

Surfacic networks

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Abstract

Surfacic networks are structures built upon a 2D manifold. Many systems, including transportation networks and various urban networks, fall into this category. The fluctuations of node elevations imply significant deviations from typical plane networks and require specific tools to understand their impact. Here, we present such tools, including lazy paths that minimize elevation differences, graph arduousness which measures the tiring nature of shortest paths (SPs), and the excess effort, which characterizes positive elevation variations along SPs. We illustrate these measures using toy models of surfacic networks and empirically examine pedestrian networks in selected cities. Specifically, we examine how changes in elevation affect the spatial distribution of betweenness centrality. We also demonstrate that the excess effort follows a nontrivial power law distribution, with an exponent that is not universal, which illustrates that there is a significant probability of encountering steep slopes along SPs, regardless of the elevation difference between the starting point and the destination. These findings highlight the significance of elevation fluctuations in shaping network characteristics. Surfacic networks offer a promising framework for comprehensively analyzing and modeling complex systems that are situated on or constrained to a surface environment.

Significance Statement

Networks on nonflat surfaces, such as transportation and urban systems in hilly places, require specialized analyses due to elevation fluctuations. We introduce metrics like lazy paths and graph arduousness to quantify path difficulty and the excess effort which characterizes positive elevation variations along shortest paths (SPs). By analyzing simple models and real-life pedestrian networks, we show how elevation variations affect betweenness centrality, path ruggedness, and the overall network efficiency. The additional effort required to travel from one point to another follows a broad distribution, indicating the notable occurrence that many SPs entail significant extra exertion to ascend steep slopes. Understanding how elevation fluctuations influence network navigation is essential for analyzing and modeling these systems that are situated on surfaces.

Introduction

Surfacic networks are defined by a set of nodes and edges that are embedded in a 2D manifold. This manifold could be the plane in the case of usual “plane networks” (i.e. embedded in a plane, but not necessarily planar), or a sphere, or any other rugged surface that correspond to the topography of a place. More generally, this surface could be Earth’s surface, biological membranes, or even computational surfaces. As such, they potentially constitute a fundamental concept in various fields, including geography, computer graphics, materials science, and biology. Potential applications could be found in Geographic Information Systems (GIS) applications where surface networks are used to model transportation networks, utility networks, and other spatially distributed systems on the Earth’s

surface. It is worth noting here that surfacic networks have to be differentiated from “surface network” in transport planning/geography used for transport networks that are not air or sub-surface networks. Note that the elevation of a node could also in principle represent another quantity such as the gross domestic product (GDP), average income, etc. and that surfacic networks could be used in some abstract space. Surfacic networks can be considered as a subset of spatial networks (1). A simple example of such a network is shown in Fig. 1 (see the part on toy models for details).

Most spatial networks studied so far are embedded in a 2D plane and surfacic networks generalize this by considering non flat 2D manifolds that can have curvature fluctuations. Other spatial networks could be embedded in 3D networks (such as the

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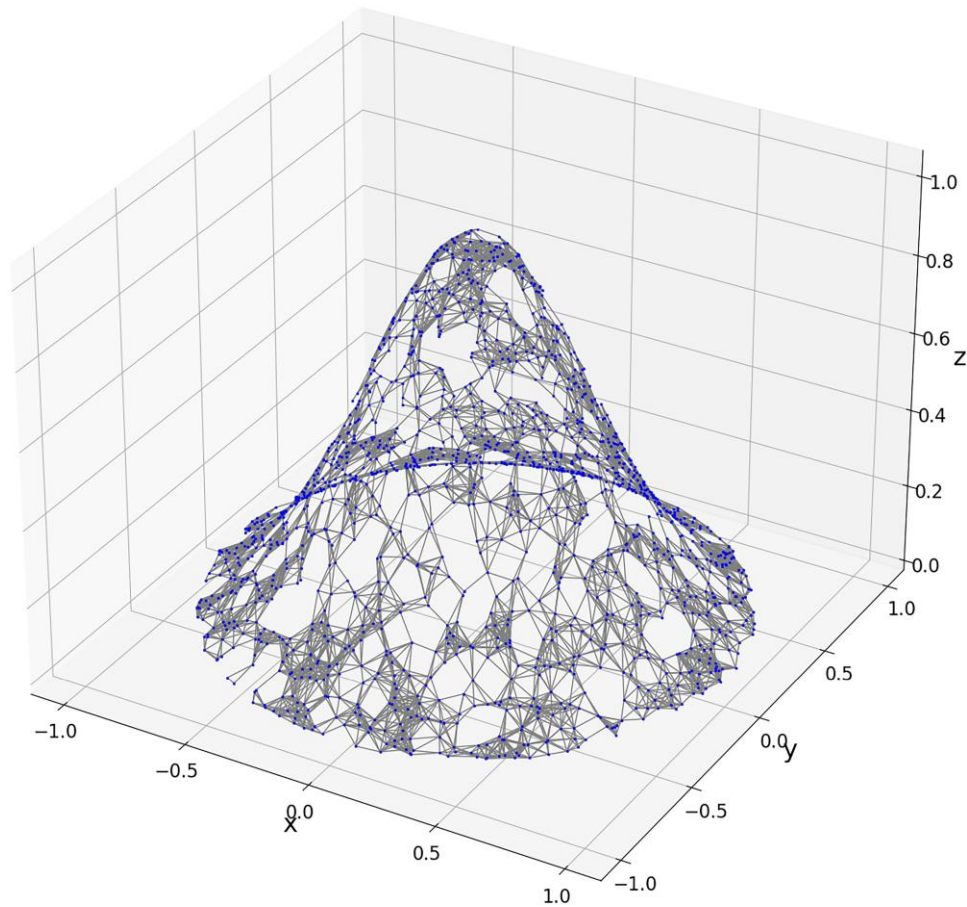


Fig. 1. Example of a surfacic network: random geometric graph on a Gaussian surface (the network is constructed here for $N = 1,500$ points and with threshold $r_0 = 0.1$, and the Gaussian surface is obtained for $z_{\max} = 1.0$ and $\sigma = 0.4$).

important case of the neuronal network (2, 3)) and could be defined as “volumetric” networks or “physical” networks (4). We therefore have the following nested inclusion between these sets of networks:

$$\text{Surfacic} \subset \text{Volumetric} \subset \text{Spatial} \subset \text{Networks}$$

We mention here the existence of the term “surface networks” that was quoted in Ref. (5) where the authors study data-driven representations for 3D triangle meshes, which are one of the prevalent objects used to represent 3D geometry.

The geometry of the surface will influence the structure and the behavior of a surfacic network. Geometric properties such as curvature, topology, and spatial constraints certainly play a crucial role in shaping the connectivity and dynamics of these networks. This is essentially due to spatial constraints imposed by the surface on which the networks are embedded. These constraints affect the arrangement of nodes and edges, as well as the navigation and flow of information or resources within the network. In contrast to spatial networks that have been thoroughly studied, the impact of the embedding topography on the network structure has not been systematically considered in the literature. There are a few exceptions such as (6–8) that considered the impact of elevation fluctuations on pedestrian paths. Other urban networks are naturally impacted by elevation fluctuations, such as the water distribution network where node elevation is an important information that has to be taken into account (9). At a more theoretical level, network geometry studies (see Ref. (10) and references therein) are probably connected to

surfacic networks. However, further studies are needed in order to exploit this possible correspondence.

A practical example is the road network that follows the topography of a city. This type of networks was extensively studied (see for example (1, 11–14) and references therein), and in this case, maps represent in general a vertical projection (“from above”) of the surfacic network, which can be very misleading (in particular in terms of the physical effort for a pedestrian for example). This type of spatial network is different from 3D spatial networks (such as the brain for example) and constitute a prime example of a surfacic network. Some specific measures will then be needed to characterize the importance of the “third” dimension. The example of road and street networks is particularly relevant for pedestrians and perhaps even more so for cyclists, where the altitude variation represents an effort for individuals. These networks are fundamental for analyzing spatial relationships, optimizing routes, and supporting decision-making in urban planning, logistics, and environmental management.

In order to quantify the impact of the surface shape on the network structure, we will introduce a set of new tools that take into account the third dimension described by the elevation of nodes. In addition to an adjacency matrix, each node is described by its coordinates (x, y, z) where the elevation (or height) z can display large fluctuations. In the general case of volumetric networks the elevation z has no constraint, while for surfacic networks, the elevation z is a function of the coordinates: $z = F(x, y)$. The function F defines the 2D manifold embedded in 3D space. In the constant case $z = \text{const.}$, we recover usual plane networks.

There are many important theoretical questions about surfacic networks. For example, we need to understand what is the impact of elevation fluctuations on the usual properties of graphs. This concerns the spatial distribution of the betweenness centrality (BC), the shape of shortest paths (SPs), etc. For this, we can monitor on toy models the evolution from a graph on a flat plane to a graph on a deformed surface. Also, it is important to understand the impact of elevation fluctuations on standard graphs such as the minimum spanning tree and other benchmarks.

We introduce a toy model where the network is constructed over a paraboloid which will allow us to discuss various properties related to SP, or to the BC. We then consider empirical examples of road and pedestrian networks in real cities, for which elevation differences are particularly relevant.

Tools and measures

The coordinates of a node i is denoted by x_i, y_i, z_i . The Euclidean distance between two adjacent nodes can be generalized under the form

$$\ell_e(i, j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \gamma(z_i - z_j)^2}. \quad (1)$$

With this definition, we can monitor the influence of elevation: for $\gamma = 0$, the network is “flat,” and for $\gamma = 1$, we consider the full impact of elevation fluctuations.

We note that surfacic networks can be seen as weighted 2D networks where nodes are weighted by their elevation, but encoding the elevation in weights could be very complex. In this respect, the network could be seen as a fitness or hidden variable model (15, 16). An important fact here is that the weights (elevation) are non-trivially correlated (depending on the surface) and depend in a complicated way on the shape of the surface. In the case where the surface is defined by a function of the form $z = F(x, y)$ (and the network connects the nodes of coordinates $(x_i, y_i, z_i = F(x_i, y_i))$), we can construct the weights that encodes the elevation. For example, if an edge e connects two nodes $M(x, y, z)$ and $M'(x', y', z')$, and if we assume that these nodes are close to each other, the weight that represents the total distance is given (if one assumes that the surface is differentiable):

$$w(e) = \sqrt{\Delta x^2 + \Delta y^2 + (\vec{\nabla} F \cdot \vec{u}_e)^2}, \quad (2)$$

where $\Delta x = x' - x$, $\Delta y = y' - y$, and $\vec{u}_e = (\Delta x, \Delta y)$. We see here that, although the representations in terms of a weighted 2D graph and a surfacic network are equivalent, the formulation with weights could rapidly become very complex.

Also, in this representation, we see that if the network is constructed on a differentiable surface $z = F(x, y)$, if $x_i \approx x_j$ and $y_i \approx y_j$, we have: $z_i \approx z_j$. We expect such a continuity property to be valid in general for real-world networks (such as road networks for example), but there could be exceptions such as in the case of pedestrian networks, where elevators for example induce discontinuities.

Excess effort

A shortest path $SP(i, j)$ from i to j incurs a total elevation difference given by

$$\sum_{e \in SP(i, j)} \Delta z(e) = z(j) - z(i) \equiv \Delta z(i, j), \quad (3)$$

where $\Delta z(e)$ is the elevation difference of edge e (in the direction of the SP $\Delta z(e) = z(\text{target}) - z(\text{source})$). Any continuous path must satisfy this equation, but if there is a discontinuity (such as an

elevator for example) we would have to include vertical edges in order to still satisfy it.

In a human navigation domain, where positive slope requires more effort, we define the elevation effort between i and j as

$$\Delta z^+(i, j) = \sum_{e \in SP(i, j)} \Delta z(e) \theta(\Delta z), \quad (4)$$

where $\theta(x)$ is the Heaviside function (we can similarly define Δz^- for negative slope segments to represent down-hill energy, time, etc. advantage).

If the SP from i to j is monotonically increasing, we have $\Delta z^+(i, j) = \Delta z(i, j)$. However in real cases, we have to go up and down so that we have to climb up a distance Δz^+ larger than the elevation difference Δz . There is therefore an excess effort $E(i, j)$ that we can measure with

$$E(i, j) = \frac{\Delta z^+(i, j)}{\Delta z(i, j)} - 1. \quad (5)$$

With these measures, we can then study various network statistics such as its average, distribution, etc.

Lazy paths and graph arduousness

In general, the SP on a spatial network is computed using as the weight the Euclidean length of a link: $w(e) = \ell_1(e) = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. The (weighted) SP is then the one that minimizes the sum of $w(e)$ along it.

In surfacic networks, however the elevation of a node is important. We can therefore define SPs so that they weight elevation difference, rather than minimizing total distance only. In particular for pedestrians, it makes sense to avoid paths with a large (positive) elevation difference. We therefore assign the following weight to an edge

$$w(e) = \begin{cases} \ell_1(e) + \mu \Delta z(e) & \text{if } \Delta z(e) > 0 \\ \ell_1(e) & \text{if } \Delta z(e) < 0, \end{cases} \quad (6)$$

where $\Delta z(e) = z(\text{end node}) - z(\text{starting node})$ and where $\mu > 0$ is a parameter. We then look for the optimal path that connects two nodes such that the total weight $W = \sum_{e \in \text{path}} w(e)$ is minimum. We note that with this choice of weight, smoother slopes are always favored. However, in reality, a pedestrian might prefer a path with a steeper slope followed by a longer flat section (see Fig. S1). It would be interesting to explore more complex weight functions that account for this behavior.

The parameter μ governs the relative weight of the elevation effort and the length of the trip. For $\mu = 0$, this optimal path is the usual SP that minimizes the total distance. When $\mu \gg 1$, the optimal path essentially minimizes the excess effort. The choice $\mu = 1$ corresponds then to the case where we choose a longer path if the detour is of the order (or smaller) than the excess effort difference. More precisely, assume there is a path 1 characterized by a total weight $W_1 = L_1 + \mu \Delta z_1$ and another path 2 by $W_2 = L_2 + \mu \Delta z_2$. Assume that $L_1 > L_2$. If $\Delta z_1 > \Delta z_2$, there is no ambiguity the SP has also the smallest excess effort and the optimal path is L_2 . In contrast, if $\Delta z_1 < \Delta z_2$, we are in an ambiguous case: the path L_1 is longer but it has a smaller excess effort. The resulting optimal path will then depend on the value of μ , and the path 1 will be chosen if $W_1 < W_2$ which implies that $L_1 - L_2 < \mu(\Delta z_2 - \Delta z_1)$ which indeed corresponds to the fact that what we lose in detour, we gain in excess effort. Larger values of μ would give more weight to the elevation effort. We show in the SI (see Fig. S2) the evolution of the arduousness for different values of μ . Here and in the following, we will use $\mu = 1$.

We thus look for the path that minimizes the total weight with weights given by Eq. 6. The optimal path is then the shortest one that minimizes the total elevation effort, which we coin “lazy path,” and we denote its length by $L_{\text{lazy}}(i, j)$ which is given by

$$L_{\text{lazy}}(i, j) = \sum_{e \in \text{LP}(i, j)} \ell_1(e), \quad (7)$$

where $\text{LP}(i, j)$ is the set of links belonging to the lazy path from i to j . We leave $\Delta z < 0$ unweighted in this experiment, but note that we could assign a negative weight to represent any advantage conferred by a “down-slope” link. In pedestrian networks, a negative Δz confers energy, time, and psychological advantage in navigating a surfacic network. This may not be the same in other domains. Note that this lazy distance naturally induces directionality in the network: $L_{\text{lazy}}(i, j) \neq L_{\text{lazy}}(j, i)$.

We denote by $L_{\text{tot}}(i, j)$ the total length of the SP between i and j , which minimizes the following expression

$$L_{\text{tot}}(i, j) = \sum_{e \in \text{SP}(i, j)} \ell_1(e), \quad (8)$$

where $\text{SP}(i, j)$ is the set of edges of the SP between i and j . Note that the expressions in Eqs. 7 and 8 both represent a total length. However, for Eq. 7, it corresponds to the total length of the path that minimizes $\sum_e w(e)$, while for Eq. 8, the path minimizes $\sum_e \ell_1(e)$.

We then construct the ratio of these two lengths

$$A(i, j) = \frac{L_{\text{lazy}}(i, j)}{L_{\text{tot}}(i, j)} - 1. \quad (9)$$

For a flat network, we have $A(i, j) = 0$. If for a pair (i, j) , we have a large value of A , it means that the lazy path is much longer than the SP and that elevation is critical. We can average this quantity $A(i, j)$ over all pairs of nodes and obtain the average “arduousness” of the graph G

$$A(G) = \frac{1}{N(N-1)} \sum_{i \neq j} A(i, j). \quad (10)$$

The arduousness of a flat network is defined as $A(\text{flat}) = 0$. Higher values of A indicate greater significance of elevation, implying that, without a detour, one must contend with substantial elevation differences. We note that in other fields such as optimization on nonconvex problems, we could imagine that arduousness might provide a measure of search difficulty in data with many local optima.

Toy models

We introduce the concept of surfacic networks using pedestrian paths through a topographical city, but there are certainly many models of surfacic networks that can be imagined. Potentially, any network model can be embedded on any surface, leading to an infinite number of possible models. Here, we explore three simple models that allows to investigate different aspects of surfacic networks. First, we consider a network constructed over a paraboloid, mimicking cases where the topography has a single peak. The main parameter is then the height of the peak (rescaled by the typical size of the area), and we can discuss various properties when this maximum height is varied. In the second “Gaussian” model, the width of the peak can also be monitored. Finally, we consider another random null model where the elevation of a node is a random variable. This allows us to investigate the impact of elevation fluctuations on various aspects of surfacic networks.

For practical purposes and expositional simplicity, we construct the toy networks as a function of the underlying topography. We could have merely layered a network topology on a topographical surface.

The parabolic model

The idea of this model is to mimic a network that is defined on a surface which has one main “peak.” We thus assume that the x_i, y_i coordinates of the nodes are random distributed in the plane (typically in the disk of radius $R = 1$), and that for each node i , the elevation z_i is given as a function of their (random) planar coordinates x_i, y_i by

$$z_i = z_{\text{max}}(1 - x_i^2 - y_i^2). \quad (11)$$

More generally, for a surface defined by $z = f(x, y)$, we would choose $z_i = f(x_i, y_i)$ with random x_i and y_i . We can then choose any rule to construct the network on top of this surface. Here, we choose to construct a random geometric graph (RGG) (17) where two nodes i and j are connected only if their distance is less than a threshold $\ell_1(i, j) \leq r_0$. We choose r_0 such that the average degree is 6 ensuring the existence of a large giant component (see Ref. (1) and references therein). We note here that Krioukov et al. (18) considered a RGG built on a hyperbolic geometry, a first example of a surfacic network (see Ref. (19) for a recent review on RGG).

In this simple case considered here, we thus have a paraboloid embedding the network which allows us to investigate the impact of elevation by varying z_{max} from 0 at the minimum (which corresponds to a flat network) to z_{max} at the maximum, which is much larger than the typical size R . For this model, we will discuss its arduousness, excess effort, and the spatial distribution of BC.

Arduousness

We compute the arduousness (Eq. 10) for the parabolic graph when z_{max} varies from 0 (flat network) to z_{max} large (in our experiment, we choose $z_{\text{max}} = 5$). We obtain the result shown in Fig. 2A. We observe on this figure the presence of a maximum (and we expect this feature to be quite general): when z_{max} is small, the arduousness is small as they are many SPs that do not necessitate to climb large elevation differences. For very large z_{max} , most SPs avoid the peak of the mountain and the lazy and SPs are very similar. There is therefore a maximum value $z_{\text{max}} = z^*$ for which many SPs actually go through the peak and entail climbing many edges with a relatively large Δz . Around this point z^* , there is a large difference between SP and lazy path.

Excess effort

We compute the average excess effort (E) and obtain the distribution $P(E)$ shown in Fig. 2B. We observe on this figure that this distribution has a power law tail of the form $P(E) \sim E^{-\gamma}$ with $\gamma \approx 2$. This behavior indicates that there is a small but non negligible probability of finding a pair of nodes such that E is very large (up to 10^4). Large values of E typically occur when the distance $\ell_1(i, j)$ is large (i.e. nodes i and j are far apart) while their elevation difference is small. In such cases, the SP traverses nodes with significant elevation changes, leading to a large Δz^+ , and consequently, a very high value of E .

We also measure the distribution of the angle θ of edges (see Fig. 2C). The angle for an edge e connecting nodes $M = (x, y, z)$ and $M' = (x', y', z')$ is defined as $\theta = \text{atan}(z' - z) / [(x' - x)^2 + (y' - y)^2]^{1/2}$. It is here an indication of the slopes experienced by SPs. In the case of the parabolic model

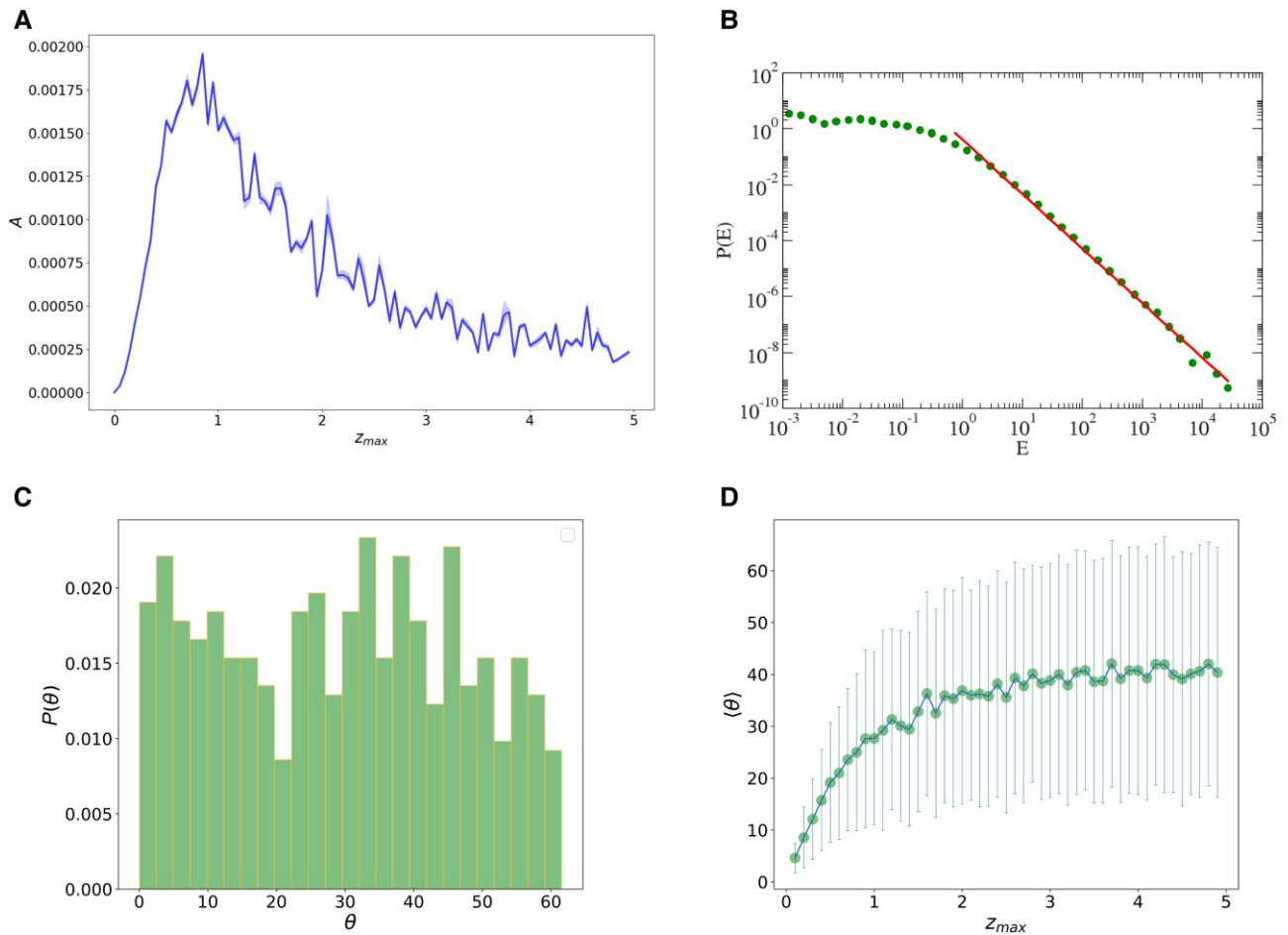


Fig. 2. The parabolic model. A) Arduousness of the graph computed for the parabolic model when z_{\max} is varied. B) Distribution $P(E)$ of the excess effort (for $z_{\max} = 1$, $N = 200$ and 1,000 configurations). The straight line is a power law on the tail with exponent 2.0 ($r^2 = 0.99$). C) Distribution $P(\theta)$ of the angle of edges ($N = 200$, 1,000 configurations, $z_{\max} = 1$). D) Average slope vs. z_{\max} ($N = 200$, 1,000 configurations).

find a roughly uniform distribution. The average angle (θ) of edges depends however on z_{\max} , and we show the result on Fig. 2D. We observe a rapid increase when z_{\max} goes to 1, followed by a plateau for a value (θ) $\approx 40^\circ$. It is interesting that (θ) never crosses this value: when z_{\max} becomes too large, SPs naturally avoid the steep edges on paths that are also too long.

The spatial distribution of the BC

BC is an important measure in networks and represents a reasonable proxy for the traffic on a link (see for example (1)). It points to structurally important nodes that can be considered as bottlenecks in a network. For node i , it is defined as

$$g(i) = \frac{1}{N(N-1)} \sum_{s \neq t} \frac{\sigma_{st}(i)}{\sigma_{st}}, \quad (12)$$

where σ_{st} is the number of (weighted) SPs between s and t and $\sigma_{st}(i)$ is the number of (weighted) SPs between s and t that go through the node i (N is the number of nodes in the graph). In this context, SPs are determined based on a specific weight (referred to as weighted BC by some authors, but here simply as BC), where we consider the length of an edge as its weight.

For most spatial networks, there is some correlation (on average) between BC and distance to the center (1). For example, on a regular 1D lattice of length L , BC is $g(x) = x(L-x)$ for $x \in [0, L]$

and has a maximum at $L/2$ (note that for a disordered planar network, the BC can be more complicated, see for example (1, 20)). We expect that elevation will be an important factor governing the spatial distribution of the BC, and it is interesting to study the correlation between the BC and the elevation. We note that some analysis of flows on a pedestrian surfacic network, can be found in Ref. (6) where the authors test the association between various BC metrics and pedestrian flow counts.

For flat (isotropic) networks, BC decreases with the distance d from the center. We can then plot the average BC vs. the distance d for different values of z_{\max} (the peak of the parabolic surface). The result is shown in Fig. 3A.

For small z_{\max} , we recover the usual flat network behavior: high BC nodes are close to the center of the graph (which corresponds to the center of the disk here). When z_{\max} increases, we observe that there is a crossover to another regime where the large BC nodes are actually close to the boundary where the elevation is small. This can be confirmed by visual inspection on the two cases presented in Fig. 3B and C where large BC nodes are shown in yellow (this figure is shown for a random geometric graph with a large average degree; for smaller value of (k) the phenomenon exists but with a smaller amplitude).

We provide a simple hand-waving argument to estimate when this crossover happens (for a disk of radius $R = 1$). We consider two

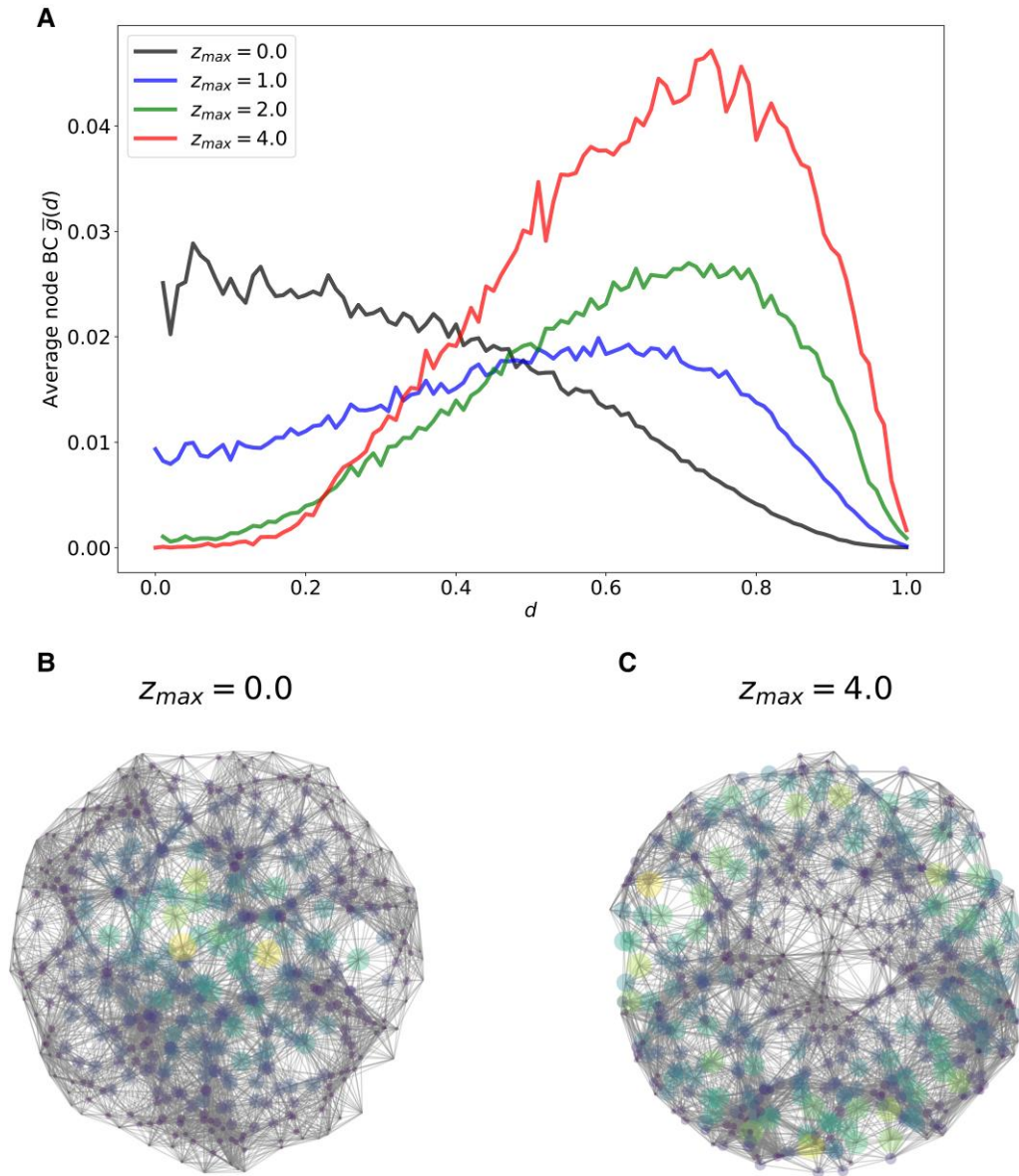


Fig. 3. A) Spatial distribution of the BC for parabolic networks: we plot here the average BC versus the distance to the center for different values of z_{max} (computed for 100 nodes and averaged over 100 configurations). B, C) Spatial distribution for BC for a flat network (left, $z_{max} = 0.0$) and a nonflat network with a maximum at the center (right, $z_{max} = 4.0$). The color code and size of nodes is proportional to the BC (Calculation are done on random geometric graphs of size $N = 100, 200$ configurations, and average degree $\langle k \rangle = 12$).

diametrically opposed nodes A and B with both having a small elevation and located at a radial distance of order $\sim R = 1$ from the center. The length $\ell_S(A, B)$ of the path going through the center (i.e. the “peak” of the mountain) is given by the arc length of the parabola from $-R$ to $+R$ which is given by

$$\begin{aligned} \ell_S(A, B) &= 2 \int_0^R \sqrt{1 + (2z_{max}\tau)^2} d\tau \\ &= \sqrt{1 + 4z_{max}^2} \\ &\quad + \frac{1}{2z_{max}} \log \left[2z_{max} + \sqrt{1 + 4z_{max}^2} \right]. \end{aligned} \quad (13)$$

The equatorial path (i.e. that avoids the mountain peak and goes around) is of length $\ell_E(A, B) = \pi R = \pi$. For z_{max} small, it is shorter to go straight from A to B: $\ell_S < \ell_E$. When z_{max} increases ℓ_S increases and at a certain point ℓ_S and ℓ_E become equal. This happens for $z_{max} = z^* \approx 1$. Above this value z^* , it becomes more optimal to

avoid the peak of the mountain and in this case the small elevation nodes become central.

The Gaussian model

Next, we experiment with a Gaussian shaped surface (see an example in Fig. 4A). As in the parabolic case, we choose random coordinates (x_i, y_i) and compute the surfacic network as a function of an underlying topography

$$z_i = z_{max} e^{-(x_i^2 + y_i^2)/2\sigma^2}. \quad (14)$$

As in the parabolic case, we can monitor the effect of the peak height z_{max} but also the width of this peak with σ . For a small σ the peak is very steep, while for a large σ (compared to the system size R), the slope of paths towards the peak of the mountain are shallow.

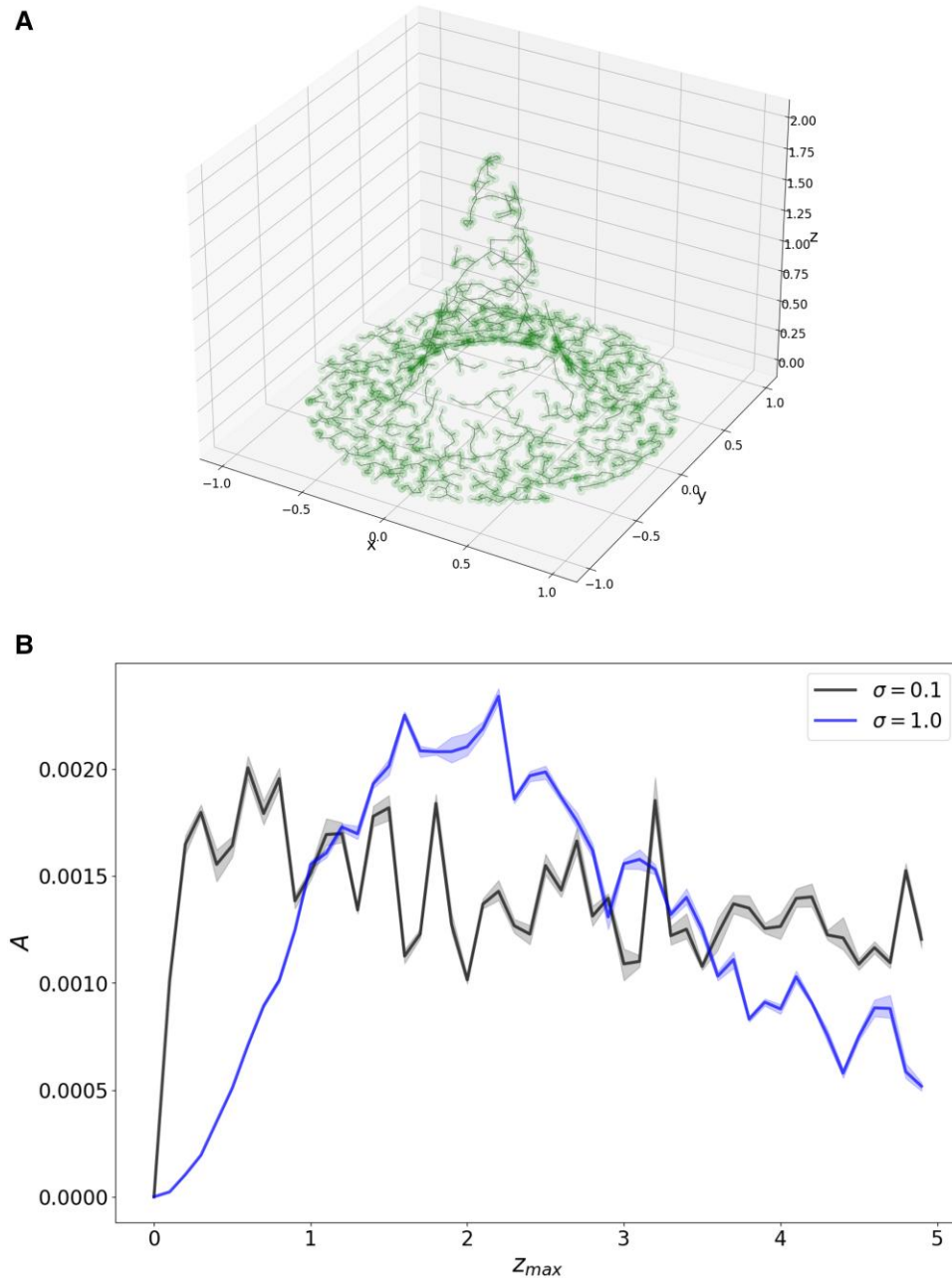


Fig. 4. A) Example of a Gaussian shaped surfacic network (here an Euclidean minimum spanning tree for $N = 1,500$ nodes and constructed over a Gaussian surface with $z_{\max} = 2/0$, and $\sigma = 0.2$). B) Arduousness computed for the Gaussian model when z_{\max} is varied and for two different values of $\sigma = 0.1$ and 1 (results are computed for $N = 200$ nodes, 100 configurations, and for 100×100 pairs of nodes).

We compute the arduousness in this Gaussian case for two different values of $\sigma = 0.1$ and $\sigma = 1.0$. The results are shown in Fig. 4B. We observe that for $\sigma = 1$ we recover a behavior similar to the one obtained in the parabolic case. However, when σ is too small (here $\sigma = 0.1$), the peak is very narrow and the influence of z_{\max} is limited as few SPs actually go over it (and go around it), leading to an almost constant arduousness.

The random model

A simple model for studying the impact of fluctuations on various properties of surfacic networks can be defined as follows. We start

from a flat graph G where nodes have coordinates (x_i, y_i) , and we assign to each node i a random elevation z_i defined as

$$z_i = \bar{z} + \sigma \zeta_i, \quad (15)$$

where ζ_i is a random number of order 1, and σ determines the scale of the fluctuations. We simulated this model for a uniformly distributed random set of points. Here, we focus on just two different aspects: the structure of SPs and the minimum spanning tree.

With this model, we can observe how elevation fluctuations alter SPs. This is important from a practical point of view for city maps: if the SPs are very different, it means that we should be careful when using the view from above (or the projection of the map) for

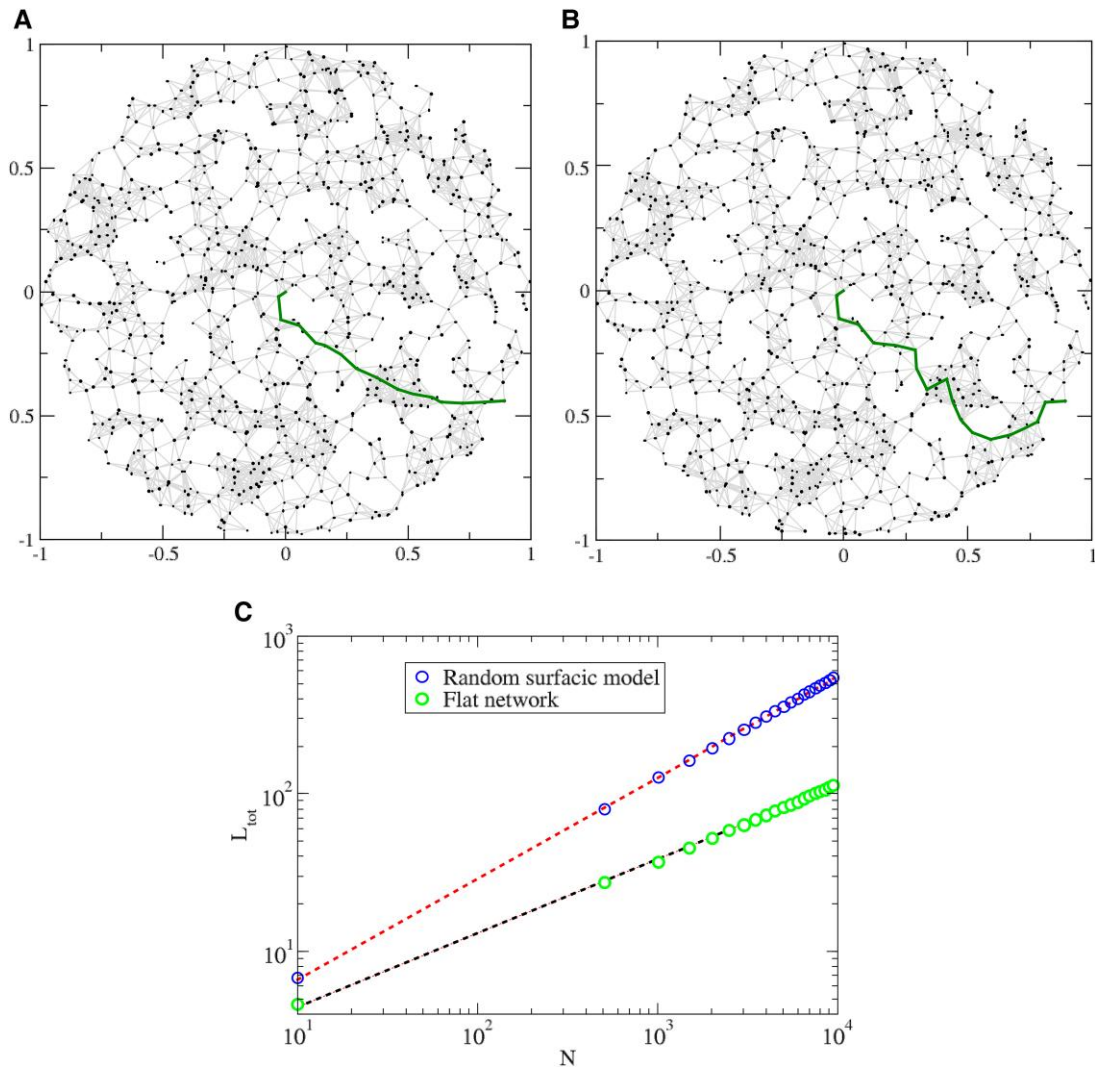


Fig. 5. A, B) Example of the deformation of a SP. On the left (A), we show the SP computed for the flat graph, and on the right (B), we show the shortest (between the same pair of nodes) computed for the surfacic network. C) Scaling of the total length with the number of nodes. Dotted lines are power law fits of the form $L_{\text{tot}} \sim N^\tau$. For the flat network, we obtain $\tau \approx 0.46$ (in agreement with the standard result \sqrt{N} and for the random model, we obtain $\tau \approx 0.64$ a 3D behavior (with $r^2 = 0.99$ in both cases). These results are obtained for an average over 100 configurations and the random surfacic network is obtained for $\sigma = 1$.

navigating in the city. We tested this on a random geometric graph (see Refs. (1, 21) and references therein) constructed over a set of points in the 2D plane and where we assign to each point (x_i, y_i) , a random elevation z_i . Results are shown in Fig. 5A and B. We have highlighted in this figure a pair of nodes and the corresponding SP. On the Fig. 5A, we consider the flat graph, and in the Fig. 5B, we take into account the (random) elevation. We observe that in the presence of elevation, the SP is very different and typically avoids large elevation nodes, as expected from the BC analysis.

There are many other aspects that we could study. In particular, we could study the Euclidean minimum spanning tree (EMST) for this model and characterize the importance of elevation fluctuations in the structure of the EMST (see Fig. S3 that highlights the impact of elevation on the EMST structure). It is well known (22) that for flat networks, the total length L_{tot} of an EMST scales as $L_{\text{tot}} \sim N^\tau$ with $\tau = 1/2$. Results presented in Fig. 5C show that for the surfacic model with $z_i = \sigma u_i$ (where u_i is a random number in $[-1, 1]$), the exponent is different $\tau \approx 0.64$ (for $\sigma = 1$). For a dimensional network, we expect that the typical distance between nodes is of order $1/N^{1/d}$ which leads

to a total length scaling as $N^{1-1/d}$. The result obtained here thus corresponds to $d = 3$, showing that the fluctuations in this model are enough to destroy the surfacic feature of the network which now resembles more a 3D network. Real-world surfaces does not display this sort of large fluctuations, but we believe that the scaling of the MST on surfaces deserves probably further study, with possible crossovers for some models from a 2D to a 3D behavior.

Empirical analysis

Pedestrian (and road) networks are typical examples of surfacic networks where elevation plays a critical role. Elevation directly influences the accessibility of various urban points, shaping numerous aspects of a city's spatial economy, including land values, the viability of commercial centers, and the balance between jobs and housing. Despite its importance, research on pedestrian networks remains less developed than that on road networks (23), even though it is a vital component of urban infrastructure. Notably, elevation affects pedestrian speeds,

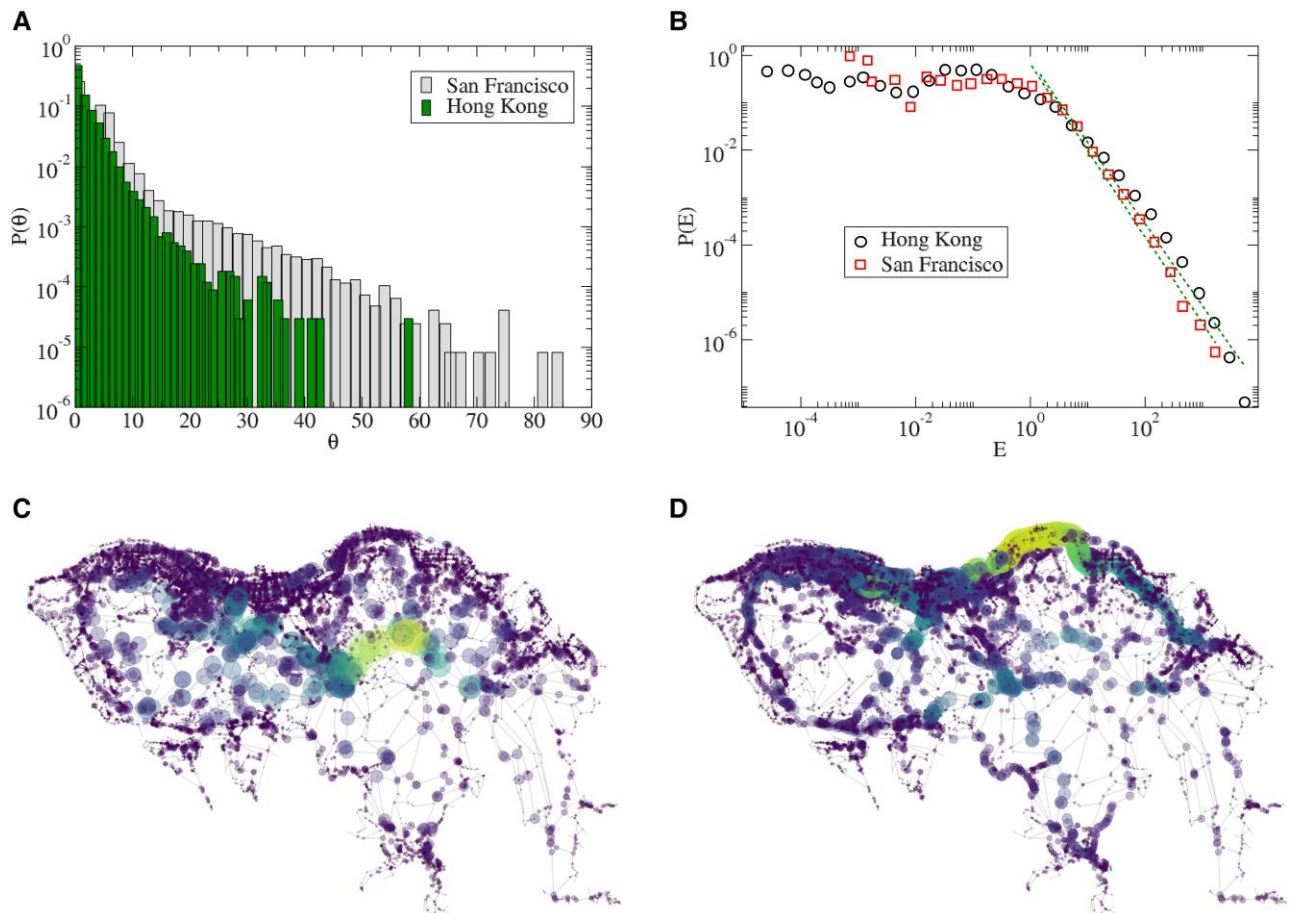


Fig. 6. A) Slope distribution for edges for the two networks of Hong Kong and San Francisco. B) Excess effort distribution $P(E)$ for Hong Kong and San Francisco (computed for 500×500 pairs of nodes). The dotted lines represent power law fits (with exponents $\gamma = 1.72$ and 1.82). C, D) Distribution of the BC in Hong Kong (the size and color depend on the BC: the larger it is or the brighter it is and the larger the BC). On C), we show the result computed on the 2D plane graph (neglecting the elevation) and on D), we show the result when elevation is taken into account.

influencing the geometry of minimum-time paths. Early discussions on footpaths in hilly terrains were initiated in Ref. (24) and later revisited in Ref. (25). For flat cities (i.e. with small fluctuations of the elevation—for example, the maximum elevation for Paris is 130 m), elevation difference is irrelevant. This is in contrast with other cities constructed over hilly surfaces. This is for example the case of Hong Kong island, a densely inhabited urban space rising from sea-level to over 500 m and having very many steeply sloped roads and paths. We will also consider the case of San Francisco, which is interesting in the sense that most hilly cities have curving streets to accommodate the topography, but San Francisco is an outlier case where the road grid was laid over the top of the hills, so some parts of the city have particularly steep streets. These two examples will help us to illustrate our measures and results. These networks are extracted from the extensive dataset provided in Ref. (14). More specifically, geopackages are provided for each city, and the graph was topologically simplified such that nodes represent dead-ends and junctions, but full correct edge geometry is maintained. This graph's edges contain attributes for “length” (representing Haversine distances between original unsimplified nodes, then summed when graph was simplified), and also a 3D length that represents 3D Euclidean distances between projected original unsimplified nodes, then summed when graph was simplified (see the SI for more details about the network construction and the corresponding pseudocode).

Elevation and slope distributions, excess effort

Simple statistics for pedestrian networks can be measured using the extensive dataset provided in Refs. (14, 26) such as the elevation distribution, fluctuations across cities in the world, etc. (see Fig. S4 for the elevation distribution for cities in different countries). Beside standard statistical indicators, we compute for a given city the average elevation of nodes and their Gini coefficient: if all the elevations are (almost) equal the Gini coefficient is (close to) zero and if a few nodes have a very different elevation than the rest, the Gini coefficient will be close to one. The result is shown for a few countries in the Fig. S5. We observe that the average elevation \bar{z} decreases with the Gini coefficient G : cities with large fluctuations are found at a lower altitude (a rough fit gives $\bar{z} \sim 1/G$). This might seem counter-intuitive but is because, as with the case of Hong Kong island, coastal and deltaic cities often spread into surrounding hilly terrain. On the other hand, high altitude cities are often built on mountain plateaux (such as Mexico City).

The slope distribution is also an interesting indicator of the intensity of elevation fluctuations experienced on a surfacic network. We show the result for the two different cities Hong Kong and San Francisco in Fig. 6A.

Excess effort

We computed the excess effort for the two cities and the results are shown in Fig. 6B. We observe that for $E > 1$, the tail can

reasonably be fitted by a power law of the form $P(E) \sim E^{-\gamma}$ with $\gamma = 1.72$ for Hong Kong and $\gamma = 1.82$ for San Francisco. These values are close to each other, but significantly different from the parabolic model for which we obtain $\gamma = 2$. The occurrence of pairs of nodes for which the excess effort is very large is thus more likely in these two cities compared to the parabolic model. This is somehow expected as these real-world surfacic networks display more than one peak and are more rugged. The results are shown here for 500×500 pairs of nodes randomly chosen. We plotted (see Fig. S6) the distribution for different sizes (up to $1,000 \times 1,000$) which displays a quick convergence with size.

BC: spatial distribution

As expected from the theoretical considerations and results obtained for toy models (see above for results on the parabolic model), when the elevation of a node is too large it becomes something to avoid. As a consequence, SPs avoid elevation peaks and low elevation nodes become more central than in a flat city.

This is confirmed in the specific case of Hong Kong where we show both the BC map (see Fig. 6C and D) in the 2D case (elevation is not taken into account) and the full 3D case (with elevation). The results confirm our theoretical analysis: in the 2D case, central nodes (in the spatial sense) are also the ones with the largest BC and when we take elevation into account we observe that large BC nodes are on the boundary of the island (especially the northern one where the density of roads is larger).

Discussion

We have described a class of networks that has not been formally considered before, which we call surfacic network. For these networks, the shape of the surface governs the elevation fluctuations and therefore movement through the network, in our example pedestrian experience and behavior. The difference between non-surfacic and surfacic version of a network is crucial in general for any network where there is a cost associated with an elevation difference, or indeed a benefit.

We illustrate using an urban pedestrian network, with our results showing that elevation changes the network dynamics of city movement potential. This category of networks potentially represents a powerful framework for analyzing and modeling complex systems that are located on, or confined to, some kind of surface.

A more definitive and comprehensive characterization of supply networks is needed. While this article has provided a minimalistic approach, a thorough mapping of surfacic network properties onto established graph-theoretic concepts would be valuable. A potential direction is linking geometric aspects, such as elevation-based surfaces $z = f(x, y)$, with graph theoretical properties of networks. Specifically, surfacic networks require tools to measure flatness, effort, connectivity, and weighted distance on a surface where elevation is systematically related to network traversal, setting them apart from other networks where proximity in z is unrelated to traversal effort. Second, the impact of directionality in surfacic networks needs further exploration. Defining surfacic networks as directed graphs (digraphs) opens up a rich area of study. Pedestrian networks, for instance, may be navigated differently based on direction, and revisiting the current work with directionality in mind could reveal new insights. A digraph model could capture differences in path efficiency, excess effort, and weighted centrality depending on direction, prompting questions such as whether a lazy path uphill is equivalent to one downhill, and under what circumstances they differ.

Potentially, surfacic networks represent a versatile and interdisciplinary concept with applications across various fields. For example, they might be used to model chemical processes on biological structures, such as trees, forests or coral reefs, where topography influences performance (e.g. by governing exposure to light). Surfacic network measures that we have defined, such as excess effort, would seem to have potential in measuring benefits and costs of traversing such networks, for example, in building physical infrastructure, or in expending energy when moving through a surfacic network; and benefits of capitalizing on local slopes, such as when using gravity in hydrological systems engineering, or estimating profit potential of alternative paths in a network representation of a financial derivative instruments. By capturing explicit relationships between network topology and an underlying geometric topography, these networks provide valuable insights into the structure, behavior, and dynamics of complex systems situated on surfaces, paving the way for advances in science, engineering, and technology.

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Supplementary Material

Supplementary material is available at PNAS Nexus online.

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Author Contributions

M.B. conceptualization, investigation, visualization, methodology, writing-original draft, writing-review and editing; G.B. conceptualization, investigation, visualization, methodology, writing-original draft, writing-review and editing; A.C. conceptualization, investigation, visualization, methodology, writing-original draft, writing-review and editing; C.W. conceptualization, investigation, visualization, methodology, writing-original draft; writing-review and editing.

Data Availability

All the data used in this article are publicly available at <https://www.openstreetmap.org/> and at <https://dataverse.harvard.edu/dataverse/global-urban-street-networks>. The codes (in python) used for the analysis and visualization are available on the open repository zenodo at the address <https://zenodo.org/records/14557635> with the DOI: 10.5281/zenodo.14557635.

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