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Research article

Impacts of magnetic field and thermal radiation on squeezing flow and heat transfer of third grade nanofluid between two disks embedded in a porous medium

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ABSTRACT

In this present study, the impacts of magnetic field and thermal radiation on squeezing flow and heat transfer of third grade nanofluid between two disks embedded in a porous medium with temperature jump boundary conditions is analyzed using differential transformation method. The results of the approximate analytical solutions are verified using a fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method. From the analysis, the results of the two methods show excellent agreements. Also, the parametric studies using the approximate analytical solutions show that for a suction parameter greater than zero, the radial velocity of the lower disc increases while that of the upper disc decreases as a result of a corresponding increase in the viscosity of the fluid from the lower squeezing disc to the upper disc. For an increasing magnetic field parameter, the radial velocity of the lower disc decreases while that of the upper disc increases. As the third grade fluid parameter increases, there is a reduction in the fluid viscosity thereby increasing resistance between the fluid molecules. Also, it is found that as the radiation parameter increases, rate of heat transfer to the third grade fluid increases. There is a recorded decrease in the fluid temperature profile as the Prandtl number increases due to decrease in the thermal diffusivity of the third grade fluid. The agreement of the results of the present study and the experimental work shows the validation of the models used in this work to study the flow behaviour of the fluid. It is envisaged that the present work will increase the understanding of the flow behaviour of third grade nanofluid and heat transfer processes as evident in coal slurries, polymer solutions, textiles, ceramics, catalytic reactors, oil recovery applications etc.

1. Introduction

The various applications of non-Newtonian fluids such as in catalytic reactors, oil recovery applications, power transmission, nasogastric tubes, hydraulic lifts, electric motors etc. have in recent times continued to arouse the interests of fluid dynamics and thermal engineering researchers. Also, the continuous wide areas of engineering, industrial and biological applications of fluid and heat transfer flow between two parallel discs or surfaces call for renewed studies and analysis the flow and heat transfer problems. In the past few decades, different studies have been presented on flow, heat and mass transfer analyses of non-Newtonian fluid. In such studies, Mustafa et al. [1] presented transient heat transfer analysis of squeezing fluid flow through two parallel surfaces. Hayat et al. [2] examined the flow of a second grade fluid being squeezed by two parallel discs and presented with proper comparison the behaviour of the second grade fluid considering or neglecting magnetic effect. In an attempt to solve the extension of Hayat et al model considering suction and injection on magneto-hydrodynamic squeezing flow, Domairry and Aziz [3] adopted a semi-analytical method, homotopy perturbation method to obtain a symbolic solution for investigating and predicting the influence of suction and injection on magnetohydrodynamic (MHD) squeezing flow under standard conditions. A similar study was performed by Siddiqui et al. [4] using two parallel plates with the squeezing viscous fluid under transient condition. In the same year, Rashidi et al. [5] approached the problem using different analytical schemes. Incorporating nanotechnology into the problem of squeezing flow, Khan and Aziz [6, 7], investigated on the effect of natural convection on fluids with nanoparticles. In an extended study, the influence of porosity on the squeezing flow of nanofluids was considered. Another study on squeezing nanofluid flow between two vertical plates was

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performed by Kuznestov and Nield [8]. The authors investigated the boundary layer of the squeezing process by using natural convection principle of fluid flow to generate different governing models. Hashimi et al. [9] utilized the models and obtained analytical solutions to the governing equations with the assumption of no slip and no temperature jump in the solid-liquid interface. Although, there some other studies presented on the magnetohydrodynamics flow of nanofluids [10, 11, 12, 13, 14], these studies are based on the assumption of no slip and no temperature jump. However, the two annulled assumptions have to a reasonable extent influence on the flow and heat transfer processes in under investigation. In other to provide an extension to the works of the previous studies by considering slip condition, an appreciable number of studies have been presented on squeezable fluids of different nano-particles sizes, different concentration, different stretching effect and different phases by taking slip into account [15, 16, 17, 18].

Past research works have presented magnetohydrodynamic fluid flow considering porous medium, nanofluid and some other factors capable of reshaping the squeezing process from idealization into reality [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. Flow analyses of third grade fluids such as slurry flows, dilute polymer solution (polyisobutane, methylmethacrystalate in n-buthyl acetate), molten plastics, food rheology polymers mixed with melts, manufacturing oils and polymer melts like high viscosity silicon oils, blood, etc. have been subjects of great interests to various fluid mechanics researchers. Therefore, in recent times, the flow and heat transfer characteristics of third grade nanofluid in pipes and channels have been analyzed [61, 62, 63]. Following the findings of Fosdick and Rajagopal [64] that third grade fluid gives different properties to those of fluids such as Newtonian and second grade fluids, Majhi and Nair [65] examined the shearing stresses at the wall of a fluid flow process considering a third grade fluid as the working fluid. The authors compared their obtained results to the results of numerical methods in the work of Massoudi and Christie [66] and an excellent agreement was reached by the two sets of results. In another work, a third grade fluid with constant viscosity was analyzed by Yurusoy and Pakdemirli [67]. The authors developed approximate analytical solutions and verify their work with the numerical solution of the work of Vajrevelu et al. [68]. The research work gave interesting results and found great application in rotary devices. Other studies on third grade fluids includes the fluctuating fluid flow by Hayat et al. [69], boundary layer analysis by Muhammet [70], heat transfer analysis by Yurusoy [71], entropy generation analysis by Pakdemirli et al. [72] and partial slip analysis by Sajid et al. [73]. Different schemes have also been employed to efficiently solve the resulting ordinary differential equations associated with squeezing fluid flow between two parallel surfaces as presented by [74, 75, 76, 77, 78, 79, 80]. In very recent study, Hayat et al. [81] utilized homotopy analysis method analyze the axisymmetric squeezing flow and heat transfer of third grade nanofluid under convective conditions. However, to the best of our knowledge, the study of magnetohydrodynamic unsteady squeezing flow of third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump condition using differential transformation method (DTM) has not been considered and analyzed in literature. Therefore, in this work, differential transformation method is applied to carry out nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and magnetic field with temperature-jump boundary conditions. The method as applied in the equation converges very fast and is very efficient for the handling of both ordinary and partial differential equations. In order to verify the approximate analytical solution, a fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method is used. The results of the two methods show excellent agreements. Also, the influences of various parameters on the flow and heat transfer processes are studied and discussed.

2. Problem formulation

Consider an unsteady axisymmetrically flow of third grade nanofluid through two parallel disks as shown in Figure 1. As presented in the Figure, the upper disk is moving towards a stationary lower disk subjected to a uniform magnetic field strength applied perpendicular to the disks. It is assumed that the disks are maintained at constant temperature while the fluid structure is everywhere in thermodynamic equilibrium.

For an incompressible homogeneous thermodynamically compatible third grade fluid, the Cauchy stress tensor, $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = p\mathbf{I} + \mu\mathbf{A_1} + \alpha_1\mathbf{A_2} + \alpha_2\mathbf{A_1^2} + \beta_1\mathbf{A_3} + \beta_2(\mathbf{A_1A_2} + \mathbf{A_2A_1}) + \beta_3\left(\mathbf{trA_1^2}\right)\mathbf{A_1}. \quad (1)$$

This Rivilin Erickson tensor (A_1, A_2 and A_3) is a temporal evolution of strain rate tensor such that the derivative rotate and translate with the flow field and can be derived from

$$\mathbf{A_1} = (grad\ \mathbf{V}) + (grad\ \mathbf{V})^T, \quad (2)$$

$$\mathbf{A_n} = \frac{d\mathbf{A_{n-1}}}{dt} + \mathbf{A_{n-1}}(grad\ \mathbf{V}) + (grad\ \mathbf{V})^T\mathbf{A_{n-1}}, \quad n \geq 1 \quad (3)$$

For the motions of the fluid, Clausius-Duhem inequality must be satisfied [64]. The conditions that fluid motions are thermodynamically compatible, the Clausius-Duhem inequality is satisfied and the assumption that the Helmholtz free energy is minimum when the fluid is locally at rest (stable) are given as

$$\mu \geq 0, \alpha_1 \geq 0, \beta_1 = \beta_2 = 0, \beta_3 \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta \geq 0 \quad (4)$$

The equation for the velocity and temperature fields are given as

$$\mathbf{V} = (\overline{u}(t,r,z), 0, \overline{w}(t,r,z)) \text{ and } \mathbf{T} = \overline{T}(t,r,z) \quad (5)$$

Using Eqs. (1), (2), (3), (4), and (5), the algebraic forms of the conservation equations can be developed as;

$$\frac{\partial\overline{u}}{\partial r} + \frac{\overline{u}}{r} + \frac{\partial\overline{w}}{\partial z} = 0 \quad (6)$$

$$\rho_{nf}\left(\frac{\partial\overline{u}}{\partial t} + \overline{u}\frac{\partial\overline{u}}{\partial r} + \overline{w}\frac{\partial\overline{u}}{\partial z}\right) = \frac{\partial\tau_{rr}}{\partial r} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} - \frac{\sigma_{nf}B_0^2\overline{u}}{1-ct} - \frac{\mu_{nf}}{K}\overline{u} \quad (7)$$

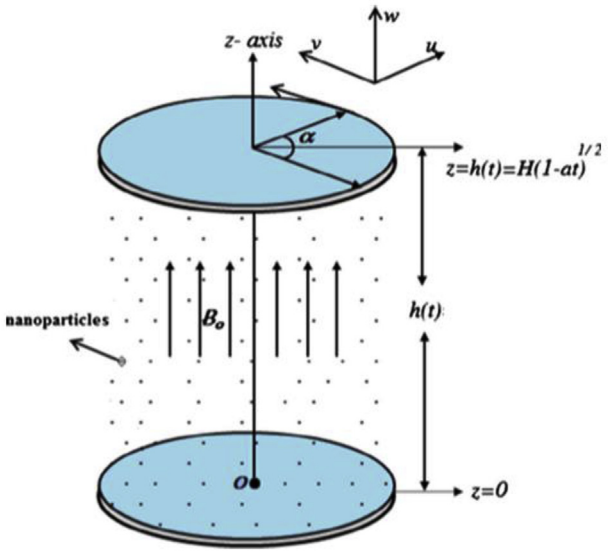

Figure 1. The squeezing flow of Third grade nanofluid between parallel circular plate.

$$\rho_{nf}\left(\frac{\partial \overline{w}}{\partial t}+\overline{u}\frac{\partial \overline{w}}{\partial r}+\overline{w}\frac{\partial \overline{w}}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{\partial \tau_{rz}}{\partial z}-\frac{\mu_{nf}}{K}\overline{w} \tag{8}$$

where

$$\tau_{rr}=-p+2\mu_{nf}\frac{\partial \overline{u}}{\partial r}+2(\alpha_1)_{nf}\left[\overline{u}\frac{\partial^2 \overline{u}}{\partial r^2}+\overline{w}\frac{\partial^2 \overline{u}}{\partial r\partial z}+2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\frac{\partial \overline{w}}{\partial r}\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)\right]$$
$$+(\alpha_2)_{nf}\left[4\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2\right]+4(\beta_3)_{nf}\frac{\partial \overline{u}}{\partial r}\left[2\frac{\overline{u}^2}{r^2}+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2+\left\{2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\left(\frac{\partial \overline{w}}{\partial z}\right)^2\right\}\right] \tag{9}$$

$$\tau_{\theta\theta}=-p+2\mu_{nf}\frac{\overline{u}}{r}+2(\alpha_1)_{nf}\left[\frac{\overline{u}}{r}\frac{\partial \overline{u}}{\partial r}+\frac{\overline{w}}{r}\frac{\partial \overline{u}}{\partial z}+\frac{\overline{u}^2}{r^2}\right]+4(\alpha_2)_{nf}\frac{\overline{u}^2}{r^2}$$
$$+4(\beta_3)_{nf}\frac{\overline{u}}{r}\left[2\frac{\overline{u}^2}{r^2}+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2+\left\{2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\left(\frac{\partial \overline{w}}{\partial r}\right)^2\right\}\right] \tag{10}$$

$$\tau_{zz}=-p+2\mu_{nf}\frac{\partial \overline{w}}{\partial z}+2(\alpha_1)_{nf}\left[\overline{u}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\overline{w}\frac{\partial^2 \overline{w}}{\partial z^2}+2\left(\frac{\partial \overline{w}}{\partial r}\right)^2+\frac{\partial \overline{u}}{\partial z}\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)\right]$$
$$+(\alpha_2)_{nf}\left[4\left(\frac{\partial \overline{w}}{\partial r}\right)^2+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2\right]+4(\beta_3)_{nf}\left(\frac{\partial \overline{w}}{\partial z}\right)\left[2\frac{\overline{u}^2}{r^2}+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2+\left\{2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\left(\frac{\partial \overline{w}}{\partial z}\right)^2\right\}\right] \tag{11}$$

$$\tau_{rz}=\mu_{nf}\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)+(\alpha_1)_{nf}\left[\left(\overline{u}\frac{\partial}{\partial r}+\overline{w}\frac{\partial}{\partial z}\right)\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)+\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}+\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}+3\left(\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}\right)\right]+2(\alpha_2)_{nf}\left(\frac{\partial \overline{u}}{\partial r}+\frac{\partial \overline{w}}{\partial z}\right)\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)$$
$$+2(\beta_3)_{nf}\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)\left[2\frac{\overline{u}^2}{r^2}+\left(\frac{\partial \overline{u}}{\partial z}+\frac{\partial \overline{w}}{\partial r}\right)^2+\left\{2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\left(\frac{\partial \overline{w}}{\partial z}\right)^2\right\}\right] \tag{12}$$

After substitution of Eqs. (9), (10), (11), and (12) into above momentum equations in Eqs. (7) and (8) and expansion of the resulting equations, one arrives at Eqs. (13) and (14)

$$\rho_{nf}\left(\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial r}+\overline{w}\frac{\partial \overline{u}}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{\partial^2 \overline{u}}{\partial z^2}+\frac{1}{r}\frac{\partial \overline{u}}{\partial r}-\frac{\overline{u}}{r^2}\right)+$$
$$(\alpha_1)_{nf}\begin{bmatrix}-2\frac{\overline{u}^2}{r^3}-\frac{2}{r^2}\frac{\partial \overline{u}}{\partial t}+\frac{4}{r}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+3\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}+5\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+3\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+2\frac{\overline{w}}{r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+\frac{2}{r}\left(\frac{\partial \overline{w}}{\partial r}\right)^2\\ -2\frac{\overline{u}}{r^2}\frac{\partial \overline{u}}{\partial r}+4\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}+\overline{u}\frac{\partial^3 \overline{u}}{\partial r\partial z^2}+\overline{w}\frac{\partial^3 \overline{w}}{\partial r\partial z^2}+3\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial z^2}\\ -2\frac{\overline{w}}{r^2}\frac{\partial \overline{u}}{\partial z}+\frac{\partial^3 \overline{u}}{\partial z^2\partial t}+\overline{w}\frac{\partial^3 \overline{u}}{\partial z^3}+4\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+2w\frac{\partial^3 \overline{u}}{\partial r^2\partial z}+\frac{2}{r}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+4\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}\\ +10\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{2}{r}\frac{\partial^2 \overline{u}}{\partial r\partial t}+\frac{\partial^3 \overline{w}}{\partial r\partial z\partial t}+\overline{u}\frac{\partial^3 \overline{u}}{\partial r^2\partial z}+2\frac{\partial^3 \overline{u}}{\partial r^2\partial t}+2u\frac{\partial^3 \overline{u}}{\partial r^3}+2\frac{\overline{u}}{r}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}\end{bmatrix}$$
$$+(\alpha_2)_{nf}\begin{bmatrix}-4\frac{\overline{u}^2}{r^3}+\frac{4}{r}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}+2\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial z^2}+4\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial \overline{w}}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial \overline{u}}{\partial z}\right)^2+8\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r^2}\\ +2\frac{\partial^2 \overline{w}}{\partial r\partial z}\frac{\partial \overline{w}}{\partial z}+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\frac{2}{r}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}\end{bmatrix}$$
$$+(\beta_3)_{nf}\begin{bmatrix}-8\frac{\overline{u}^3}{r^4}+8\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{8}{r}\left(\frac{\partial \overline{u}}{\partial r}\right)^3+24\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{4}{r}\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{u}}{\partial z}\right)^2+\frac{4}{r}\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{8}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{u}}{\partial r}\\ +16\frac{\overline{u}}{r^2}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\frac{8}{r}\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{w}}{\partial z}\right)^2+8\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+8\left(\frac{\partial \overline{w}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial r^2}-4\frac{\overline{u}}{r^2}\left(\frac{\partial \overline{w}}{\partial r}\right)^2+16\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}\\ +4\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{w}}{\partial r\partial z}+8\frac{\partial^2 \overline{u}}{\partial r^2}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+8\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+4\left(\frac{\partial \overline{w}}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial r\partial z}-8\frac{\overline{u}}{r^2}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+4\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial z^2}\\ +4\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial r\partial z}+4\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{u}}{\partial z^2}+16\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+8\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+4\left(\frac{\partial \overline{w}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial z^2}+4\frac{\overline{u}}{r^2}\left(\frac{\partial \overline{u}}{\partial z}\right)^2\\ +4\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial r^2}+6\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial z^2}+8\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+6\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial z^2}-8\frac{\overline{u}}{r^2}\left(\frac{\partial \overline{w}}{\partial z}\right)^2+6\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial r\partial z}\\ -8\frac{\overline{u}^2}{r^3}\frac{\partial \overline{u}}{\partial r}+6\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial r\partial z}+16\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+12\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}+12\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+4\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial r^2}\end{bmatrix}-\frac{\sigma_{nf}B_0^2\overline{u}}{1-ct}-\frac{\mu_{nf}}{K}\overline{u} \tag{13}$$

$$\rho_{nf}\left(\frac{\partial \overline{w}}{\partial t}+\overline{u}\frac{\partial \overline{w}}{\partial r}+\overline{w}\frac{\partial \overline{w}}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial \overline{w}}{\partial r}+\frac{\partial^2 \overline{w}}{\partial r^2}+\frac{\partial^2 \overline{w}}{\partial z^2}\right)$$

$$+(\alpha_1)_{nf}\left(\begin{array}{l}\frac{\overline{u}}{r}\frac{\partial^2 \overline{w}}{\partial r^2}+\frac{\overline{w}}{r}\frac{\partial^2 \overline{w}}{\partial z^2}+\frac{\partial^3 \overline{w}}{\partial r^2\partial t}+\frac{\overline{w}}{r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\frac{3}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}+\frac{3}{r}\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}\\ +\frac{1}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial z}+2\frac{\partial^3 \overline{w}}{\partial z^2\partial t}+\overline{w}\frac{\partial^3 \overline{u}}{\partial r\partial z^2}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+3\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}\\ +4\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+\frac{1}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}+\overline{u}\frac{\partial^3 \overline{w}}{\partial r^3}+\frac{\overline{u}}{r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+u\frac{\partial^3 \overline{u}}{\partial r^2\partial z}+\overline{w}\frac{\partial^3 \overline{w}}{\partial r^3\partial z}\\ +3\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r^2}+3\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{\partial^3 \overline{u}}{\partial r\partial z\partial t}+5\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}\\ +\frac{1}{r}\frac{\partial^2 \overline{w}}{\partial r\partial t}+\frac{1}{r}\frac{\partial^2 \overline{u}}{\partial z\partial t}+2\overline{w}\frac{\partial^3 \overline{w}}{\partial z^3}+2\overline{u}\frac{\partial^3 \overline{w}}{\partial r\partial z^2}+8\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+4\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}\\ +4\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}.\end{array}\right)$$

$$+(\alpha_2)_{nf}\left(\begin{array}{l}\frac{2}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}+4\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}\\ +2\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+2\frac{\partial^2 \overline{u}}{\partial r^2}\frac{\partial \overline{w}}{\partial r}+8\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+\frac{2}{r}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}+\frac{2}{r}\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}\\ +\frac{2}{r}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}+2\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}+2\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r^2}\end{array}\right)$$

$$+(\beta_3)_{nf}\left[\begin{array}{l}\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+\frac{4}{r}\frac{\partial \overline{w}}{\partial r}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+4\frac{\overline{u}^2}{r^3}\frac{\partial \overline{w}}{\partial r}+8\frac{\overline{u}}{r^2}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}\\ +\frac{4}{r}\frac{\partial \overline{u}}{\partial z}\left(\frac{\partial \overline{w}}{\partial z}\right)^2+4\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{u}}{\partial r\partial z}+\frac{6}{r}\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial \overline{w}}{\partial r}+\frac{4}{r}\frac{\partial \overline{w}}{\partial r}\left(\frac{\partial \overline{w}}{\partial z}\right)^2\\ +4\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial r\partial z}+4\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial r^2}+4\frac{\overline{u}^2}{r^3}\frac{\partial \overline{u}}{\partial z}+8\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}\\ +8\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{6}{r}\frac{\partial \overline{u}}{\partial z}\left(\frac{\partial \overline{w}}{\partial r}\right)^2+8\frac{\overline{u}}{r^2}\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}+4\left(\frac{\partial \overline{w}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial r\partial z}\\ +8\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r^2}+\frac{2}{r}\left(\frac{\partial \overline{u}}{\partial z}\right)^3+\frac{2}{r}\left(\frac{\partial \overline{w}}{\partial r}\right)^3+4\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{w}}{\partial r^2}+8\frac{\overline{u}^2}{r^2}\frac{\partial^2 \overline{w}}{\partial z^2}\\ +4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial r^2}+6\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{u}}{\partial r\partial z}+6\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial r^2}-8\frac{\overline{u}^2}{r^3}\frac{\partial \overline{w}}{\partial r}\\ +8\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}+8\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial z^2}+6\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{u}}{\partial r\partial z}-8\frac{\overline{u}^2}{r^3}\frac{\partial \overline{u}}{\partial z}\\ +6\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial r^2}+8\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+4\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial z^2}+16\frac{\overline{u}}{r^2}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}\\ +12\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+16\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}+4\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial^2 \overline{w}}{\partial z^2}+8\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial z^2}\\ +24\left(\frac{\partial \overline{w}}{\partial z}\right)^2\frac{\partial^2 \overline{w}}{\partial z^2}+12\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+8\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}+8\frac{\partial^2 \overline{u}}{\partial z^2}\frac{\partial \overline{w}}{\partial z}\frac{\partial \overline{w}}{\partial r}\\ +18\frac{\partial \overline{w}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}.\end{array}\right]-\frac{\mu_{nf}}{K}\overline{w} \tag{14}$$

And the energy equation is given as

$$
\begin{aligned}
&(\rho C_p)_{nf}\left(\frac{\partial \overline{T}}{\partial t}+\overline{u}\frac{\partial \overline{T}}{\partial r}+\overline{w}\frac{\partial \overline{T}}{\partial z}\right)=k\left(\frac{\partial^2 \overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{T}}{\partial r}+\frac{\partial^2 \overline{T}}{\partial z^2}\right)+\mu\left[2\left(\frac{\partial \overline{u}}{\partial r}\right)^2+2\frac{\overline{u}^2}{r^2}+\left(\frac{\partial \overline{u}}{\partial z}\right)^2+2\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+2\left(\frac{\partial \overline{w}}{\partial z}\right)^2+\left(\frac{\partial \overline{w}}{\partial r}\right)^2\right]\\
&+(\alpha_1)_{nf}\left[\begin{array}{l}
2\frac{\overline{u}^3}{r^3}+4\left(\frac{\partial \overline{u}}{\partial r}\right)^3+2\frac{\overline{u}}{r^2}\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r^2}+\overline{w}\frac{\partial u}{\partial z}\frac{\partial^2 \overline{u}}{\partial z^2}+\overline{u}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r^2}+\overline{u}\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial r\partial z}+\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{u}}{\partial z\partial t}+2\frac{\overline{uw}}{r^2}\frac{\partial \overline{u}}{\partial z}+\overline{u}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}\\
+\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z\partial t}+\overline{w}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{u}}{\partial z^2}+3\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{w}}{\partial r}\right)^2+4\left(\frac{\partial \overline{w}}{\partial z}\right)^3+3\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial \overline{w}}{\partial z}+2\overline{u}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\overline{w}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+3\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial \overline{w}}{\partial z}\\
+2\overline{u}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+\overline{w}\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial z}+3\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial \overline{w}}{\partial z}+3\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{u}}{\partial z}\right)^2+6\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}+2\frac{\overline{u}^2}{r^2}\frac{\partial \overline{u}}{\partial r}+\overline{w}\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial z}+2\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial t}\\
+\frac{\partial \overline{u}}{\partial z}\frac{\partial^2 \overline{w}}{\partial r\partial t}+\frac{\partial \overline{w}}{\partial r}\frac{\partial^2 \overline{w}}{\partial r\partial t}+6\frac{\partial \overline{u}}{\partial r}\frac{\partial u}{\partial z}\frac{\partial \overline{w}}{\partial r}+2\overline{u}\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r^2}+2\overline{w}\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z^2}+2\overline{w}\frac{\partial \overline{u}}{\partial r}\frac{\partial^2 \overline{u}}{\partial r\partial z}+2\frac{\partial \overline{w}}{\partial z}\frac{\partial^2 \overline{w}}{\partial z\partial t}.
\end{array}\right]\\
&+(\alpha_2)_{nf}\left[4\frac{\overline{u}^3}{r^3}+3\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{u}}{\partial z}\right)^2+3\frac{\partial \overline{u}}{\partial r}\left(\frac{\partial \overline{w}}{\partial r}\right)^2+6\frac{\partial \overline{u}}{\partial r}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+6\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\frac{\partial \overline{w}}{\partial z}+4\left(\frac{\partial \overline{u}}{\partial r}\right)^3+3\left(\frac{\partial \overline{u}}{\partial z}\right)^2\frac{\partial \overline{w}}{\partial z}+4\left(\frac{\partial \overline{w}}{\partial z}\right)^3+3\left(\frac{\partial \overline{w}}{\partial r}\right)^2\frac{\partial \overline{w}}{\partial z}\right]\\
&+(\beta_3)_{nf}\left[\begin{array}{l}
2\left(\frac{\partial \overline{w}}{\partial r}\right)^4+8\frac{\overline{u}^4}{r^4}+8\left(\frac{\partial \overline{u}}{\partial r}\right)^2\left(\frac{\partial \overline{u}}{\partial z}\right)^2+2\left(\frac{\partial \overline{u}}{\partial z}\right)^4+16\frac{\overline{u}^2}{r^2}\left(\frac{\partial \overline{u}}{\partial r}\right)^2+8\left(\frac{\partial \overline{u}}{\partial r}\right)^4+16\frac{\overline{u}^2}{r^2}\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+8\left(\frac{\partial \overline{u}}{\partial z}\right)^3\frac{\partial \overline{w}}{\partial r}+8\frac{\overline{u}^2}{r^2}\left(\frac{\partial \overline{w}}{\partial r}\right)^2\\
+8\frac{\overline{u}^2}{r^2}\left(\frac{\partial \overline{u}}{\partial z}\right)^2+8\frac{\partial \overline{u}}{\partial z}\left(\frac{\partial \overline{w}}{\partial r}\right)^3+12\left(\frac{\partial \overline{u}}{\partial z}\right)^2\left(\frac{\partial \overline{w}}{\partial r}\right)^2+16\left(\frac{\partial \overline{u}}{\partial r}\right)^2\left(\frac{\partial \overline{w}}{\partial z}\right)^2+8\left(\frac{\partial \overline{u}}{\partial r}\right)^2\left(\frac{\partial \overline{w}}{\partial r}\right)^2+8\left(\frac{\partial \overline{w}}{\partial r}\right)^2\left(\frac{\partial \overline{w}}{\partial z}\right)^2+16\frac{\overline{u}^2}{r^2}\left(\frac{\partial \overline{w}}{\partial z}\right)^2\\
+16\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}\left(\frac{\partial \overline{w}}{\partial z}\right)^2+8\left(\frac{\partial \overline{u}}{\partial z}\right)^2\left(\frac{\partial \overline{w}}{\partial z}\right)^2+16\left(\frac{\partial \overline{u}}{\partial r}\right)^2\frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{w}}{\partial r}+8\left(\frac{\partial \overline{w}}{\partial z}\right)^4
\end{array}\right]\\
&+\frac{16\sigma_{nf}{}^{*}T_h^3}{3(k_1{}^{*})_{nf}}\left(\frac{\partial^2 \overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \overline{T}}{\partial r}+\frac{\partial^2 \overline{T}}{\partial z^2}\right)+\frac{\sigma_{nf}B_0{}^2\overline{u}^2}{1-ct}
\end{aligned}
\tag{15}
$$

The related boundary conditions are given as

$$
\begin{aligned}
&\overline{u}(r,z,t)=U_w=\frac{ar}{2(1-ct)},\quad \overline{w}(r,z,t)=-w_0,\quad -k_{nf}\frac{\partial \overline{T}}{\partial z}=H_1\left(T_f-\overline{T}\right)\ \text{at}\ z=0,\\
&\overline{u}(r,z,t)=0,\quad \overline{w}(r,z,t)=\frac{\partial h}{\partial t}=-\frac{c}{2}\sqrt{\frac{v}{a(1-ct)}},\quad -k_{nf}\frac{\partial \overline{T}}{\partial z}=H_2(\overline{T}-T_h)\ \text{at}\ z=h(t),
\end{aligned}
\tag{16}
$$

where $w_0>0$ represents suction.

The physical and thermal properties in Eqs. (7), (8), (9), (10), (11), (12), (13), (14), and (15) are given as

$$\rho_{nf}=\rho_s\varphi+\rho_f(1-\varphi) \tag{17}$$

$$\left(\rho c_p\right)_{nf}=\left(\rho c_p\right)_s\varphi+\left(\rho c_p\right)_f(1-\varphi) \tag{18}$$

$$(\rho\beta)_{nf}=(\rho\beta)_s\varphi+(\rho\beta)_f(1-\varphi) \tag{19}$$

$$\mu_{nf}=\mu_f(1-\varphi)^{-2.5} \tag{20}$$

$$\sigma_{nf}=\sigma_f\left[1+\left\{3\left\{\frac{\sigma_s}{\sigma_f}-1\right\}\varphi\left\{\left\{\frac{\sigma_s}{\sigma_f}+2\right\}\varphi-\left\{\frac{\sigma_s}{\sigma_f}-1\right\}\varphi\right\}^{-1}\right\}\right], \tag{21}$$

$$k_{nf}=k_f\left[k_s+2k_f-2\varphi\left(k_f-k_s\right)\right]\left[k_s+2k_f+\varphi\left(k_f-k_s\right)\right]^{-1} \tag{22}$$

$$\frac{\partial q_r}{\partial y}=-\frac{4\sigma}{3K}\frac{\partial \overline{T}^4}{\partial y}\cong-\frac{16\sigma T_s^3}{3K}\frac{\partial^2 \overline{T}}{\partial y^2}\ (\text{using Rosseland's approximation}) \tag{23}$$

The nanofluid in this present study contains pure water as the base fluid while Copper (II) Oxide (CuO) as the nanoparticles. Tables 1 and 2 present the physical and thermal properties of the base fluid and the nanoparticles, respectively.

Using the following similarity variables for the transformations of the system of partial differential equations to system of ordinary differential equations:

$$\eta=\frac{z}{h(t)},\quad \overline{u}=U_w f'(\eta),\quad \overline{w}=-\sqrt{\frac{av}{1-ct}}f(\eta),\quad \theta=\frac{\overline{T}-T_h}{T_f-T_h} \tag{24}$$

We obtain the following system of ordinary differential equations,

$$
\begin{aligned}
&f^{iv}+ff'''-\frac{Sq}{2}(3f''-\eta f''')-\left(M^2+\frac{1}{Da}\right)f''\\
&+\alpha\left(-2f''f'''-f'f^{iv}-ff^{v}+\frac{Sq}{2}\left(5f^{iv}-\eta f^{v}\right)\right)-\gamma\left(2f''f'''+f'f^{iv}\right)\\
&+\beta\left(7f''^3+24f'f''f'''+3f'^2f^{iv}+\mathrm{Re}\left(3f''f'''^2+\frac{3}{2}f'''^2f^{iv}\right)\right)=0,
\end{aligned}
\tag{25}
$$

$$(1+Rd)\theta'' + \Pr\left(\theta' f - \frac{Sq}{2}\eta\theta'\right) + PrEc\left\{\begin{array}{l} M^2 f'^2 + \frac{6}{\mathrm{Re}} f'^2 + f''^2 + \alpha\left(\begin{array}{l} \frac{1}{\mathrm{Re}}\left(-6f'^3 - 6ff'f''\right) \\ -f'f''^2 - ff''f''' + \frac{Sq}{2}\left(3f''^2 + \eta f''f'''\right) + \frac{Sq}{\mathrm{Re}}\left(6f'^2 + 3\eta f'f''\right) \end{array}\right) \\ -3\gamma\left(\frac{2f'^3}{\mathrm{Re}} + \frac{f'f''^2}{2}\right) + \beta\left(\frac{18}{\mathrm{Re}} f'^4 + 6f'^2 f''^2 + \frac{\mathrm{Re}}{2} f''^4\right) \end{array}\right\} = 0 \tag{26}$$

The corresponding boundary conditions are

$$f(0) = A, f(1) = \frac{Sq}{2},\ f'(0) = 1, f'(1) = 0, \tag{27}$$

$$\theta'(0) = \Upsilon_1(\theta(0) - 1),\ \theta'(1) = -\Upsilon_2\theta(1) \tag{28}$$

where

$$\begin{array}{l} \alpha = \frac{(\alpha_1)_{nf} a}{\mu_{nf}(1-ct)}, \beta = \frac{2(\beta_3)_{nf} a^2}{\mu_{nf}(1-ct)^2}, \gamma = \frac{(\alpha_2)_{nf} a}{\mu_{nf}(1-ct)}, Re = \frac{ar^2}{2\nu_{nf}(1-ct)}, \\ Pr = \frac{\mu_{nf} C_{p,nf}}{k_{nf}}, Ec = \frac{U_w^2}{C_{p,nf}(T_f - T_h)}, M^2 = \frac{\sigma_{nf} B_0^2}{\rho_{nf} a}, Sq = \frac{c}{a}, \\ A = \sqrt{\frac{(1-ct)}{av}} w_0, Rd = \frac{16\sigma_{nf}^* T_h^3}{3k_{1,nf}^* K}, \Upsilon_1 = \frac{h_1 h(t)}{K}, \Upsilon_2 = \frac{h_2 h(t)}{K}, \end{array} \tag{29}$$

3. Method of solutions using differential transformation method

It is evident that the nonlinearities in the governing equations of flow and heat transfer in Eqs. (25) and (26) make the development of exact analytical solutions for the equations very difficult, if not to say impossible. Considering the comparative advantages, provision of acceptable analytical results with convenient convergence and stability coupled with relative simplicity of differential transform method, the system of nonlinear differential equations in Eqs. (25) and (26) are solved using differential transformation method. The basic definition, procedure and properties can be found in our previous publications [52, 57].

Following our previous studies [57], the differential transformation or the recursive relations for the governing Eqs. (25) and (26) are;

Table 1. Physical and thermal properties of the base fluid.

Base fluid	ρ (kg/m^3)	c_p (J/kgK)	k (W/mK)
Pure water	997.1	4179	0.613

Table 2. Physical and thermal properties of nanoparticles.

Nanoparticles	ρ (kg/m^3)	c_p (J/kgK)	k (W/mK)
Copper (II) Oxide (CuO)	783	540	18

$$
\begin{pmatrix}
(k+1)(k+2)(k+3)(k+4)F_{k+4} + \sum_{l=0}^{k}(l+1)(l+2)(l+3)F_{l+3}F_{k-l} \\
-1/2Sq\left(3(k+1)(k+2)F_{k+2} - \sum_{l=0}^{k}(l+1)(l+2)(l+3)F_{l+3}\delta(k-1-l)\right) \\
-\left(M^2 + Da^{-1}\right)(k+1)(k+2)F_{k+2} \\
+\alpha\begin{pmatrix}
-2\sum_{l=0}^{k}(l+1)(l+2)(l+3)F_{l+3}(k-l+1)(k-l+2)F_{k-l+2} \\
-\sum_{l=0}^{k}(l+1)(l+2)(l+3)(l+4)F_{l+4}(k-l+1)F_{k-l+1} \\
-\sum_{l=0}^{k}(l+1)(l+2)(l+3)(l+4)(l+5)F_{l+5}F_{k-l} \\
+1/2Sq\begin{pmatrix}
5(k+1)(k+2)(k+3)(k+4)F_{k+4} \\
-\sum_{l=0}^{k}(l+1)(l+2)(l+3)(l+4)(l+5)F_{l+5}\delta(k-1-l)
\end{pmatrix}
\end{pmatrix} \\
-y\begin{pmatrix}
2\sum_{l=0}^{k}(l+1)(l+2)(l+3)F_{l+3}(k-l+1)(k-l+2)F_{k-l+2} \\
+\sum_{l=0}^{k}(l+1)(l+2)(l+3)(l+4)F_{l+4}(k-l+1)F_{k-l+1}
\end{pmatrix} \\
+7\beta\sum_{p=0}^{k}\begin{pmatrix}
\sum_{l=0}^{p}(l+1)(l+2)F_{l+2}(p-l+1)(p-l+2) \\
F_{p-l+2}(k-p+1)(k-p+2)F_{k-p+2}
\end{pmatrix} \\
+24\sum_{p=0}^{k}\begin{pmatrix}
\sum_{l=0}^{p}(l+1)(l+2)(l+3)F_{l+3} \\
(p-l+1)(p-l+2)F_{p-l+2}(k-p+1)F_{k-p+1}
\end{pmatrix} \\
+3\sum_{p=0}^{k}\begin{pmatrix}
\sum_{l=0}^{p}(l+1)(l+2)(l+3)(l+4)F_{l+4} \\
(p-l+1)F_{p-l+1}(k-p+1)F_{k-p+1}
\end{pmatrix} \\
+R_e\begin{pmatrix}
3\sum_{p=0}^{k}\begin{pmatrix}
\sum_{l=0}^{p}(l+1)(l+2)(l+3)F_{l+3}(p-l+1) \\
F_{p-l+3}(k-p+1)(k-p+2)F_{k-p+2}
\end{pmatrix} \\
+3/2\sum_{p=0}^{k}\begin{pmatrix}
\sum_{l=0}^{p}(l+1)(l+2)(l+3)(l+4)F_{l+4} \\
(p-l+1)(p-l+2)F_{p-l+2}(k-p+1)(k-p+2)F_{k-p+2}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix} = 0 \tag{30}
$$

$$
\begin{aligned}
&(1+Rd)(k+1)(k+2)\theta_{k+2}+\Pr\left(\sum_{l=0}^{k}(l+1)\theta_{l+1}F_{k-l}-1/2Sq\sum_{l=0}^{k}(l+1)\theta_{l+1}\delta(k-1-l)\right)\\
&+\Pr Ec\left(\begin{array}{l}
M^2\sum_{l=0}^{k}(l+1)F_{l+1}(k-l+1)F_{k-l+1}+6\dfrac{\sum_{l=0}^{k}(l+1)F_{l+1}(k-l+1)F_{k-l+1}}{R_e}\\
+\sum_{l=0}^{k}(l+1)(l+2)F_{l+2}(k-l+1)(k-l+2)F_{k-l+2}+\\
\alpha\left(\begin{array}{l}
-6\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)F_{l+1}(p-l+1)F_{p-l+1}(k-p+1)F_{k-p+1}\right)\\
\dfrac{-6\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)F_{l+2}(p-l+1)F_{p-l+1}F_{k-p}\right)}{R_e}\\
-\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)F_{l+2}(p-l+1)(p-l+2)F_{p-l+2}(k-p+1)F_{k-p+1}\right)\\
-\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)(l+3)F_{l+3}(p-l+1)(p-l+2)F_{p-l+2}F_{k-p}\right)\\
+1/2Sq\left(\begin{array}{l}
3\sum_{l=0}^{k}(l+1)(l+2)F_{l+2}(k-l+1)(k-l+2)F_{k-l+2}\\
+\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)(l+3)F_{l+3}(p-l+1)(p-l+2)F_{p-l+2}\delta(k-1-p)\right)
\end{array}\right)\\
+\dfrac{Sq\left(6\sum_{l=0}^{k}(l+1)F_{l+1}(k-l+1)F_{k-l+1}+3\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)F_{l+2}(p-l+1)F_{p-l+1}\delta(k-1-p)\right)\right)}{R_e}
\end{array}\right)\\
-3y\left(\begin{array}{l}
2\dfrac{\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)F_{l+1}(p-l+1)F_{p-l+1}(k-p+1)F_{k-p+1}\right)}{R_e}\\
+1/2\sum_{p=0}^{k}\left(\sum_{l=0}^{p}(l+1)(l+2)F_{l+2}(p-l+1)(p-l+2)F_{p-l+2}(k-p+1)F_{k-p+1}\right)
\end{array}\right)+\\
\beta\left(\begin{array}{l}
18\dfrac{\sum_{p=0}^{k}\left(\sum_{n=0}^{p}\left(\sum_{l=0}^{n}(l+1)F_{l+1}(n-l+1)F_{n-l+1}(p-n+1)F_{p-n+1}(k-p+1)F_{k-p+1}\right)\right)}{R_e}\\
+6\sum_{p=0}^{k}\left(\sum_{n=0}^{p}\left(\sum_{l=0}^{n}(l+1)(l+2)F_{l+2}(n-l+1)(n-l+2)F_{n-l+2}(p-n+1)F_{p-n+1}(k-p+1)F_{k-p+1}\right)\right)\\
+1/2R_e\sum_{p=0}^{k}\left(\sum_{n=0}^{p}\left(\begin{array}{l}\sum_{l=0}^{n}(l+1)(l+2)F_{l+2}(n-l+1)(n-l+2)F_{n-l+2}\\(p-n+1)(p-n+2)F_{p-n+2}(k-p+1)(k-p+2)F_{k-p+2}\end{array}\right)\right)
\end{array}\right)
\end{array}\right)=0
\end{aligned}
\tag{31}
$$

where, $\delta(k)=\begin{cases}1 k=0\\ 0 k\neq 0\end{cases}$

After that, the other boundary conditions are represented at first. These unknowns follow the series solution and are obtained using the actual boundary condition associated with the governing equation and re-substituted back into the series solution.

Transforming and representing the boundary conditions,

$$F_0 = A, F_1 = 1,\ F_2 = C, F_3 = E, \ldots . \theta[0] = G, \theta[1] = H, \tag{32}$$

$$F_4 = -\frac{1}{12Da\left(12C^2R_e\beta + 5Sq\alpha - 2\alpha + 6\beta - 2y + 2\right)}\begin{pmatrix} 36CDaE^2R_e\beta + 56C^3Da\beta - 120ADa\alpha F_5 \\ -24CDaE\alpha + 288CDaE\beta - 24CDaEy \\ -2CDaM^2 + 6ADaE - 3CDaSq - 2C \end{pmatrix} \tag{33}$$

$$\theta_2 = \frac{\Pr}{2R_e(1+Rd)}\begin{pmatrix} -8C^4EcR_e{}^2\beta + 12ACEEcR_e\alpha - 6C^2EcR_eSq\alpha \\ +4C^2EcR_e\alpha - 24C^2EcR_e\beta + 6C^2EcR_ey \\ +12ACEc\alpha - 4C^2EcR_e - EcM^2R_e - AHR_e \\ -6EcSq\alpha + 6Ec\alpha - 18Ec\beta + 6Ecy - 6Ec \end{pmatrix} \tag{34}$$

$$F_5 = \frac{1}{60Da\begin{pmatrix} 72C^4R_e^2\beta^2 + 108ACER_e\alpha\beta + 54C^2R_eSq\alpha\beta \\ -36C^2R_e\alpha\beta + 72C^2R_e\beta^2 - 24C^2R_e\beta y - 12AC\alpha^2 \\ +120AC\alpha\beta - 12AC\alpha y + 24C^2R_e\beta + 10Sq^2\alpha^2 \\ +2A^2\alpha - 14Sq\alpha^2 + 27Sq\alpha\beta - 9Sq\alpha y + 9Sq\alpha + 4\alpha^2 \\ -18\alpha\beta + 6\alpha y + 18\beta^2 - 12\beta y + 2y^2 - 6\alpha + 12\beta - 4y + 2 \end{pmatrix}}$$

$$\begin{pmatrix} 1296C^2DaE^3R_e^2\beta^2 - 3456C^4DaER_e\beta^2 + 4320AC^2DaR_e\alpha\beta F_6 \\ -1080C^2DaE^2R_e\alpha\beta + 12528C^2DaE^2R_e\beta^2 - 1080C^2DaE^2R_e\beta \\ y - 72C^2DaEM^2R_e\beta - 270DaE^3R_eSq\alpha\beta + 360ACDaE^2R_e\beta \\ -336C^4Da\alpha\beta + 3360C^4Da\beta^2 - 336C^4Da\beta y - 126C^2DaER_eSq\beta \\ -2700C^2DaESq\alpha\beta + 108DaE^3R_e\alpha\beta - 324DaE^3R_e\beta^2 + 108DaE^3R_e\beta y \\ +56AC^3Da\beta + 1800ADaSq\alpha^2F_6 - 36C^2DaER_e\beta + 144C^2DaE\alpha^2 \\ -2088C^2DaE\alpha\beta + 288C^2DaE\alpha y + 14040C^2DaE\beta^2 - 2088C^2Da \\ E\beta y + 144C^2DaEy^2 + 12C^2DaM^2\alpha - 120C^2DaM^2\beta + 12C^2DaM^2y \\ -108DaE^3R_e\beta + 180DaE^2Sq\alpha^2 - 2160DaE^2Sq\alpha\beta + 180DaE^2Sq\alpha y \\ +15DaEM^2Sq\alpha - 60ACDaE\alpha + 648ACDaE\beta - 60ACDaEy \\ -2ACDaM^2 - 720ADa\alpha^2F_6 + 2160ADa\alpha\beta F_6 - 720ADa\alpha yF_6 \\ +18C^2DaSq\alpha - 180C^2DaSq\beta + 18C^2DaSqy - 72C^2ER_e\beta \\ -72DaE^2\alpha^2 + 1080DaE^2\alpha\beta - 144DaE^2\alpha y - 2592DaE^2\beta^2 \\ +1080DaE^2\beta y - 72DaE^2y^2 - 6DaEM^2\alpha + 18DaEM^2\beta \\ +15DaESq^2\alpha + 6A^2DaE - 3ACDaSq + 720ADa\alpha F_6 \\ +72DaE^2\alpha - 864DaE^2\beta + 72DaE^2y + 6DaEM^2 - 21DaESq\alpha \\ +18DaESq\beta - 6DaESqy + 12C^2\alpha - 120C^2\beta + 12C^2y + 6DaESq \\ +6DaE\alpha - 18DaE\beta + 6DaEy + 15ESq\alpha - 2AC - 6DaE \\ -6E\alpha + 18E\beta - 6Ey + 6E - 6DaEM^2y - 1080C^2DaE\beta \end{pmatrix} \tag{35}$$

4. Numerical procedure for the analysis of the governing equation

For the purpose of verifying the solutions of DTM for the system of the governing equations in Eqs. (25) and (26), a numerical scheme based on fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method is developed. In order to apply the numerical scheme, the fourth-order and second-order ordinary differential equations in Eqs. (25) and (26), respectively, are decomposed into a system of first-order differential equations as follows:

$$f' = p, \tag{36}$$

$$f'' = p' = q, \tag{37}$$

$$f''' = q' = w, \tag{38}$$

$$f^{iv} = w' = z, \tag{39}$$

$$z = fw - \frac{Sq}{2}(3q - \eta w) - \left(M^2 + \frac{1}{Da}\right)q + \alpha\left(-2qw - pz - fz' + \frac{Sq}{2}(5z - \eta z')\right) + \gamma(2qw + pz) + \beta\left(7q^2 + 24pqw + 3p^2z + \mathrm{Re}\left(3qw^2 + \frac{3}{2}q^2z\right)\right) = 0, \tag{40}$$

From Eq. (40), it could be established that

$$z' = \frac{\left\{fw - \frac{Sq}{2}(3q - \eta w) - \left(M^2 + \frac{1}{Da}\right)q + \alpha\left(-2qw - pz + \frac{5zSq}{2}\right) - \gamma(2qw + pz) + \beta\left(7q^2 + 24pqw + 3p^2z + \mathrm{Re}\left(3qw^2 + \frac{3}{2}q^2z\right)\right) - z\right\}}{\alpha\left(f + \frac{Sq\eta}{2}\right)} \tag{41}$$

The above Eqs. (36), (37), (38), (39), (40), and (41) can be written as

$$a(\eta, f, p, q, w, z) = p, \tag{42}$$

$$b(\eta, f, p, q, w, z) = q, \tag{43}$$

$$c(\eta,f,p,q,w,z)=w, \tag{44}$$

$$d(\eta,f,p,q,w,z)=z \tag{45}$$

$$e(\eta,f,p,q,z)=\frac{\left\{fw-\frac{Sq}{2}(3q-\eta w)-\left(M^2+\frac{1}{Da}\right)q+\alpha\left(-2qw-pz+\frac{5zSq}{2}\right)-\gamma(2qw+pz)+\beta\left(7q^2+24pqw+3p^2z+\mathrm{Re}\left(3qw^2+\frac{3}{2}q^2z\right)\right)-z\right\}}{\alpha\left(f+\frac{Sq\eta}{2}\right)} \tag{46}$$

The iterative scheme of the fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) for the above system of first-order equations is given as

$$f_{i+1}=f_i+h\left(\frac{2835}{27648}k_1+\frac{18575}{48384}k_3+\frac{13525}{55296}k_4+\frac{277}{14336}k_5+\frac{1}{4}k_6\right) \tag{47}$$

$$p_{i+1}=p_i+h\left(\frac{2835}{27648}l_1+\frac{18575}{48384}l_3+\frac{13525}{55296}l_4+\frac{277}{14336}l_5+\frac{1}{4}l_6\right) \tag{48}$$

$$q_{i+1}=q_i+h\left(\frac{2835}{27648}m_1+\frac{18575}{48384}m_3+\frac{13525}{55296}m_4+\frac{277}{14336}m_5+\frac{1}{4}m_6\right) \tag{49}$$

$$w_{i+1}=w_i+h\left(\frac{2835}{27648}n_1+\frac{18575}{48384}n_3+\frac{13525}{55296}n_4+\frac{277}{14336}n_5+\frac{1}{4}n_6\right) \tag{50}$$

$$z_{i+1}=z_i+h\left(\frac{2835}{27648}r_1+\frac{18575}{48384}r_3+\frac{13525}{55296}r_4+\frac{277}{14336}r_5+\frac{1}{4}r_6\right) \tag{51}$$

where

$$k_1=a(\eta_i,f_i,p_i,q_i,w_i,z_i)$$

$$l_1=b(\eta_i,f_i,p_i,q_i,w_i,z_i)$$

$$m_1=c(\eta_i,f_i,p_i,q_i,w_i,z_i)$$

$$n_1=d(\eta_i,f_i,p_i,q_i,w_i,z_i)$$

$$r_1=e(\eta_i,f_i,p_i,q_i,w_i,z_i)$$

$$k_2=a\left(\eta_i+\frac{1}{5}h,\ f_i+\frac{1}{5}k_1h,\ p_i+\frac{1}{5}l_1h,\quad q_i+\frac{1}{5}m_1h,\quad w_i+\frac{1}{5}n_1h,\quad z_i+\frac{1}{5}r_1h\right)$$

$$l_2=b\left(\eta_i+\frac{1}{5}h,\ f_i+\frac{1}{5}k_1h,\ p_i+\frac{1}{5}l_1h,\quad q_i+\frac{1}{5}m_1h,\quad w_i+\frac{1}{5}n_1h,\quad z_i+\frac{1}{5}r_1h\right)$$

$$m_2=c\left(\eta_i+\frac{1}{5}h,\ f_i+\frac{1}{5}k_1h,\ p_i+\frac{1}{5}l_1h,\quad q_i+\frac{1}{5}m_1h,\quad w_i+\frac{1}{5}n_1h,\quad z_i+\frac{1}{5}r_1h\right).$$

$$n_2=d\left(\eta_i+\frac{1}{5}h,\ f_i+\frac{1}{5}k_1h,\ p_i+\frac{1}{5}l_1h,\quad q_i+\frac{1}{5}m_1h,\quad w_i+\frac{1}{5}n_1h,\quad z_i+\frac{1}{5}r_1h\right)$$

$$r_2=e\left(\eta_i+\frac{1}{5}h,\ f_i+\frac{1}{5}k_1h,\ p_i+\frac{1}{5}l_1h,\quad q_i+\frac{1}{5}m_1h,\quad w_i+\frac{1}{5}n_1h,\quad z_i+\frac{1}{5}r_1h\right)$$

$$k_3=a\begin{pmatrix}\eta_i+\frac{3}{10}h,\ f_i+\frac{3}{40}k_1h+\frac{9}{40}k_2h, p_i+\frac{3}{40}l_1h+\frac{9}{40}l_2h,\\ q_i+\frac{3}{40}m_1h+\frac{9}{40}m_2h,\quad w_i+\frac{3}{40}n_1h+\frac{9}{40}n_2h,\quad z_i+\frac{3}{40}r_1h+\frac{9}{40}r_2h\end{pmatrix}$$

$$l_3=b\begin{pmatrix}\eta_i+\frac{3}{10}h,\ f_i+\frac{3}{40}k_1h+\frac{9}{40}k_2h,\quad p_i+\frac{3}{40}l_1h+\frac{9}{40}l_2h,\\ q_i+\frac{3}{40}m_1h+\frac{9}{40}m_2h,\quad w_i+\frac{3}{40}n_1h+\frac{9}{40}n_2h,\quad z_i+\frac{3}{40}r_1h+\frac{9}{40}r_2h\end{pmatrix}$$

$$m_3=c\begin{pmatrix}\eta_i+\frac{3}{10}h,\ f_i+\frac{3}{40}k_1h+\frac{9}{40}k_2h,\quad p_i+\frac{3}{40}l_1h+\frac{9}{40}l_2h,\\ q_i+\frac{3}{40}m_1h+\frac{9}{40}m_2h,\quad w_i+\frac{3}{40}n_1h+\frac{9}{40}n_2h,\quad z_i+\frac{3}{40}r_1h+\frac{9}{40}r_2h\end{pmatrix}$$

$$n_3=d\begin{pmatrix}\eta_i+\frac{3}{10}h,\ f_i+\frac{3}{40}k_1h+\frac{9}{40}k_2h,\quad p_i+\frac{3}{40}l_1h+\frac{9}{40}l_2h,\\ q_i+\frac{3}{40}m_1h+\frac{9}{40}m_2h,\quad w_i+\frac{3}{40}n_1h+\frac{9}{40}n_2h,\quad z_i+\frac{3}{40}r_1h+\frac{9}{40}r_2h\end{pmatrix}$$

$$r_3=e\begin{pmatrix}\eta_i+\frac{3}{10}h,\ f_i+\frac{3}{40}k_1h+\frac{9}{40}k_2h, p_i+\frac{3}{40}l_1h+\frac{9}{40}l_2h,\\ q_i+\frac{3}{40}m_1h+\frac{9}{40}m_2h,\quad w_i+\frac{3}{40}n_1h+\frac{9}{40}n_2h,\quad z_i+\frac{3}{40}r_1h+\frac{9}{40}r_2h\end{pmatrix}$$

$$k_4=a\begin{pmatrix}\eta_i+\frac{3}{5}h,\ f_i+\frac{3}{10}k_1h-\frac{9}{10}k_2h+\frac{6}{5}k_3h,\quad p_i+\frac{3}{10}l_1h-\frac{9}{10}l_2h+\frac{6}{5}l_3h,\\ q_i+\frac{3}{10}m_1h-\frac{9}{10}m_2h+\frac{6}{5}m_3h,\quad w_i+\frac{3}{10}n_1h-\frac{9}{10}n_2h+\frac{6}{5}n_3h,\\ z_i+\frac{3}{10}r_1h-\frac{9}{10}r_2h+\frac{6}{5}r_3h\end{pmatrix}.$$

$$l_4=b\begin{pmatrix}\eta_i+\frac{3}{5}h, f_i+\frac{3}{10}k_1h-\frac{9}{10}k_2h+\frac{6}{5}k_3h,\ p_i+\frac{3}{10}l_1h-\frac{9}{10}l_2h+\frac{6}{5}l_3h,\\ q_i+\frac{3}{10}m_1h-\frac{9}{10}m_2h+\frac{6}{5}m_3h,\ w_i+\frac{3}{10}n_1h-\frac{9}{10}n_2h+\frac{6}{5}n_3h,\\ z_i+\frac{3}{10}r_1h-\frac{9}{10}r_2h+\frac{6}{5}r_3h\end{pmatrix}$$

$$m_4=c\begin{pmatrix}\eta_i+\frac{3}{5}h, f_i+\frac{3}{10}k_1h-\frac{9}{10}k_2h+\frac{6}{5}k_3h,\ p_i+\frac{3}{10}l_1h-\frac{9}{10}l_2h+\frac{6}{5}l_3h,\\ q_i+\frac{3}{10}m_1h-\frac{9}{10}m_2h+\frac{6}{5}m_3h,\ w_i+\frac{3}{10}n_1h-\frac{9}{10}n_2h+\frac{6}{5}n_3h,\\ z_i+\frac{3}{10}r_1h-\frac{9}{10}r_2h+\frac{6}{5}r_3h\end{pmatrix}$$

Table 3. Numerical values of skin friction for different parameter

							RKFM [81]	HAM	DTM
A	γ	β	Re	M	S_q	A	$-\frac{1}{2}Re C_f$	$-\frac{1}{2}Re C_f$	$-\frac{1}{2}Re C_f$
0.01	0.10	0.10	1.00	0.40	1.00	0.01	3.88243	3.88243	3.88243
0.02							3.95950	3.95950	3.95950
0.03							4.03655	4.03655	4.03655
0.01	0.11						3.85249	3.85249	3.85249
	0.12						3.82255	3.82255	3.82255
	0.13						3.79260	3.79260	3.79260
	0.10	0.20					4.63884	4.63884	4.63884
		0.25					5.02298	5.02298	5.02298
		0.29					5.33196	5.33196	5.33196
		0.10	0.10				3.85868	3.85868	3.85868
			0.20				3.86116	3.86116	3.86116
			0.30				3.86368	3.86368	3.86368
			1.00	0.50			3.90088	3.90088	3.90088
				0.60			3.92337	3.92337	3.92337
				0.70			3.94985	3.94985	3.94985
				0.40	0.70		6.41631	6.41631	6.41631
					0.75		6.00413	6.00413	6.00413
					0.80		5.58953	5.58953	5.58953
					1.00	0.02	4.06568	4.06568	4.06568
						0.03	4.24817	4.24817	4.24817
						0.04	4.42993	4.42993	4.42993

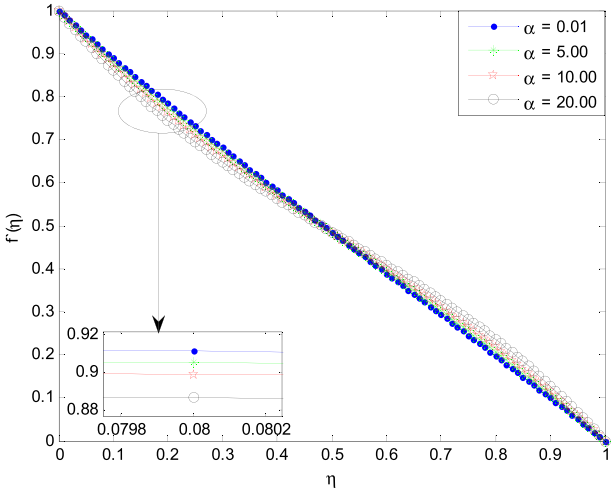

Figure 2. Effect of α on dimensionless velocity profile.

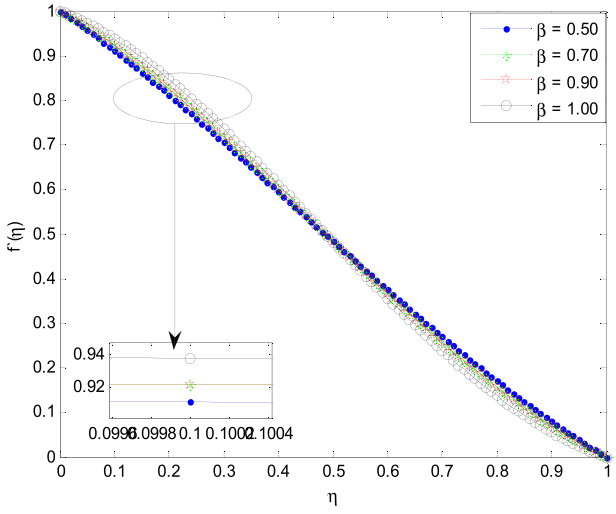

Figure 3. Effect of β on dimensionless velocity profile.

$$n_4 = d\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

$$r_4 = e\begin{pmatrix} \eta_i + \frac{3}{5}h,\ f_i + \frac{3}{10}k_1h - \frac{9}{10}k_2h + \frac{6}{5}k_3h,\ p_i + \frac{3}{10}l_1h - \frac{9}{10}l_2h + \frac{6}{5}l_3h, \\ q_i + \frac{3}{10}m_1h - \frac{9}{10}m_2h + \frac{6}{5}m_3h,\ w_i + \frac{3}{10}n_1h - \frac{9}{10}n_2h + \frac{6}{5}n_3h, \\ z_i + \frac{3}{10}r_1h - \frac{9}{10}r_2h + \frac{6}{5}r_3h \end{pmatrix}$$

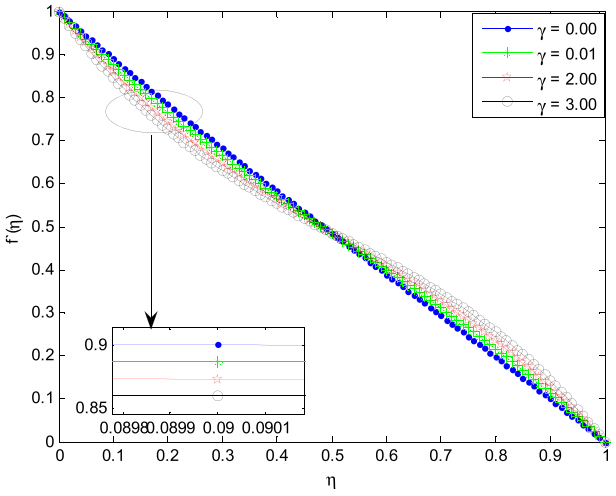

Figure 4. Effect of γ on dimensionless velocity profile.

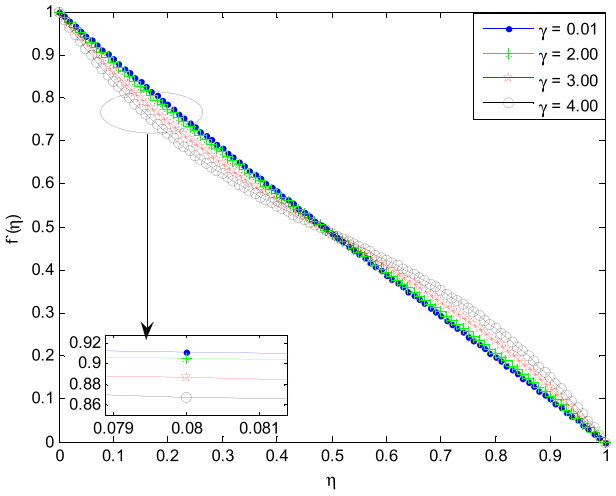

Figure 5. Effect of γ on dimensionless velocity profile.

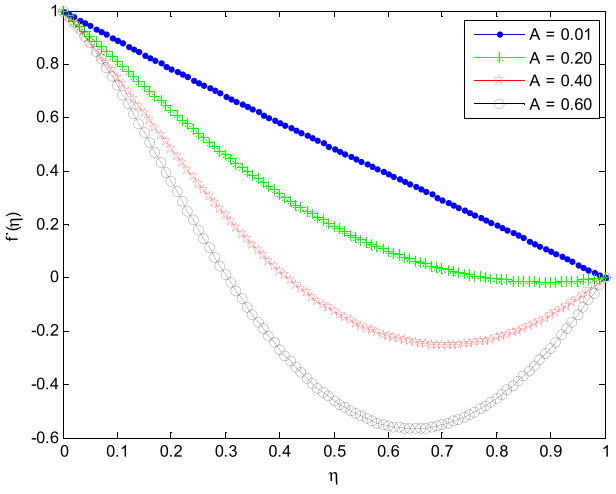

Figure 6. Effect of suction term on dimensionless velocity profile.

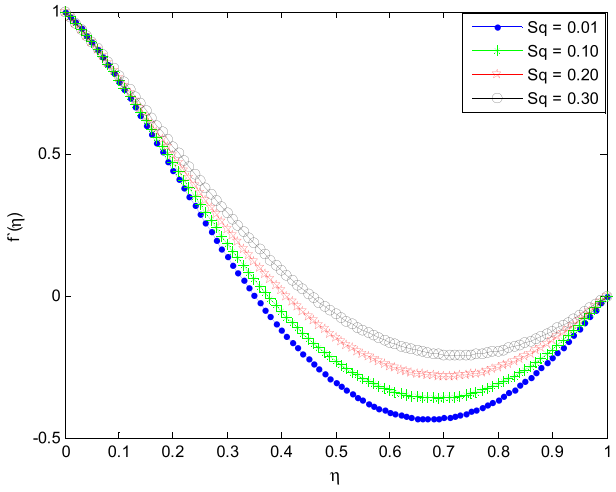

Figure 7. Effect of squeezing term on dimensionless velocity profile.

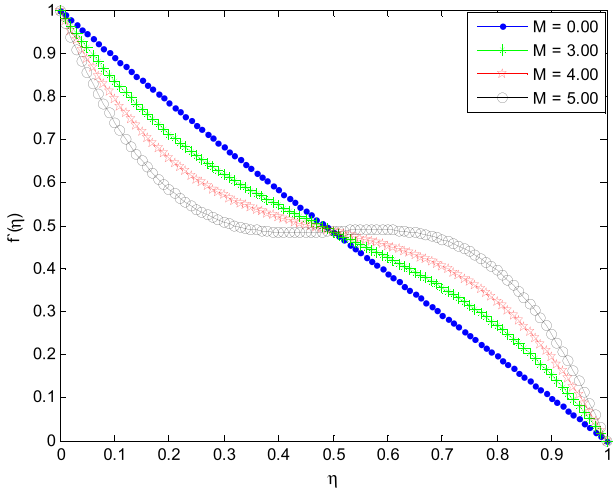

Figure 8. Effect of Hartman number on dimensionless velocity profile.

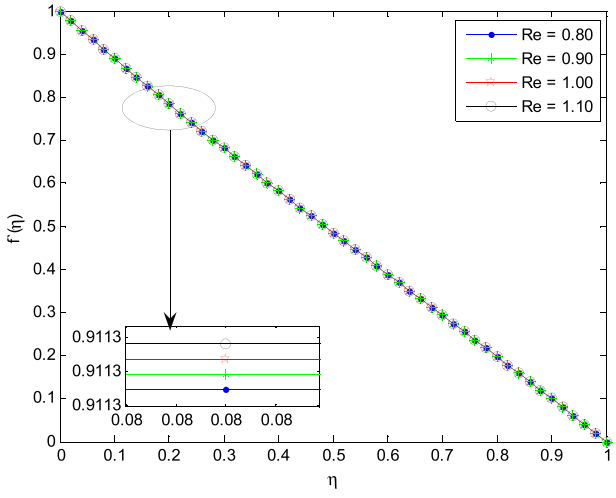

Figure 9. Effect of Reynold's number on dimensionless. velocity profile.

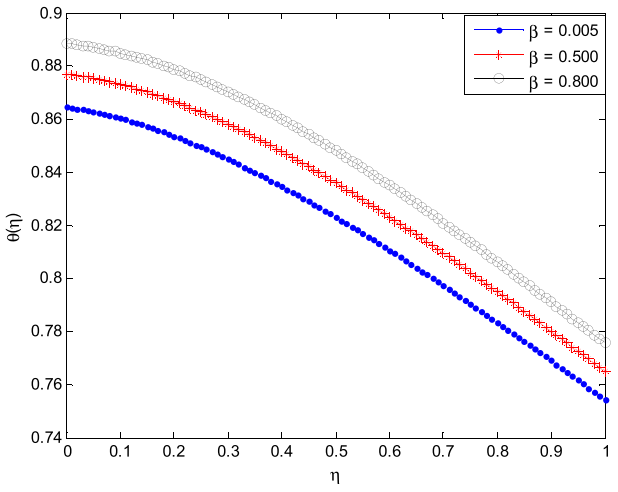

Figure 10. Effect of the third grade fluid parameter on Dimensionless temperature profile.

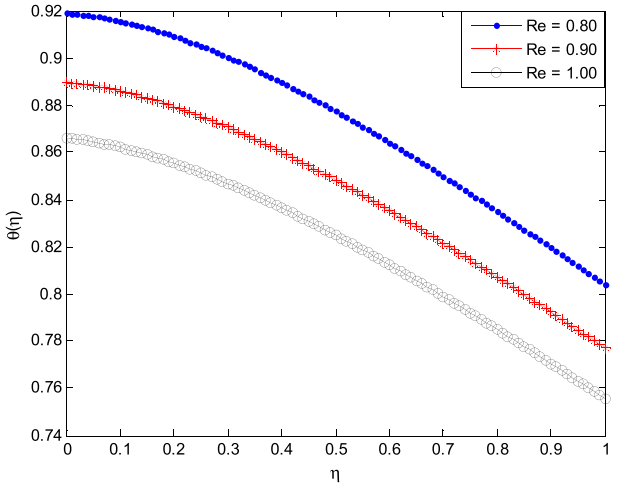

Figure 11. Effect of Reynold's number on dimensionless Dimensionless temperature profile.

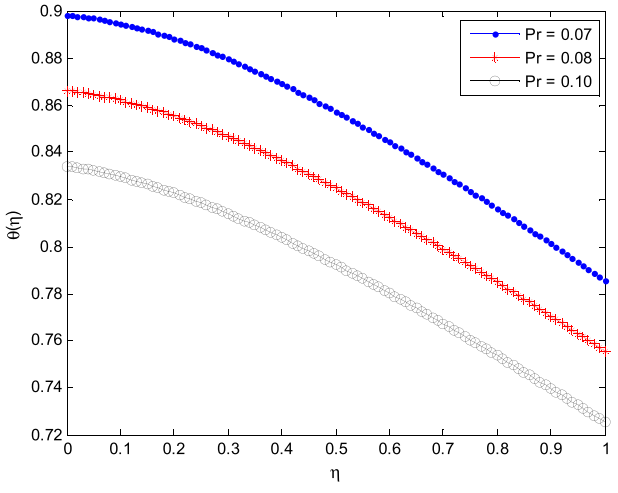

Figure 12. Effect of Prandtl number on dimensionless temperature profile.

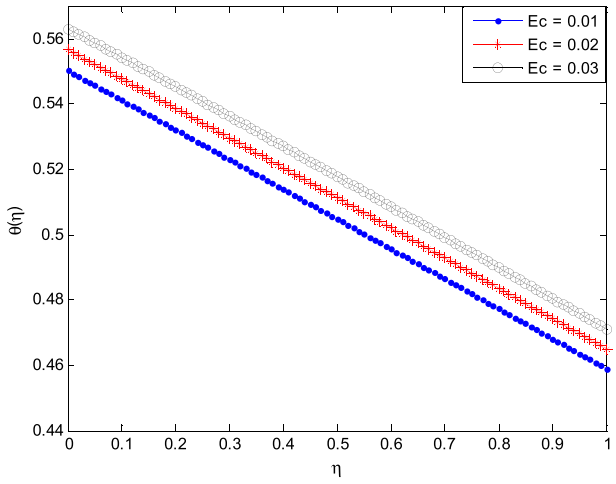

Figure 13. Effect of Eckert number on dimensionless temperature profile.

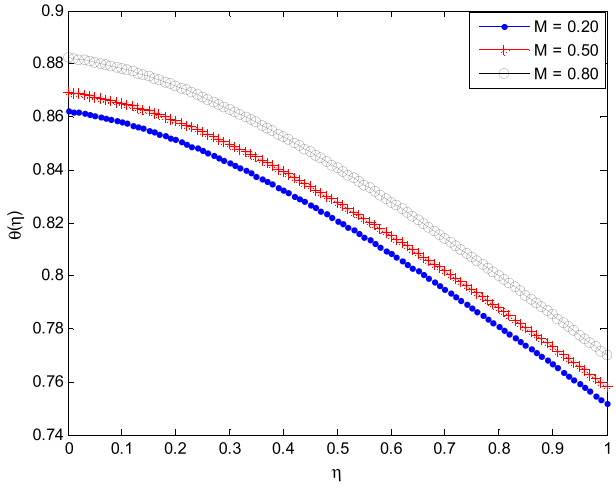

Figure 14. Effect of Hartman number on dimensionless temperature profile.

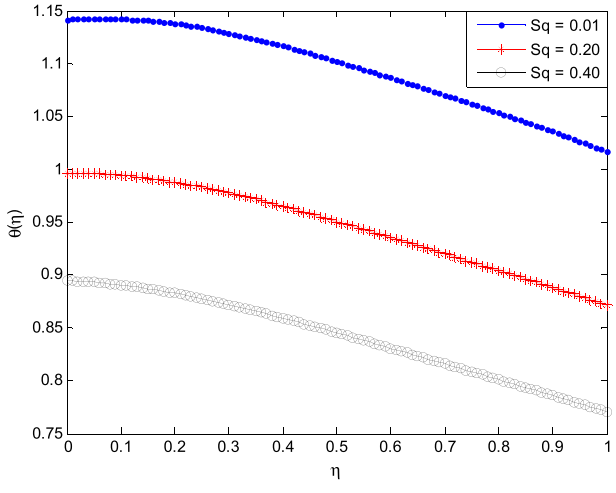

Figure 15. Effect of squeezing parameter on dimensionless temperature profile.

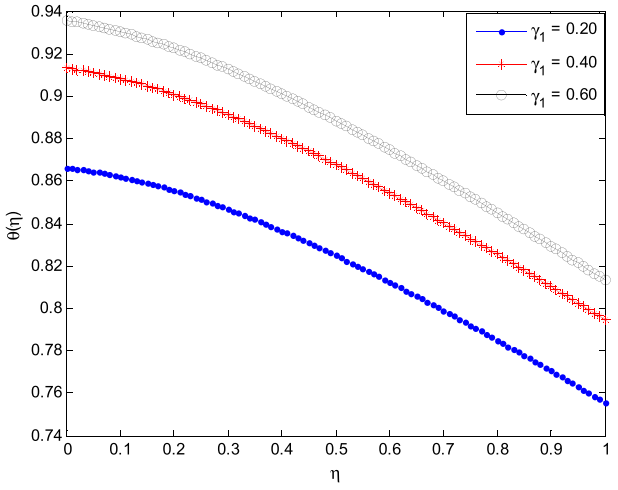

Figure 16. Effect of the first thermal Biot number on dimensionless temperature profile.

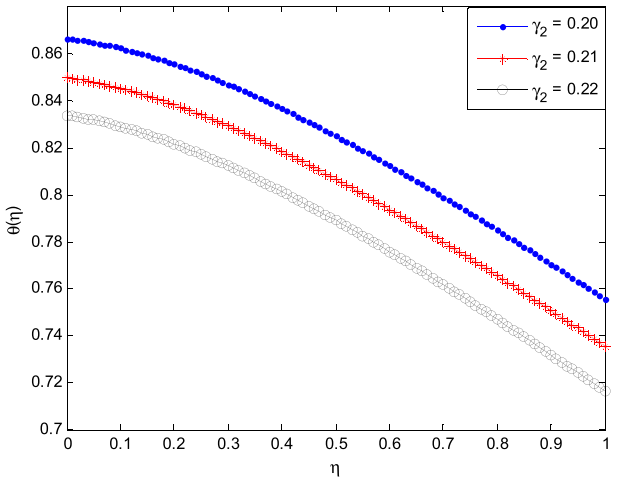

Figure 17. Effect of the second thermal Biot number on dimensionless temperature profile.

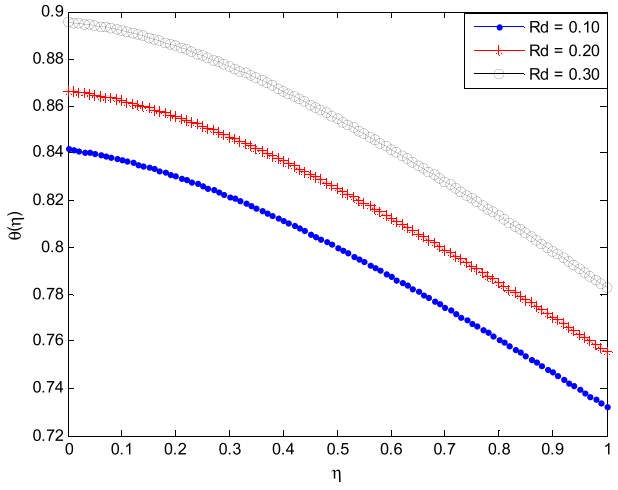

Figure 18. Effect of Radiation term on dimensionless temperature profile.

$$k_5 = a\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\quad p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\quad w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$l_5 = b\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h,\quad p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h,\quad w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$m_5 = c\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h, & p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h, & w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$n_5 = c\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h, & p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h, & w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$r_5 = d\begin{pmatrix} \eta_i + h,\ f_i - \frac{11}{54}k_1h + \frac{5}{2}k_2h - \frac{70}{27}k_3h + \frac{35}{27}k_4h, & p_i - \frac{11}{54}l_1h + \frac{5}{2}l_2h - \frac{70}{27}l_3h + \frac{35}{27}l_4h, \\ q_i - \frac{11}{54}m_1h + \frac{5}{2}m_2h - \frac{70}{27}m_3h + \frac{35}{27}m_4h, & w_i - \frac{11}{54}n_1h + \frac{5}{2}n_2h - \frac{70}{27}n_3h + \frac{35}{27}n_4h, \\ z_i - \frac{11}{54}r_1h + \frac{5}{2}r_2h - \frac{70}{27}r_3h + \frac{35}{27}r_4h \end{pmatrix}$$

$$k_6 = a\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$m_6 = c\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$l_6 = b\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

$$n_6 = d\begin{pmatrix} \eta_i + \frac{7}{8}h,\ f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}$$

Table 4. Numerical values of Nusselt number for different parameter

									RKFM	HAM [81]	DTM
α	γ	β	Sq	Pr	Ec	γ_1	γ_2	Rd	Nu	Nu	Nu
0.01	0.10	0.10	1.00	0.10	0.10	0.20	0.20	0.20	0.092034	0.092034	0.092034
0.02									0.091945	0.091945	0.091945
0.03									0.091856	0.091856	0.091856
0.01	0.11								0.092149	0.092149	0.092149
	0.12								0.092265	0.092265	0.092265
	0.13								0.092380	0.092380	0.092380
	0.10	0.20							0.088806	0.088806	0.088806
		0.25							0.087186	0.087186	0.087186
		0.29							0.085563	0.085563	0.085563
		0.10	0.70						0.093319	0.093319	0.093319
			0.75						0.092502	0.092502	0.092502
			0.80						0.093528	0.093528	0.093528
			1.00	0.11					0.090331	0.090331	0.090331
				0.12					0.088628	0.088628	0.088628
				0.13					0.086926	0.086926	0.086926
				0.10	0.15				0.083306	0.083306	0.083306
					0.25				0.065851	0.065851	0.065851
					0.35				0.048396	0.048396	0.048396
					0.10	0.30			0.108540	0.108540	0.108540
						0.40			0.119230	0.119230	0.119230
						0.50			0.126720	0.126720	0.126720
						0.20	0.70		0.090240	0.090240	0.090240
							0.80		0.092731	0.092731	0.092731
							0.90		0.094174	0.094174	0.094174
							0.20	0.30	0.101120	0.101120	0.101120
								0.40	0.110210	0.110210	0.110210
								0.50	0.119300	0.119300	0.119300

$$r_6 = e\begin{pmatrix} \eta_i + \frac{7}{8}h, f_i + \frac{1631}{55296}k_1h + \frac{175}{512}k_2h + \frac{575}{13824}k_3h + \frac{44275}{110592}k_4h + \frac{253}{4096}k_5h, \\ p_i + \frac{1631}{55296}l_1h + \frac{175}{512}l_2h + \frac{575}{13824}l_3h + \frac{44275}{110592}l_4h + \frac{253}{4096}l_5h, \\ q_i + \frac{1631}{55296}m_1h + \frac{175}{512}m_2h + \frac{575}{13824}m_3h + \frac{44275}{110592}m_4h + \frac{253}{4096}m_5h, \\ n_i + \frac{1631}{55296}w_1h + \frac{175}{512}w_2h + \frac{575}{13824}w_3h + \frac{44275}{110592}w_4h + \frac{253}{4096}w_5h, \\ z_i + \frac{1631}{55296}r_1h + \frac{175}{512}r_2h + \frac{575}{13824}r_3h + \frac{44275}{110592}r_4h + \frac{253}{4096}r_5h \end{pmatrix}.$$

Using the above fifth-order Runge-Kutta Fehlberg method coupled with shooting method, computer programs are written in MATLAB for the numerical solutions of Eqs. (25) and (26). The numerical results for a step size, $h = 0.01$ are compared with the results of the DTM as presented in Table 3.

5. Fluid flow and heat transfer parameters of engineering interests

In the fluid flow and heat transfer analysis, there are some important parameters which are of great interests in the engineering analysis of the thermal-fluidic studies. These set of important considerations include skin friction coefficient and Nusselt number.

5.1. Skin friction

The local skin friction coefficient at lower disk is

$$C_f = \frac{\tau_w|_{x=0}}{\frac{1}{2}\rho U_w^2} \tag{52}$$

Using the similarity variables in Eq. (24) and the dimensionless parameters in Eq. (29), one obtains the local skin friction coefficient as

$$C_f = \sqrt{2}\mathrm{Re}^{-0.5}\begin{pmatrix} 2f''(0) + \alpha(3Sqf''(0) - 2Af'''(0) + 2f''(0)) \\ -2\gamma f''(0) + \beta\left(\frac{\mathrm{Re}}{4}f'''^3(0) + 6f''(0)\right) \end{pmatrix} \tag{53}$$

5.2. Nusselt number

The local Nusselt number at the disk is

$$Nu = \frac{h(t)q_w}{K(T_f - T_h)}|_{z=0}. \tag{54}$$

where wall heat flux is defined as:

$$q_w|_{z=0} = -K\frac{\partial T}{\partial z}|_{z=0} + q_r|_{z=0}. \tag{55}$$

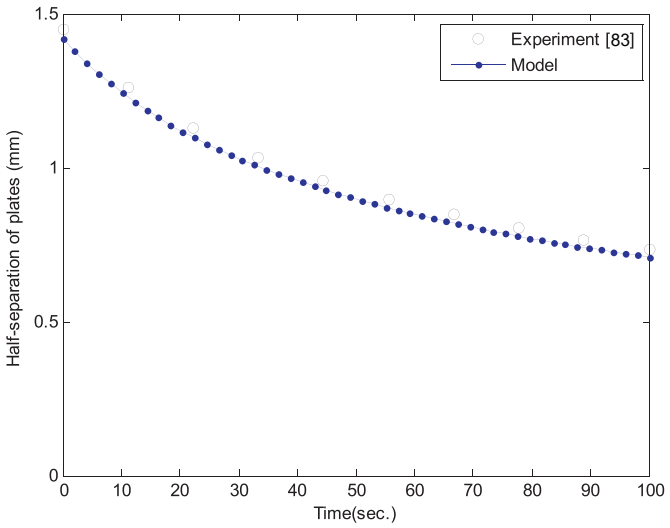

Figure 19. Comparison of experimental and the present model results.

Also, with the help of the similarity variables in Eq. (24) and the dimensionless parameters in Eq. (29), we arrived at the Nusselt number as

$$Nu = (1 + Rd)\theta'(0) \tag{56}$$

6. Results and discussion

The results of differential transformation method (DTM) and the developed fifth-order Runge-Kutta Fehlberg method (RKFM) coupled with shooting method are presented in Table 3. Parametric studies are carried out and the influences of various parameters on the flow and heat transfer processes are established as shown in Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, and 18 (see Table 4).

6.1. Effect of the third grade fluid parameters on dimensionless velocity profile

The impacts of the flow parameters on the dimensionless velocity profile of the fluid are shown in Figures 1, 2, 3, and 4. It is evident that an increase in the fluid flow parameters of the squeezing flow causes a corresponding increase in the fluid flow velocity. This is because the fluid flow parameters in question vary inversely with the viscosity of the fluid being squeezed. As these parameters increase, the viscosity of the fluid decreases and consequently increases the velocity of the fluid as the molecules of the squeezing flow are free to move with less restriction. These parameters can be used as a monitoring agent as they directly affect the viscosity of the third grade squeezing fluid.

6.2. Impacts of suction and squeezing parameters on dimensionless velocity profile

Figures 6 and 7 present the impacts of suction and squeezing parameters on dimensionless velocity profile. Figure 6 shows how the suction parameter affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a suction parameter greater than zero, the radial velocity of the lower disc increases while that of the upper disc decreases as a result of a corresponding increase in the viscosity of the third grade fluid from the lower squeezing disc to the upper disc. Figure 7 depicts the impact of the squeezing parameter on the fluid flow between the parallel discs. The figure shows that an increase in the squeezing parameter causes a corresponding increase in the squeezing rate. This is because as the squeezing parameter increases, the radial velocity of the squeezing discs increases there by generating a driving compressive rotary force on the fluid flowing between the two parallel discs.

6.3. Impacts of Hartman and Reynold's numbers on dimensionless velocity profile

Figures 8 and 9 illustrates the influence of Hartman number and Reynold's numbers on dimensionless velocity profile. Figure 8 shows how the Hartman number affects the velocity of the squeezing discs for an increasing value of the fluid parameters. It is obvious that for a Hartman number greater than zero or for an increasing Hartman number, the radial velocity of the lower disc decreases while that of the upper disc increases as a result of a corresponding increase in the viscosity of the third grade fluid from the upper disc to the lower squeezing disc. As the Hartman number becomes large, it automatically raises the magnetic field as a result of a corresponding increase in the Lorentz force, hence decreases the flow velocity while increase in Reynold's number increases the velocity of the fluid as shown in Figure 9.

6.4. Impacts of the third grade fluid parameter and Reynold's number on dimensionless temperature profile

Figures 10 and 11 depict the influence of the third grade fluid parameter and Reynold's number on dimensionless temperature profile. It has been ascertained that an increase in the third grade fluid parameter causes reduction in the fluid viscosity thereby increasing resistance between the fluid molecules. However, as the Reynold's number associated with the third grade fluid increases, a decreasing effect is noticed in the dimensionless temperature profile. This is because there is a reduction in the convective capability of a high velocity fluid as compare to that with a moderate velocity.

6.5. Effects of Prandtl and Eckert numbers on dimensionless temperature profile

Figures 12 and 13 show the influence of Prandtl and Eckert numbers on dimensionless temperature profile. Figure 12 depicts a decrease in temperature profile as the Prandtl number increases. This is because an increase in the Prandtl number reduces thermal diffusivity thereby reducing the temperature profile. In Figure 13, Eckert number is observed to have a linear increasing property on the dimensionless temperature profile. This is because of the increase in the total kinetic energy of the fluid which correspondingly elevate the fluid temperature.

6.6. Impacts of Hartman number and squeezing parameter on dimensionless temperature profile

Figures 14 and 15 present the influence of Hartman number and Squeezing parameter on dimensionless temperature profile. It is clear that as the Hartman number becomes large, it automatically raises the magnetic field as a result of an increase in the Lorentz force. This makes the temperature profile to increase as the Hartman number increases. Considering Figure 15, a rapid increase in the dimensionless temperature profile is noticed for a large value of squeezing parameter as a result of a driving compressive rotary force which generates a noticeable heating effect thereby increasing the temperature profile.

6.7. Effects of the thermal Biot number and radiation term on dimensionless temperature profile

Figures 16, 17, and 18 depict the influence of the thermal Biot number and Radiation term on dimensionless temperature profile. In Figures 16 and 17, the two thermal Biot number have opposing effect on the dimensionless temperature profile but the cooling effect generated by the first Biot number is more that the temperature rising effect obtained from the second even for the same range of values. As a result, these

parameters can serve as a control for temperature monitoring. However, in Figure 18, as the radiation parameter increases, the dimensionless temperature profile increases. This is because, an increase in the radiation property causes a reduction in the absorptivity and consequently increases the rate of heat transfer to the third grade fluid.

6.8. Validation of the developed models and the solutions

Although, experimental studies on flow analyses of third grade fluids are not common, there are various examples of fluid which exhibit the characteristics of third grades fluids have been given in literature. These include dilute polymer solution (polyisobutane, methylmethacrystalate in n-buthyl acetate), molten plastics, food rheology polymers mixed with melts, manufacturing oils and polymer melts like high viscosity silicon oils, blood, slurries etc. Zhang et al. [82] and Grimm [83] presented experimental studies on the flow analyses of polymer melts and liquids, respectively. The comparison of the present study and the experimental work of Zhang et al. [83] is shown in Figure 19 as shown. The agreement of the results of the present study and the experimental work shows the validation of the models used in this work to study the flow behaviour of the fluid.

7. Conclusion

In this study, differential transformation method has been applied to carry out nonlinear analysis of unsteady squeezing flow and heat transfer of a third grade nanofluid between two parallel disks embedded in a porous medium under the influences of thermal radiation and temperature jump boundary conditions. The developed flow and thermal models were also solved numerically using a fifth-order Runge-Kutta Fehlberg method (Cash-Karp Runge-Kutta) coupled with shooting method. Also, the influences of various flow and heat transfer parameters were investigated. The agreement of the results of the present study and the experimental work shows the validation of the models used in this work to study the flow behaviour of the fluid. Important significance of study includes the study of flow and heat transfer of third grade fluid as applied in energy conservation, coal slurries, polymer solutions, textiles, ceramics, catalytic reactors, oil recovery applications, friction reduction and micro mixing biological samples.

Declarations

Author contribution statement

M. G. Sobamowo: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.

A. A. Yinusa: Contributed reagents, materials, analysis tools or data; Wrote the paper.

S.T. Aladenusi: Contributed reagents, materials, analysis tools or data.

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The authors declare no conflict of interest.

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