

$$q_t + 6qq_x + q_{xxx} = 0,$$

the Gross-Pitaevskii (GP) equation⁶⁻⁸:

$$i\hbar u_t = -\frac{\hbar^2}{2m} \nabla^2 u + \bar{V}_d(x)u + \frac{4\pi\hbar^2 a_s}{m} |u|^2 u, x \in R^d, t \geq 0,$$

and the nonlinear Schrödinger (NLS) equation⁹:

$$iq_t + q_{xx} \pm 2q^2 q^* = 0.$$

Among them, the NLS equation is widely used in the study of nonlinear optics, Bose-Einstein condensation and other physics. For the multicomponent BECs, most of them are studied by the CNLS equations. Qin and other scholars gave the four-component BECs with repulsive interactions to establish the four-component CNLS equations¹⁰.

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When the CNLS equations are given, it is particularly important work to solve their soliton solutions. The basic idea of Darboux transformation: It uses a seed solution of the nonlinear evolution equation and the solution of its Lax pairs, then obtains the new solution of the nonlinear evolution equation and the corresponding solution of its Lax pairs via the algebraic algorithm and differential operation^{11,12}. Li and other researchers use Darboux transformation to solve the two-dimensional NLS equation with PT symmetry potential¹³. In addition, the NLS equation can be solved by Hirota bilinear method¹⁴, inverse scattering method¹⁵, Bäcklund transform method¹⁶, and so on. Subsequently, some researchers used Darboux transformation to find the exact solution of the CNLS equations. Priya et.al use the generalized Darboux transformation to obtain the N-order rogue wave solutions of the CNLS equations with cross-phase modulation and four-wave mixing terms¹⁷⁻¹⁹. Some soliton methods are proposed to solve the NLS equation and CNLS equations with a (space, time)-modulated external potential in Ref.²⁰⁻²³. The physical exact nonlinear solutions of the (n+1)-dimensional Schrödinger are obtained using RB sub-ODE and He's semi-inverse techniques, and several types of solutions, bright optical and dark solitons are derived in²⁴. Some extract new solutions of space-time stochastic fractional nonlinear Schrödinger equation with spatiotemporal dispersion are presented, and some new stochastic solutions with physical parameters are constructed via exponential distribution in²⁵.

In this paper, we mainly use Darboux transform method to investigate the four-component CNLS equations, which is a tedious process compared with the two-component and three-component CNLS equations. By using the Lax pair already given in the literature, we construct the matrix Lax pairs in the spectral problem, then the dark-bright-bright-bright soliton solutions and dark-dark-bright-bright soliton solutions of the four-component CNLS equations are calculated via the zero seed solution and the non-zero seed solution, the dynamical behaviors of the soliton solutions are shown on complex wave backgrounds.

Results

Darboux transformation for four-component CNLS equations. Since most of the CNLS equations are non-productible and contain imaginary numbers, the CNLS equations with some components are also more computationally intensive, so some traditional methods are not effective in solving this problem. In the following, with the help of the Lax pairs in the literature [9], the exact solutions of the four-component CNLS equations are calculated by constructing Darboux transformation, then the interactions between the solitary solutions are presented on complex wave backgrounds.

We consider the following four-component CNLS equations⁹:

$$\begin{cases} iq_{1,t} + \frac{1}{2}q_{1,xx} - (|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_1 = 0, \\ iq_{2,t} + \frac{1}{2}q_{2,xx} - (|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_2 = 0, \\ iq_{3,t} + \frac{1}{2}q_{3,xx} - (|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_3 = 0, \\ iq_{4,t} + \frac{1}{2}q_{4,xx} - (|q_1|^2 + |q_2|^2 + |q_3|^2 + |q_4|^2)q_4 = 0, \end{cases} \tag{1}$$

the multicomponent BECs provide a good platform for the investigation of vector solitons both theoretically and experimentally due to the abundance of intra- and interatomic interactions. And many more dark vector solitons have been experimentally observed in multicomponent repulsive BECs, the experimental observations of the collisions of bright-dark-bright solitons are realized in three-component BECs with repulsive interactions²⁶.

The Lax pairs of Eq. (1) take the following forms:

$$\psi_x = U(\lambda; Q)\psi = (i\lambda\sigma_3 + iQ)\psi = \begin{pmatrix} i\lambda & -iq_1^* & -iq_2^* & -iq_3^* & -iq_4^* \\ iq_1 & -i\lambda & 0 & 0 & 0 \\ iq_2 & 0 & -i\lambda & 0 & 0 \\ iq_3 & 0 & 0 & -i\lambda & 0 \\ iq_4 & 0 & 0 & 0 & -i\lambda \end{pmatrix} \psi, \tag{2}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 0 & -q_1^* & -q_2^* & -q_3^* & -q_4^* \\ q_1 & 0 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 & 0 \\ q_3 & 0 & 0 & 0 & 0 \\ q_4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\varphi_t = V(\lambda; Q)\psi = [i\lambda^2\sigma_3 + i\lambda Q - \frac{1}{2}(i\sigma_3 Q^2 - \sigma_3 Q_x)]\psi = \begin{pmatrix} i\lambda^2 + \frac{1}{2}i(q_1^*q_1 + q_2^*q_2 + q_3^*q_3 + q_4^*q_4) & -i\lambda q_1^* - \frac{1}{2}q_{1,x}^* & -i\lambda q_2^* - \frac{1}{2}q_{2,x}^* & -i\lambda q_3^* - \frac{1}{2}q_{3,x}^* & -i\lambda q_4^* - \frac{1}{2}q_{4,x}^* \\ i\lambda q_1 - \frac{1}{2}q_{1,x} & -i\lambda^2 - \frac{1}{2}iq_1q_1^* & -\frac{1}{2}iq_1q_2^* & -\frac{1}{2}iq_1q_3^* & -\frac{1}{2}iq_1q_4^* \\ i\lambda q_2 - \frac{1}{2}q_{2,x} & -\frac{1}{2}iq_2q_1^* & -i\lambda^2 - \frac{1}{2}iq_2q_2^* & -\frac{1}{2}iq_2q_3^* & -\frac{1}{2}iq_2q_4^* \\ i\lambda q_3 - \frac{1}{2}q_{3,x} & -\frac{1}{2}iq_3q_1^* & -\frac{1}{2}iq_3q_2^* & -i\lambda^2 - \frac{1}{2}iq_3q_3^* & -\frac{1}{2}iq_3q_4^* \\ i\lambda q_4 - \frac{1}{2}q_{4,x} & -\frac{1}{2}iq_4q_1^* & -\frac{1}{2}iq_4q_2^* & -\frac{1}{2}iq_4q_3^* & -i\lambda^2 - \frac{1}{2}iq_4q_4^* \end{pmatrix} \psi. \tag{3}$$

Here * is the complex conjugate, $q_i (i = 1, 2, 3, 4)$ is a function of the variable space x and time variable t , the λ is the universal parameter.

Constructing Darboux transformation of Eq. (1) by introducing the transformation matrix T :

$$\tilde{\psi} = T\psi, T = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} \end{pmatrix}. \tag{4}$$

Using the compatibility, we can yield:

$$\psi_x = \tilde{U}\psi, \tilde{U} = \left(T_x + i\lambda T\sigma_3 + iT\tilde{Q} \right) T^{-1}, \tag{5}$$

and

$$\psi_t = \tilde{V}\psi, \tilde{V} = \left(T_t + i\lambda^2 T\sigma_3 + i\lambda T\tilde{Q} - \frac{1}{2}iT\sigma_3\tilde{Q}^2 + \frac{1}{2}T\sigma_3\tilde{Q}_x \right) T^{-1}. \tag{6}$$

If two Lax pairs \tilde{U}, \tilde{V} and U, V have the same types in systems (5) and (6), the $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)^T$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)^T$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)^T$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)^T$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)^T$ are the basic solutions of Eqs. (2) and (3), then we give the following linear algebraic systems:

$$\begin{cases} \sum_{i=0}^{N-1} \left(A_{11}^{(i)} + A_{12}^{(i)}M_j^{(1)} + A_{13}^{(i)}M_j^{(2)} + A_{14}^{(i)}M_j^{(3)} + A_{15}^{(i)}M_j^{(4)} \right) \lambda_j^i = -\lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{21}^{(i)} + A_{22}^{(i)}M_j^{(1)} + A_{23}^{(i)}M_j^{(2)} + A_{24}^{(i)}M_j^{(3)} + A_{25}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(1)}\lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{31}^{(i)} + A_{32}^{(i)}M_j^{(1)} + A_{33}^{(i)}M_j^{(2)} + A_{34}^{(i)}M_j^{(3)} + A_{35}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(2)}\lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{41}^{(i)} + A_{42}^{(i)}M_j^{(1)} + A_{43}^{(i)}M_j^{(2)} + A_{44}^{(i)}M_j^{(3)} + A_{45}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(3)}\lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{51}^{(i)} + A_{52}^{(i)}M_j^{(1)} + A_{53}^{(i)}M_j^{(2)} + A_{54}^{(i)}M_j^{(3)} + A_{55}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(4)}\lambda_j^N. \end{cases} \tag{7}$$

with

$$\begin{aligned} M_j^{(1)} &= \frac{\varphi_2 + v_j^{(11)}\phi_2 + v_j^{(12)}\eta_2 + v_j^{(13)}\xi_2 + v_j^{(14)}\varepsilon_2}{\varphi_1 + v_j^{(11)}\phi_1 + v_j^{(12)}\eta_1 + v_j^{(13)}\xi_1 + v_j^{(14)}\varepsilon_1}, \\ M_j^{(2)} &= \frac{\varphi_3 + v_j^{(21)}\phi_3 + v_j^{(22)}\eta_3 + v_j^{(23)}\xi_3 + v_j^{(24)}\varepsilon_3}{\varphi_1 + v_j^{(21)}\phi_1 + v_j^{(22)}\eta_1 + v_j^{(23)}\xi_1 + v_j^{(24)}\varepsilon_1}, \\ M_j^{(3)} &= \frac{\varphi_4 + v_j^{(31)}\phi_4 + v_j^{(32)}\eta_4 + v_j^{(33)}\xi_4 + v_j^{(34)}\varepsilon_4}{\varphi_1 + v_j^{(31)}\phi_1 + v_j^{(32)}\eta_1 + v_j^{(33)}\xi_1 + v_j^{(34)}\varepsilon_1}, \\ M_j^{(4)} &= \frac{\varphi_5 + v_j^{(41)}\phi_5 + v_j^{(42)}\eta_5 + v_j^{(43)}\xi_5 + v_j^{(44)}\varepsilon_5}{\varphi_1 + v_j^{(41)}\phi_1 + v_j^{(42)}\eta_1 + v_j^{(43)}\xi_1 + v_j^{(44)}\varepsilon_1}, \quad 0 \leq j \leq 5N, \end{aligned} \tag{8}$$

here λ_j and $v_j^{(k)}$ are given the right parameters, so that the Eqs. (7) and (8) are not zero. We consider a 5×5 matrix T , and T is of the form as following:

$$\begin{cases} T_{11} = \lambda^N + \sum_{i=0}^{N-1} A_{11}^{(i)}\lambda^i, T_{12} = \sum_{i=0}^{N-1} A_{12}^{(i)}\lambda^i, T_{13} = \sum_{i=0}^{N-1} A_{13}^{(i)}\lambda^i, T_{14} = \sum_{i=0}^{N-1} A_{14}^{(i)}\lambda^i, T_{15} = \sum_{i=0}^{N-1} A_{15}^{(i)}\lambda^i, \\ T_{21} = \sum_{i=0}^{N-1} A_{21}^{(i)}\lambda^i, T_{22} = \lambda^N + \sum_{i=0}^{N-1} A_{22}^{(i)}\lambda^i, T_{23} = \sum_{i=0}^{N-1} A_{23}^{(i)}\lambda^i, T_{24} = \sum_{i=0}^{N-1} A_{24}^{(i)}\lambda^i, T_{25} = \sum_{i=0}^{N-1} A_{25}^{(i)}\lambda^i, \\ T_{31} = \sum_{i=0}^{N-1} A_{31}^{(i)}\lambda^i, T_{32} = \sum_{i=0}^{N-1} A_{32}^{(i)}\lambda^i, T_{33} = \lambda^N + \sum_{i=0}^{N-1} A_{33}^{(i)}\lambda^i, T_{34} = \sum_{i=0}^{N-1} A_{34}^{(i)}\lambda^i, T_{35} = \sum_{i=0}^{N-1} A_{35}^{(i)}\lambda^i, \\ T_{41} = \sum_{i=0}^{N-1} A_{41}^{(i)}\lambda^i, T_{42} = \sum_{i=0}^{N-1} A_{42}^{(i)}\lambda^i, T_{43} = \sum_{i=0}^{N-1} A_{43}^{(i)}\lambda^i, T_{44} = \lambda^N + \sum_{i=0}^{N-1} A_{44}^{(i)}\lambda^i, T_{45} = \sum_{i=0}^{N-1} A_{45}^{(i)}\lambda^i, \\ T_{51} = \sum_{i=0}^{N-1} A_{51}^{(i)}\lambda^i, T_{52} = \sum_{i=0}^{N-1} A_{52}^{(i)}\lambda^i, T_{53} = \sum_{i=0}^{N-1} A_{53}^{(i)}\lambda^i, T_{54} = \sum_{i=0}^{N-1} A_{54}^{(i)}\lambda^i, T_{55} = \lambda^N + \sum_{i=0}^{N-1} A_{55}^{(i)}\lambda^i, \end{cases} \tag{9}$$

where N is a natural number, the $A_{mn}^{(i)}(m, n = 1, 2, 3, 4, 5, i \geq 0)$ are the functions of x and t . By calculations, we can obtain ΔT as following

$$\Delta T = \prod_{j=1}^{5N} (\lambda - \lambda_j), \tag{10}$$

which proves that $\lambda_j (j = 1 \leq j \leq 5N,)$ are $5N$ roots of ΔT . Based on these above conditions, we will proof that \tilde{U} and V have the same forms with U and V , respectively.

Proposition 1 *The matrix \tilde{U} defined by Eq. (5) has the same type as U , we have*

$$\tilde{U} = \begin{pmatrix} i\lambda & -i\tilde{q}_1^* & -i\tilde{q}_2^* & -i\tilde{q}_3^* & -i\tilde{q}_4^* \\ i\tilde{q}_1 & -i\lambda & 0 & 0 & 0 \\ i\tilde{q}_2 & 0 & -i\lambda & 0 & 0 \\ i\tilde{q}_3 & 0 & 0 & -i\lambda & 0 \\ i\tilde{q}_4 & 0 & 0 & 0 & -i\lambda \end{pmatrix}, \tag{11}$$

the relations between the new solution and the old solution of Eq. (5) are given:

$$\begin{cases} \tilde{q}_1 = q_1 + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = q_2 + 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = q_3 + 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = q_4 + 2A_{51}^{(N-1)}. \end{cases} \tag{12}$$

Proof Setting $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_x + TA)T^* = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) & B_{14}(\lambda) & B_{15}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda) & B_{24}(\lambda) & B_{25}(\lambda) \\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) & B_{34}(\lambda) & B_{35}(\lambda) \\ B_{41}(\lambda) & B_{42}(\lambda) & B_{43}(\lambda) & B_{44}(\lambda) & B_{45}(\lambda) \\ B_{51}(\lambda) & B_{52}(\lambda) & B_{53}(\lambda) & B_{54}(\lambda) & B_{55}(\lambda) \end{pmatrix}, \tag{13}$$

it is easy to verify that $B_{sl} (1 \leq s, l \leq 5)$ are $5N$ or $5N + 1$ degree polynomials of λ . Through calculation, Eq. (13) can be obtained in the following form:

$$(T_x + TU)T^* = (\Delta T)P(\lambda), \tag{14}$$

where

$$P(\lambda) = \begin{pmatrix} P_{11}^{(1)}\lambda + P_{11}^{(0)} & P_{12}^{(0)} & P_{13}^{(0)} & P_{14}^{(0)} & P_{15}^{(0)} \\ P_{21}^{(0)} & P_{22}^{(1)}\lambda + P_{22}^{(0)} & P_{23}^{(0)} & P_{24}^{(0)} & P_{25}^{(0)} \\ P_{31}^{(0)} & P_{32}^{(0)} & P_{33}^{(1)}\lambda + P_{33}^{(0)} & P_{34}^{(0)} & P_{35}^{(0)} \\ P_{41}^{(0)} & P_{42}^{(0)} & P_{43}^{(0)} & P_{44}^{(1)}\lambda + P_{44}^{(0)} & P_{45}^{(0)} \\ P_{51}^{(0)} & P_{52}^{(0)} & P_{53}^{(0)} & P_{54}^{(0)} & P_{55}^{(1)}\lambda + P_{55}^{(0)} \end{pmatrix}, \tag{15}$$

and $P_{mn}^{(k)} (m, n = 1, 2, 3, 4, 5, k = 0, 1)$ satisfy the functions without λ . Based on Eq. (14), the system is obtained as following

$$(T_x + TU) = P(\lambda)T. \tag{16}$$

Through comparing the $N + 1, N, N - 1$ -th powers of λ in Eq. (16), we have the following systems:

$$\begin{cases} P_{11}^{(1)} = i, P_{11}^{(0)} = 0, P_{12}^{(0)} = -iq_1^* - 2iA_{12}^{(N-1)}, P_{13}^{(0)} = -iq_2^* - 2iA_{13}^{(N-1)}, \\ P_{14}^{(0)} = -iq_3^* - 2iA_{14}^{(N-1)}, P_{15}^{(0)} = -iq_4^* - 2iA_{15}^{(N-1)}, \\ P_{21}^{(0)} = iq_1 + 2iA_{21}^{(N-1)}, P_{22}^{(1)} = -i, P_{22}^{(0)} = 0, P_{23}^{(0)} = 0, P_{24}^{(0)} = 0, P_{25}^{(0)} = 0, \\ P_{31}^{(0)} = iq_2 + 2iA_{31}^{(N-1)}, P_{32}^{(0)} = 0, P_{33}^{(1)} = -i, P_{33}^{(0)} = 0, P_{34}^{(0)} = 0, P_{35}^{(0)} = 0, \\ P_{41}^{(0)} = iq_3 + 2iA_{41}^{(N-1)}, P_{42}^{(0)} = 0, P_{43}^{(0)} = 0, P_{44}^{(1)} = -i, P_{44}^{(0)} = 0, P_{45}^{(0)} = 0, \\ P_{51}^{(0)} = iq_4 + 2iA_{51}^{(N-1)}, P_{52}^{(0)} = 0, P_{53}^{(0)} = 0, P_{54}^{(0)} = 0, P_{55}^{(1)} = -i, P_{55}^{(0)} = 0. \end{cases} \tag{17}$$

In this section, we set that the new matrix \tilde{U} has same type with U , which means that they have the same structures only q, q^* of U transformed into \tilde{q}, \tilde{q}^* of \tilde{U} . After detailed calculation, we compare the ranks of λ^N , and get the objective equations as following:

$$\begin{cases} \tilde{q}_1 = q_1 + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = q_2 + 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = q_3 + 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = q_4 + 2A_{51}^{(N-1)}, \end{cases} \tag{18}$$

from Eqs. (17) and (18), we know that $\tilde{U} = P(\lambda)$. The proof is completed. □

Proposition 2 Under the transformation (6), the matrix \tilde{V} defined by Eq. (6) has the same form as V , that is,

$$\tilde{V} = \begin{pmatrix} -i\lambda^2 + \frac{1}{2}i\tau & -i\lambda\tilde{q}_1^* - \frac{1}{2}\tilde{q}_{1,x}^* & -i\lambda\tilde{q}_2^* - \frac{1}{2}\tilde{q}_{2,x}^* & -i\lambda\tilde{q}_3^* - \frac{1}{2}\tilde{q}_{3,x}^* & -i\lambda\tilde{q}_4^* - \frac{1}{2}\tilde{q}_{4,x}^* \\ i\lambda\tilde{q}_1 - \frac{1}{2}\tilde{q}_{1,x} & -i\lambda^2 - \frac{1}{2}i\tilde{q}_1\tilde{q}_1^* & -\frac{1}{2}i\tilde{q}_1\tilde{q}_2^* & -\frac{1}{2}i\tilde{q}_1\tilde{q}_3^* & -\frac{1}{2}i\tilde{q}_1\tilde{q}_4^* \\ i\lambda\tilde{q}_2 - \frac{1}{2}\tilde{q}_{2,x} & -\frac{1}{2}i\tilde{q}_2\tilde{q}_1^* & -i\lambda^2 - \frac{1}{2}i\tilde{q}_2\tilde{q}_2^* & -\frac{1}{2}i\tilde{q}_2\tilde{q}_3^* & -\frac{1}{2}i\tilde{q}_2\tilde{q}_4^* \\ i\lambda\tilde{q}_3 - \frac{1}{2}\tilde{q}_{3,x} & -\frac{1}{2}i\tilde{q}_3\tilde{q}_1^* & -\frac{1}{2}i\tilde{q}_3\tilde{q}_2^* & -i\lambda^2 - \frac{1}{2}i\tilde{q}_3\tilde{q}_3^* & -\frac{1}{2}i\tilde{q}_3\tilde{q}_4^* \\ i\lambda\tilde{q}_4 - \frac{1}{2}\tilde{q}_{4,x} & -\frac{1}{2}i\tilde{q}_4\tilde{q}_1^* & -\frac{1}{2}i\tilde{q}_4\tilde{q}_2^* & -\frac{1}{2}i\tilde{q}_4\tilde{q}_3^* & -i\lambda^2 - \frac{1}{2}i\tilde{q}_4\tilde{q}_4^* \end{pmatrix}, \tag{19}$$

where $\tau = \tilde{q}_1^*\tilde{q}_1 + \tilde{q}_2^*\tilde{q}_2 + \tilde{q}_3^*\tilde{q}_3 + \tilde{q}_4^*\tilde{q}_4$, the relations between the new solutions and the old solutions of Eq. (6) are given:

$$\begin{cases} \tilde{q}_1 = q_1 + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = q_2 + 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = q_3 + 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = q_4 + 2A_{51}^{(N-1)}. \end{cases} \tag{20}$$

Proof Setting $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_t + TV)T^* = \begin{pmatrix} C_{11}(\lambda) & C_{12}(\lambda) & C_{13}(\lambda) & C_{14}(\lambda) & C_{15}(\lambda) \\ C_{21}(\lambda) & C_{22}(\lambda) & C_{23}(\lambda) & C_{24}(\lambda) & C_{25}(\lambda) \\ C_{31}(\lambda) & C_{32}(\lambda) & C_{33}(\lambda) & C_{34}(\lambda) & C_{35}(\lambda) \\ C_{41}(\lambda) & C_{42}(\lambda) & C_{43}(\lambda) & C_{44}(\lambda) & C_{45}(\lambda) \\ C_{51}(\lambda) & C_{52}(\lambda) & C_{53}(\lambda) & C_{54}(\lambda) & C_{55}(\lambda) \end{pmatrix}. \tag{21}$$

It is easy to verify that $C_{sl} (1 \leq s, l \leq 5)$ are $5N$ or $5N + 1$ degree polynomials of λ .

Through some calculations, the $\lambda_j (j = 1 \leq j \leq 5,)$ are the roots of $C_{sl} (1 \leq s, l \leq 5)$. Thus, Eq. (21) has the following structure

$$(T_t + TV)T^* = (\Delta T)Q(\lambda), \tag{22}$$

where

$$Q(\lambda) = \begin{pmatrix} \tau_1 & Q_{12}^{(1)}\lambda + Q_{12}^{(0)} & Q_{13}^{(1)}\lambda + Q_{13}^{(0)} & Q_{14}^{(1)}\lambda + Q_{14}^{(0)} & Q_{15}^{(1)}\lambda + Q_{15}^{(0)} \\ Q_{21}^{(1)}\lambda + Q_{21}^{(0)} & \tau_2 & Q_{23}^{(1)}\lambda + Q_{23}^{(0)} & Q_{24}^{(1)}\lambda + Q_{24}^{(0)} & Q_{25}^{(1)}\lambda + Q_{25}^{(0)} \\ Q_{31}^{(1)}\lambda + Q_{31}^{(0)} & Q_{32}^{(1)}\lambda + Q_{32}^{(0)} & \tau_3 & Q_{34}^{(1)}\lambda + Q_{34}^{(0)} & Q_{35}^{(1)}\lambda + Q_{35}^{(0)} \\ Q_{41}^{(1)}\lambda + Q_{41}^{(0)} & Q_{42}^{(1)}\lambda + Q_{42}^{(0)} & Q_{43}^{(1)}\lambda + Q_{43}^{(0)} & \tau_4 & Q_{45}^{(1)}\lambda + Q_{45}^{(0)} \\ Q_{51}^{(1)}\lambda + Q_{51}^{(0)} & Q_{52}^{(1)}\lambda + Q_{52}^{(0)} & Q_{53}^{(1)}\lambda + Q_{53}^{(0)} & Q_{54}^{(1)}\lambda + Q_{54}^{(0)} & \tau_5 \end{pmatrix}, \tag{23}$$

with $\tau_1 = Q_{11}^{(2)}\lambda^2 + Q_{11}^{(1)}\lambda + Q_{11}^{(0)}$, $\tau_2 = Q_{22}^{(2)}\lambda^2 + Q_{22}^{(1)}\lambda + Q_{22}^{(0)}$, $\tau_3 = Q_{33}^{(2)}\lambda^2 + Q_{33}^{(1)}\lambda + Q_{33}^{(0)}$, $\tau_4 = Q_{44}^{(2)}\lambda^2 + Q_{44}^{(1)}\lambda + Q_{44}^{(0)}$, $\tau_5 = Q_{55}^{(2)}\lambda^2 + Q_{55}^{(1)}\lambda + Q_{55}^{(0)}$, and $Q_{mn}^{(k)} (m, n = 1, 2, 3, 4, 5, k = 0, 1, 2)$ satisfy the functions without λ . We obtain the following system from Eq. (22)

$$T_t + TV = Q(\lambda)T. \tag{24}$$

By comparing the $N + 2, N + 1, N$ -th powers of λ in Eq. (24), we have the following system

$$\begin{cases} Q_{11}^{(2)} = i, Q_{11}^{(1)} = 0, Q_{11}^{(0)} = \frac{1}{2}i(\tilde{q}_1^* \tilde{q}_1 + \tilde{q}_2^* \tilde{q}_2 + \tilde{q}_3^* \tilde{q}_3 + \tilde{q}_4^* \tilde{q}_4), Q_{1k}^{(1)} = -iq_{k-1}^* - 2iA_{1k}^{(N-1)}, \\ Q_{1k}^{(0)} = -\frac{1}{2}q_{k-1,x}^* - A_{1k,x}^{(N-1)} + 2iA_{1k}^{(N-2)}, Q_{k1}^{(1)} = iq_{k-1} + 2iA_{k1}^{(N-1)}, \\ Q_{k1}^{(0)} = -\frac{1}{2}q_{k-1,x} - A_{k1,x}^{(N-1)} - 2iA_{k1}^{(N-2)}, Q_{22}^{(2)} = Q_{33}^{(2)} = Q_{44}^{(2)} = Q_{55}^{(2)} = -i, \\ Q_{kk}^{(1)} = 0, Q_{hh}^{(0)} = -\frac{1}{2}iq_{h-1}q_{h-1}^* - iq_{h-1}^*A_{h1}^{(N-1)} - iq_{h-1}A_{1h}^{(N-1)} - 2iA_{h1}^{(N-1)}A_{1h}^{(N-1)}, \\ Q_{kh}^{(0)} = -\frac{1}{2}iq_{k-1}q_{h-1}^* - iq_{h-1}^*A_{k1}^{(N-1)} - iq_{k-1}A_{1h}^{(N-1)} - 2iA_{k1}^{(N-1)}A_{1h}^{(N-1)}, \\ Q_{hk}^{(0)} = -\frac{1}{2}iq_{h-1}q_{k-1}^* - iq_{k-1}^*A_{h1}^{(N-1)} - iq_{h-1}A_{1k}^{(N-1)} - 2iA_{h1}^{(N-1)}A_{1k}^{(N-1)}, \end{cases} \tag{25}$$

where $k = 2, 3, 4, 5, h = 2, 3, 4, 5$, and $h = h, kh$ and hk are referring to $k < h$.

In the section, we set that the new matrix \tilde{V} has same type with V , which means that they have the same structures only q, q^* of V transformed into \tilde{q}, \tilde{q}^* of \tilde{V} . After detailed calculations, we compare the ranks of λ^N , and get the objective equations as following:

$$\begin{cases} \tilde{q}_1 = q_1 + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = q_2 + 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = q_3 + 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = q_4 + 2A_{51}^{(N-1)}, \end{cases} \tag{26}$$

according to Eqs. (25) and (26), we know that $\tilde{V} = Q(\lambda)$. The proof is completed. □

Conclusion and discussion

In this paper, the four-component CNLS equations are investigated, which describe the wave propagations of multicomponent Bose-Einstein condensates. Starting from its Lax pair with initial non-zero plane-wave conditions, we find the vector soliton solutions (non-zero backgrounds) and soliton-like solutions (zero backgrounds) with the aid of the modified Darboux transformation. Then the N -soliton solutions of the four-component CNLS equations in the case of zero-seeded and non-zero-seeded solutions are considered. And we construct three kinds of solutions including bright-bright-bright soliton solutions, dark-bright-bright soliton solutions and dark-dark -bright-bright soliton solutions. The 1-soliton solution and 2-soliton solution of the

four-component CNLS equation are calculated using the zero seed solution and non-zero seed solution, the dynamical behaviors of the soliton solutions are shown on complex wave backgrounds.

Very recently, the non-degenerate bright solitons are discussed in the N -component coupled systems with attractive interactions based Hirota bilinear method. Our method can also be extended to N -component coupled systems with repulsive interactions. Moreover, some numerical simulations may show that these soliton states are robust against small deviations and weak noises. Therefore, there are many possibilities to observe them in real experiments.

Methods

N -soliton solutions for the four-component coupled NLS equation on complex wave backgrounds. *N -soliton solutions on zero wave backgrounds.* In order to obtain the N -soliton solution formula of four-component CNLS equations on zero wave backgrounds with Darboux transformation, we firstly give a set of seed solutions $q_1 = q_2 = q_3 = q_4 = 0$ and substitute these solutions into Eqs. (4) and (5), which can get five basic solutions for CNLS equations:

$$\varphi = \begin{pmatrix} e^{i\lambda x+i\lambda^2 t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \phi = \begin{pmatrix} 0 \\ e^{-i\lambda x-i\lambda^2 t} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ e^{-i\lambda x-i\lambda^2 t} \\ 0 \\ 0 \end{pmatrix}, \xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\lambda x-i\lambda^2 t} \\ 0 \end{pmatrix}, \varepsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{-i\lambda x-i\lambda^2 t} \end{pmatrix}. \tag{27}$$

Substituting Eq. (27) into Eq. (8), we give rise to

$$\begin{aligned} M_j^{(1)} &= e^{-2(i\lambda x+i\lambda^2 t+F_{1,j})}, M_j^{(2)} = e^{-2(i\lambda x+i\lambda^2 t+F_{2,j})}, \\ M_j^{(3)} &= e^{-2(i\lambda x+i\lambda^2 t+F_{3,j})}, M_j^{(4)} = e^{-2(i\lambda x+i\lambda^2 t+F_{4,j})}, \end{aligned} \tag{28}$$

with $e^{(-2F_{1,j})} = v_j^{(11)}, e^{(-2F_{2,j})} = v_j^{(22)}, e^{(-2F_{3,j})} = v_j^{(33)}, e^{(-2F_{4,j})} = v_j^{(44)}$.

In order to obtain N -soliton solutions of Eq. (1) with zero wave backgrounds, we consider matrix T as following:

$$T = \begin{pmatrix} \lambda^N + \sum_{i=0}^{N-1} A_{11}\lambda_j^i & \sum_{i=0}^{N-1} A_{12}\lambda_j^i & \sum_{i=0}^{N-1} A_{13}\lambda_j^i & \sum_{i=0}^{N-1} A_{14}\lambda_j^i & \sum_{i=0}^{N-1} A_{15}\lambda_j^i \\ \sum_{i=0}^{N-1} A_{21}\lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{22}\lambda_j^i & \sum_{i=0}^{N-1} A_{23}\lambda_j^i & \sum_{i=0}^{N-1} A_{24}\lambda_j^i & \sum_{i=0}^{N-1} A_{25}\lambda_j^i \\ \sum_{i=0}^{N-1} A_{31}\lambda_j^i & \sum_{i=0}^{N-1} A_{32}\lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{33}\lambda_j^i & \sum_{i=0}^{N-1} A_{34}\lambda_j^i & \sum_{i=0}^{N-1} A_{35}\lambda_j^i \\ \sum_{i=0}^{N-1} A_{41}\lambda_j^i & \sum_{i=0}^{N-1} A_{42}\lambda_j^i & \sum_{i=0}^{N-1} A_{43}\lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{44}\lambda_j^i & \sum_{i=0}^{N-1} A_{45}\lambda_j^i \\ \sum_{i=0}^{N-1} A_{51}\lambda_j^i & \sum_{i=0}^{N-1} A_{52}\lambda_j^i & \sum_{i=0}^{N-1} A_{53}\lambda_j^i & \sum_{i=0}^{N-1} A_{54}\lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{55}\lambda_j^i \end{pmatrix}, \tag{29}$$

and

$$\begin{cases} \sum_{i=0}^{N-1} \left(A_{11}^{(i)} + A_{12}^{(i)}M_j^{(1)} + A_{13}^{(i)}M_j^{(2)} + A_{14}^{(i)}M_j^{(3)} + A_{15}^{(i)}M_j^{(4)} \right) \lambda_j^i = -\lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{21}^{(i)} + A_{22}^{(i)}M_j^{(1)} + A_{23}^{(i)}M_j^{(2)} + A_{24}^{(i)}M_j^{(3)} + A_{25}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(1)} \lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{31}^{(i)} + A_{32}^{(i)}M_j^{(1)} + A_{33}^{(i)}M_j^{(2)} + A_{34}^{(i)}M_j^{(3)} + A_{35}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(2)} \lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{41}^{(i)} + A_{42}^{(i)}M_j^{(1)} + A_{43}^{(i)}M_j^{(2)} + A_{44}^{(i)}M_j^{(3)} + A_{45}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(3)} \lambda_j^N, \\ \sum_{i=0}^{N-1} \left(A_{51}^{(i)} + A_{52}^{(i)}M_j^{(1)} + A_{53}^{(i)}M_j^{(2)} + A_{54}^{(i)}M_j^{(3)} + A_{55}^{(i)}M_j^{(4)} \right) \lambda_j^i = -M_j^{(4)} \lambda_j^N. \end{cases} \tag{30}$$

The N -soliton solutions of Eq. (1) on zero wave backgrounds are obtained according to Cramer's rule in Eq. (30) as following forms:

$$\begin{cases} \tilde{q}_1 = 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = 2A_{51}^{(N-1)}, \end{cases} \tag{31}$$

where

$$A_{21}^{(N-1)} = \frac{\Delta A_{21}^{(N-1)}}{\Delta}, A_{31}^{(N-1)} = \frac{\Delta A_{31}^{(N-1)}}{\Delta}, A_{41}^{(N-1)} = \frac{\Delta A_{41}^{(N-1)}}{\Delta}, A_{51}^{(N-1)} = \frac{\Delta A_{51}^{(N-1)}}{\Delta}, \tag{32}$$

with

$$\begin{aligned}
 \Delta &= \begin{pmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \dots & \lambda_1^{(N-1)} & \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \dots & \lambda_2^{(N-1)} & \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{5N,1}} & e^{\theta_{5N,2}} & e^{\theta_{5N,3}} & e^{\theta_{5N,4}} & \dots & \lambda_{5N}^{(N-1)} & \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix}, \\
 \Delta A_{21}^{(N-1)} &= \begin{pmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \dots & -\lambda_1^{(N)} & \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \dots & -\lambda_2^{(N)} & \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{5N,1}} & e^{\theta_{5N,2}} & e^{\theta_{5N,3}} & e^{\theta_{5N,4}} & \dots & -\lambda_{5N}^{(N)} & \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix}, \\
 \Delta A_{31}^{(N-1)} &= \begin{pmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \dots & -\lambda_1^{(N)} & \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \dots & -\lambda_2^{(N)} & \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{5N,1}} & e^{\theta_{5N,2}} & e^{\theta_{5N,3}} & e^{\theta_{5N,4}} & \dots & -\lambda_{5N}^{(N)} & \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix}, \\
 \Delta A_{41}^{(N-1)} &= \begin{pmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \dots & -\lambda_1^{(N)} & \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \dots & -\lambda_2^{(N)} & \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{5N,1}} & e^{\theta_{5N,2}} & e^{\theta_{5N,3}} & e^{\theta_{5N,4}} & \dots & -\lambda_{5N}^{(N)} & \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix}, \\
 \Delta A_{51}^{(N-1)} &= \begin{pmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \dots & -\lambda_1^{(N)} & \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \dots & -\lambda_2^{(N)} & \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{5N,1}} & e^{\theta_{5N,2}} & e^{\theta_{5N,3}} & e^{\theta_{5N,4}} & \dots & -\lambda_{5N}^{(N)} & \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},
 \end{aligned} \tag{33}$$

and $\theta_{1,1} = -2(i\lambda_1 x + i\lambda_1^2 t + F_{1,1})$, $\theta_{1,2} = -2(i\lambda_1 x + i\lambda_1^2 t + F_{2,1})$, $\theta_{1,3} = -2(i\lambda_1 x + i\lambda_1^2 t + F_{3,1})$, $\theta_{1,4} = -2(i\lambda_1 x + i\lambda_1^2 t + F_{4,1})$, $\theta_{2,1} = -2(i\lambda_2 x + i\lambda_2^2 t + F_{1,2})$, $\theta_{2,2} = -2(i\lambda_2 x + i\lambda_2^2 t + F_{2,2})$, $\theta_{2,3} = -2(i\lambda_2 x + i\lambda_2^2 t + F_{3,2})$, $\theta_{2,4} = -2(i\lambda_2 x + i\lambda_2^2 t + F_{4,2})$, $\theta_{5N,1} = -2(i\lambda_{5N} x + i\lambda_{5N}^2 t + F_{1,5N})$, $\theta_{5N,2} = -2(i\lambda_{5N} x + i\lambda_{5N}^2 t + F_{2,5N})$, $\theta_{5N,3} = -2(i\lambda_{5N} x + i\lambda_{5N}^2 t + F_{3,5N})$, $\theta_{5N,4} = -2(i\lambda_{5N} x + i\lambda_{5N}^2 t + F_{4,5N})$.

In order to obtain 1-soliton solution of the four-component CNLS equations on zero wave backgrounds, we consider $N = 1$ in Eqs. (29) and (30), and obtain the matrix T

$$T = \begin{pmatrix} \lambda + A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} & A_{14}^{(0)} & A_{15}^{(0)} \\ A_{21}^{(0)} & \lambda + A_{22}^{(0)} & A_{23}^{(0)} & A_{24}^{(0)} & A_{25}^{(0)} \\ A_{31}^{(0)} & A_{32}^{(0)} & \lambda + A_{33}^{(0)} & A_{34}^{(0)} & A_{35}^{(0)} \\ A_{41}^{(0)} & A_{42}^{(0)} & A_{43}^{(0)} & \lambda + A_{44}^{(0)} & A_{45}^{(0)} \\ A_{51}^{(0)} & A_{52}^{(0)} & A_{53}^{(0)} & A_{54}^{(0)} & \lambda + A_{55}^{(0)} \end{pmatrix}, \tag{34}$$

and

$$\begin{cases} A_{11}^{(0)} + A_{12}^{(0)} M_j^{(1)} + A_{13}^{(0)} M_j^{(2)} + A_{14}^{(0)} M_j^{(3)} + A_{15}^{(0)} M_j^{(4)} = -\lambda_j, \\ A_{21}^{(0)} + A_{22}^{(0)} M_j^{(1)} + A_{23}^{(0)} M_j^{(2)} + A_{24}^{(0)} M_j^{(3)} + A_{25}^{(0)} M_j^{(4)} = -M_j^{(1)} \lambda_j, \\ A_{31}^{(0)} + A_{32}^{(0)} M_j^{(1)} + A_{33}^{(0)} M_j^{(2)} + A_{34}^{(0)} M_j^{(3)} + A_{35}^{(0)} M_j^{(4)} = -M_j^{(2)} \lambda_j, \\ A_{41}^{(0)} + A_{42}^{(0)} M_j^{(1)} + A_{43}^{(0)} M_j^{(2)} + A_{44}^{(0)} M_j^{(3)} + A_{45}^{(0)} M_j^{(4)} = -M_j^{(3)} \lambda_j, \\ A_{51}^{(0)} + A_{52}^{(0)} M_j^{(1)} + A_{53}^{(0)} M_j^{(2)} + A_{54}^{(0)} M_j^{(3)} + A_{55}^{(0)} M_j^{(4)} = -M_j^{(4)} \lambda_j. \end{cases} \tag{35}$$

According to Eq. (35) and Cramer's rule, we can derive the systems

$$\begin{aligned}
 \Delta_1 &= \begin{pmatrix} 1 & e^{-2\rho_{1,1}} & e^{-2\rho_{2,1}} & e^{-2\rho_{3,1}} & e^{-2\rho_{4,1}} \\ 1 & e^{-2\rho_{1,2}} & e^{-2\rho_{2,2}} & e^{-2\rho_{3,2}} & e^{-2\rho_{4,2}} \\ 1 & e^{-2\rho_{1,3}} & e^{-2\rho_{2,3}} & e^{-2\rho_{3,3}} & e^{-2\rho_{4,3}} \\ 1 & e^{-2\rho_{1,4}} & e^{-2\rho_{2,4}} & e^{-2\rho_{3,4}} & e^{-2\rho_{4,4}} \\ 1 & e^{-2\rho_{1,5}} & e^{-2\rho_{2,5}} & e^{-2\rho_{3,5}} & e^{-2\rho_{4,5}} \end{pmatrix}, \\
 \Delta A_{21}^{(0)} &= \begin{pmatrix} -\lambda_1 e^{-2\rho_{1,1}} & e^{-2\rho_{1,1}} & e^{-2\rho_{2,1}} & e^{-2\rho_{3,1}} & e^{-2\rho_{4,1}} \\ -\lambda_2 e^{-2\rho_{1,2}} & e^{-2\rho_{1,2}} & e^{-2\rho_{2,2}} & e^{-2\rho_{3,2}} & e^{-2\rho_{4,2}} \\ -\lambda_3 e^{-2\rho_{1,3}} & e^{-2\rho_{1,3}} & e^{-2\rho_{2,3}} & e^{-2\rho_{3,3}} & e^{-2\rho_{4,3}} \\ -\lambda_4 e^{-2\rho_{1,4}} & e^{-2\rho_{1,4}} & e^{-2\rho_{2,4}} & e^{-2\rho_{3,4}} & e^{-2\rho_{4,4}} \\ -\lambda_5 e^{-2\rho_{1,5}} & e^{-2\rho_{1,5}} & e^{-2\rho_{2,5}} & e^{-2\rho_{3,5}} & e^{-2\rho_{4,5}} \end{pmatrix}, \\
 \Delta A_{31}^{(0)} &= \begin{pmatrix} -\lambda_1 e^{-2\rho_{2,1}} & e^{-2\rho_{1,1}} & e^{-2\rho_{2,1}} & e^{-2\rho_{3,1}} & e^{-2\rho_{4,1}} \\ -\lambda_2 e^{-2\rho_{2,2}} & e^{-2\rho_{1,2}} & e^{-2\rho_{2,2}} & e^{-2\rho_{3,2}} & e^{-2\rho_{4,2}} \\ -\lambda_3 e^{-2\rho_{2,3}} & e^{-2\rho_{1,3}} & e^{-2\rho_{2,3}} & e^{-2\rho_{3,3}} & e^{-2\rho_{4,3}} \\ -\lambda_4 e^{-2\rho_{2,4}} & e^{-2\rho_{1,4}} & e^{-2\rho_{2,4}} & e^{-2\rho_{3,4}} & e^{-2\rho_{4,4}} \\ -\lambda_5 e^{-2\rho_{2,5}} & e^{-2\rho_{1,5}} & e^{-2\rho_{2,5}} & e^{-2\rho_{3,5}} & e^{-2\rho_{4,5}} \end{pmatrix}, \\
 \Delta A_{41}^{(0)} &= \begin{pmatrix} -\lambda_1 e^{-2\rho_{3,1}} & e^{-2\rho_{1,1}} & e^{-2\rho_{2,1}} & e^{-2\rho_{3,1}} & e^{-2\rho_{4,1}} \\ -\lambda_2 e^{-2\rho_{3,2}} & e^{-2\rho_{1,2}} & e^{-2\rho_{2,2}} & e^{-2\rho_{3,2}} & e^{-2\rho_{4,2}} \\ -\lambda_3 e^{-2\rho_{3,3}} & e^{-2\rho_{1,3}} & e^{-2\rho_{2,3}} & e^{-2\rho_{3,3}} & e^{-2\rho_{4,3}} \\ -\lambda_4 e^{-2\rho_{3,4}} & e^{-2\rho_{1,4}} & e^{-2\rho_{2,4}} & e^{-2\rho_{3,4}} & e^{-2\rho_{4,4}} \\ -\lambda_5 e^{-2\rho_{3,5}} & e^{-2\rho_{1,5}} & e^{-2\rho_{2,5}} & e^{-2\rho_{3,5}} & e^{-2\rho_{4,5}} \end{pmatrix}, \\
 \Delta A_{51}^{(0)} &= \begin{pmatrix} -\lambda_1 e^{-2\rho_{4,1}} & e^{-2\rho_{1,1}} & e^{-2\rho_{2,1}} & e^{-2\rho_{3,1}} & e^{-2\rho_{4,1}} \\ -\lambda_2 e^{-2\rho_{4,2}} & e^{-2\rho_{1,2}} & e^{-2\rho_{2,2}} & e^{-2\rho_{3,2}} & e^{-2\rho_{4,2}} \\ -\lambda_3 e^{-2\rho_{4,3}} & e^{-2\rho_{1,3}} & e^{-2\rho_{2,3}} & e^{-2\rho_{3,3}} & e^{-2\rho_{4,3}} \\ -\lambda_4 e^{-2\rho_{4,4}} & e^{-2\rho_{1,4}} & e^{-2\rho_{2,4}} & e^{-2\rho_{3,4}} & e^{-2\rho_{4,4}} \\ -\lambda_5 e^{-2\rho_{4,5}} & e^{-2\rho_{1,5}} & e^{-2\rho_{2,5}} & e^{-2\rho_{3,5}} & e^{-2\rho_{4,5}} \end{pmatrix}.
 \end{aligned} \tag{36}$$

here $\rho_{i,1} = i\lambda_1 x + i\lambda_1^2 t + F_{i,1}$, $\rho_{i,2} = i\lambda_2 x + i\lambda_2^2 t + F_{i,2}$, $\rho_{i,3} = i\lambda_3 x + i\lambda_3^2 t + F_{i,3}$, $\rho_{i,4} = i\lambda_4 x + i\lambda_4^2 t + F_{i,4}$, $\rho_{i,5} = i\lambda_5 x + i\lambda_5^2 t + F_{i,5}$, $i = 1, 2, 3, 4$.

Based on the Eq. (32), we can obtain the following systems

$$A_{21}^{(0)} = \frac{\Delta A_{21}^{(0)}}{\Delta_1}, A_{31}^{(0)} = \frac{\Delta A_{31}^{(0)}}{\Delta_1}, A_{41}^{(0)} = \frac{\Delta A_{41}^{(0)}}{\Delta_1}, A_{51}^{(0)} = \frac{\Delta A_{51}^{(0)}}{\Delta_1}, \tag{37}$$

the analytic 1-soliton solutions of the four-component CNLS equations are obtained on zero wave backgrounds by Darboux transformation as following

$$\begin{cases} \tilde{q}_1 = 2A_{21}^{(0)}, \\ \tilde{q}_2 = 2A_{31}^{(0)}, \\ \tilde{q}_3 = 2A_{41}^{(0)}, \\ \tilde{q}_4 = 2A_{51}^{(0)}. \end{cases} \tag{38}$$

In order to obtain the 2-soliton solutions of the four-component coupled NLS equations (1) on zero wave backgrounds, we consider $N = 2$ in Eqs. (29) and (30), and obtain the matrix T

$$T = \begin{pmatrix} \lambda^2 + A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & A_{14}^{(1)} & A_{15}^{(1)} \\ A_{21}^{(1)} & \lambda^2 + A_{22}^{(1)} & A_{23}^{(1)} & A_{24}^{(1)} & A_{25}^{(1)} \\ A_{31}^{(1)} & A_{32}^{(1)} & \lambda^2 + A_{33}^{(1)} & A_{34}^{(1)} & A_{35}^{(1)} \\ A_{41}^{(1)} & A_{42}^{(1)} & A_{43}^{(1)} & \lambda^2 + A_{44}^{(1)} & A_{45}^{(1)} \\ A_{51}^{(1)} & A_{52}^{(1)} & A_{53}^{(1)} & A_{54}^{(0)} & \lambda^2 + A_{55}^{(1)} \end{pmatrix}, \tag{39}$$

and

$$\begin{cases} A_{11}^{(1)} + A_{12}^{(1)} M_j^{(1)} + A_{13}^{(1)} M_j^{(2)} + A_{14}^{(1)} M_j^{(3)} + A_{15}^{(1)} M_j^{(4)} = -\lambda_j^2, \\ A_{21}^{(1)} + A_{22}^{(1)} M_j^{(1)} + A_{23}^{(1)} M_j^{(2)} + A_{24}^{(1)} M_j^{(3)} + A_{25}^{(1)} M_j^{(4)} = -M_j^{(1)} \lambda_j^2, \\ A_{31}^{(1)} + A_{32}^{(1)} M_j^{(1)} + A_{33}^{(1)} M_j^{(2)} + A_{34}^{(1)} M_j^{(3)} + A_{35}^{(1)} M_j^{(4)} = -M_j^{(2)} \lambda_j^2, \\ A_{41}^{(1)} + A_{42}^{(1)} M_j^{(1)} + A_{43}^{(1)} M_j^{(2)} + A_{44}^{(1)} M_j^{(3)} + A_{45}^{(1)} M_j^{(4)} = -M_j^{(3)} \lambda_j^2, \\ A_{51}^{(1)} + A_{52}^{(1)} M_j^{(1)} + A_{53}^{(1)} M_j^{(2)} + A_{54}^{(1)} M_j^{(3)} + A_{55}^{(1)} M_j^{(4)} = -M_j^{(4)} \lambda_j^2. \end{cases} \tag{40}$$

According to Eq. (40) and Cramer's rule, we can derive the systems

$$\begin{aligned}
 \Delta_2 &= \begin{vmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & \lambda_1 & \lambda_1 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & \lambda_2 & \lambda_2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{10,1}} & e^{\theta_{10,2}} & e^{\theta_{10,3}} & e^{\theta_{10,4}} & \lambda_{10} & \lambda_{10} e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\
 \Delta A_{21}^{(1)} &= \begin{vmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & -\lambda_1^2 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & -\lambda_2^2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{10,1}} & e^{\theta_{10,2}} & e^{\theta_{10,3}} & e^{\theta_{10,4}} & -\lambda_{10}^2 e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\
 \Delta A_{31}^{(1)} &= \begin{vmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & -\lambda_1^2 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & -\lambda_2^2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{10,1}} & e^{\theta_{10,2}} & e^{\theta_{10,3}} & e^{\theta_{10,4}} & -\lambda_{10}^2 e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\
 \Delta A_{41}^{(1)} &= \begin{vmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & -\lambda_1^2 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & -\lambda_2^2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{10,1}} & e^{\theta_{10,2}} & e^{\theta_{10,3}} & e^{\theta_{10,4}} & -\lambda_{10}^2 e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\
 \Delta A_{51}^{(1)} &= \begin{vmatrix} 1 & e^{\theta_{1,1}} & e^{\theta_{1,2}} & e^{\theta_{1,3}} & e^{\theta_{1,4}} & -\lambda_1^2 e^{\theta_{1,4}} & \lambda_1 e^{\theta_{1,1}} & \lambda_1 e^{\theta_{1,2}} & \lambda_1 e^{\theta_{1,3}} & \lambda_1 e^{\theta_{1,4}} \\ 1 & e^{\theta_{2,1}} & e^{\theta_{2,2}} & e^{\theta_{2,3}} & e^{\theta_{2,4}} & -\lambda_2^2 e^{\theta_{2,4}} & \lambda_2 e^{\theta_{2,1}} & \lambda_2 e^{\theta_{2,2}} & \lambda_2 e^{\theta_{2,3}} & \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & e^{\theta_{10,1}} & e^{\theta_{10,2}} & e^{\theta_{10,3}} & e^{\theta_{10,4}} & -\lambda_{10}^2 e^{\theta_{10,4}} & \lambda_{10} e^{\theta_{10,1}} & \lambda_{10} e^{\theta_{10,2}} & \lambda_{10} e^{\theta_{10,3}} & \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \tag{41}
 \end{aligned}$$

where $\theta_{j,i} = -2(i\lambda_j x + i\lambda_j^2 t + F_{(i,j)})$, $i = 1, 2, 3, 4, j = 1, 2, \dots, 10$.

Based on Eq. (32), we can obtain the following systems

$$A_{21}^{(1)} = \frac{\Delta A_{21}^{(1)}}{\Delta_2}, A_{31}^{(1)} = \frac{\Delta A_{31}^{(1)}}{\Delta_2}, A_{41}^{(1)} = \frac{\Delta A_{41}^{(1)}}{\Delta_2}, A_{51}^{(1)} = \frac{\Delta A_{51}^{(1)}}{\Delta_2}, \tag{42}$$

the analytic 2-soliton solutions of the four-component CNLS equation are obtained on zero wave backgrounds by Darboux transformation as following

$$\begin{cases} \tilde{q}_1 = 2A_{21}^{(1)}, \\ \tilde{q}_2 = 2A_{31}^{(1)}, \\ \tilde{q}_3 = 2A_{41}^{(1)}, \\ \tilde{q}_4 = 2A_{51}^{(1)}. \end{cases} \tag{43}$$

N-soliton solutions on zero and plane wave backgrounds. The *N*-soliton solution of the zero seed solution case is one of the simplest cases, however, the *N*-soliton solution formula of non-zero seed solution is more complicated with Darboux transformation. Because the non-zero solution must satisfy Eq. (1), we firstly give a set of seed solutions $q_1 = e^{-it}$, $q_2 = q_3 = q_4 = 0$ and substitute these solutions into Eqs. (4) and (5), which can get the basic solutions for CNLS equations:

$$\begin{aligned}
 \varphi &= \begin{pmatrix} He^{(\sqrt{1-\lambda^2})x+it} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \phi = \begin{pmatrix} 0 \\ K(\lambda + i\sqrt{1-\lambda^2})e^{(\sqrt{1-\lambda^2})x-it} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ e^{-i\lambda x - i\lambda^2 t} \\ 0 \\ 0 \end{pmatrix}, \\
 \xi &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\lambda x - i\lambda^2 t} \\ 0 \end{pmatrix}, \varepsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{-i\lambda x - i\lambda^2 t} \end{pmatrix}. \tag{44}
 \end{aligned}$$

with $H = \sqrt{\frac{\lambda^2 + \frac{1}{2}}{\lambda^2 - \frac{1}{2}}} K$.

Substituting Eq. (44) into Eq. (8), we give rise to

$$\begin{aligned}
 M_j^{(1)} &= \frac{K(\lambda + i\sqrt{1-\lambda^2})}{H} e^{-2it + G_{1,j}}, M_j^{(2)} = \frac{1}{H} e^{(-i\lambda - \sqrt{1-\lambda^2})x - (i\lambda^2 + i)t + G_{2,j}}, \\
 M_j^{(3)} &= \frac{1}{H} e^{(-i\lambda - \sqrt{1-\lambda^2})x - (i\lambda^2 + i)t + G_{3,j}}, M_j^{(4)} = \frac{1}{H} e^{(-i\lambda - \sqrt{1-\lambda^2})x - (i\lambda^2 + i)t + G_{4,j}}, \tag{45}
 \end{aligned}$$

where $e^{G_{1,j}} = v_j^{(11)}$, $e^{G_{2,j}} = v_j^{(22)}$, $e^{G_{3,j}} = v_j^{(33)}$, $e^{G_{4,j}} = v_j^{(44)}$, $1 \leq j \leq 5N$.

In order to obtain the *N*-soliton solutions of Eq. (1) on zero and plane wave backgrounds, we consider matrix *T* as following:

$$T = \begin{pmatrix} \lambda^N + \sum_{i=0}^{N-1} A_{11} \lambda_j^i & \sum_{i=0}^{N-1} A_{12} \lambda_j^i & \sum_{i=0}^{N-1} A_{13} \lambda_j^i & \sum_{i=0}^{N-1} A_{14} \lambda_j^i & \sum_{i=0}^{N-1} A_{15} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{21} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{22} \lambda_j^i & \sum_{i=0}^{N-1} A_{23} \lambda_j^i & \sum_{i=0}^{N-1} A_{24} \lambda_j^i & \sum_{i=0}^{N-1} A_{25} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{31} \lambda_j^i & \sum_{i=0}^{N-1} A_{32} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{33} \lambda_j^i & \sum_{i=0}^{N-1} A_{34} \lambda_j^i & \sum_{i=0}^{N-1} A_{35} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{41} \lambda_j^i & \sum_{i=0}^{N-1} A_{42} \lambda_j^i & \sum_{i=0}^{N-1} A_{43} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{44} \lambda_j^i & \sum_{i=0}^{N-1} A_{45} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{51} \lambda_j^i & \sum_{i=0}^{N-1} A_{52} \lambda_j^i & \sum_{i=0}^{N-1} A_{53} \lambda_j^i & \sum_{i=0}^{N-1} A_{54} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{55} \lambda_j^i \end{pmatrix}, \tag{46}$$

and

$$\begin{cases} \sum_{i=0}^{N-1} (A_{11}^{(i)} + A_{12}^{(i)} M_j^{(1)} + A_{13}^{(i)} M_j^{(2)} + A_{14}^{(i)} M_j^{(3)} + A_{15}^{(i)} M_j^{(4)}) \lambda_j^i = -\lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{21}^{(i)} + A_{22}^{(i)} M_j^{(1)} + A_{23}^{(i)} M_j^{(2)} + A_{24}^{(i)} M_j^{(3)} + A_{25}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(1)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{31}^{(i)} + A_{32}^{(i)} M_j^{(1)} + A_{33}^{(i)} M_j^{(2)} + A_{34}^{(i)} M_j^{(3)} + A_{35}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(2)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{41}^{(i)} + A_{42}^{(i)} M_j^{(1)} + A_{43}^{(i)} M_j^{(2)} + A_{44}^{(i)} M_j^{(3)} + A_{45}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(3)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{51}^{(i)} + A_{52}^{(i)} M_j^{(1)} + A_{53}^{(i)} M_j^{(2)} + A_{54}^{(i)} M_j^{(3)} + A_{55}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(4)} \lambda_j^N. \end{cases} \tag{47}$$

The N-soliton solutions of Eq. (1) on zero and plane wave backgrounds are obtained according to Cramer’s rule in Eq. (47) as following forms:

$$\begin{cases} \tilde{q}_1 = e^{-it} + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = 2A_{51}^{(N-1)}, \end{cases} \tag{48}$$

where

$$A_{21}^{(N-1)} = \frac{\Delta A_{21}^{(N-1)}}{\Delta}, \quad A_{31}^{(N-1)} = \frac{\Delta A_{31}^{(N-1)}}{\Delta}, \quad A_{41}^{(N-1)} = \frac{\Delta A_{41}^{(N-1)}}{\Delta}, \quad A_{51}^{(N-1)} = \frac{\Delta A_{51}^{(N-1)}}{\Delta}, \tag{49}$$

with

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & \dots & \lambda_1^{(N-1)} \omega_1 \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & \dots & \lambda_2^{(N-1)} \omega_2 \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{5N} e^{\theta_{5N,1}} & \frac{1}{H} e^{\theta_{5N,2}} & \frac{1}{H} e^{\theta_{5N,3}} & \frac{1}{H} e^{\theta_{5N,4}} & \dots & \lambda_{5N}^{(N-1)} \omega_{5N} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{H} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{vmatrix}, \\ \Delta A_{21}^{(N-1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \dots & -\omega_1 \lambda_1^{(N)} e^{\theta_{1,1}} & \omega_1 \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \dots & -\omega_2 \lambda_2^{(N)} e^{\theta_{2,1}} & \omega_2 \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{5N} e^{\theta_{5N,1}} & \frac{1}{H} e^{\theta_{5N,2}} & \dots & -\omega_{5N} \lambda_{5N}^{(N)} e^{\theta_{5N,1}} & -\omega_{5N} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{H} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{vmatrix}, \\ \Delta A_{31}^{(N-1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \dots & -\frac{1}{H} \lambda_1^{(N)} e^{\theta_{1,2}} & \omega_1 \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \dots & -\frac{1}{H} \lambda_2^{(N)} e^{\theta_{2,2}} & \omega_2 \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{5N} e^{\theta_{5N,1}} & \frac{1}{H} e^{\theta_{5N,2}} & \dots & -\frac{1}{H} \lambda_{5N}^{(N)} e^{\theta_{5N,2}} & \omega_{5N} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{H} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{vmatrix}, \\ \Delta A_{41}^{(N-1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \dots & -\frac{1}{H} \lambda_1^{(N)} e^{\theta_{1,3}} & \omega_1 \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \omega_1 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \dots & -\frac{1}{H} \lambda_2^{(N)} e^{\theta_{2,3}} & \omega_2 \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_1 e^{\theta_{5N,1}} & \frac{1}{H} e^{\theta_{5N,2}} & \dots & -\frac{1}{H} \lambda_{5N}^{(N)} e^{\theta_{5N,3}} & \omega_{5N} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{H} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{vmatrix}, \\ \Delta A_{51}^{(N-1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \dots & -\frac{1}{H} \lambda_1^{(N)} e^{\theta_{1,4}} & \omega_1 \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \dots & -\frac{1}{H} \lambda_2^{(N)} e^{\theta_{2,4}} & \omega_2 \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{5N} e^{\theta_{5N,1}} & \frac{1}{H} e^{\theta_{5N,2}} & \dots & -\frac{1}{H} \lambda_{5N}^{(N)} e^{\theta_{5N,4}} & \omega_{5N} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{H} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{vmatrix}, \end{aligned} \tag{50}$$

with $\omega_j = \frac{K}{H} (\lambda_j + i\sqrt{1 - \lambda_j^2})$, where i is an imaginary number, then $\theta_{j,1} = -2it + G_{(1,j)}$, $\theta_{j,i} = (-i\lambda_j - \sqrt{1 - \lambda_j^2})x - (i\lambda_j^2 + i)t + G_{(1,j)}$, where the ranges of the subscripts j, i of θ are $1 \leq i \leq 4, 1 \leq j \leq 5N$.

In order to obtain 1-soliton solutions of the four-component coupled NLS equations on zero and plane wave backgrounds, we consider $N = 1$ in Eqs. (46) and (47), and obtain the matrix T

$$T = \begin{pmatrix} \lambda + A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} & A_{14}^{(0)} & A_{15}^{(0)} \\ A_{21}^{(0)} & \lambda + A_{22}^{(0)} & A_{23}^{(0)} & A_{24}^{(0)} & A_{25}^{(0)} \\ A_{31}^{(0)} & A_{32}^{(0)} & \lambda + A_{33}^{(0)} & A_{34}^{(0)} & A_{35}^{(0)} \\ A_{41}^{(0)} & A_{42}^{(0)} & A_{43}^{(0)} & \lambda + A_{44}^{(0)} & A_{45}^{(0)} \\ A_{51}^{(0)} & A_{52}^{(0)} & A_{53}^{(0)} & A_{54}^{(0)} & \lambda + A_{55}^{(0)} \end{pmatrix}, \tag{51}$$

and

$$\begin{cases} A_{11}^{(0)} + A_{12}^{(0)} M_j^{(1)} + A_{13}^{(0)} M_j^{(2)} + A_{14}^{(0)} M_j^{(3)} + A_{15}^{(0)} M_j^{(4)} = -\lambda_j, \\ A_{21}^{(0)} + A_{22}^{(0)} M_j^{(1)} + A_{23}^{(0)} M_j^{(2)} + A_{24}^{(0)} M_j^{(3)} + A_{25}^{(0)} M_j^{(4)} = -M_j^{(1)} \lambda_j, \\ A_{31}^{(0)} + A_{32}^{(0)} M_j^{(1)} + A_{33}^{(0)} M_j^{(2)} + A_{34}^{(0)} M_j^{(3)} + A_{35}^{(0)} M_j^{(4)} = -M_j^{(2)} \lambda_j, \\ A_{41}^{(0)} + A_{42}^{(0)} M_j^{(1)} + A_{43}^{(0)} M_j^{(2)} + A_{44}^{(0)} M_j^{(3)} + A_{45}^{(0)} M_j^{(4)} = -M_j^{(3)} \lambda_j, \\ A_{51}^{(0)} + A_{52}^{(0)} M_j^{(1)} + A_{53}^{(0)} M_j^{(2)} + A_{54}^{(0)} M_j^{(3)} + A_{55}^{(0)} M_j^{(4)} = -M_j^{(4)} \lambda_j. \end{cases} \tag{52}$$

According to Eq. (52) and Cramer’s rule, we can derive the systems

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} \\ 1 & \omega_3 e^{\theta_{3,1}} & \frac{1}{H} e^{\theta_{3,2}} & \frac{1}{H} e^{\theta_{3,3}} & \frac{1}{H} e^{\theta_{3,4}} \\ 1 & \omega_4 e^{\theta_{4,1}} & \frac{1}{H} e^{\theta_{4,2}} & \frac{1}{H} e^{\theta_{4,3}} & \frac{1}{H} e^{\theta_{4,4}} \\ 1 & \omega_5 e^{\theta_{5,1}} & \frac{1}{H} e^{\theta_{5,2}} & \frac{1}{H} e^{\theta_{5,3}} & \frac{1}{H} e^{\theta_{5,4}} \end{vmatrix}, \quad \Delta A_{21}^{(0)} = \begin{vmatrix} -\omega_1 \lambda_1 e^{\theta_{1,1}} & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} \\ -\omega_2 \lambda_2 e^{\theta_{2,1}} & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} \\ -\omega_3 \lambda_3 e^{\theta_{3,1}} & \omega_3 e^{\theta_{3,1}} & \frac{1}{H} e^{\theta_{3,2}} & \frac{1}{H} e^{\theta_{3,3}} & \frac{1}{H} e^{\theta_{3,4}} \\ -\omega_4 \lambda_4 e^{\theta_{4,1}} & \omega_4 e^{\theta_{4,1}} & \frac{1}{H} e^{\theta_{4,2}} & \frac{1}{H} e^{\theta_{4,3}} & \frac{1}{H} e^{\theta_{4,4}} \\ -\omega_5 \lambda_5 e^{\theta_{5,1}} & \omega_5 e^{\theta_{5,1}} & \frac{1}{H} e^{\theta_{5,2}} & \frac{1}{H} e^{\theta_{5,3}} & \frac{1}{H} e^{\theta_{5,4}} \end{vmatrix}, \\ \Delta A_{31}^{(0)} &= \begin{vmatrix} -\frac{1}{H} \lambda_1 e^{\theta_{1,2}} & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} \\ -\frac{1}{H} \lambda_2 e^{\theta_{2,2}} & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} \\ -\frac{1}{H} \lambda_3 e^{\theta_{3,2}} & \omega_3 e^{\theta_{3,1}} & \frac{1}{H} e^{\theta_{3,2}} & \frac{1}{H} e^{\theta_{3,3}} & \frac{1}{H} e^{\theta_{3,4}} \\ -\frac{1}{H} \lambda_4 e^{\theta_{4,2}} & \omega_4 e^{\theta_{4,1}} & \frac{1}{H} e^{\theta_{4,2}} & \frac{1}{H} e^{\theta_{4,3}} & \frac{1}{H} e^{\theta_{4,4}} \\ -\frac{1}{H} \lambda_5 e^{\theta_{5,2}} & \omega_5 e^{\theta_{5,1}} & \frac{1}{H} e^{\theta_{5,2}} & \frac{1}{H} e^{\theta_{5,3}} & \frac{1}{H} e^{\theta_{5,4}} \end{vmatrix}, \quad \Delta A_{41}^{(0)} = \begin{vmatrix} -\frac{1}{H} \lambda_1 e^{\theta_{1,3}} & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} \\ -\frac{1}{H} \lambda_2 e^{\theta_{2,3}} & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} \\ -\frac{1}{H} \lambda_3 e^{\theta_{3,3}} & \omega_3 e^{\theta_{3,1}} & \frac{1}{H} e^{\theta_{3,2}} & \frac{1}{H} e^{\theta_{3,3}} & \frac{1}{H} e^{\theta_{3,4}} \\ -\frac{1}{H} \lambda_4 e^{\theta_{4,3}} & \omega_4 e^{\theta_{4,1}} & \frac{1}{H} e^{\theta_{4,2}} & \frac{1}{H} e^{\theta_{4,3}} & \frac{1}{H} e^{\theta_{4,4}} \\ -\frac{1}{H} \lambda_5 e^{\theta_{5,3}} & \omega_5 e^{\theta_{5,1}} & \frac{1}{H} e^{\theta_{5,2}} & \frac{1}{H} e^{\theta_{5,3}} & \frac{1}{H} e^{\theta_{5,4}} \end{vmatrix}, \\ \Delta A_{51}^{(0)} &= \begin{vmatrix} -\frac{1}{H} \lambda_1 e^{\theta_{1,4}} & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} \\ -\frac{1}{H} \lambda_2 e^{\theta_{2,4}} & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} \\ -\frac{1}{H} \lambda_3 e^{\theta_{3,4}} & \omega_3 e^{\theta_{3,1}} & \frac{1}{H} e^{\theta_{3,2}} & \frac{1}{H} e^{\theta_{3,3}} & \frac{1}{H} e^{\theta_{3,4}} \\ -\frac{1}{H} \lambda_4 e^{\theta_{4,4}} & \omega_4 e^{\theta_{4,1}} & \frac{1}{H} e^{\theta_{4,2}} & \frac{1}{H} e^{\theta_{4,3}} & \frac{1}{H} e^{\theta_{4,4}} \\ -\frac{1}{H} \lambda_5 e^{\theta_{5,4}} & \omega_5 e^{\theta_{5,1}} & \frac{1}{H} e^{\theta_{5,2}} & \frac{1}{H} e^{\theta_{5,3}} & \frac{1}{H} e^{\theta_{5,4}} \end{vmatrix}. \end{aligned} \tag{53}$$

here $\omega_j = \frac{K}{H} (\lambda_j + i\sqrt{1 - \lambda_j^2})$, the i is an imaginary number, then $\theta_{j,1} = -2it + G_{(1,j)}$, $\theta_{j,i} = (-i\lambda_j - \sqrt{1 - \lambda_j^2})x - (i\lambda_j^2 + i)t + G_{(1,j)}$, where the ranges of the subscripts j and i of θ are $1 \leq i \leq 4$, $1 \leq j \leq 5$.

Based on Eq. (49), we can obtain the following systems

$$A_{21}^{(0)} = \frac{\Delta A_{21}^{(0)}}{\Delta_1}, A_{31}^{(0)} = \frac{\Delta A_{31}^{(0)}}{\Delta_1}, A_{41}^{(0)} = \frac{\Delta A_{41}^{(0)}}{\Delta_1}, A_{51}^{(0)} = \frac{\Delta A_{51}^{(0)}}{\Delta_1}, \tag{54}$$

the analytic 1-soliton solutions of the four-component CNLS equations are obtained on zero and plane wave backgrounds by Darboux transformation as following

$$\begin{cases} \tilde{q}_1 = e^{-it} + 2A_{21}^{(0)}, \\ \tilde{q}_2 = 2A_{31}^{(0)}, \\ \tilde{q}_3 = 2A_{41}^{(0)}, \\ \tilde{q}_4 = 2A_{51}^{(0)}. \end{cases} \tag{55}$$

In order to obtain 2-soliton solutions of the four-component CNLS equations on zero and plane wave backgrounds, we consider $N = 2$ in Eqs. (46) and (47), and obtain the matrix T

$$T = \begin{pmatrix} \lambda^2 + A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} & A_{14}^{(1)} & A_{15}^{(1)} \\ A_{21}^{(1)} & \lambda^2 + A_{22}^{(1)} & A_{23}^{(1)} & A_{24}^{(1)} & A_{25}^{(1)} \\ A_{31}^{(1)} & A_{32}^{(1)} & \lambda^2 + A_{33}^{(1)} & A_{34}^{(1)} & A_{35}^{(1)} \\ A_{41}^{(1)} & A_{42}^{(1)} & A_{43}^{(1)} & \lambda^2 + A_{44}^{(1)} & A_{45}^{(1)} \\ A_{51}^{(1)} & A_{52}^{(1)} & A_{53}^{(1)} & A_{54}^{(0)} & \lambda^2 + A_{55}^{(1)} \end{pmatrix}, \tag{56}$$

and

$$\begin{cases} A_{11}^{(1)} + A_{12}^{(1)} M_j^{(1)} + A_{13}^{(1)} M_j^{(2)} + A_{14}^{(1)} M_j^{(3)} + A_{15}^{(1)} M_j^{(4)} = -\lambda_j^2, \\ A_{21}^{(1)} + A_{22}^{(1)} M_j^{(1)} + A_{23}^{(1)} M_j^{(2)} + A_{24}^{(1)} M_j^{(3)} + A_{25}^{(1)} M_j^{(4)} = -M_j^{(1)} \lambda_j^2, \\ A_{31}^{(1)} + A_{32}^{(1)} M_j^{(1)} + A_{33}^{(1)} M_j^{(2)} + A_{34}^{(1)} M_j^{(3)} + A_{35}^{(1)} M_j^{(4)} = -M_j^{(2)} \lambda_j^2, \\ A_{41}^{(1)} + A_{42}^{(1)} M_j^{(1)} + A_{43}^{(1)} M_j^{(2)} + A_{44}^{(1)} M_j^{(3)} + A_{45}^{(1)} M_j^{(4)} = -M_j^{(3)} \lambda_j^2, \\ A_{51}^{(1)} + A_{52}^{(1)} M_j^{(1)} + A_{53}^{(1)} M_j^{(2)} + A_{54}^{(1)} M_j^{(3)} + A_{55}^{(1)} M_j^{(4)} = -M_j^{(4)} \lambda_j^2. \end{cases} \tag{57}$$

According to Eq. (57) and Cramer’s rule, we can derive the systems

$$\begin{aligned} \Delta_2 &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & \lambda_1 & \omega_1 \lambda_1 e^{\theta_{1,1}} & \frac{1}{H} \lambda_1 e^{\theta_{1,2}} & \frac{1}{H} \lambda_1 e^{\theta_{1,3}} & \frac{1}{H} \lambda_1 e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & \lambda_2 & \omega_2 \lambda_2 e^{\theta_{2,1}} & \frac{1}{H} \lambda_2 e^{\theta_{2,2}} & \frac{1}{H} \lambda_2 e^{\theta_{2,3}} & \frac{1}{H} \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{10} e^{\theta_{10,1}} & \frac{1}{H} e^{\theta_{10,2}} & \frac{1}{H} e^{\theta_{10,3}} & \frac{1}{H} e^{\theta_{10,4}} & \lambda_{10} & \omega_{10} \lambda_{10} e^{\theta_{10,1}} & \frac{1}{H} \lambda_{10} e^{\theta_{10,2}} & \frac{1}{H} \lambda_{10} e^{\theta_{10,3}} & \frac{1}{H} \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\ \Delta A_{21}^{(1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & -\omega - 1\lambda_1^2 e^{\theta_{1,1}} & \omega_1 \lambda_1 e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1 e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & -\omega_2 \lambda_2^2 e^{\theta_{2,1}} & \omega_2 \lambda_2 e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{10} e^{\theta_{10,1}} & \frac{1}{H} e^{\theta_{10,2}} & \frac{1}{H} e^{\theta_{10,3}} & \frac{1}{H} e^{\theta_{10,4}} & -\omega_{10} \lambda_{10}^2 e^{\theta_{10,1}} & \omega_{10} \lambda_{10} e^{\theta_{10,1}} & \dots & \frac{1}{H} \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\ \Delta A_{31}^{(1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & -\frac{1}{H} \lambda_1^2 e^{\theta_{1,2}} & \omega_1 \lambda_1 e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1 e^{\theta_{1,4}} \\ 1 & \omega_1 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & -\frac{1}{H} \lambda_2^2 e^{\theta_{2,2}} & \omega_2 \lambda_2 e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_1 e^{\theta_{10,1}} & \frac{1}{H} e^{\theta_{10,2}} & \frac{1}{H} e^{\theta_{10,3}} & \frac{1}{H} e^{\theta_{10,4}} & -\frac{1}{H} \lambda_{10}^2 e^{\theta_{10,2}} & \omega_{10} \lambda_{10} e^{\theta_{10,1}} & \dots & \frac{1}{H} \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\ \Delta A_{41}^{(1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & -\frac{1}{H} \lambda_1^2 e^{\theta_{1,3}} & \omega_1 \lambda_1 e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1 e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & -\frac{1}{H} \lambda_2^2 e^{\theta_{2,3}} & \omega_2 \lambda_2 e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{10} e^{\theta_{10,1}} & \frac{1}{H} e^{\theta_{10,2}} & \frac{1}{H} e^{\theta_{10,3}} & \frac{1}{H} e^{\theta_{10,4}} & -\frac{1}{H} \lambda_{10}^2 e^{\theta_{10,3}} & \omega_{10} \lambda_{10} e^{\theta_{10,1}} & \dots & \frac{1}{H} \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \\ \Delta A_{51}^{(1)} &= \begin{vmatrix} 1 & \omega_1 e^{\theta_{1,1}} & \frac{1}{H} e^{\theta_{1,2}} & \frac{1}{H} e^{\theta_{1,3}} & \frac{1}{H} e^{\theta_{1,4}} & -\frac{1}{H} \lambda_1^2 e^{\theta_{1,4}} & \omega_1 \lambda_1 e^{\theta_{1,1}} & \dots & \frac{1}{H} \lambda_1 e^{\theta_{1,4}} \\ 1 & \omega_2 e^{\theta_{2,1}} & \frac{1}{H} e^{\theta_{2,2}} & \frac{1}{H} e^{\theta_{2,3}} & \frac{1}{H} e^{\theta_{2,4}} & -\frac{1}{H} \lambda_2^2 e^{\theta_{2,4}} & \omega_2 \lambda_2 e^{\theta_{2,1}} & \dots & \frac{1}{H} \lambda_2 e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \omega_{10} e^{\theta_{10,1}} & \frac{1}{H} e^{\theta_{10,2}} & \frac{1}{H} e^{\theta_{10,3}} & \frac{1}{H} e^{\theta_{10,4}} & -\frac{1}{H} \lambda_{10}^2 e^{\theta_{10,4}} & \omega_{10} \lambda_{10} e^{\theta_{10,1}} & \dots & \frac{1}{H} \lambda_{10} e^{\theta_{10,4}} \end{vmatrix}, \end{aligned} \tag{58}$$

where $\omega_j = \frac{\kappa}{H} (\lambda_j + i\sqrt{1 - \lambda_j^2})$, the i is an imaginary number, then $\theta_{j,1} = -2it + G_{(1,j)}$, $\theta_{j,i} = (-i\lambda_j - \sqrt{1 - \lambda_j^2})x - (i\lambda_j^2 + i)t + G_{(1,j)}$, the ranges of the subscripts j and i of θ are $1 \leq i \leq 4, 1 \leq j \leq 10$

Based on the Eq. (49), we can obtain the following systems

$$A_{21}^{(1)} = \frac{\Delta A_{21}^{(1)}}{\Delta_2}, A_{31}^{(1)} = \frac{\Delta A_{31}^{(1)}}{\Delta_2}, A_{41}^{(1)} = \frac{\Delta A_{41}^{(1)}}{\Delta_2}, A_{51}^{(1)} = \frac{\Delta A_{51}^{(1)}}{\Delta_2}, \tag{59}$$

the analytic 2-soliton solutions of four-component CNLS equations are obtained on zero and plane wave backgrounds by Darboux transformation as following

$$\begin{cases} \tilde{q}_1 = e^{-it} + 2A_{21}^{(1)}, \\ \tilde{q}_2 = 2A_{31}^{(1)}, \\ \tilde{q}_3 = 2A_{41}^{(1)}, \\ \tilde{q}_4 = 2A_{51}^{(1)}. \end{cases} \tag{60}$$

We can obtain a new class of dark-bright-bright-bright soliton solutions(60), which admits the one-valley dark soliton in component q_1 and triple-hump bright solitons in the other three components.

N-soliton solutions on complex wave backgrounds. Next we make two non-zero seed solutions, so that we can get the novel N-soliton solutions on complex wave backgrounds. In order to obtain the N-soliton solution formula of the four-component coupled NLS equations on complex wave backgrounds with Darboux transformation, we give a set of seed solutions $q_1 = q_2 = e^{-2it}, q_3 = q_4 = 0$ and substitute these solutions into Eqs. (4) and (5), which can get basic solutions for CNLS equations:

$$\varphi = \begin{pmatrix} Le^{(\sqrt{-\lambda^2+2})x+5it} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \phi = \begin{pmatrix} 0 \\ e^{(\sqrt{-\lambda^2+2})x+it} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 0 \\ 0 \\ e^{(\sqrt{-\lambda^2+2})x+it} \\ 0 \\ 0 \end{pmatrix},$$

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{-i\lambda x - i\lambda^2 t} \\ 0 \end{pmatrix}, \varepsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{-i\lambda x - i\lambda^2 t} \end{pmatrix}.$$
(61)

with $L = \frac{4i\lambda}{(i\lambda - \sqrt{-\lambda^2+2})(\lambda^2-2)}$.

Substituting Eq. (61) into Eq. (8), we give rise to

$$M_j^{(1)} = \frac{1}{L} e^{-4it+G_{1j}}, M_j^{(2)} = \frac{1}{L} e^{-4it+G_{2j}},$$

$$M_j^{(3)} = \frac{1}{L} e^{(-i\lambda - \sqrt{-\lambda^2+2})x - (i\lambda^2+5i)t+G_{3j}}, M_j^{(4)} = \frac{1}{L} e^{(-i\lambda - \sqrt{-\lambda^2+2})x - (i\lambda^2+5i)t+G_{4j}},$$
(62)

where $e^{G_{1j}} = v_j^{(11)}$, $e^{G_{2j}} = v_j^{(22)}$, $e^{G_{3j}} = v_j^{(33)}$, $e^{G_{4j}} = v_j^{(44)}$, $1 \leq j \leq 5N$.

In order to obtain the N-soliton solution of Eq. (1) on complex wave backgrounds, we consider matrix T as follow:

$$T = \begin{pmatrix} \lambda^N + \sum_{i=0}^{N-1} A_{11} \lambda_j^i & \sum_{i=0}^{N-1} A_{12} \lambda_j^i & \sum_{i=0}^{N-1} A_{13} \lambda_j^i & \sum_{i=0}^{N-1} A_{14} \lambda_j^i & \sum_{i=0}^{N-1} A_{15} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{21} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{22} \lambda_j^i & \sum_{i=0}^{N-1} A_{23} \lambda_j^i & \sum_{i=0}^{N-1} A_{24} \lambda_j^i & \sum_{i=0}^{N-1} A_{25} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{31} \lambda_j^i & \sum_{i=0}^{N-1} A_{32} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{33} \lambda_j^i & \sum_{i=0}^{N-1} A_{34} \lambda_j^i & \sum_{i=0}^{N-1} A_{35} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{41} \lambda_j^i & \sum_{i=0}^{N-1} A_{42} \lambda_j^i & \sum_{i=0}^{N-1} A_{43} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{44} \lambda_j^i & \sum_{i=0}^{N-1} A_{45} \lambda_j^i \\ \sum_{i=0}^{N-1} A_{51} \lambda_j^i & \sum_{i=0}^{N-1} A_{52} \lambda_j^i & \sum_{i=0}^{N-1} A_{53} \lambda_j^i & \sum_{i=0}^{N-1} A_{54} \lambda_j^i & \lambda^N + \sum_{i=0}^{N-1} A_{55} \lambda_j^i \end{pmatrix},$$
(63)

and

$$\begin{cases} \sum_{i=0}^{N-1} (A_{11}^{(i)} + A_{12}^{(i)} M_j^{(1)} + A_{13}^{(i)} M_j^{(2)} + A_{14}^{(i)} M_j^{(3)} + A_{15}^{(i)} M_j^{(4)}) \lambda_j^i = -\lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{21}^{(i)} + A_{22}^{(i)} M_j^{(1)} + A_{23}^{(i)} M_j^{(2)} + A_{24}^{(i)} M_j^{(3)} + A_{25}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(1)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{31}^{(i)} + A_{32}^{(i)} M_j^{(1)} + A_{33}^{(i)} M_j^{(2)} + A_{34}^{(i)} M_j^{(3)} + A_{35}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(2)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{41}^{(i)} + A_{42}^{(i)} M_j^{(1)} + A_{43}^{(i)} M_j^{(2)} + A_{44}^{(i)} M_j^{(3)} + A_{45}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(3)} \lambda_j^N, \\ \sum_{i=0}^{N-1} (A_{51}^{(i)} + A_{52}^{(i)} M_j^{(1)} + A_{53}^{(i)} M_j^{(2)} + A_{54}^{(i)} M_j^{(3)} + A_{55}^{(i)} M_j^{(4)}) \lambda_j^i = -M_j^{(4)} \lambda_j^N. \end{cases}$$
(64)

The N-soliton solutions of Eq. (1) are obtained according to Cramer's rule on complex wave backgrounds in Eq. (64) as following forms:

$$\begin{cases} \tilde{q}_1 = e^{-2it} + 2A_{21}^{(N-1)}, \\ \tilde{q}_2 = e^{-2it} + 2A_{31}^{(N-1)}, \\ \tilde{q}_3 = 2A_{41}^{(N-1)}, \\ \tilde{q}_4 = 2A_{51}^{(N-1)}, \end{cases}$$
(65)

where

$$A_{21}^{(N-1)} = \frac{\Delta A_{21}^{(N-1)}}{\Delta}, A_{31}^{(N-1)} = \frac{\Delta A_{31}^{(N-1)}}{\Delta}, A_{41}^{(N-1)} = \frac{\Delta A_{41}^{(N-1)}}{\Delta}, A_{51}^{(N-1)} = \frac{\Delta A_{51}^{(N-1)}}{\Delta},$$
(66)

among them

$$\Delta = \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} e^{\theta_{1,4}} & \dots & \lambda_1^{(N-1)} \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_2} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} e^{\theta_{2,4}} & \dots & \lambda_2^{(N-1)} \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_{5N}} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} e^{\theta_{5N,4}} & \dots & \lambda_{5N}^{(N-1)} \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},$$

$$\Delta A_{21}^{(N-1)} = \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} e^{\theta_{1,2}} & \dots & -\frac{1}{L_1} \lambda_1^{(N)} e^{\theta_{1,1}} & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_2} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} e^{\theta_{2,2}} & \dots & -\frac{1}{L_2} \lambda_2^{(N)} e^{\theta_{2,1}} & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_{5N}} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} e^{\theta_{5N,2}} & \dots & -\frac{1}{L_{5N}} \lambda_{5N}^{(N)} e^{\theta_{5N,1}} & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},$$

$$\Delta A_{31}^{(N-1)} = \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} e^{\theta_{1,2}} & \dots & -\frac{1}{L_1} \lambda_1^{(N)} e^{\theta_{1,2}} & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_2} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} e^{\theta_{2,2}} & \dots & -\frac{1}{L_2} \lambda_2^{(N)} e^{\theta_{2,2}} & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_{5N}} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} e^{\theta_{5N,2}} & \dots & -\frac{1}{L_{5N}} \lambda_{5N}^{(N)} e^{\theta_{5N,2}} & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},$$

$$\Delta A_{41}^{(N-1)} = \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \dots & -\frac{1}{L_1} \lambda_1^{(N)} e^{\theta_{1,3}} & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_1} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \dots & -\frac{1}{L_2} \lambda_2^{(N)} e^{\theta_{2,3}} & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_1} e^{\theta_{5N,1}} & \frac{1}{L_{5N}} e^{\theta_{5N,2}} & \dots & -\frac{1}{L_{5N}} \lambda_{5N}^{(N)} e^{\theta_{5N,3}} & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},$$

$$\Delta A_{51}^{(N-1)} = \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \dots & -\frac{1}{L_1} \lambda_1^{(N)} e^{\theta_{1,4}} & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,1}} & \dots & \frac{1}{L_1} \lambda_1^{(N-1)} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \dots & -\frac{1}{L_2} \lambda_2^{(N)} e^{\theta_{2,4}} & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,1}} & \dots & \frac{1}{L_2} \lambda_2^{(N-1)} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_{5N}} e^{\theta_{5N,1}} & \frac{1}{L_{5N}} e^{\theta_{5N,2}} & \dots & -\frac{1}{L_{5N}} \lambda_{5N}^{(N)} e^{\theta_{5N,4}} & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,1}} & \dots & \frac{1}{L_{5N}} \lambda_{5N}^{(N-1)} e^{\theta_{5N,4}} \end{pmatrix},$$

w i t h $L_j = \frac{4i\lambda_j}{(i\lambda_j - \sqrt{-\lambda_j^2 + 2})(\lambda_j^2 - 2)}$, $\theta_{1,1} = -4it + G_{(1,1)}$, $\theta_{1,2} = -4it + G_{(2,1)}$,

$\theta_{1,4} = \left(-i\lambda - \sqrt{-\lambda^2 + 2} \right) x - (i\lambda^2 + 5i)t + G_{4,1}$, $\theta_{2,1} = -4it + G_{(1,2)}$, $\theta_{2,2} = -4it + G_{(2,2)}$,

$\theta_{2,4} = \left(-i\lambda - \sqrt{-\lambda^2 + 2} \right) x - (i\lambda^2 + 5i)t + G_{4,2}$, $\theta_{5N,1} = -4it + G_{(1,5N)}$, $\theta_{5N,2} = -4it + G_{(2,5N)}$,

$\theta_{5N,4} = \left(-i\lambda - \sqrt{-\lambda^2 + 2} \right) x - (i\lambda^2 + 5i)t + G_{4,5N}$.

In order to obtain 1-soliton solutions of the four-component coupled NLS equations on complex wave backgrounds, we consider $N = 1$ in Eqs. (63) and (64), and obtain the matrix T

$$T = \begin{pmatrix} \lambda + A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} & A_{14}^{(0)} & A_{15}^{(0)} \\ A_{21}^{(0)} & \lambda + A_{22}^{(0)} & A_{23}^{(0)} & A_{24}^{(0)} & A_{25}^{(0)} \\ A_{31}^{(0)} & A_{32}^{(0)} & \lambda + A_{33}^{(0)} & A_{34}^{(0)} & A_{35}^{(0)} \\ A_{41}^{(0)} & A_{42}^{(0)} & A_{43}^{(0)} & \lambda + A_{44}^{(0)} & A_{45}^{(0)} \\ A_{51}^{(0)} & A_{52}^{(0)} & A_{53}^{(0)} & A_{54}^{(0)} & \lambda + A_{55}^{(0)} \end{pmatrix},$$

and

$$\begin{cases} A_{11}^{(0)} + A_{12}^{(0)} M_j^{(1)} + A_{13}^{(0)} M_j^{(2)} + A_{14}^{(0)} M_j^{(3)} + A_{15}^{(0)} M_j^{(4)} = -\lambda_j, \\ A_{21}^{(0)} + A_{22}^{(0)} M_j^{(1)} + A_{23}^{(0)} M_j^{(2)} + A_{24}^{(0)} M_j^{(3)} + A_{25}^{(0)} M_j^{(4)} = -M_j^{(1)} \lambda_j, \\ A_{31}^{(0)} + A_{32}^{(0)} M_j^{(1)} + A_{33}^{(0)} M_j^{(2)} + A_{34}^{(0)} M_j^{(3)} + A_{35}^{(0)} M_j^{(4)} = -M_j^{(2)} \lambda_j, \\ A_{41}^{(0)} + A_{42}^{(0)} M_j^{(1)} + A_{43}^{(0)} M_j^{(2)} + A_{44}^{(0)} M_j^{(3)} + A_{45}^{(0)} M_j^{(4)} = -M_j^{(3)} \lambda_j, \\ A_{51}^{(0)} + A_{52}^{(0)} M_j^{(1)} + A_{53}^{(0)} M_j^{(2)} + A_{54}^{(0)} M_j^{(3)} + A_{55}^{(0)} M_j^{(4)} = -M_j^{(4)} \lambda_j. \end{cases}$$

According to Eq. (69) and Cramer's rule, we can derive the systems:

$$\begin{aligned}
 \Delta_1 &= \begin{pmatrix} 1 & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,4}} \\ 1 & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{1}{L_5} e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,4}} \end{pmatrix}, \\
 \Delta A_{21}^{(0)} &= \begin{pmatrix} -\frac{1}{L_1} \lambda_1 e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,4}} \\ -\frac{1}{L_2} \lambda_2 e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{L_5} \lambda_5 e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,4}} \end{pmatrix}, \\
 \Delta A_{31}^{(0)} &= \begin{pmatrix} -\frac{1}{L_1} \lambda e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,4}} \\ -\frac{1}{L_2} \lambda e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{L_5} \lambda e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,4}} \end{pmatrix}, \\
 \Delta A_{41}^{(0)} &= \begin{pmatrix} -\frac{1}{L_1} \lambda_1 e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,4}} \\ -\frac{1}{L_2} \lambda_2 e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{L_5} \lambda_5 e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,4}} \end{pmatrix}, \\
 \Delta A_{51}^{(0)} &= \begin{pmatrix} -\frac{1}{L_1} \lambda_1 e^{\theta_{1,4}} & \frac{1}{L_1} e^{\theta_{1,1}} & \frac{1}{L_1} e^{\theta_{1,2}} & \frac{1}{L_1} e^{\theta_{1,3}} & \frac{1}{L_1} e^{\theta_{1,4}} \\ -\frac{1}{L_2} \lambda_2 e^{\theta_{2,4}} & \frac{1}{L_2} e^{\theta_{2,1}} & \frac{1}{L_2} e^{\theta_{2,2}} & \frac{1}{L_2} e^{\theta_{2,3}} & \frac{1}{L_2} e^{\theta_{2,4}} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{L_5} \lambda_5 e^{\theta_{5,4}} & \frac{1}{L_5} e^{\theta_{5,1}} & \frac{1}{L_5} e^{\theta_{5,2}} & \frac{1}{L_5} e^{\theta_{5,3}} & \frac{1}{L_5} e^{\theta_{5,4}} \end{pmatrix}. \tag{70}
 \end{aligned}$$

with $L_j = \frac{4i\lambda_j}{(i\lambda_j - \sqrt{-\lambda_j^2 + 2})(\lambda_j^2 - 2)}$, $\theta_{1,1} = -4it + G_{(1,1)}$, $\theta_{1,2} = -4it + G_{(2,1)}$, $\theta_{1,3} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{3,1}$, $\theta_{1,4} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{4,1}$, $\theta_{2,1} = -4it + G_{(1,2)}$, $\theta_{2,2} = -4it + G_{(2,2)}$, $\theta_{2,3} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{3,2}$, $\theta_{2,4} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{4,2}$, $\theta_{5,1} = -4it + G_{(1,5)}$, $\theta_{5,2} = -4it + G_{(2,5)}$, $\theta_{5,3} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{3,5}$, $\theta_{5,4} = (-i\lambda - \sqrt{-\lambda^2 + 2})x - (i\lambda^2 + 5i)t + G_{4,5}$.

Based on Eq. (70), we can obtain the following systems

$$A_{21}^{(0)} = \frac{\Delta A_{21}^{(0)}}{\Delta_1}, A_{31}^{(0)} = \frac{\Delta A_{31}^{(0)}}{\Delta_1}, A_{41}^{(0)} = \frac{\Delta A_{41}^{(0)}}{\Delta_1}, A_{51}^{(0)} = \frac{\Delta A_{51}^{(0)}}{\Delta_1}, \tag{71}$$

the analytic 1-soliton solutions of four-component CNLS equations on complex wave backgrounds are obtained by Darboux transformation as following

$$\begin{cases} \tilde{q}_1 = e^{-2it} + 2A_{21}^{(0)}, \\ \tilde{q}_2 = e^{-2it} + 2A_{31}^{(0)}, \\ \tilde{q}_3 = 2A_{41}^{(0)}, \\ \tilde{q}_4 = 2A_{51}^{(0)}. \end{cases} \tag{72}$$

The collision properties between dark-dark-bright-bright solitons (72) are considered, and the vector solitons are expected to be much more abundant than those of previously reported vector soliton collisions. Some soliton solutions and interactions of four-component CNLS equations on complex wave backgrounds are considered, which can describe the four-component BECs. Furthermore, some dark soliton solutions can exist widely in the collision processes. We investigate the wave structures of four-component NLS equations with mixed nonzero and zero wave conditions, which can be used to describe the interactions of many bodies in differential fields. And, we find some dark soliton solutions (non-zero backgrounds) and soliton-like solutions (zero backgrounds) with the aid of the Darboux transformation.

Data availability

All data generated or analysed during this study are included in this published article.

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References

1. London, F. On the Bose-Einstein condensation. *Phys. Rev.* **45**, 947–954 (1938).

2. Jie, L. *Bose-Einstein Condensate Dynamics-Nonlinear Attempts, Coherence and Instability* (Beijing Science Press, 2009).
3. Hulin, D., Mysyrowicz, A. & Guillaume, C. B. Evidence for Bose-Einstein statistics in an exciton gas. *Phys. Rev. Lett.* **45**(24), 1970–1973 (1980).
4. Anderson, M. H., Ensher, J. R., Matthews, M. R., Weiman, C. E. & Cornell, E. A. Observation of Bose-Einstein condensation in a dilute atomic vapor. *Science* **269**, 198–201 (1995).
5. Korteweg, D. J. & Vries, G. D. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Lond. Edinb. Dubl. Philos. Mag. J. Sci.* **39**(240), 422–443 (1895).
6. Ruprecht, P. A., Holland, M. J., Burnett, K. & Edwards, M. Time-dependent solution of nonlinear Schrödinger equation for Bose-condensed trapped neutral atoms. *Phys. Rev. A.* **51**(6), 4704–4711 (1995).
7. Cerimele, M. M. *et al.* Numerical solution of the Gross-Pitaevskii equation using an explicit finite-difference scheme: An application to trapped Bose-Einstein condensates. *Phys. Rev. E.* **62**(1), 1382 (2000).
8. Schlein, B. Dynamics of Bose-Einstein condensates. *New Trends Math. Phys.* **2009**, 556–589 (2009).
9. Lathio, C. G., Mohamadou, A., Kofane, T. C. & Porsezian, K. Exact quasi-soliton solutions and soliton interaction for the inhomogeneous coupled Hirota Max well Cbloch equations. *J. Optic.* **12**(8), 085202 (2010).
10. Qin, Y. H., Zhao, L. C., Yang, Z. Q. & Ling, L. M. Multivalley dark solitons in multicomponent Bose-Einstein condensates with repulsive interactions. *Phys. Rev. E.* **104**, 014201 (2021).
11. Zhou, Z. X. On the Darboux transformation for 1+2-dimensional equations. *Lett. Math. Phys.* **16**(1), 9–17 (1988).
12. Li, Y. S. *Solitons and Accumulable Systems* (Shanghai Science and Technology Education Press, 1999).
13. Li, Y. Q. *et al.* Dromion structures in the 2+1-dimensional nonlinear Schrödinger equation with a parity-time-symmetric potential. *App. Math. Lett.* **47**, 8–12 (2015).
14. Hirota, R. & Ito, M. Resonance of solitons in one dimension. *J. Phys. Soc. Jpn.* **52**(3), 744–748 (1983).
15. Ablowitz, M. J. & Clarkson, P. A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991).
16. Gu, C. H. & Hu, H. S. A unified explicit form of Bäcklund transformations for generalized hierarchies of KdV equations. *Lett. Math. Phys.* **11**(4), 325–335 (1986).
17. Priya, N. V. & Senthilvelan, M. Generalized Darboux transformation and N-th order rogue wave solution of a general coupled nonlinear Schrödinger equations. *Commun. Nonlinear. Sci. Numer. Simul.* **20**(2), 401–420 (2015).
18. Manakov, S. V. On the theory of two-dimensional stationary self-focusing electromagnetic waves. *Soviet Phys. JETP* **38**, 248–253 (1974).
19. Lv, X. & Peng, M. S. Painlevé-integrability and explicit solutions of the general two-coupled nonlinear Schrödinger system in the optical fiber communications. *Nonlinear Dyn.* **73**(1–2), 405–410 (2013).
20. Yu, F. J., Liu, C. P. & Li, L. Broken and unbroken solutions and dynamic behaviors for the mixed local-nonlocal Schrödinger equation. *Appl. Math. Lett.* **117**, 107075 (2021).
21. Li, L., Yu, F. J. & Duan, C. N. A generalized nonlocal Gross-Pitaevskii (NGP) equation with an arbitrary time-dependent linear potential. *Appl. Math. Lett.* **110**, 106584 (2020).
22. Li, L., Liu, Y. Y. & Yu, F. J. Some reverse space(RS) rational solutions for the nonlocal coupled nonlinear Schrödinger equations on the plane wave backgrounds. *Appl. Math. Lett.* **129**, 107976 (2022).
23. Yu, F. J. & Fan, R. Nonstandard bilinearization and interaction phenomenon for PT-symmetric coupled nonlocal nonlinear Schrödinger equations. *Appl. Math. Lett.* **103**, 106209 (2020).
24. Abdelwahed, H. G., El-Shewy, E. K., Abdelrahman, A. E. & Alsarhana, A. F. On the physical nonlinear (n+1)-dimensional Schrödinger equation applications. *Results Phys.* **21**, 103798 (2021).
25. Abdelrahman, A. E., Sohaly, M. A. & Alharbi, Y. F. Fundamental stochastic solutions for the conformable fractional NLSE with spatiotemporal dispersion via exponential distribution. *Phys. Scr.* **96**, 125223 (2021).
26. Lannig, S. *et al.* Collisions of three-component vector solitons in Bose-Einstein condensates. *Phys. Rev. Lett.* **125**, 170401 (2020).

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Competing interests

The authors declare no competing interests.

Additional information

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