

# Social Distancing in Infection Control\*

## A Lattice Model Demonstration

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The article illustrates the efficacy of proper social distancing methods during disease outbreaks. We inspect the outbreak by counting the number of ways  $n$  individuals are infected, given an initially infected patient introduced in a lattice ( $c_n$ ). Using self-avoiding walks on lattices where the infection spreads according to the SIR (Susceptible, Infected, Removed) model, we were able to show an exponential decay of the ratio of initial and final values of  $c_n$  after the distancing methods were employed.

### 1. Introduction

Modelling infection spread over biological networks using computations (both deterministic and stochastic) with simple random walks, self-avoiding random walks, etc., has been substantially explored before. Mostly questions were raised along the lines of estimating whether a substantial part of a given network could be infected, assuming a random node to be initially infected. Bauer and Lizier [1] present a more nuanced investigation regarding the role of a single individual on the spreading dynamics rather than the global properties of the network.

Herein, we shall set up a rather similar walk-counting method to assess the efficacy of social distancing and quarantine on influencing the infection spread. Unlike [1], however, our enquiry involves inspecting ramifications of the topological properties of the network and how they are affected when measures such as social distancing are imposed.



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#### Keywords

Infectious diseases, nCoV, self-avoiding Walks.

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## 2. Motivations and Setup

General epidemic spreading models on network structures identify a complex network with a graph  $\Gamma = (\mathbb{V}, \mathbb{E})$  (here  $\mathbb{V}$  and  $\mathbb{E}$  are the vertex and edge sets respectively). It is natural to consider the spreading process in terms of *infection walks* in the corresponding graph. A **walk** of length  $N$  is a map from  $S = \{1, \dots, N\}$  into  $\mathbb{V}$ .

$$\gamma : S \longrightarrow \mathbb{V}$$

$$n \longmapsto \gamma_n$$

Such that  $\gamma_i, \gamma_{i+1} \in \mathbb{E} \quad \forall i$ . A walk is said to be **self-avoiding** if  $\gamma$  is injective.

An infinite graph  $\Gamma = (\mathbb{V}, \mathbb{E})$  is a **lattice** if it is connected, transitive and locally finite. For the sake of simplicity, we shall only look at walks on lattices, in particular the  $\mathbb{Z}^d$  lattice.

Notice that this would immediately restrict the number of susceptible individuals a given node can infect, and we would have to consider each individual identical in terms of spreading influence in the network.

Motivated by the behaviour of the present COVID-19 outbreak, we limit ourselves to the SIR model of epidemic spreading. When considering infection walks on lattices, a certain node can only infect its immediate neighbours. Given that an arbitrary node can infect precisely  $2d$  neighbours (an assumption made to fix the number of people an average individual is likely to infect), our network resembles the  $\mathbb{Z}^d$  lattice.

When considering infection walks on lattices, a certain node can only infect its immediate neighbours.

Proceeding in terms of the SIR model, we consider as susceptible those nodes which have not been infected by the walk. The infected nodes infect their neighbours, continuing the walk, and are either classified as *deceased* or *recovered* (with immunity) within a specific duration required by the disease. These deceased or immune nodes comprise the effectively *removed* individuals. This final idiosyncrasy in the model tacitly encodes the injectivity of



the walk, which pinpoints self-avoiding walks as our primary object of study.

### 3. Self-avoiding Walks on Lattices

The model is defined as in [3], assigning equal probability to all paths of length  $n$  starting from the origin and without self-intersections. This family of probability measures is not consistent as  $n$  is varied, and thus does not define a stochastic process; the model is combinatorial in nature.

The number of  $n$ -step self-avoiding walks from the origin is denoted by  $c_n$ . According to our previous setup,  $c_n$  represents given sufficient time- the number of possible ways  $n$  individuals could be infected given a particular node is infected initially (without loss of generality, the origin).

The simplest bounds on the behaviour of  $c_n$  are obtained as follows. An upper bound on  $c_n$  is given by the number of walks which have no immediate reversals, or in other words which never visit the same site at times  $i$  and  $i + 2$ . Avoiding immediate reversals allows  $2d$  choices for the initial step, and  $2d - 1$  choices for the  $n - 1$  remaining steps, for a total of  $2d(2d - 1)^{n-1}$ . For a lower bound we simply count the number of walks in which each step is in one of the  $d$  positive coordinate directions. Such walks are necessarily self-avoiding. Thus we have

$$d^n \leq c_n \leq 2d(2d - 1)^{n-1}.$$

For better estimates, we look at one further construction. By  $SAW_n$  we denote the collection of Self-avoiding walks of length  $n$  starting from the origin, with the clear consequence that  $c_n = |SAW_n|$

The **concatenation map** provides an intuitive way to construct a self-avoiding walk of length  $m + n$  from smaller self-avoiding walks of lengths  $m$  and  $n$  respectively:

$$\begin{aligned} \text{Concat} : SAW_m \times SAW_n &\longrightarrow SAW_{m+n} \\ (\gamma^{(1)}, \gamma^{(2)}) &\longmapsto \text{Concat}(\gamma^{(1)}, \gamma^{(2)}) \end{aligned}$$

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Given by

$$\text{Concat}(\gamma^{(1)}, \gamma^{(2)})(k) = \gamma^{(1)}(k) \quad k = 1, \dots, n$$

$$\text{Concat}(\gamma^{(1)}, \gamma^{(2)})(k) = \gamma^{(1)}(n) + \gamma^{(2)}(k - n) - \gamma^{(2)}(1)$$

for  $k = n + 1, \dots, n + m$ .

Since the map so defined ensures the existence of  $n$  and  $m$  step self-avoiding walks for every corresponding  $n + m$  step walk that the former pair concatenates to form, the map is clearly onto. It is not one-one since  $\text{Concat}(\gamma^{(1)}, \gamma^{(2)})$  produces a walk whose first  $n$  steps and final  $m$  steps are self avoiding, but may nevertheless intersect itself globally.

The construction of such a map is revealing about the sequence  $c_n$ . Since the concatenation map is not injective, but simply surjective, we have

$$|\text{SAW}_{m+n}| \leq |\text{SAW}_m \times \text{SAW}_n| = |\text{SAW}_m| |\text{SAW}_n|$$

$$\text{i.e.} \quad c_{m+n} \leq c_m c_n \quad \forall m, n \in \mathbb{N}.$$

Therefore,  $\log c_n$  is a *subadditive* sequence:  $\log c_{n+m} \leq \log c_n + \log c_m$ . We make use of the folklore subadditivity lemma[2]:

**Lemma.** *If  $a_i \in \mathbb{R}$  is a subadditive sequence,  $(a_n)_{n \geq 1}$  converges to  $\inf_{k \geq 1} \frac{a_k}{k}$*

*Proof:* It suffices to show that

$$\limsup \frac{a_n}{n} \leq \frac{a_k}{k} \quad \text{for every } k.$$

We fix  $k$  and let

$$A_k = \max_{1 \leq r \leq k} a_r.$$

Given a positive integer  $n$  we let  $j$  denote the largest integer which is strictly less than  $\frac{n}{k}$ . Then  $n = jk + r$  for some integer  $r$  with  $1 \leq r \leq k$ . Using subadditivity, we have

$$a_n \leq ja_k + a_r \leq \frac{n}{k} a_k + A_k.$$



Dividing through by  $n$  and taking the  $\limsup_{n \geq 1}$  then gives the lemma.  $\square$

The **connective constant** of the lattice is defined as

$$\mu = \exp(\inf_{N \geq 1} N^{-1} \log c_N),$$

and the lemma implies that  $\log \mu = \lim_{n \rightarrow \infty} n^{-1} \log c_n$ . Hence we have the better lower bound:

$$\mu^n \leq c_n \quad \forall n \in \mathbb{N}.$$

#### 4. Effects of Social Distancing

Practices such as social distancing and quarantine are bound to reduce the number of susceptible nodes that can be infected by a given node. In other words, the degree of each vertex is reduced, altering the dimension of the lattice. In an effective quarantine, the number of susceptible individuals that a given person is in contact with should be diminished considerably.

If the exposure of an individual can be curtailed by even half of its original value, we have the following effect.

If the dimension drops from  $d$  to  $d/2$ , the number of ways  $n$  nodes are infected before and after (say  $c_n$  and  $c'_n$ ) can be compared from the bounds obtained previously.

$$c'_n \leq d(d-1)^{n-1} \leq d^n \leq \mu^n \leq c_n.$$

Now clearly  $\frac{d}{\mu} < 1$  This shows that

$$\frac{c'_n}{c_n} \leq \frac{d^n}{c_n} \leq \left(\frac{d}{\mu}\right)^n,$$

which decays exponentially with  $n$ .

Evidently, even a social distancing regime which prohibits the number of interactions of an individual to half its usual value shows an exponential decay of the ratio of the number of ways  $n$  people can be infected in a network. Infection models generally examine the likelihood of a disease outbreak becoming an

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epidemic. Consequently, the feasibility of longer infection walks primarily dictates whether an outbreak can infect a sizeable proportion of the network, or in other words, become an epidemic. In case of viral outbreaks such as the novel coronavirus which show features of the SIR model of infection spreading and might likewise yield to our setup of self-avoiding walks, restrictions such as social distancing provide good results if executed correctly.

This present study can be expanded considerably by taking into account an average of how many other people a given individual is likely to infect. That average, which is certainly disease-specific, would then indicate the dimension of the lattice to be considered.

Further enquiries could be carried out along the lines of the role of the connective constant of the lattice in concern. Particularly, we have not explored whether a regime of social distancing customs could be so designed as to allow every infected node to be connected to other susceptible nodes as described by a specific lattice other than  $\mathbb{Z}^d$ , and whether the connective constant of the new lattice affects the ratio of infection walks in favour of the corresponding observations in  $\mathbb{Z}^d$ .

### Suggested Reading

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