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Optimizing the equity reassignment process: A novel application for family businesses

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ABSTRACT

Purpose: This paper aims to support the equity reassignment process of large family business conglomerates, which can be complex due to both the nature and number of companies involved and specific owner requirements. Addressing these issues is especially important in the context of family businesses, since a successful reassignment might resolve and prevent family conflicts.

Design/methodology/approach: The paper presents a model that determines the optimal reassignment in terms of a specific owner's preferences. This model can also handle different types of requirements, including accounting for equity and intra-loan partition between owners and controlling for liquidity, capital structure, and transaction costs. The model also considers risk diversification for each member's fortune by considering the uncertainty involved in the future value of each firm, which can change at any point depending on industry and market conditions. The methodology not only finds the optimal solution in terms of a specific target, but it allows for post-optimal analysis so that owners can obtain important insights in terms of the costs involved in adding each requirement to the model.

Findings/Results: The model was successfully applied in a real case study. The tool played a primary role in identifying a new equity distribution for a family holding structure composed of 4 members and 26 companies. In the first step, the model derived an optimal solution in terms of the target chosen by the owners, but it did not fully satisfy all members. However, owners were able to come to a decision regarding final reassignment after doing a sensible post-optimal analysis.

Originality/value: Previous research has focused on analyzing the special characteristics of family-run businesses and how they differ with respect to non-family-run businesses in terms of performance, governance, and management, among other things. However, this paper is the first referring to the process of ownership reassignment and to use an optimization model in its methodology. It is also the first study that bridges the gaps between the disciplines of portfolio optimization, corporate finance, and family business.

1. Introduction

Ownership and leadership succession is a central topic of discussion in family business. According to the review made in Benavides-Velasco et al. (2013), succession is a main theme in family business research, followed by corporate governance and management and organizational theory.[1] Papers about succession, such as Cabrera-Suárez et al. (2001) Sharma et al. (2003) and Breton-Miller et al. (2004), examine the factors that determine successful ownership transition, discuss conflicts between generations and propose planning for a proper succession. Significant problems can arise when businesses redistribute ownership among descendants. A common policy is to distribute stakes equally, because it is considered the "fair" thing to do. However, this solution might not be efficient in terms of what is best for the business or what the descendants prefer. Heirs might have different capabilities, preferences, management styles, and tolerance for risk, and hence might prefer to exchange or buy/sell stakes between them. Moreover, some members might not want to share ownership to prevent future familiar conflicts. With respect to this issue, Harvey and Evans (1994) divide family business conflicts into different categories, which differ depend-

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[1] For details about corporate governance, see Lubatkin et al. (2005) and Miller et al. (2007). For details about management and organizational theory, see Chrisman et al. (2005), Westhead and Howorth (2006) and Danes et al. (2007).

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ing in the level of complexity and interaction between family owners, external stakeholders and the same business. They show how conflicts emerge on each development stage of the business and propose a resolution process to cope with conflicts at each level. Vilaseca (2002) studies the conflict of interests between family shareholders and the management in Spanish companies. He concludes that owners should be employed in the business, because it increases their commitment. He also finds that owners' commitment is positively related with the time the CEO has been working in the firm and negatively related with the number of shareholders. These results validate the idea that splitting business relationships is sensible when some members are not involved, especially when there are differences in management style with active members.

When the number of companies and owners involved is small, reassignment can be done easily. However, when dealing with large family multibusinesses groups, ownership reassignment can become a complex problem. In addition to owners' requirements and preferences, it is necessary to consider the differences between companies in terms of asset composition, exposure to market conditions, cash flow patterns, and capital structure, among other areas. In such cases, high-quality solutions can only be achieved with the help of a quantitative decision-making tool. Note that this equity reassignment problem also applies in situations involving combining ownership, such as conglomerate mergers; it is necessary to look carefully at synergies, economies of scale, and diversification gains.[2]

The value of each company is a crucial input in equity reallocation because it determines the wealth of each owner. When this equity is not traded on market, its value must be estimated.[3] Conventional methodologies for doing so are based on estimates of companies' future cash flows or current EBITDA numbers[4] (see details in Damodaran (2012) and Larrabee and Voss (2012)). The variability of future cash flows increases the estimation error of these models. Such valuation uncertainty can motivate reassignment, even for owners who have no conflicts with one another. Since each member's wealth can take on different values, a compromise might emerge to balance their wealth and its volatility via reassignment.

There is plenty of research examining the performance differences between family-run and non-family-run business, with mixed results. Covering 350 papers on this subject, the survey in Pindado and Requejo (2015) mentions the advantages and disadvantages of family-controlled firms that could explain these results. Family business don't have the incentive to make decisions for a short term benefit, have less agency costs on average and can be more competitive when reputation matters. That's why evidence in Anderson and Reeb (2003) don't come as a surprise. They show that, under some founding-family ownership bounds, S&P 500 family firms perform better than non-family firms. However, family-controlled firms have intra-family conflicts, parental altruism and nepotism. For example, Pérez-González (2006) show that when a family member is chosen as a chief executive officer (CEO), then return on assets and market-to-book ratios decline with respect to firms where the CEO chosen is not member of the family.[5] There is also research that study the behavioral differences in terms of diversification. For example, Gomez-Mejia et al. (2010) determine in a sample composed of 360 companies that family firms diversify less, both domestically and internationally than non-family firms. Previous findings are also supported by the results in Sanchez-Bueno and Usero (2014). In this paper, they also discover that family businesses increase international diversification when the second largest shareholder is a financial company.

However, no previous research addresses the reassignment problem, and moreover, no one uses any mathematical model when proposing a recommendation or solution. The reassignment problem has not been discussed in operations research literature either, which confirms that no optimization model has been used for this purpose. The reason for this may be related to some known problems. Since the value of each company is uncertain, the equity assignment problem resembles the portfolio optimization problem, first described in Markowitz (1952). One can consider the stakes to be the weights each owner possesses in a portfolio of uncertain equity and their aim to find the composition best suited to their risk profile. In the context of the business reassignment problem, this translates in finding the equity distribution that reduces the risk of each member's fortune. Analogously, reassignment can be related to portfolio rebalancing, as each member aims to change the equity stakes (weights) in their portfolios.[6] Thus, many of the metrics and concepts made for portfolio optimization can be applied to the reassignment problem as well.

The objective of this paper is to present a general optimization model that can help identify new and/or better equity reassignment options. The paper is organized as follows: Section 2 explains the model and the inclusion of different requirements as constraints. Section 3 presents the case study data, specifically, a family holding company consisting of 4 owners and 26 companies operating in 5 different industries. Finally, Section 4 concludes and presents possible extensions of the research.

2. Model

2.1. Business reassignment model

Suppose that a holding company includes a set of J companies owned by a group of M members. Denote α_{ij} the current stake (%) of member i in company j. The sum of the stakes in each company is one, i.e., $\sum_{i\in M} \alpha_{ij} = 1,\ j \in J$.

Let $\bar{v}_j$ be the estimated value of company j. With this data in hand, one can calculate the fortune of each member of the holding company, which equals the total value of their stakes in all the companies. Let the vector α_i denote the current stakes of member i in the companies. The estimated wealth or equity $\bar{E}_i$ of member i is

$$\bar{E}_i := \sum_{j\in J} \alpha_{ij}\bar{v}_j = \bar{v}'\alpha_i \tag{1}$$

The idea behind the Business Reassignment Problem (BRP) model is to obtain new values x_i for the stakes of each member. The BRP can be written as follows:

$$\min_{x_i \geq 0}\ f(x_1, x_2 .., x_M) \tag{2}$$

$$s.t.:\ \bar{v}' x_i = \bar{v}'\alpha_i \quad i \in M \tag{3}$$

$$\sum_{i\in M} x_i = \vec{1} \tag{4}$$

$$(x_1, x_2 ..., x_M) \in \Phi \tag{5}$$

Constraint (3) maintains the prior estimated wealth after reallocation. Equation (4) is the natural constraint for any assignment: The sum of the stakes in each company must be one. The BRP should only account for members who want changes in their stakes. If some members do not participate in the reassignment process, constraint (4) can be written as $\sum_{i\in M} x_i = e$, where $e \leq \vec{1}$ represents the cumulative ownership in each company of all members involved in the reallocation. Set

[2] Reviews and analyses of merger performances are discussed in Andrade et al. (2001), Leland (2007), and Ismail et al. (2011).

[3] For details on private equity, see Klier (2009), Harris et al. (2014), and Phalippou and Gottschalg (2009).

[4] EBITDA stands for earnings before interest, taxes, depreciation and amortization.

[5] See references in Bertrand and Schoar (2006) and Pindado and Requejo (2015) to look for more evidence about performance differences between family business and non-family business.

[6] Woodside-Oriakhi et al. (2013), Cariño and Turner (1998) and Fang et al. (2006), among others, provide examples of portfolio rebalancing.

Φ in (5) contains all the specific requirements of the owners in a particular case. Finally, the target function (2) is a metric representing what owners want to improve in the new configuration. It must be defined by the owners and hence changes from case to case. Although the model can be applied to any business reassignment, it is particularly suitable for family business. Therefore, many of the examples of constraints in the set Φ and objective functions explained in the paper are inspired by these companies.

The BRP model captures individual rationality because it guarantees each owner will keep at least the current value of her or his fortune. It considers efficiency, since reassignment quality is being quantified through the objective function (2), which considers owner preferences and aims. The model also allows for "fairness" in the sense of respecting everyone's preferences and rights equally. This principle is a main concern in family business succession, and thus the objective function and requirements included in (5) must respect it. As observed in the real case study, the model enables us to perform a post-optimal analysis, which plays an important role in finding a reassignment solution that leaves all member satisfied. Finally, the BRP ensures incentive compatibility if parameters related to the equity value are estimated without the vested interest of the owners. The level of commitment and involvement of each owner to the family business determines the level of information asymmetry. To solve this agency problem, the equity valuation process of all companies must be performed before implementing the BRP. Each owner must agree to and respect the information revealed in the valuation process before proceeding to the redistribution of shares. To reduce information asymmetries, every member can ask for external advice on asset valuation. Moreover, members can agree to hire a third party to lead the ownership reassignment process. Cesaroni and Sentuti (2017) explain that external advisory should consider hard and soft issues for successful successions in family business.

2.2. Target examples

2.2.1. Risk

One possible objective function relates to risk exposure, or the variability of each member's fortune. Each company has a particular asset composition and belongs to a different industry, and thus has a unique risk profile. The BRP can account for this risk by considering the value of each company as random and model dependent. Analogous to assets in a portfolio problem, it is therefore possible to include risk by measuring the variance of the value of each company. Moreover, it is possible to account for the fact that companies share common sources of risk by estimating the correlations of the values between companies. The wealth of each fortune can be thought of as a portfolio of assets, and thus it is plausible to estimate its risk by estimating its variance.

Denote the random variable v_j as the equity value of company j. Let $\bar{v}_j$ be the estimated value derived from v_j and C be the covariance matrix between the values of each company. The variance V_i of a member's wealth is

$$V_i := Var(E_i) = Var(v' x_i) = x_i' C x_i \tag{6}$$

where $Var()$ denotes the variance. The difference with respect to the mean-variance portfolio problem is that the objective function of the BRP should include the preferences of all members. Objective functions of the type $\min \sum_{i\in M} V_i = \sum_{i\in M} x_i' C x_i$ are efficient in terms of maximizing diversification gains, but that might not be "fair" in the sense of how risk reduction is distributed among owners. One possible solution is to seek a new assignment where V_i does not differ significantly across members, i.e., to attain homogeneity in terms of risk. Thus, a target for (2) can be to minimize the highest V_i

$$\min \max_{i\in M} V_i = \min \max_{i\in M} x_i' C x_i \tag{7}$$

In portfolio problems, Sharpe (1994) defines the Sharpe ratio (SR), which compares the expected return of a portfolio to its volatility in order to evaluate performance in terms of the mean-variance compromise. A low (high) SR indicates poor (good) performance. In the BRP, the SR can be applied to standardize the estimated value of its fortune by its estimated volatility. The SR for each owner can be defined as

$$SR_i := \frac{\bar{E}_i}{\sqrt{V_i}} = \frac{\bar{v}' x_i}{\sqrt{x_i' C x_i}} \tag{8}$$

Thus, another possibility is to seek a new assignment where SR_i does not differ significantly across members, i.e., to attain homogeneity in terms of risk and wealth compromise. As such, a target for (2) is to maximize the lowest SR_i

$$\max_{x_i \geq 0} \min_{i\in M} SR_i = \max_{x_i \geq 0} \min_{i\in M} \frac{\bar{v}' x_i}{\sqrt{x_i' C x_i}} = \max_{x_i \geq 0} \min_{i\in M} \frac{\bar{v}' \alpha_i}{\sqrt{x_i' C x_i}} \tag{9}$$

The latter objective function is equivalent to

$$\min_{x_i \geq 0} \max_{i\in M} \frac{\sqrt{x_i' C x_i}}{\bar{v}' \alpha_i} \tag{10}$$

The benefit of writing the target as in (10) is that $\bar{v}' \alpha_i$ is an input. Hence, the problem can be solved by second-order cone programming when Φ is a convex set. Note that in a portfolio problem, investors with a higher tolerance for risk are willing to accept risky portfolios when rewards are higher. The BRP can allow us to compensate a higher level of risk with more wealth if constraint (3) is relaxed. To estimate the expected values and the covariance matrix, the free cash flows method allows us to simulate different cash flow patterns by changing some key drivers, such as growth, cost of capital, operating margins, or reinvestment rates. These key drivers can be used across companies simultaneously to consider possible correlations.

2.2.2. Leverage

Another target relates to the leverage each member holds as a result of the reassignment. Companies can have varying leverage within a conglomerate; it is possible that one member could end up with stakes in the ones with the most leverage and have higher financial exposure and thus a higher cost of debt. In the case of family business, Romano et al. (2001) find that variables like size, family control, and business planning influence the financial decisions within these companies. They also show that they rely more on family loans as a source of finance than non-family-owned companies. Gottardo and Maria Moisello (2014) provide evidence that the capital structure of family firms is more leveraged than non-family firms in medium-large companies only. They also find a positive correlation between leverage and the level of involvement of the family in management.

One way to account for the capital structure or leverage of each member is to calculate the debt-to-equity ratio. To do so, let d_j be the total debt of company j. The debt of member i is defined as the sum of the debt of each company weighted by the stakes of i

$$TD_i := \sum_{j\in J} d_j x_{ij} = d' x_i \tag{11}$$

The estimated leverage ratio of each member is defined by dividing their debt by their wealth

$$LR_i := \frac{TD_i}{\bar{E}_i} = \frac{d' x_i}{\bar{v}' x_i} \tag{12}$$

One option is to seek a new assignment where LR_i does not differ significantly across members, i.e., to attain homogeneity in leverage levels. This can be achieved by maximizing the lowest leverage ratio across members

$$\min_{x_i \geq 0} \max_{i\in M} LR_i = \min_{x_i \geq 0} \max_{i\in M} \frac{d' x_i}{\bar{v}' x_i} = \min_{x_i \geq 0} \max_{i\in M} \frac{d' x_i}{\bar{v}' \alpha_i} \tag{13}$$

The last equality is valid in the BRP because of equation (3), and thus the objective function is linear. In fact, when Φ is a convex set, the problem is convex. Note that the BRP can include for risk and leverage simultaneously. In such cases, we can leave one of the functions (10) or (13) as a target and control for the other as a constraint. The case study in section 3 illustrates how to include both measures.

Proposition 1. *Without set Φ, the optimal solution when using any of the targets (10) or (13) is given by*

$$x_i^* = \frac{\bar{v}'\alpha_i}{\bar{v}'e}e \quad i \in M \tag{14}$$

Proof. When the objective function is (10) and (13), and in absence of specific constraints, BRP equals to

$$\min \quad z \tag{15}$$

$$s.t.: z \geq g_i(x_i) \quad i \in M \tag{16}$$

$$\bar{v}'x_i = \bar{v}'\alpha_i \quad i \in M \tag{17}$$

$$\sum_{i\in M} x_i = \vec{e}, \quad x_i \geq 0 \quad i \in M \tag{18}$$

where $g_i(x) = \frac{\sqrt{x'Cx}}{\bar{v}'\alpha_i}$ for target (10) and $g_i(x) = \frac{d'x_i}{\bar{v}'\alpha_i}$ for target (13). The Lagrangian function is

$$z + \sum_{i\in M} \lambda_i(g_i(x_i) - z) + \rho'(\sum_{i\in M} x_i - e) + \sum_{i\in M} \beta_i(\bar{v}'x_i - \bar{v}'\alpha_i) - \sum_{i\in M} \mu_i x_i, \tag{19}$$

with $\lambda, \rho, \beta, \mu$ Lagrangian multipliers. It follows that the optimal solution satisfies $z = g_i(x_i)$ $i \in M$. We see this first from the KKT conditions

$$\sum_{i\in M} \lambda_i = 1, \quad \lambda_i(z - g_i(x_i)) = 0 \quad i \in M. \tag{20}$$

Then

$$\sum_{i\in M} \lambda_i g_i(x_i) = z. \tag{21}$$

Second, also from the KKT conditions,

$$\lambda_i \bigtriangledown g_i(x_i) + \rho + \beta_i \bar{v} - \mu_i = \vec{0} \quad i \in M. \tag{22}$$

Multiplying equation (22) by x_i' and adding the terms (note that $\mu_i' x_i = 0$), then

$$\sum_{i\in M} \lambda_i \bigtriangledown g_i(x_i)' x_i + \rho' e + \bar{v} \sum_{i\in M} \beta_i \alpha_i = 0. \tag{23}$$

For both ratios, $\bigtriangledown g_i(x)'x = g_i(x)$. Then equation (23) becomes

$$\sum_{i\in M} \lambda_i g_i(x_i) + \rho' e + \bar{v} \sum_{i\in M} \beta_i \alpha_i = 0. \tag{24}$$

Plugging in (21),

$$z = -(\rho' e + \bar{v} \sum_{i\in M} \beta_i \alpha_i). \tag{25}$$

If $z > g_i(x_i)$, then $\lambda_i = 0$ and hence $\rho + \beta_i \bar{v} \geq 0$. Multiplying the latter by α_i and adding it up, we get $\rho' e + \bar{v} \sum_{i\in M} \beta_i \alpha_i \geq 0$, which means that $z \leq 0$.

But $g_i(x_i) \geq 0$, hence $z = g_i(x_i) = 0$. This is a contradiction for BRP, since $SR_i > 0$ when using the Sharpe ratio. When using the leverage ratio, this means that $d'e = 0$ a contradiction. Then $z = g_i(x_i)$. Note that the solution in equation (14) satisfies

$$\frac{x_i}{\bar{v}'\alpha_i} = \frac{x_h}{\bar{v}'\alpha_h} \quad i, h \in M. \tag{26}$$

It is easy to see that the function $\tilde{g}_i(x) := g_i(x)\bar{v}'\alpha_i$ is homogeneous of degree 1 for both ratios. Solutions that satisfy condition (26) are then optimal, since

$$g_i(x_i) = \frac{\tilde{g}(x_i)}{\bar{v}'\alpha_i} = \frac{1}{\bar{v}'\alpha_i}\tilde{g}(\frac{\bar{v}'\alpha_i}{\bar{v}'\alpha_h}x_h) = \frac{1}{\bar{v}'\alpha_i}\frac{\bar{v}'\alpha_i}{\bar{v}'\alpha_h}\tilde{g}(x_h) = g_h(x_h). \quad \square \tag{27}$$

The above result is intuitive. The reassignment of two members is linearly related and scaled by a constant factor that relates to the total wealth of each member. For example, if one member's equity is two times the other's, then that member's stakes will be exactly two times the other's in each company. Since volatility and debt are homogeneous functions of degree 1, all members have the same ratios.

2.3. Examples constraints defining Φ

2.3.1. Ownership partition

Ownership partition occurs when some members do not want to share stakes in some companies or any company. As previously noted, this can happen when members do not share the same management style and prefer to end business relationships in order to maintain friendships. Therefore, we must assign such companies to only one member. In those cases, the following binary variable has to be added:

$$y_{ij} = \begin{cases} 1, & \text{if } x_{ij} > 0. \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

To illustrate the constraints that should be included in the model, suppose that member i and l do not want to share stakes in company j. The following constraints must be included:

$$y_{ij} + y_{lj} = 1 \tag{29}$$

$$y_{hj} \geq x_{hj} \quad h: i, l \tag{30}$$

$$y_{hj} \in \{0, 1\} \quad h: i, l \tag{31}$$

The first constraint (29) ensures that member i does not own any part of company j when member l does, and vice versa. Constraints (30)–(31) define the relationship between y_{hj} and x_{hj}, i.e., when $x_{hj} > 0$, then $y_{hj} = 1$.

2.3.2. Intra-group debt

When companies' activities depend on each other, as in a holding structure, there might be accounts payable, and consequently, accounts receivable between them. According to Buchuk et al. (2014), companies within Chilean business groups that use intra-group loans invest more and have higher ROE than other firms. Gopalan et al. (2007) show that intra-group loans provide a financial advantage because such an arrangement reduces the probability that a company will go bankrupt. Intra-group debt produces debt between owners. In the context of family business, if members decide to split ownership, they may also decide to dissolve or bound their internal debt, even if doing so causes them to lose the benefits found in empirical studies. To include this requirement in the model, let a_{jk} be an account receivable/payable of j to h (receivable if $a_{jk} > 0$ and payable otherwise). The total net amount of these accounts A_{il} between i and l with any stake assignment is:

$$A_{il} := \sum_{j\in J, k\in J} x_{ij} a_{jk} x_{lk} \tag{32}$$

One possibility is to add the following constraint:

$$|A_{il}| \leq A_{max} \tag{33}$$

where A_{max} represents the current debt between members, or a smaller amount if a reduction is desired. Note that $a_{jk} = -a_{kj}$, i.e., an account receivable of j to h is an account payable of h to j. As expected, the net sum of the accounts in the holding structure is zero, i.e., $\sum_{j\in J, k\in J} a_{jk} = 0$. With these properties, then:

$$A_{li} = \sum_{j\in J, k\in J} x_{lj} a_{jk} x_{ik} = -\sum_{j\in J, k\in J} x_{lj} a_{kj} x_{ik} = -\sum_{j\in J, k\in J} x_{ik} a_{kj} x_{lj} = -A_{il} \tag{34}$$

Hence, $A_{ii} = 0$, as expected, and $\sum_{i\in M, l\in M} A_{il} = 0$ for any stake's assignment. We note that solution (14) in Proposition 1 has the property $A_{il} = 0$; in fact:

$$A_{il} = \sum_{j\in J,k\in J} \frac{\bar{v}'\alpha_i}{\bar{v}'e} e_j a_{jk} \frac{\bar{v}'\alpha_l}{\bar{v}'e} e_k = \frac{\bar{v}'\alpha_i}{\bar{v}'e}\frac{\bar{v}'\alpha_l}{\bar{v}'e} \sum_{j\in J,k\in J} e_j a_{jk} e_k$$
$$= \frac{\bar{v}'\alpha_i}{\bar{v}'e}\frac{\bar{v}'\alpha_l}{\bar{v}'e} A_{ee} = 0 \tag{35}$$

2.3.3. Liquidity

In order to avoid having one member receive stakes from the most liquid companies and leave mainly non-current assets to the others, the asset composition of each owner's stake must be controlled. With respect to family business, Kuan et al. (2011) examine the relationship between corporate governance and cash holdings in Taiwan companies. Their results show that family-run firms hold more cash than non-family-run firms. They also find that more independent boards of directors, and less collateral used by owners for personal borrowing, increases cash reserves and reduces agency costs of cash holdings in family-controlled firms. Steijvers and Niskanen (2013) discover that companies managed by founder CEOs hold less cash than those managed by descendant CEOs, especially when ownership is concentrated among few members. De Visscher et al. (2011) also provide detailed documentation on liquidity management for family-run firms.

To control for liquidity in the BRP, we add a constraint to bound the working capital of each member. Denoting wc_j as the working capital of company j, the working capital WC_i of member i can be defined as the weighted sum of the working capital of each company times the stakes of the member:

$$WC_i := wc' x_i \tag{36}$$

One option to address working capital issues is to include the constraint $WC_i \geq 0$ to ensure that members have no working capital issues. To do so, the company's current assets is bounded to be at least its current liabilities. Another option is to have the same working capital for each member, proportioned by each member's wealth, that is, $\frac{WC_i}{\bar{v}'\alpha_i}$, which is constant for every $i \in M$. Note that solution (14) satisfies both types of constraints when the working capital of the holding structure is positive. In such a case,

$$WC_i = \frac{\bar{v}'\alpha_i}{\bar{v}'e} wc' e \tag{37}$$

2.3.4. Transaction costs

The exchange of shares during the reassignment process involves paying more than common fees. It might also involve paying personal capital gains taxes in each company where exchange happens. This issue is reminiscent of what happens in asset allocation problems, where managers limit the turnover of their portfolio at each rebalancing period, especially for active strategies. However, reassignment is done only once in the BRP problem, so it becomes a less important issue. In cases where owners want to handle transaction costs endogenously, one option is to include a constraint based on the work done in Perold (1984), Li et al. (2000), and Yoshimoto (1996) for portfolio problems. For example, add

$$\|x - \alpha\|_p \leq TC \tag{38}$$

where TC is proportional to the maximum cost allowed and $\|x-\alpha\|_p$ represents the difference between the current and new assignment, which is proportional to the ownership exchange and thus to the transaction costs involved. For example, when $p = \{1, 2\}$

$$\|x - \alpha\|_1 = \sum_{i\in M,j\in J} \left| x_{ij} - \alpha_{ij} \right| \tag{39}$$

$$\|x - \alpha\|_2 = \sum_{i\in M,j\in J} (x_{ij} - \alpha_{ij})^2 \tag{40}$$

2.3.5. Preserve the subsidiary structure

Some companies might be subsidiaries of other companies. As such, the member's ownership of that subsidiary depends on the share owned in these larger companies. Up to this point, reassignments could be completed without considering the current share of the company in a subsidiary. This means that a new assignment could break the current dependency among these companies. Suppose that company j has a share β_{jk} of company k. Then, the constraint that maintains this dependency is:

$$x_{ik} = \beta_{jk} x_{ij} \tag{41}$$

In summary, the formulation of the BRP depends on what owners are looking for and what are their specific requirements, which can change according to the context of the company and its owners. This section provides examples on how to include common topics of family business successions, such as leverage, liquidity, co-ownership, intra-group loans etc. Since each requirement consists in a set of equations, it is easy to see its impact in the quality of the solution achieved, by adding or removing the equations related to this requirement. The following section shows how those examples are applied on a real case study.

3. Results & discussion

Four family members, denoted F, I, S, and A in order to maintain anonymity, own 26 private companies. These companies operate in these five sectors: agribusiness (AG), finance (FI), food processing (FP), forestry (FR), and transportation (T). Some of the holding company's minor partners decided not take part in the reassignment process and therefore played no further role. I will refer to the holding company, then, as FISA.

FP companies are the primary and largest firms in this holding company, and this is the area in which the family has the most expertise, e.g., in the baking industry. The FI firms also control FISA finances, such as debt financing and risk and asset management. Only one firm operates in the T sector; it owns a fleet of trucks dedicated to cargo. The FR companies deal in radiata pine and eucalyptus plantations, which are processed and exported as wood. The AG companies plant mainly avocado and walnut trees.

In the early 1990s, each member inherited equal ownership of 8 companies, which all operated in the FP, FI, and T sectors. At that time, everyone decided to appoint member I to manage the holding company, as they had been the most involved member prior to receiving their inheritance. While I managed the company, the others acted as passive investors, simply participating in an annual board meeting. Over the next 25 years the company grew, and the holding firm acquired and created 18 more companies, regularly giving owners the option to choose their stakes. Member I, as the active manager, raised most of the capital for new and leveraged acquisitions. Recently, conflicting ideas about future investments and management styles have emerged among the owners. Moreover, other members wanted to become active managers, specifically F and S. This situation started to affect the holding company; new investment opportunities were missed and important capital expenditures delayed because of the lack of consensus. As this conflict also damaged the members' familial relationships, they had no choice but to reassign and split equity.

3.1. Equity valuation and composition before reassignment

The primary task before reassignment was to determine what was actually being reassigned. Up to this point, no accurate valuation had been conducted for any of the companies. Thus all members got involved in the equity and debt valuation process of each company. Different methods were used, depending on the type of firm. For example, for firms with larger historical information about cash flows, the model based on estimating future cash flows was used, as in Damodaran (2012). For the financial sector and the youngest companies, either equity value or recent appraisals by lenders was considered. For valuations using the cash flow model, key drivers of the valuation of

Table 1
Descriptive statistics for equity value by sector, in million USD. The equity value is derived from 1000 scenarios, that is, 1000 combinations of key driver values. Excess kurtosis is the amount over that of a standard normal distribution (3). The Sharpe ratio (SR) is the mean-to-volatility ratio. In summary, 58% of conglomerate equity value was in agribusiness and finance. In proportion to the mean, those two sectors are also more volatile, i.e., they have lower SRs.

	AG	FI	FP	FR	T	Total
Mean	8.5	14.2	10.3	8.2	1.3	42.5
Standard deviation	1.6	3.0	3.3	0.8	0.0	4.8
Median	8.4	14.2	10.5	8.2	1.3	42.6
Sharpe ratio	5.2	4.7	3.12	10.34	202.5	9
Percentile 5%	5.8	9.3	4.9	7.0	1.3	34.9
Percentile 95%	11.4	19.4	15.6	9.6	1.3	50.6
Excess kurtosis	−0.1	0.1	0.0	−0.1	0.2	−0.1

Table 2
Debt value by sector in million USD. The most leveraged sectors are agribusiness and finance. They carry 90% of total debt. The Leverage ratio compares mean equity to debt.

	AG	FI	FP	FR	T	Total
Debt	13.1	15.7	3.3	0.0	0.0	32.1
Leverage ratio	1.53	1.10	0.32	0.00	0.00	0.76

each company were determined, such as growth rates, operating margins, reinvestment rates, and cost of capital. In this way, it was easy to simulate different cash flow patterns and produce values for different scenarios in the future. Some drivers affected more than one company, such that values between companies could be correlated. To support the results, the members also hired external business valuation experts to assess the largest companies. Tables 1, 2 present the equity and debt valuation results, respectively, grouped by sector. It is important to note that each member had to sign a document indicating their agreement with the final values in order to continue with the reallocation process. The reason was to ensure that no member could refute the estimations for their own reasons after the reassignment.

Table 3 and Fig. 1 show the debt and equity distributions of each owner in each sector prior to reallocation. After analyzing these results, all members demanded the preservation of the previous compromise regarding their leverage ratios, as defined in (12). The members also decided to use target (10), i.e., to improve the Sharpe ratio homogeneously among members. As indicated above, F and S wanted to become active managers and decided to collaborate. A, satisfied with I's management, decided to maintain passive ownership. Hence, companies whose assets cannot be easily divided, mainly because they are illiquid, must be reassigned to one of these two groups (going forward, group 1 consists of members A and I; group 2, F and S). There are 17 "non-splittable" companies, and the model had to include a binary variable for each one.

In terms of the accounts payable and receivable between companies, the financial sector firms and firms with larger and more stable cash flows usually funded new firms that did not have enough cash flow to satisfy their working capital and capital expenditure needs. Before reallocation, group 1 owed group 2 US$ 1.6 million. FISA wanted to reduce this amount, if possible, in order to avoid future conflicts. The mature companies in the holding structure have higher cash flows and more liquidity than the other firms. Thus, to avoid working capital problems, the model bounded the working capital of each member, as defined in (36), to ensure positive amounts. The model had to include additional constraints related to specific requirements. For example, group 2 did not want stakes in some AG companies because they are outside their area of expertise and geographically distant. The final model used for FISA was the following:

- Set M = {F,I,S,A}
- Set NS: Non-splittable companies NS ⊂ J = {1..26}

Table 3
Owner's mean equity and deviation results (in million USD) before reassignment after simulating 1000 scenarios of key driver values. The Sharpe ratio (SR) is the mean-to-volatility ratio of equity. As expected, member I had the best SR but also the highest leverage ratio: new and leveraged acquisitions in which member I had higher stakes were in sectors with higher SRs. The other members had similar ratios.

	A	I	F	S	Total
Equity (mean)	8.2	11.8	8.4	8.1	36.4
Equity (volatility)	2.3	2.8	2.3	2.3	9.7
Debt	6.4	10.3	6.7	6.3	29.7
Sharpe ratio	3.5	4.2	3.6	3.5	3.8
Leverage ratio	78%	88%	80%	78%	82%

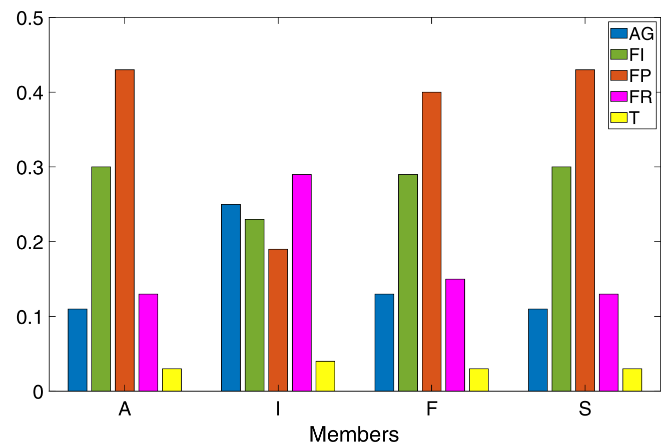

Fig. 1. Equity composition of each member in each sector before reassignment. Member I had higher stakes in agribusiness, forestry, and transportation than did the other members. The allocations of members A, F, and S were similar in composition and wealth.

- Set G: Companies in the AG sector in which group 2 does not want stakes G ⊂ J

$$\min_{\substack{x_i \geq 0 \\ y_{1j}, y_{2j} \in \{0,1\}}} \max_{i \in M} \frac{x_i' C x_i}{\alpha_i' C \alpha_i} \tag{42}$$

$$s.t.: \vec{v}' x_i = \vec{v}' \alpha_i \quad i \in M \tag{43}$$

$$\sum_{i \in M} x_i = \vec{1} \tag{44}$$

$$d' x_i = d' \alpha_i \quad i \in M \tag{45}$$

$$y_{1j} + y_{2j} = 1 \quad j \in NS \tag{46}$$

$$y_{1j} \geq x_{Ij} + x_{Aj} \quad j \in NS \tag{47}$$

$$y_{2j} \geq x_{Fj} + x_{Sj} \quad j \in NS \tag{48}$$

$$|A_{A,F} + A_{A,S} + A_{I,F} + A_{I,S}| \leq A_{max} \tag{49}$$

$$wc' x_i \geq 0 \quad i \in M \tag{50}$$

$$x_{Fj} = x_{Sj} = 0 \quad j \in G \tag{51}$$

Target (42) aims to improve everyone's Sharpe ratio equally. If this value is smaller than 1, a reassignment that improves everyone's Sharpe ratio is feasible, but if the value is greater than 1, such a reallocation is not possible. Not turning this aim into a hard constraint allows for a solution in the latter case. Set Φ is composed of all the specific requirements from equations (45) to (51). Constraints (45) ensure that each member keeps their current leverage ratio. Split equations in (46) to (48) assign non-splittable companies to either group one or group two. Constraint (49) bounds the accounts payable receivable/payable between groups to a user-defined number A_{max}. Recall that terms A_{il} explicitly depend on the stakes (see equation (32)). Constraints (50) require positive working capital for every member. Finally, equations in

(51) prevent the assignment of stakes in some AG companies to members of group 2.

Note that the model does not include the constraint (38) for transaction costs. The Chilean law and tax regulations allow owners to exchange shares between one another without having to pay capital gain taxes. Thus, in this case transaction costs are negligible when compared to the equity of each member. The implementation of the final reassignment was conducted by legal and tax experts who took care of all legal arrangements, such as the dissolution and creation of new firms and the exchange of debt guarantees, among other steps. This process is not detailed in this paper.

3.2. Application of the BRP model

This section explains how the model was used to find a solution that satisfied all members. The above model is a mixed integer non-linear programming problem. Therefore, the Artelys-Knitro 10.1 solver was used. A feasible solution meeting all constraints existed, yet it would have caused F and S to worsen their SRs by 33%. Naturally, they did not like this solution and demanded alternatives. So the next step was to quantify the improvement in the Sharpe ratios produced from removing one or more constraints. Removing all constraints in Φ except for (45) resulted in all members being assigned the same ratios, except for member I, who experienced a slight increase of 4%. Although this solution was not practical, it revealed important information: Any reassignment with additional constraints was likely to decrease someone's SR.

The next step was to conduct a sensitivity analysis for each constraint. The idea was to find a solution that satisfied as many requirements as possible but resulted in a similar outcome to removing all constraints in Φ. After trying different options, it was found that the target was very sensitive to the most valuable company in the FP sector (70% of total value). This company (which we will refer to as FP1) is non-splittable. By dropping the equations in (46) to (48) for FP1 and including the remaining constraints, the worst SR decrease was only 2%. Moreover, if the only constraints in Φ besides (45) are the equations in (46) to (48) for FP1, the worst SR decrease was 22%. Therefore, any reallocation that included constraints (46) to (48) for FP1 (and any other set of constraints) produced a decrease between 22% and 33%.

Group 1 inquired how much reassignment could be improved by removing the constraints related to the AG sector (51), which were imposed by group 2. When those equations were removed, the worst decrease was 27%, a 6% improvement with respect to the solution with all constraints (a 33% decrease). Relatedly, if the only constraints included in Φ were (45), (46) to (48) (applied only to FP1) and (51), then the largest decrease was approximately 33%, which is the same value obtained when including all constraints.

In view of the previous results, the members had one real alternative to the case with all constraints: They could share ownership in FP1. In that case, the largest decrease in SR was 2%. As shown before, this was close to the results obtained when removing all constraints in Φ, where everyone has a minimum of a 0% increase. Any other solution implied a decrease of at least 22%, which approximates the decrease produced by the solution with all constraints. Moreover, group 1 could have argued that the real decrease with all constraints was 27%, since the gap between 27% and 33% was due to the demands of group 2. A and I were fine with both alternatives. Thus, F and S (group 2) had to decide whether to share ownership of FP1 (each owning 20%) with members A and I or accept a more volatile fortune. This volatility increase was approximately US $1 million each. Ultimately, group 2 preferred to share FP1. Table 4 and Fig. 2 provide more details about the selected assignment.

Table 4
Owner's mean equity and deviation (in million USD) after reassignment. S experienced the largest decrease in SR, 2%.

	A	I	F	S	Total
Equity (mean)	8.2	11.8	8.4	8.1	36.4
Equity (volatility)	2.4	2.7	2.3	2.3	9.7
Sharpe ratio	3.5	4.4	3.5	3.4	3.8
Sharpe ratio increase (%)	−1	3	1	−2	0

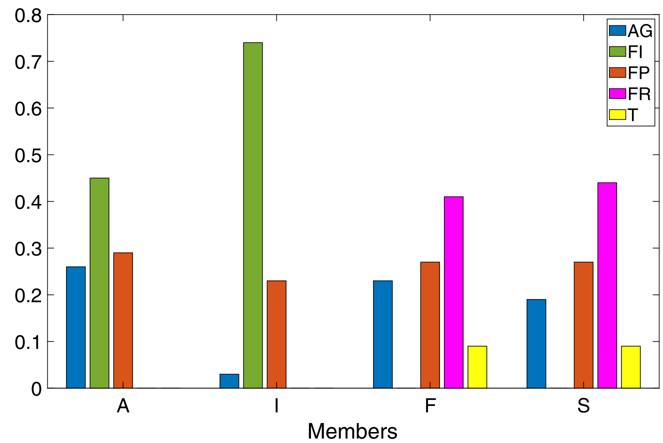

Fig. 2. Equity of each member in each sector after reassignment. The results differed from the initial assignment (Fig. 1). The members of group 1 had allocations in only three sectors and mainly in the food processing sector. The members of group 2 had a similar composition and maintained interests in agribusiness, despite the constraints added by this group in this sector. The allocations in the finance sector were similar across members.

4. Conclusion

This paper proposes a new approach for reassigning equity within family businesses. The methodology is based on a optimization model, which can handle different objective values and requirements of the business group owners. The latter includes partial equity and intra-loan partition between owners and homogeneity in liquidity and capital structure measures among members. The model also considers the risk of each company within its value and how to diversify the uncertainty of each owner's fortune by applying ideas from mean-variance portfolio optimization. The paper is a novel contribution in that it combines disciplines that have not been considered together in previous research. The business reassignment process needs to use corporate finance knowledge for equity and debt valuation of each company and to know what to control in the reassignment. It also demands knowledge of how family businesses are run in order to understand the concerns and possible conflicts between stockholders. It requires knowledge about mathematical programming and portfolio optimization to translate the requirements into a model that can be solved by computational algorithms.

The methodology played a primary role in reassigning stakes in a family business holding company. One of the keys was to convince owners that the valuation process of each company must come before the reassignment. The satisfaction of each owner in terms of the quality of the reassignment depended heavily on having their unanimous consent regarding the equity and debt value of each company. To attain this, the process considered different valuation methods depending on the nature and historical characteristics of the company. Discussions emerged when two valuation methodologies gave different estimates, especially between members that wanted to dissolve business relationships. That's why the existence of a neutral party in charge of the process was crucial for overcoming these problems. The fact that the optimization model handles uncertainty also played an important role, because it downplayed the need for an exact estimation. Participants knew that variations from mean values are quantified and penalized by the model. After resolving the equity valuation question, the real case study shows how the BRP not only identified a reassignment that a manual approach could not achieve given the problem's complexity, but allowed for the evaluation of alternatives in a post-optimal analysis and the quantification of the costs of each requirement (constraint) included in the objective function.

A possible extension is to address the uncertainty of the equity value explicitly in the optimization problem, as was done for portfolio optimization. Analogous to the mean-variance portfolio problem, one of the drawbacks of the BRP is that the risk involved in the equity valuation uncertainty is only measured by the variance. There are other ways of including higher moments in its probability distribution.[7] One option is to build a two-stage stochastic programming model that assigns ownership in a first stage before revealing the realized equity for each member in the second stage.[8] The benefit of this approach is that it allows for other types of probability distribution for the drivers generating the value for each company. Thus, the model can choose reassignment that behaves homogeneously across members, even for extreme scenarios. Another improvement might be to include dependency between the value of a company and its owner, and to consider synergies (penalties) when companies belong (do not belong) to the same owner.

Declarations

Author contribution statement

Lorenzo Reus: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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[7] See seminal publications of Rockafellar et al. (2000), Black and Litterman (1992), and Goldfarb and Iyengar (2003) to see how portfolios are constructed without assuming normality of returns.

[8] See Mulvey and Shetty (2004), Fábián (2008), and Topaloglou et al. (2008) for applications of stochastic programming in portfolio optimization.