Research article

The generalized relativistic harmonic oscillator with a point interaction

Y. Chargui [b], K. Abdel-Rahman [a,*], F. Abdel-Ilah [a]

[a] *Department of Physics, Faculty of Science, University of Khartoum, Soudan*
[b] *Department of Physics, College of Science and Arts at ArRass, Qassim University, Saudi Arabia*

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ABSTRACT

We study the perturbation of the one-dimensional generalized relativistic harmonic oscillator (GRHO) by a Lorentz scalar delta-shaped interaction. By exactly solving the implied Dirac equation, we show that the presence of the singular potential brings about drastic changes in the structure of the energy spectrum of the system. Particularly, an apparent anomaly of doubly degenerate energy levels is noted when the strength of the local term becomes infinite and energy eigenvalues in the range $[-mc^2, mc^2]$ are obtained for some negative values of the delta-coupling and for all settings of the oscillator parameters.

1. Introduction

No doubt, exactly-solvable models in quantum mechanics, both non-relativistic (Schrödinger equation) and relativistic (e.g., Klein-Gordon and Dirac equations), are crucial for the understanding of physics. In particular, they serve as paradigms to develop models for handling more complicated physical problems, and, at the same time, they are very interesting from a purely mathematical perspective, since the exact solvability accounts for a certain hidden symmetry of the considered system. Therefore, finding new exact solutions in quantum mechanics remains always inviting and desirable. The harmonic oscillator is one of the most important exactly-solvable models in non-relativistic quantum mechanics. As a matter of fact every potential curve can be approximated as a harmonic potential around a stable equilibrium point. Then, the need for a relativistic analog of this system has led researchers to formulate the so-called Dirac oscillator (DO) model. This system was first introduced by Ito et al. [1]. The underlying idea is to replace the momentum operator **p** in the free Dirac equation by $\mathbf{p} - im\omega\mathbf{r}$, where **r** is the position vector, m is the mass of the particle and ω is the oscillator frequency. The attention to the problem was renewed later by Moshinsky and Szczepaniak [2], who gave it the name Dirac oscillator, since its non-relativistic limit reproduces the usual harmonic oscillator with a strong spin-orbit coupling. In recent years, the DO has attracted much attention and the several aspects of the model has been considered. We quote for instance, the quantum super-symmetry of the DO [3, 4], the equivalence between the two-dimensional DO and the Jaynes-Cummings model of quantum optics [5], quantum deformations of the

DO [6, 7], and the thermodynamic properties of the DO, found to be relevant to the studies on quark-gluon plasma models [8, 9]. In addition, the DO has been successfully applied as a quark confining potential in QCD [10] as well as it has been connected with many new phenomena in condensed matter physics, such as the quantum Hall effect and fractional statistics [11, 12]. Moreover the Dirac oscillator, very important remains its successful use. Notably, the first experimental realization of the one-dimensional DO [13] has greatly enhanced its potential of application.

Given its pertinence as argued above, different exact solvable generalizations of the DO have been proposed in the literature. Of special interest is the model obtained by adding a scalar and a vector harmonic oscillator potentials, of equal strengths, to the basic DO coupling. This system, known as the generalized relativistic harmonic oscillator (GRHO) [14, 15], can be used to describe various phenomena in the nuclei such as the possible existence of antinucleons [14]. Thus, when particularly the physics of nucleons at small distances is concerned, it would be interesting to analyze the physical effects brought, when a local potential is added in the vicinity of the coordinate origin, to the GRHO model. Such effects have been already investigated within analogous situations in the non-relativistic [16, 17] as well as relativistic [18, 19] cases.

In the present paper, we consider the problem of the one-dimensional (1D) GRHO perturbed by a Lorentz scalar delta-shaped potential. We organize our paper as follows: in section 2, we solve the Dirac equation for the 1D GRHO and we obtained the equation for determining the exact energy eigenvalues. Our results are shown to be a generalization to

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* Corresponding author.
E-mail address: kukabdrahman@gmail.com (K. Abdel-Rahman).

those obtained in the literature from studying similar perturbations of the relativistic harmonic oscillator [18] and the Dirac oscillator [19]. The exact normalized eigenfunctions of the model are also worked out. In section 3 we present our conclusion.

2. Wave functions and energy levels

In $(1+1)$-dimensional space-time, the stationary equation describing the GRHO, with a rest mass m in the presence of a Lorentz scalar delta-potential is given by

$$\alpha\left(-i\hbar c\frac{\partial}{\partial x}-i\beta m\omega cx\right)\psi(x)+\beta\left[mc^2+S(x)+\hbar cg\delta(x)\right]\psi(x)$$
$$=[E-V(x)]\psi(x) \qquad (1)$$

where ω is the frequency characterizing the Dirac string coupling (pseudoscalar potential), $S(x)$ and $V(x)$ are the scalar and the vector harmonic potentials, respectively, g is a dimensionless parameter describing the strength of the delta-function potential, while ψ is a two-component spinor corresponding to the energy E.

It is worth noting that, for the Dirac equation in $(1+1)$-dimensions, the chiral transformed spinor is defined by: $\psi_C:=\alpha\psi$ [15]. Since α anticommutes with β, the equation of ψ_C is given by

$$\alpha\left(-i\hbar c\frac{\partial}{\partial x}+i\beta m\omega cx\right)\psi_C(x)-\beta\left[mc^2+S(x)+\hbar cg\delta(x)\right]\psi_C(x)$$
$$=[E-V(x)]\psi_C(x) \qquad (2)$$

Thus the effect of the chiral transformation on Eq. (1) is to invert the sign of the mass as well as of the scalar potential and the Dirac string coupling, in addition to interchanging the upper and lower components of the spinor ψ. This symmetry will be exploited later to relate the two possible configurations of the GRHO.

Next, using the standard representation of the Dirac matrices: $\alpha=\sigma_1$ and $\beta=\sigma_3$, with σ_i the usual 2×2 Pauli matrices, Eq. (1) reduces, for $x\neq 0$, to the following coupled system of first-order differential equations

$$\left(-i\hbar c\frac{\partial}{\partial x}+im\omega cx\right)\varphi(x)=\left[E-mc^2-\Sigma(x)\right]\chi(x) \qquad (3)$$

$$\left(i\hbar c\frac{\partial}{\partial x}+im\omega cx\right)\chi(x)=-\left[E+mc^2-\Delta(x)\right]\varphi(x) \qquad (4)$$

where χ and φ are upper and lower components of the spinor ψ, respectively, while $\Sigma=V+S$ and $\Delta=V-S$. Furthermore, to construct the GRHO we have two possible settings: either $\Sigma=kx^2$ and $\Delta=0$, or $\Sigma=0$ and $\Delta=kx^2$, where k is a real parameter. However, as we have remarked above, these two cases are indeed connected each other by the chiral transformation, since inverting the since of S induces the changes $\Sigma\to\Delta$ and $\Delta\to\Sigma$. Hence, the results for the case $\Sigma=0$ can be immediately obtained from the case $\Delta=0$ by just inverting the sign of m and g in the relevant expressions, and vice versa. Therefore, in what follows, we will restrict our attention to the case $\Sigma=kx^2$ and $\Delta=0$. Being so, in order to solve the problem, two situations have to be distinguished: the first is when $E=-mc^2$, for which the solution (usually called the isolated solution) can be derived directly from the first-order equations (3)-(4), and the second is when $E\neq-mc^2$, for which the system (3)-(4) can be transformed into a Sturm-Liouville problem before being solved.

Furthermore, before proceeding further, we should remember that the solution of Eq. (1) goes through the use the appropriate jump condition for the total spinor around the origin. As the non local interaction is introduced in the Dirac equation as a Lorentz scalar potential, the two-component wave function of the Dirac particle $\psi(x)$ has to fulfill the following boundary condition (BC) [20, 21]

$$\psi\left(0^+\right)=\begin{pmatrix}\cosh g & i\sinh g\\-i\sinh g & \cosh g\end{pmatrix}\psi\left(0^-\right) \qquad (5)$$

• Solution for $E=-mc^2$.

In this case Eqs.(3)-(4) can be simplified to

$$\left(-i\hbar c\frac{\partial}{\partial x}+im\omega cx\right)\varphi(x)=-\left(2mc^2+kx^2\right)\chi(x) \qquad (6)$$

$$\left(i\hbar c\frac{\partial}{\partial x}+im\omega cx\right)\chi(x)=0 \qquad (7)$$

Solving first for χ we get

$$\chi_\pm(x)=C_\pm e^{-\kappa x^2/2} \qquad (8)$$

where $\kappa=m\omega/\hbar$ and $C_\pm$ are normalization constants, while the sign $+$ $(-)$ selects the solution in the region $x>0$ $(x<0)$. The corresponding solutions for φ are then given by

$$\varphi_\pm(x)=e^{\kappa x^2/2}\left[\tilde{C}_\pm\int^x(2mc^2+kx^2)e^{-\kappa x^2}+C'_\pm\right] \qquad (9)$$

where $\tilde{C}_\pm=C_\pm/i\hbar$ and $C'_\pm$ are integration constants. Then, the requirement that the spinor ψ be normalizable leads to the following expressions for $\varphi_\pm$

$$\varphi_\pm(x)=\tilde{C}_\pm\left\{\sqrt{\frac{\pi}{\kappa}}\left(mc^2+\frac{k}{4\kappa}\right)e^{\kappa x^2/2}\left[\mathrm{erf}\left(\sqrt{\kappa}x\right)\mp 1\right]-\frac{k}{2\kappa}xe^{-\kappa x^2/2}\right\} \qquad (10)$$

where $\mathrm{erf}(z)$ is the error function [22]. Then, demanding that the components of ψ fulfill the BC (5) entails the following equations for $C_\pm$

$$C_+=(\cosh g+v\sinh g)C_-,\quad vC_+=-(\sinh g+v\cosh g)C_- \qquad (11)$$

with

$$v=\sqrt{\frac{\pi}{\hbar^2c^2\kappa}}\left(mc^2+\frac{k}{4\kappa}\right)$$

In order to get a non trivial solution, the determinant of system (11) should vanish which leads to $\tanh g=-2v/\left(1+v^2\right)$. This means that the isolated solution, with an energy $E=-mc^2$, would exist only for particular choices of the parameters ω, k and g, such that the last condition is fulfilled.

• Solution for $E\neq-mc^2$.

In this case, decoupling the system (3)-(4) results in the following Schrödinger-like equation for the component χ:

$$\frac{\partial^2\chi(x)}{\partial x^2}+\left(\mathcal{E}-\frac{\rho^2}{4}x^2\right)\chi(x)=0 \qquad (12)$$

where we have used the auxiliary notations

$$\mathcal{E}=\frac{E^2-m^2c^4+m\omega\hbar c^2}{\hbar^2c^2},\quad\rho=\frac{2}{\hbar}\sqrt{m^2\omega^2+\left(E/c^2+m\right)k} \qquad (13)$$

At this stage, it is important to note that, in order that Eq. (12) defines a non-relativistic 1D harmonic oscillator, ρ should necessarily be real and positive. This requirement entails the existence of either a lower or an upper bound on the energy depending on whether k is positive or negative, respectively. But in the two cases this bound is given by

$$E_b\equiv-\left(1+\frac{1}{\tilde{k}}\right)mc^2 \qquad (14)$$

with $\tilde{k}=k/m\omega^2$. And for a given value of $|k|$, the possible energy lower and the upper bounds are symmetric around the value $E=-mc^2$. Note also that, while a normalizable solution of Eq. (12) for $\rho=0$ always exists, as can be easily checked, because of the BC imposed by the delta potential, having a bound state with exactly an energy E_b will be conditioned by the value of g, as we shall see later.

We proceed now to solving Eq. (12) for $\rho>0$. To this end we introduce a new variable: $z=\sqrt{\rho}x$, which brings Eq. (12) to the form

$$\frac{\partial^2 \chi(z)}{\partial z^2} + \left(\lambda + \frac{1}{2} - \frac{1}{4}z^2\right)\chi(z) = 0 \tag{15}$$

where the parameter λ is defined as $\lambda = \mathcal{E}/\rho - 1/2$. The general solution of this differential equation is well known: that is a linear combination of two parabolic cylinder functions $D_\lambda(z)$ and $D_\lambda(-z)$. Furthermore, since $\chi(z)$ must vanish as $|z| \to \infty$ to ensure the normalizability of the total wave function, it follows from the asymptotic formulae [22]:

$$D_\lambda(z) \underset{z \to +\infty}{\sim} z^\lambda e^{-z^2/4},$$

$$D_\lambda(z) \underset{z \to -\infty}{\sim} -\frac{\sqrt{2\pi}}{\Gamma(-\lambda)} e^{i\pi\lambda} z^{-(\lambda+1)} e^{z^2/4} \tag{16}$$

that the solution $\chi(z)$ of Eq. (15) should have the form

$$\chi_\pm(z) = A_\pm D_\lambda(\pm z) \tag{17}$$

where $A_\pm$ are normalization constants and the sign $+$ $(-)$ selects the solution in the region $x > 0$ $(x < 0)$. As for the corresponding lower component $\varphi_\pm(z)$, they are directly deduced from $\chi_\pm(z)$, respectively, using Eq. (4). In term of the new variable z the latter becomes

$$\varphi_\pm(z) = -i\frac{\hbar c \sqrt{\rho}}{E + mc^2}\left[\frac{\partial \chi_\pm(z)}{\partial z} + \xi z \chi_\pm(z)\right] \tag{18}$$

where $\xi = m\omega/\hbar\rho$. Notice that the derivatives of the parabolic cylinder functions; $D'_\lambda(\pm z)$, can be calculated using the relation

$$D'_\lambda(z) = -\frac{1}{2}z D_\lambda(z) + \lambda D_{\lambda-1}(z) \tag{19}$$

where the prime indicates differentiation with respect to z. Then, the resulting expressions of $\varphi_\pm(z)$ read

$$\varphi_\pm(z) = -i\frac{\hbar c \sqrt{\rho}}{E + mc^2} A_\pm \left[\left(\xi - \frac{1}{2}\right)z D_\lambda(\pm z) \pm \lambda D_{\lambda-1}(\pm z)\right] \tag{20}$$

Demanding now that the spinor ψ satisfies the BC (5) leads to the two following algebraic equations for the constants $A_\pm$

$$D_\lambda(0) A_+ = \left[D_\lambda(0)\cosh g - \eta D_{\lambda-1}(0)\sinh g\right] A_-,$$

$$\eta D_{\lambda-1}(0) A_+ = \left[D_\lambda(0)\sinh g - \eta D_{\lambda-1}(0)\cosh g\right] A_- \tag{21}$$

where

$$\eta = \frac{\hbar c(2\mathcal{E} - \rho)}{2\sqrt{\rho}(E + mc^2)} \tag{22}$$

Again, we require the determinant of the system (21) to vanish in order to get nontrivial solutions, and this leads to the following quantization condition

$$\tanh g = \frac{2\eta r t}{r^2 + \eta^2 t^2} \tag{23}$$

with the notations

$$r = 2^{\lambda/2}\sqrt{\pi}/\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right), \quad t = 2^{(\lambda-1)/2}\sqrt{\pi}/\Gamma\left(1 - \frac{\lambda}{2}\right) \tag{24}$$

and where we have made use of the formula $D_\lambda(0) = 2^{\lambda/2}\sqrt{\pi}/\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)$. The transcendental equation (23) determines the dependence of the oscillator energy on its different parameters. It has to be solved numerically in order to obtain the possible energy eigenvalues. Before proceeding to analyze the spectroscopy of the system, let us examine some particular cases of this Eq. (23).

Firstly, for $g = 0$, we can easily check that we recover the energy eigenvalues of the usual GRHO in one spatial dimension. As a matter of fact, for $g = 0$, Eq. (23) becomes equivalent to the conditions $r = 0$ or $t = 0$, which implies that either $\lambda = 2k + 1$ or $\lambda = 2(k+1)$, respectively, with k a non-negative integer (at these points $\Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)$ or $\Gamma\left(1 - \frac{\lambda}{2}\right)$ has simple poles). This finally results in the quantization condition $\lambda = n$, with n a strictly positive integer, and therefore in the following energy equation

$$E^2 - m^2 c^4 + m\omega\hbar c^2 = (2n+1)\hbar c\sqrt{m^2 c^2 \omega^2 + (E + mc^2)k}, \quad n = 0, 1, 2, \ldots \tag{25}$$

where the additional value $n = 0$ corresponds to $E = -mc^2$; the energy of the isolated solution. Eq. (25) agrees perfectly with the energy spectrum obtained for the non-perturbed GRHO (Eq. (44) in Ref. [15]); this verifies the coherence of our calculations. Notice also that, for $k = 0$, Eq. (25) reduces to the well-known energy spectrum of the usual 1D DO.

Secondly, for $k = 0$, the expressions of η and λ simplify to

$$\eta = \frac{E - mc^2}{\sqrt{2m\omega\hbar c^2}}, \quad \lambda = \frac{E^2 - m^2 c^4}{2m\omega\hbar c^2}$$

so that Eq. (23) reduces to the energy equation of the 1D DO perturbed by a Lorentz scalar delta-shaped potential, in agreement with the result of [19].

Thirdly, for $\omega = 0$, the expressions of η and λ become

$$\eta = \frac{1}{\sqrt{2\hbar c}}\left[\frac{E - mc^2}{[k(E + mc^2)]^{1/4}} - \frac{\hbar c k}{[k(E + mc^2)]^{3/4}}\right],$$

$$\lambda = \frac{E^2 - m^2 c^4}{2\hbar c\sqrt{k(E + mc^2)}} - \frac{1}{2}$$

Then Eq. (23) reduces to the energy equation of the relativistic harmonic oscillator with a Lorentz scalar delta-shaped term considered in [18]. Of course, in that case, bound-state solutions occur only for $k(E + mc^2) > 0$.

Next, let us analyze the general case, where the energy levels of the system are determined from Eq. (23). In Fig. 1 and Fig. 2 we show the graphical solutions of Eq. (23) for some illustrative values of ω, k and g, and we employ the notations:

$$f(\epsilon) = \frac{2\eta r t}{r^2 + \eta^2 t^2}, \quad \epsilon = E/mc^2 \tag{26}$$

Of course, except for the isolated energy $E = -mc^2$, the energy eigenvalues of the basic GRHO in one dimension are given by the zeros of $f(\epsilon)$. In the three figures, the upper (lower) half-plane corresponds to positive (negative) values of the delta-potential strength. As can be noted, the presence of the delta-potential impacts all the energy levels of the GRHO. This is analogous to what happens with the relativistic singular harmonic oscillator [18], but differs from what is observed in the non-relativistic case, where only even states of the harmonic oscillator are affected by the presence of the delta-potential [16]. More precisely, we can easily see that the point interaction shifts the energy eigenvalues of the system, alternately, upward and downward.

Furthermore, we know that for $k > 0$, the unperturbed energies, given by Eq. (25), are such $|E| > mc^2$ [15]. However, we can see from Fig. 1 that energy eigenvalues ranging between $-mc^2$ and mc^2 can now occur for some negative values of the coupling constant g. This aspect is reminiscent of that of the energy spectrum for the single delta-function potential within the Dirac equation.

Let us also remark that, what ever the value of g, the energy levels are all the more tightened as we approach the lower bound E_b just as it is the case when the delta perturbation is absent. Obviously, the eigenvalues closest to E_b are the least affected by the delta potential.

Another point to note from Figs. 1-2 is the occurrence of double degenerate energy levels in the limiting case when $|g| \to \infty$ becomes infinite. This fact, a priori anomalous for the 1D quantum mechanics, is a shared aspect of relativistic and non-relativistic harmonic oscillators when a point interaction, with infinite strength, is added. This can be understood by regarding the infinite delta-potential an impassible wall, located at $x = 0$, which separates the left region of the x-axis from the right one. Hence, the problem becomes equivalent to that of two disconnected potentials, so that we have no violation of the well-known non-degeneracy theorem in 1D quantum mechanics.
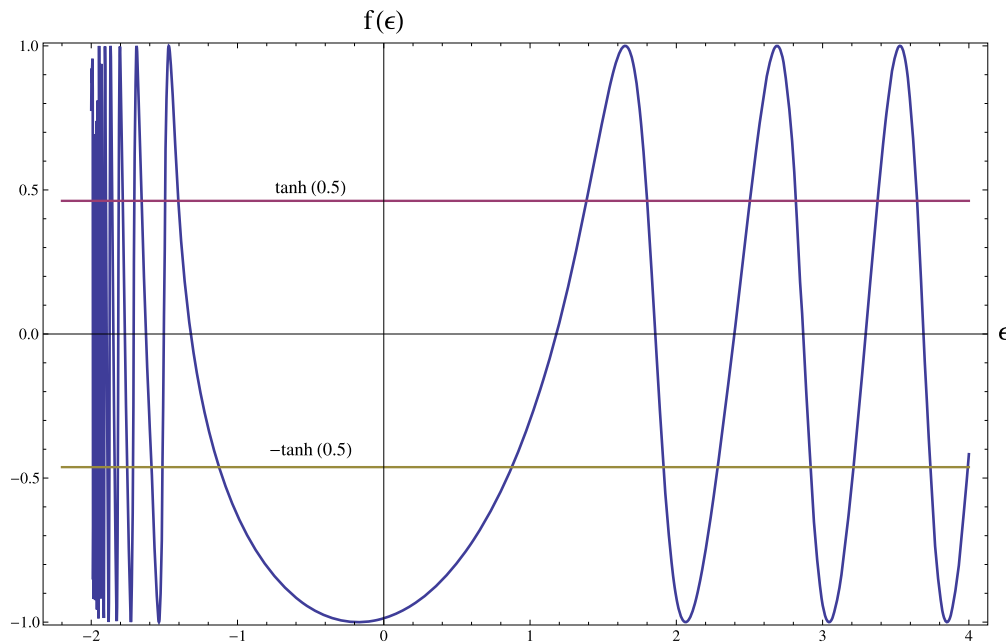
Fig. 1. Graphical solutions of Eq. (23) for $\omega = 0.5mc^2/\hbar$ and $\tilde{k} = 1$.
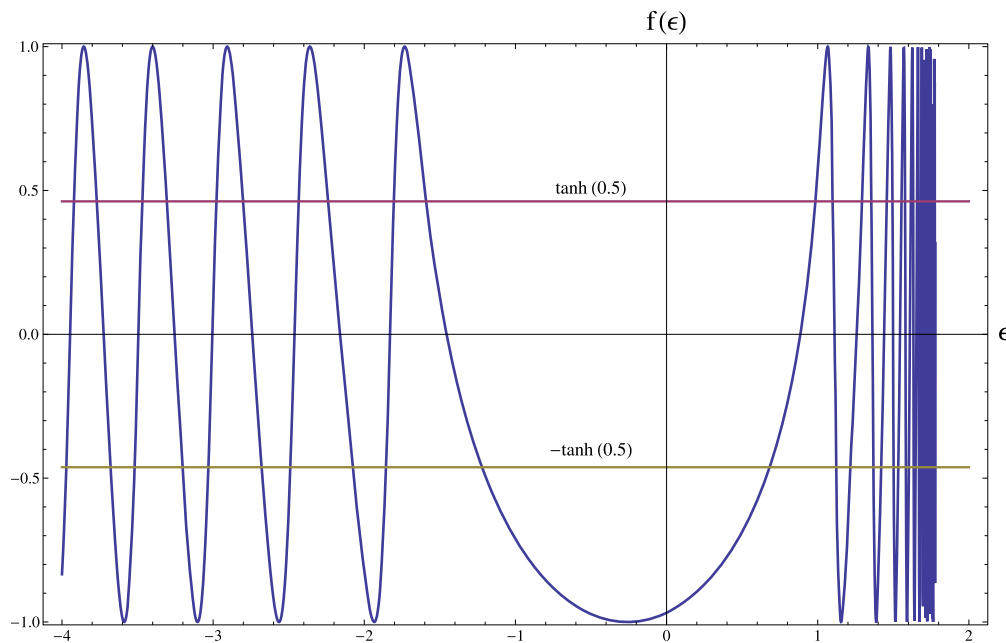

Fig. 2. Graphical solutions of Eq. (23) for $\omega = 0.5mc^2/\hbar$ and $\tilde{k} = -0.36$.

To conclude our discussion about the energy spectrum, it is worth pointing out that, according to the rule stated at the beginning of the section, the equation giving the energy eigenvalues of the 1D GRHO with $\Sigma = 0$ and $\Delta = kx^2$ could be obtained from Eq. (23) by making the substitutions $m \to -m$ and $g \to -g$. It should be also added that the isolated solution corresponding to $E = mc^2$, which occurs with the usual 1D GRHO such that $\chi(x) = \propto e^{-\kappa x^2/2}$ and $\varphi = 0$, is now banned by the presence of the delta potential. This can be easily checked by trying to impose the BC (5) to such a solution.

We end this section with the calculation of the whole normalized wave functions of the model. To do so, we begin by noting that, for a given state ψ, we have $A_+ = \pm A_-$ whatever the value of the corresponding energy. This relation ensues directly from substituting Eq. (23) into Eq. (21) and it is, indeed, consistent with the fact that the two-component spinor should satisfy the property $\psi(-x) = \beta\psi(x)$. The latter

can be easily deduced from Eq. (1). As a matter of fact, the addition of the delta-potential does not alter the symmetry of the usual GRHO with respect to the parity operation. Furthermore, the normalization constant A_+ should be obtained from the relation:

$$\int_{-\infty}^{+\infty} [\chi(x)^2 + \varphi(x)^2]dx = 1 \qquad (27)$$

Then, it would be more appropriate to rewrite the expression of $\varphi_+(x)$ as

$$\varphi_+(z) = -i\frac{\hbar c\sqrt{\rho}}{E + mc^2} A_+ \left[\left(\xi - \frac{1}{2}\right)D_{\lambda+1}(z) + \lambda\left(\xi + \frac{1}{2}\right)D_{\lambda-1}(z)\right] \qquad (28)$$

where we have made use of the following recursion relation [22]
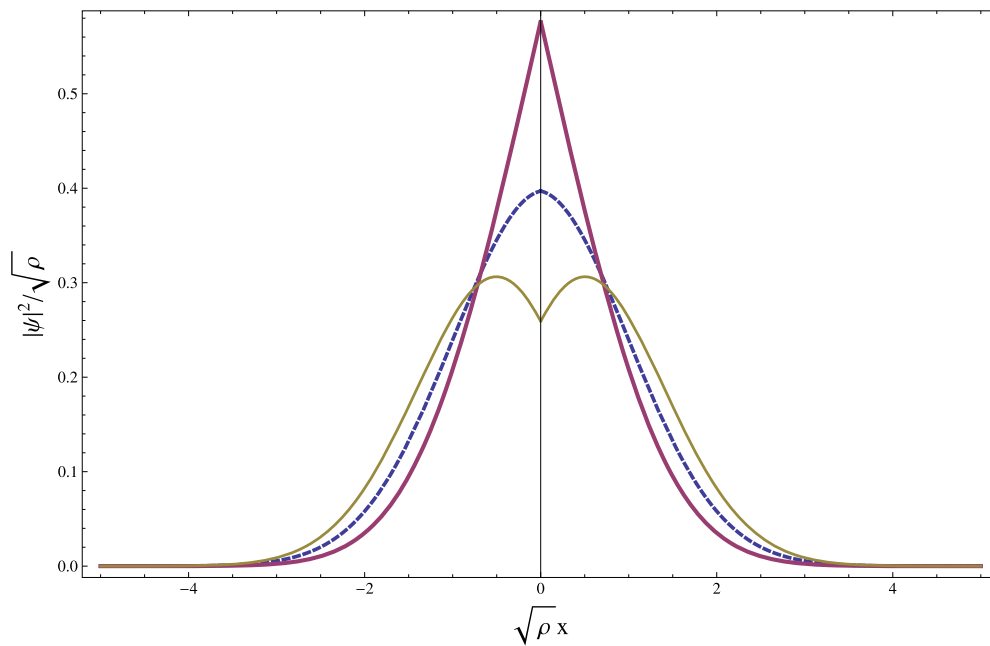
Fig. 3. Plot of the density $|\psi|^2$ associated to the first positive energy level for the usual GRHO (dashed curve) and the perturbed GRHO with two values of g: $g = 0.5$ (thin curve) and $g = -0.5$ (thick curve). The model parameters are $\omega = mc^2/\hbar$ and $\tilde{k} = 1$.
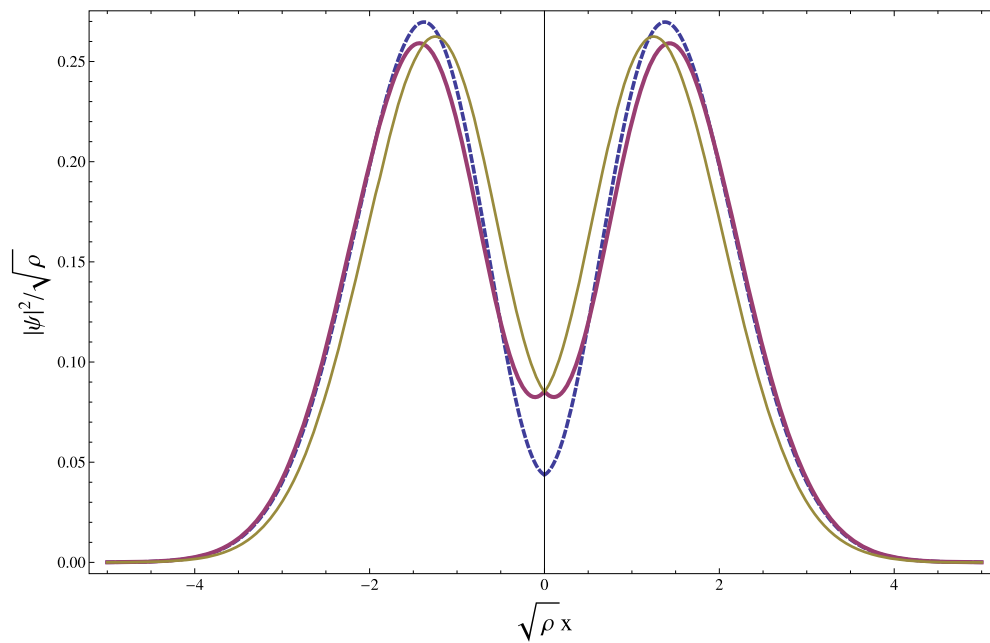
Fig. 4. Plot of the density $|\psi|^2$ associated to the second positive energy level for the usual GRHO (dashed curve) and the perturbed GRHO with two values of g: $g = 0.5$ (thin curve) and $g = -0.5$ (thick curve). The model parameters are $\omega = mc^2/\hbar$ and $\tilde{k} = 1$.

$$D_{\mu+1}(z) - zD_\mu(z) + \mu D_{\mu-1}(z) = 0$$

Next, the integrations involved in Eq. (27) can be performed with the help of the integral formulae

$$\int_0^{+\infty} D_\mu(x)^2 dx = \pi^{\frac{1}{2}} 2^{-\frac{3}{2}} \frac{\psi(\frac{1}{2} - \frac{1}{2}\mu) - \psi(-\frac{1}{2}\mu)}{\Gamma(-\mu)} \equiv h(\mu)$$

where ψ refers here to the logarithmic derivative of the gamma function, and

$$\int_0^{+\infty} D_\mu(x)D_\nu(x)dx = \frac{\pi 2^{\frac{\mu+\nu+1}{2}}}{\mu - \nu}\left[\frac{1}{\Gamma\left(\frac{1}{2} - \frac{\mu}{2}\right)\Gamma\left(-\frac{\nu}{2}\right)} - \frac{1}{\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)\Gamma\left(-\frac{\mu}{2}\right)}\right]$$

This finally gives

$$A_+ = \left(\frac{\rho}{4}\right)^{\frac{1}{4}}\left\{h(\lambda) + \frac{\eta^2}{\lambda^2}\left[\left(\xi - \frac{1}{2}\right)^2 h(\lambda + 1)\right.\right.$$

$$\left.\left. + \lambda^2\left(\xi + \frac{1}{2}\right)^2 h(\lambda - 1) + \left(\frac{1}{4} - \xi^2\right)\frac{\sqrt{\pi/2}}{\Gamma(-\lambda)}\right]\right\}^{-\frac{1}{2}} \tag{29}$$

For illustration, we compare in Fig. 3 and Fig. 4 the densities $|\psi|^2$ of the usual and the perturbed GRHO for the first two positive energy levels. As we can see, it is mainly the behavior of the density around the point $x = 0$, i.e., the center of the delta potential, which is significantly affected. For instance, from Fig. 3, it appears that, for a particle in the
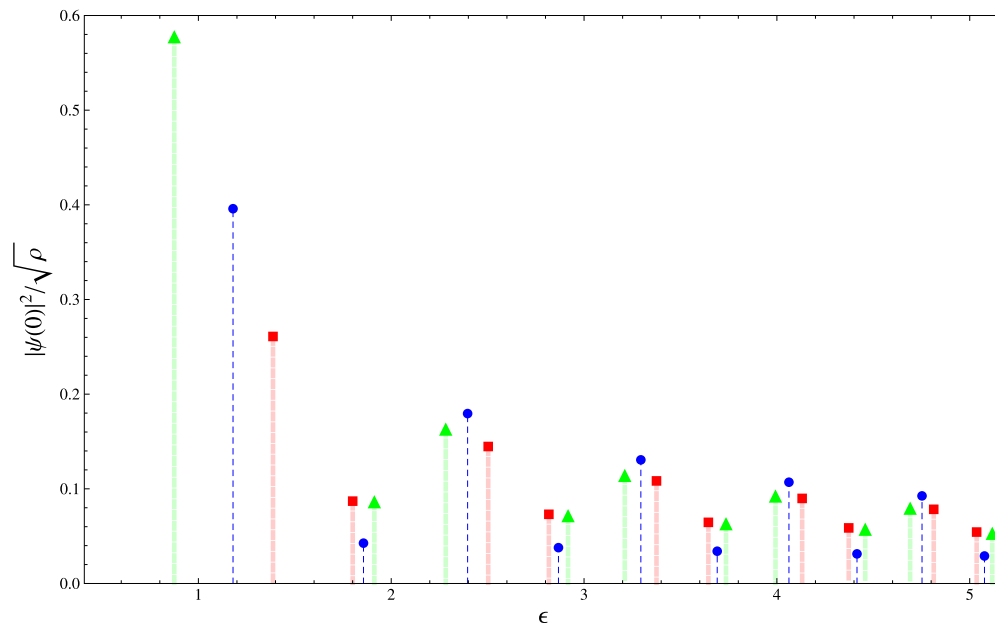
Fig. 5. Plot of $|\psi(0)|^2/\sqrt{\rho}$ for the first ten positive energy levels for the usual GRHO (blue points) and for the perturbed GRHO with two values of g: $g = 0.5$ (red points) and $g = -0.5$ (green points). The model parameters are $\omega = mc^2/\hbar$ and $\tilde{k} = 1$.

lowest positive energy state, the center becomes more or less attractive according to whether g is negative or positive. In order to shed more light on the impact of the delta potential on the oscillator density, we have depicted in Fig. 5 the value of $|\psi(0)|^2/\sqrt{\rho}$ associated to the ten first energy level for the usual and the perturbed GRHOs. From this figure we infer that, for a given positive g, if the energy eigenvalue is shifted downward (upward), that is the red point is at the left (right) of the blue one, then the density at $x = 0$ for the perturbed oscillator is increased (decreased) compared to that of the usual GRHO. However, for a given negative g, except for the first positive energy level, the situation has reversed: the density at $x = 0$ for the perturbed oscillator is above (below) that of the usual GRHO if the energy eigenvalue is shifted upward (downward), i.e., when the green point is at the right (left) of the blue one. Moreover, in the two cases, the gap between the usual and the perturbed oscillator densities is maximum for the first positive energy level.

3. Conclusion

In summary, we have solved the problem of the GRHO in one dimension in the presence of a singular delta-function potential localized at the vicinity of the origin. The equation giving the energy eigenvalues of the oscillator have been obtained as well as the associated normalized wave functions. We have found that the introduction of the point interaction affects substantially the spectroscopy of the GRHO. In particular, for an infinite strength of the delta potential, we observed the occurrence of anomalous doubly degenerate energy levels, in agreement with previous findings in the literature in analogous problems of the relativistic and non-relativistic harmonic oscillators. In addition, we found that energy eigenvalues in the range $[-mc^2, mc^2]$ can occur for some negative values of the coupling constant g. An aspect that is reminiscent of that of the energy spectrum for the single delta-function potential within the Dirac equation. Finally, the richness of our findings suggest that it would be fruitful and interesting to consider the more realistic situation of the perturbed GRHO in the three spatial dimensions. Particularly, the case of this model with $\Sigma = 0$ is related to the pseudospin symmetry in nuclei [14], and the effect of the delta potential on such a system is certainly a matter that deserves consideration.

Declarations

Author contribution statement

K. Abdelrahman, Y. Chargui: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper. F. Abdel-Ilah: Conceived and designed the analysis; Analyzed and interpreted the data.

Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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