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The Burr III-Topp-Leone-G family of distributions with applications

Fastel Chipepa [*], Broderick Oluyede, Peter O. Peter

Botswana International University of Science and Technology, P. Bag 16, Palapye, Botswana

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ABSTRACT

In this article, we present a new family of generalized distributions called the Burr III-Topp-Leone-G (BIII-TL-G). We further study in detail its structural properties including moments, probability weighted moments, distribution of order statistics, and entropy. The maximum likelihood estimation method is used to estimate the model parameters. Simulations are carried out to show the consistency and efficiency of parameter estimates and finally, real data sets are used to demonstrate the applicability of the proposed model.

1. Introduction

There is an increasing motivation in constructing new generalized families of univariate continuous distributions by adding shape parameters to a baseline distribution due to the desirable properties exhibited by the generated models. Some well-known generated families have been introduced before, such as the exponentiated-G by Gupta and Kundu [1], beta-G by Eugene et al. [2], odd Burr generalized-G by Alizadeh et al. [3], gamma-G (type I) by Zografos and Balakrishnan [4], Weibull-G by Bourguignon et al. [5], exponentiated half-logistic-G by Cordeiro et al. [6], type I half logistic-G family by Cordeiro et al. [7], exponentiated Weibull-G by Hassan et al. [8], type II half logistic-G by Hassan et al. [9] and gamma-G (type II) by Risti'c and Balakrishnan [10], among others.

In this paper, we develop and study in detail a new family of generalized distributions called the Burr III Topp-Leone-G (BIII-TL-G) distribution. Other Topp-Leone extensions in the literature include the Topp-Leone generated family of distributions by Rezaei et al. [11], Topp-Leone odd log-logistic by Brito et al. [12], odd log-logistic Topp-Leone-G by Alizadeh et al. [13], transmuted Topp-Leone-G by Yousof et al. [14] and Topp-Leone-Marshall-Olkin-G distribution by Chipepa et al. [15]. The proposed distribution provides better fit and flexibility in modeling real lifetime data sets compared to other models.

We are motivated by the desirable properties exhibited by the new distribution. The new generalized family has a good tractability property and can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) distribution. Furthermore, the new generalized family can be applied to data with monotonic or non-monotonic hazard rate functions. In addition, the new generalized family of distributions is also applicable to heavy-tailed and skewed data.

The paper is organized as follows: Section 2, presents the proposed model, series expansion of the probability density function and some statistical properties. We do inference in Section 3. Some special cases of the new family of distributions are presented in Section 4. A simulation study is presented in Section 5. Section 6 provides some demonstrations to show the usefulness of the new proposed model followed by concluding remarks in Section 7.

2. The model and statistical properties

In this paper, we make use of the method proposed by Alzaatreh et al. [16]. The method is referred to as the transformed-transformer or T-X generator for short. The T-X family of distributions has cumulative distribution function (cdf) defined by

$$F(x) = \int_{h_1}^{W(G(x;\xi))} r(t)dt, \tag{1}$$

where $r(t)$ is the probability density function (pdf) of a random variable T, $T \in [h_1, h_2]$ for $-\infty \le h_1 < h_2 < \infty$ and $W(G(x;\xi))$ is a function of the cdf of a random variable X and satisfies the conditions described by Alzaatreh et al. [16].

Moreover, Al-Shomrani et al. [17] developed a generalization of the Topp-Leone distribution which was first introduced by Topp and Leone [18]. The Topp-Leone generated family of distributions by Al-Shomrani et al. [17] has cdf and probability density function (pdf) given by

$$F_{TL-G}(x;b,\xi) = \left[1 - \overline{G}^2(x;\xi)\right]^b,$$

and

* Corresponding author.
E-mail address: chipepa.fastel@studentmail.biust.ac.bw (F. Chipepa).

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$$f_{TL-G}(x;b,\xi)=2bg(x;\xi)\overline{G}(x;\xi)\left[1-\overline{G}^2(x;\xi)\right]^{b-1},$$

respectively, for $b>0$, $\overline{G}(x;\xi)=1-G(x;\xi)$, and ξ is a vector of parameters from the baseline distribution.

We considered the generalization by Al-Shomrani et al. [17] as the baseline distribution and the function $W(G(x;\xi))=-\log(\overline{G}(x;\xi))$ and $r(t)$ being the pdf of the Burr-III distribution, so that

$$F(x;\alpha,\beta,b,\xi)=\int_0^{-\log\left(1-\left[1-\overline{G}^2(x;\xi)\right]^b\right)}\alpha\beta t^{-\beta-1}(1+t^{-\beta})^{-\alpha-1}dt$$

for $b,\alpha,\beta>0$, and ξ a parameter vector. Consequently, the cdf, pdf and hazard rate function (hrf) of the Burr III-Topp-Leone-G (BIII-TL-G) family of distributions are given by

$$F(x;\alpha,\beta,b,\xi)=\left(1+\left(-log\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]\right)^{-\beta}\right)^{-\alpha}, \tag{3}$$

$$f(x;\alpha,\beta,b,\xi)=\frac{2\alpha\beta bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]}\times\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha-1}$$

and

$$h(x;\alpha,\beta,b,\xi)=\frac{2\alpha\beta bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]}\times\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha-1}\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta-1}\times\left(1-\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha}\right)^{-1},$$

for $\alpha,\beta,b>0$ and ξ is a parameter vector from the baseline distribution.

2.1. Quantile function

We derive the quantile function of the BIII-TL-G distribution by inverting the function

$$\left(1+\left(-\log\left[1-\left(\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha}=u,$$

for $0<u<1$.

Note that

$$\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]^{-\beta}=\exp\left(1-\mu^{\frac{1}{\alpha}}\right)$$

so that

$$\left(1-\overline{G}^2(x;\xi)\right)=\left(1-\left(\exp\left(1-u^{\frac{-1}{\alpha}}\right)\right)^{\frac{-1}{\beta}}\right)^{\frac{1}{b}}.$$

The expression further simplifies to

$$G(x;\xi)=\left(1-\left(1-\left(\exp\left(1-u^{\frac{-1}{\alpha}}\right)\right)^{\frac{-1}{\beta}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}},$$

Therefore, we obtain the quantile values of the BIII-TL-G family of distributions by solving the non-linear equation

$$Q_{X_{(n)}}\left(u\right)=G^{-1}\left[\left(1-\left(1-\left(exp\left(1-u^{\frac{-1}{\alpha}}\right)\right)^{\frac{-1}{\beta}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}\right], \tag{5}$$

using iterative methods in R, MATLAB or SAS software. The median is obtained by substituting $u=\frac{1}{2}$ in equation [5] so that

$$Median\left(M_d\right)=G^{-1}\left[\left(1-\left(1-\left(exp\left(1-(1/2)^{\frac{-1}{\alpha}}\right)\right)^{\frac{-1}{\beta}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}\right]. \tag{6}$$

Furthermore, we obtain the lower and upper quartiles by substituting $u=\frac{1}{4}$ and $u=\frac{3}{4}$ in equation [5], respectively.

2.2. Expansion of density

We present the linear representation of the BIII-TL-G family of distributions in this subsection. The linear representation will aid in the derivation of other important statistical properties, for example moments and moment generating function. Note that from equation ([4]) we can write

$$\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha-1}=\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)$$

using the generalized binomial expansion, so that

$$f\left(x;\alpha,\beta,b,\xi\right)=\sum_{j=0}^{\infty}\binom{-\alpha-1}{j}\frac{2\alpha\beta bg\left(x;\xi\right)\overline{G}\left(x;\xi\right)\left(1-\overline{G}^2\left(x;\xi\right)\right)^{b-1}}{\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]}\times\left(-\log\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]\right)^{-\beta(j+1)-1}.$$

We set $y=(1-\overline{G}^2(x;\xi))^b$, and with $-\log(1-y)=\sum_{k=0}^{\infty}\frac{y^{k+1}}{k+1}$, we have

$$[-\log(1-y)]^{\delta-1}=y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

0<y<1.

Now, applying the result on power series raised to a positive integer, with $a_s=(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_sy^s\right)^m=\sum_{s=0}^{\infty}b_{s,m}y^s,$$

where

$b_{s,m}=(sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_lb_{s-l,m}$, and $b_{0,m}=a_0^m$, (see [19]), we get

$$\left(-\log\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]\right)^{-\beta(j+1)-1}=\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}b_{s,m}\binom{-\beta(j+1)-1}{m}\times\left(1-\overline{G}^2\left(x;\xi\right)\right)^{b(-\beta(j+1)-1+m+s)}$$

so that

$$f(x;\alpha,\beta,b,\xi)=\sum_{j=0}^{\infty}\sum_{m,s=0}^{\infty}b_{s,m}\binom{-\beta(j+1)-1}{m}\binom{-\alpha-1}{j}\times\frac{2\alpha\beta bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+1)+m+s)-1}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]}.$$

Also, by applying the following series expansions

$$\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]^{-1}=\sum_{z=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}\left(1-\overline{G}^2(x;\xi)\right)^{bz},$$

$$F(x;\alpha,\beta,b,\xi)=\int_0^{-\log\left(1-\left[1-\overline{G}^2(x;\xi)\right]^b\right)}\Gamma\alpha\beta t^{-\beta-1}(1+t^{-\beta})^{-\alpha-1}dt$$

$$\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+1)+m+z+s)-1}=\sum_{i=0}^{\infty}(-1)^i\overline{G}^{2i}(x;\xi)\times\binom{b(-\beta(j+1)+m+z+s)-1}{i}$$

and

$$\overline{G}^{2i+1}(x;\xi)=\sum_{p=0}^{\infty}(-1)^p\binom{2i+1}{p}G^p(x;\xi)$$

yields

$$I_R(v)=\frac{1}{1-v}\log\Bigg(\sum_{j,m,s,z,i,p=0}^{\infty}(2\alpha\beta b)^v\binom{b(-\beta(j+v)+m+z+s)-v}{i}$$
$$\times\binom{2i+v}{p}\binom{-\beta(j+v)-v}{m}\binom{-v\alpha-v}{j}\frac{(-1)^{i+p}\frac{\Gamma(z+v)}{\Gamma(v)z!}b_{s,m}}{\left[1+\frac{p}{v}\right]^v}$$
$$\times\int_0^{\infty}\left(\left[1+\frac{p}{v}\right]G(x;\xi)^{\frac{p}{v}}g(x;\xi)dx\right)^v\Bigg)$$
$$=\frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}w_p\exp((1-v)I_{REG})\right],$$

$$f(x;\alpha,\beta,b,\xi)=\sum_{j,m,s,z,i,p=0}^{\infty}(-1)^{i+p}\frac{2\alpha\beta b\Gamma(z+1)}{\Gamma(1)z!(p+1)}b_{s,m}\pi\binom{-\beta(j+1)-1}{m}$$
$$\times\binom{-\alpha-1}{j}\binom{b(-\beta(j+1)+2m+s)-1}{i}\binom{2i+1}{p}$$
$$\times(p+1)g(x;\xi)(G(x;\xi))^p$$

where $g_{p+1}(x;\xi)=(p+1)g(x;\xi)G^p(x;\xi)$ is the exponentited-G (Exp-G) distribution with power parameter $(p+1)$ and

$$w_{p+1}=\sum_{j,m,s,z,i,p=0}^{\infty}(-1)^{i+p}\frac{2\alpha\beta b\Gamma(z+1)}{\Gamma(1)z!(p+1)}b_{s,m}\binom{-\beta(j+1)-1}{m}\binom{-\alpha-1}{j}$$

It follows that the BIII-TL-G family of distributions can be expressed as an infinite linear combination of the Exp-G distribution. Therefore, other statistical properties of the BIII-TL-G family of distributions can be derived directly from those of the Exp-G distribution.

2.3. Moments, generating and characteristic functions

Let $Y_{p+1}\sim Exp-\ G(p+1)$, then using equation [7] the r^{th} raw moment, μ_r' of the BIII-TL-G family of distributions is given by

$$\mu_r'=E(X^r)=\int_{-\infty}^{\infty}x^rf(x)dx=\sum_{p=0}^{\infty}w_{p+1}E\left(Y_{p+1}^r\right)$$

where $E(Y_{p+1}^r)$ is the r^{th} raw moment of the Exp-G distribution with power parameter $(p+1)$ and w_{p+1} is defined by equation ([8]).The moment generating function (mgf) of the BIII-TL-G family of distributions is given by

$$M_X(t)=E(e^{tX})=\sum_{p=0}^{\infty}w_{p+1}M_{Y_{p+1}}(t),$$

where $M_{Y_{p+1}}(t)$ is the mgf of the Exp-G distribution with power parameter $(p+1)$.The characteristic function is given by $\varphi(t)=E(e^{itX})$, where $i=\sqrt{-1}$, so that

$$\varphi(t)=\sum_{p=0}^{\infty}w_{p+1}\varphi_{p+1}(t),$$

where $\varphi_{p+1}(t)$ is the characteristic function of Exp-G distribution and w_{p+1} is given by equation ([8]).

2.4. Distribution of order statistics

Le $X_1,X_2,\ldots,X_n$ be a random sample from the BIII-TL-G family of distributions and suppose $X_{1:n}<X_{2:n},\ldots<X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^lf(x)[F(x)]^{k+l-1}. \tag{9}$$

Substituting the pdf and cdf of the BIII-TL-G family of distributions in equation ([9]), we get

$$f(x)[F(x)]^{k+l-1}=\frac{2\alpha\beta bg\left(x;\xi\right)\overline{G}\left(x;\xi\right)\left(1-\overline{G}^2\left(x;\xi\right)\right)^{b-1}}{\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]}$$
$$\times\left(1+\left(-\log\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right)\right]^{-\beta}\right)^{-\alpha(k+l)-1}$$
$$\times\left(-\log\left[1-\left(1-\overline{G}^2\left(x;\xi\right)\right)^b\right]\right)^{-\beta-1}.$$

Using the generalized binomial expansion

$$\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta}\right)^{-\alpha(k+l)-1}=\sum_{j=0}^{\infty}\binom{-\alpha(k+l)-1}{j}$$
$$\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)^b\right)\right]\right)^{-\beta j},$$

we write

$$f(x)F(x)^{k+l-1}=\sum_{j=0}^{\infty}\binom{-\alpha(k+l)-1}{j}\frac{2\alpha\beta bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]}$$
$$\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta(j+1)-1}$$

Furthermore, applying the results as in Section 2.2, we get

$$f(x)(F(x))^{k+l-1}=\sum_{j,m,s=0}^{\infty}d_{s,m}\binom{-\beta(j+1)-1}{m}\binom{-\alpha(k+l)-1}{j}$$
$$\times\frac{2\alpha\beta bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+1)+m+s)-1}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]}.$$

Also, by applying the following generalized binomial series expansions

$$\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]=\sum_{z=0}^{\infty}\frac{\Gamma(z+1)}{\Gamma(1)z!}\left(1-\overline{G}^2(x;\xi)\right)^{bz},$$

$$\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+1)+m+z+s)-1}=\sum_{i=0}^{\infty}(-1)^i\binom{b(-\beta(j+1)+m+z+s)-1}{i}\times\overline{G}^{2i}(x;\xi)$$

and

$$\overline{G}^{2i+1}(x;\xi)=\sum_{p=0}^{\infty}(-1)^p\binom{2i+1}{p}G^p(x;\xi),$$

we obtain

$$f(x)(F(x))^{k+l-1}=\sum_{j,m,s,z,i,p=0}^{\infty}(-1)^{i+p}2\alpha\beta b\frac{\Gamma(z+1)}{\Gamma(1)z!}d_{s,m}\binom{-\beta(j+1)-1}{m}\times\binom{-\alpha(k+l)-1}{j}\binom{b(-\beta(j+1)+m+z+s)-1}{i}\binom{2i+1}{p}$$

Therefore, the distribution of the k^{th} order statistic from the BIII-TL-G family of distributions is given by

$$f_{k:n}(x)=\frac{n!}{(k-1)!(n-k)!}\sum_{j,m,s,z,i,p=0}^{\infty}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(-1)^{l+i+p}2\alpha\beta b}{(p+1)}\frac{\Gamma(z+1)}{\Gamma(1)z!}d_{s,m}\times\binom{-\beta(j+1)-1}{m}\binom{-\alpha(k+l)-1}{j}\binom{2i+1}{p}\times\binom{b(-\beta(j+1)+m+z+s)-1}{i}(p+1)g(x;\xi)G^p(x;\xi)$$

noindent where $g_{p+1}(x;\xi)=(p+1)(G(x;\xi))^pg(x;\xi)$ is the exponentited-G (Exp-G) distribution with power parameter $(p+1)$ and

$$w^{**}_{p+1}=\frac{n!}{(k-1)!(n-k)!}\sum_{j,m,s,z,i,p=0}^{\infty}\sum_{l=0}^{n-k}\binom{n-k}{l}\frac{(-1)^{l+i+p}2\alpha\beta b}{(p+1)}\frac{\Gamma(z+1)}{\Gamma(1)z!}d_{s,m}\times\binom{-\beta(j+1)-1}{m}\binom{-\alpha(k+l)-1}{j}\binom{2i+1}{p}$$

2.5. Probability Weighted Moments (PWMs)

Probability Weighted Moments (PWMs), say $\varphi_{j,i}$ of $X\sim$ BIII-TL-G (α,β,b,ξ) distribution is given by

$$\varphi_{j,i}=E(X^jF(X)^i)=\int_{-\infty}^{\infty}x^jf(x)F(x)^idx.$$

Using equation ([10]), we can write

$$f(x)F(x)^i=\sum_{j,m,s,z,i,p=0}^{\infty}(-1)^{i+p}2\alpha\beta b\frac{\Gamma(z+1)}{\Gamma(1)z!}d_{s,m}\binom{-\beta(j+1)-1}{m}\times\binom{-\alpha(i+1)-1}{j}\binom{b(-\beta(j+1)+m+z+s)-1}{i}\binom{2i+1}{p}\times g(x;\xi)(G(x;\xi))^p$$

Hence,

$$f(x)F(x)^i=\sum_{p=0}^{\infty}\eta_{p+1}h_{p+1}(x;\xi),$$

where

$$\eta_{p+1}=\sum_{j,m,s,z,i=0}^{\infty}\frac{(-1)^{i+p}2\alpha\beta b}{(p+1)}\frac{\Gamma(z+1)}{\Gamma(1)z!}d_{s,m}\binom{-\beta(j+1)-1}{m}\times\binom{-\alpha(i+1)-1}{j}\binom{b(-\beta(j+1)+m+z+s)-1}{i}\binom{2i+1}{p}$$

and $h_{p+1}(x;\xi)=(p+1)g(x;\xi)[G(x;\xi)]^p$ is an Exp-G distribution with power parameter $(p+1)$. Therefore, the PWM of the BIII-TL-G family of distributions is given by

$$\varphi_{j,i}=\sum_{k=0}^{\infty}\eta_{p+1}\int_{-\infty}^{\infty}x^jh_{p+1}(x;\xi)dx=\sum_{p=0}^{\infty}\eta_{p+1}E(H^j_{p+1}),\tag{13}$$

where H^j_{p+1} is the j^{th} power of an Exp-G distributed random variable with power parameter $(p+1)$.

2.6. R'enyi entropy

Entropy is a measure of variation of uncertainty for a random variable X with pdf $f(x)$. There are two famous measures of entropy, namely Shannon entropy [20] and R'{e}nyi entropy [21]. R'enyi entropy is defined by

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\int_0^{\infty}f^{\nu}(x)dx\right],$$

where $\nu>0$ and $\nu\neq1$. Note that

$$\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+1)+m+z+s)-1}=\sum_{i=0}^{\infty}(-1)^i\overline{G}^{2i}(x;\xi)\times\binom{b(-\beta(j+1)+m+z+s)-1}{i}$$

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^{\infty}\left[\frac{\left((2\alpha\beta b)^{v}g^{v}(x;\xi)\overline{G}^{v}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)\right)^{v(b-1)}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]^v}\times\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)-\beta\right)^{-v\alpha-v}\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-v\beta-v}\right]\right)$$

Using the generalized binomial series expansion

$$\left(1+\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)-\beta\right)^{-v\alpha-v}=\sum_{j=0}^{\infty}\binom{-v\alpha-v}{j}\times\left(-\log\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]\right)^{-\beta j},$$

and applying the results presented in Section 2.2,we get

$$I_R(\upsilon)=\frac{1}{1-\upsilon}\log\left(\sum_{j,m,s=0}^{\infty}b_{s,m}\binom{-\beta(j+\upsilon)-\upsilon}{m}\binom{-\upsilon\alpha-\upsilon}{j}\times\int_0^{\infty}\frac{(2\alpha\beta b)^{v}g^{v}(x;\xi)\overline{G}^{v}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+v)+m+s)-v}}{\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]^v}dx\right).$$

Also, by applying the following generalized binomial series expansions

$$\left[1-\left(1-\overline{G}^2(x;\xi)\right)^b\right]$$

$$\left(1-\overline{G}^2(x;\xi)\right)^{b(-\beta(j+v)+m+z+s)-v}=\sum_{i=0}^{\infty}(-1)^i\binom{b(-\beta(j+v)+m+z+s)-v}{i}\times\overline{G}^{2i}(x;\xi)$$

and

$$\overline{G}^{2i+1}(x;\xi)=\sum_{p=0}^{\infty}(-1)^p\binom{2i+1}{p}G^p(x;\xi),$$

yields

$$\frac{\partial \ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln\left(1-\overline{G}^{2}(x;\xi)\right)+\sum_{i=1}^{n}\frac{\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\ln\left(1-\overline{G}^{2}(x;\xi)\right)}{\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]}$$

$$-(-\alpha-1)\sum_{i=1}^{n}\frac{\beta\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta-1}\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\ln\left(1-\overline{G}^{2}(x;\xi)\right)}{\left(1+\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\right)\left[1-\overline{G}^{2}(1-(x;\xi))^{b}\right]}$$

$$I_R(v)=\frac{1}{1-v}\log\left(\sum_{j,m,s,z,i,p=0}^{\infty}(2\alpha\beta b)^{v}\binom{b(-\beta(j+v)+m+z+s)-v}{i}\right.$$
$$\times\binom{2i+v}{p}\binom{-\beta(j+v)-v}{m}\binom{-v\alpha-v}{j}\frac{(-1)^{i+p}\frac{\Gamma(z+v)}{\Gamma(v)z!}b_{s,m}}{\left[1+\frac{p}{v}\right]^{v}}$$
$$\left.\times\int_0^{\infty}\left(\left[1+\frac{p}{v}\right]G(x;\xi)^{\frac{p}{v}}g(x;\xi)dx\right)^{v}\right)=\frac{1}{1-v}\log\left[\sum_{p=0}^{\infty}w_p\exp((1-v)I_{REG}))\right],$$

for $v>0,\ v\neq 1$, where $I_{REG}=\frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[\frac{p}{v}+1\right](G(x;\xi))^{\frac{p}{v}}g(x;\xi)\right)^{v}dx\right]$ is the R'enyi entropy of Exp-G distribution with power parameter $\left(\frac{p}{v}+1\right)$, and

$$w_p=\sum_{j,m,s,z,i=0}^{\infty}(2\alpha\beta b)^{v}\binom{b(-\beta(j+v)+m+z+s)-v}{i}$$
$$\times\binom{2i+v}{p}\binom{-\beta(j+v)-v}{m}\binom{-v\alpha-v}{j}\frac{(-1)^{i+p}\frac{\Gamma(z+v)}{\Gamma(v)z!}b_{s,m}}{\left[1+\frac{p}{v}\right]^{v}}$$

3. Estimation

We present in this section, several methods of parameter estimation including maximum likelihood, minimum distance, weighted least squares, ordinary least squares and maximum product of spacings. The maximum likelihood technique method is studied in detail and used to estimate model parameters in this paper.

3.1. Maximum likelihood estimation

Let $X_i\sim BIII-TL-G(\alpha,\beta,b,\xi)$ with the parameter vector. The log-likelihood for a random sample of size n is given by

$$\ell(\Delta)=n\ln(2\alpha\beta b)+(b-1)\sum_{i=1}^{n}\ln\left(1-\overline{G}^{2}(x;\xi)\right)-\sum_{i=1}^{n}\ln\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]$$
$$+\sum_{i=1}^{n}\ln(g(x;\xi))+(-\beta-1)\sum_{i=1}^{n}\ln\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)$$
$$+(-\alpha-1)\sum_{i=1}^{n}\ln\left(1+\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\right)+\sum_{i=1}^{n}\ln\left[\overline{G}(x;\xi)\right]$$

The elements of the score vector is given by

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\ln\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)$$
$$-(-\alpha-1)\sum_{i=1}^{n}\frac{\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\ln\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)}{\left(1+\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\right)}$$

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum\ln\left(1+\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\right),$$

and

$$\frac{\partial\ell}{\partial\xi_k}=-(b-1)\sum_{i=1}^{n}\frac{2\overline{G}(x;\xi)\frac{\partial\overline{G}(x;\xi)}{\partial\xi_k}}{\left[1-\overline{G}^{2}(x;\xi)\right]}+\sum_{i=1}^{n}\frac{\frac{\partial\overline{G}(x;\xi)}{\partial\xi_k}}{\left(\overline{G}(x;\xi)\right)}+\sum_{i=1}^{n}\frac{\frac{\partial g(x;\xi)}{\partial\xi_k}}{(g(x;\xi))}$$
$$-\sum_{i=1}^{n}\frac{2b\left(1-\overline{G}^{2}(x;\xi)\right)^{b-1}\overline{G}(x;\xi)\frac{\partial\overline{G}(x;\xi)}{\partial\xi_k}}{\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]}+(-\alpha-1)$$
$$\times\sum_{i=1}^{n}\frac{2\beta b\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)\right]^{b}\right)^{-\beta-1}\left(1-\overline{G}^{2}(x;\xi)\right)^{b-1}\overline{G}^{2}(x;\xi)\frac{\partial\overline{G}(x;\xi)}{\partial\xi_k}}{\left(1+\left(-\log\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]\right)^{-\beta}\right)\left[1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right]}$$

These functions are not in closed form and can be solved through iterative methods using relevant software. The maximum likelihood estimates of the parameters, denoted by $\widehat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\xi_k}\right)^{T}=0$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution, where the mean vector and is the observed Fisher information matrix evaluated at, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3.2. Minimum distance methods

The minimum distance methods include minimizing the Anderson-Darling and Cram'er-von Mises goodness-of-fit statistics. The methods are based on the difference between the cdf of the BIII-TL-G distribution and the corresponding empirical distribution function.

3.2.1. Anderson-Darling method

Model parameters are obtained by minimizing the function

$$AD(\Delta|y)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)log(G(y_{i:n}|\Delta)[1-G(y_{n+1-i:n}|\Delta)]) \tag{16}$$

with respect to the model parameters. Solutions to the non-linear equations

$$\left(\frac{\partial AD(\Delta|y)}{\partial \alpha}, \frac{\partial AD(\Delta|y)}{\partial \beta}, \frac{\partial AD(\Delta|y)}{\partial b}, \frac{\partial AD(\Delta|y)}{\partial \xi_k}\right)^T = 0 \tag{17}$$

gives the estimates of the BIII-TL-G model parameters.

3.2.2. Cram 'er-von Mises Method

Cram 'er-von Mises method involves minimizing the function

$$CVM\left(\Delta|y\right) = \frac{1}{12n} + \sum_{i=1}^{n}\left(G(y_{i:n}|\Delta) - \frac{2i-1}{2n}\right)^2 \tag{18a}$$

with respect to the parameters $(\alpha, \beta, b, \xi_k)^T$.

3.3. Weighted least squares method

We obtain weighted least squares parameter estimates for the BIII-TL-G distribution by minimizing the function

$$W(\Delta|y) = \sum_{i=1}^{n} \frac{1}{Var[G(y_{i:n})]}\left(G(y_{i:n}|\Delta) - \frac{i}{n+1}\right)^2$$

where $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. The solution to the non-linear system of equations

$$\left(\frac{\partial W(\Delta|y)}{\partial \alpha}, \frac{\partial W(\Delta|y)}{\partial \beta}, \frac{\partial W(\Delta|y)}{\partial b}, \frac{\partial W(\Delta|y)}{\partial \xi_k}\right)^T = 0 \tag{20}$$

gives weighted least squares parameter estimates for the BIII-TL-G distribution.

3.4. Least squares method

Ordinary least squares parameter estimates are obtained by minimizing the function

$$Q\left(\Delta|y\right) = \sum_{i=1}^{n}\left(G(y_{i:n}|\Delta) - \frac{i}{n+1}\right)^2 \tag{21}$$

The solutions to the nonlinear equations

$$\left(\frac{\partial W(\Delta|y)}{\partial \alpha}, \frac{\partial W(\Delta|y)}{\partial \beta}, \frac{\partial W(\Delta|y)}{\partial b}, \frac{\partial W(\Delta|y)}{\partial \xi_k}\right)^T = 0 \tag{22}$$

gives the least squares parameter estimates for the BIII-TL-G distribution.

3.5. Maximum product of spacings method

The method involves finding the values of that maximizes the geometric mean of the spacings

$$GM(|\Delta|y) = \left(\prod_{i=1}^{n} D_i\right)^{\frac{1}{(n+1)}} \tag{23}$$

where, $i = 1, 2, \ldots, n$. We obtain BIII-TL-G model parameters by solving the non-linear equations

$$\left(\frac{\partial GM(\Delta|y)}{\partial \alpha}, \frac{\partial GM(\Delta|y)}{\partial \beta}, \frac{\partial GM(\Delta|y)}{\partial b}, \frac{\partial GM(\Delta|y)}{\partial \xi_k}\right)^T = 0 \tag{24}$$

using numerical methods.

4. Some special cases

We present some special cases for the BIII-TL-G family of distributions by taking the baseline distribution to be log-logistic, Weibull, and Lindley distributions, respectively. We provide plots of the density function and hazard rate function for each special case. We further provide graphical analysis of skewness and kurtosis for the special cases (parameter α was fixed and other parameters could vary).

4.1. Burr III-Topp-Leone-log-logistic (BIII-TL-LLoG) distribution

Consider the log-logistic distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda) = \lambda x^{\lambda-1}(1+x^{\lambda})^{-2}$ and $G(x;\lambda) = 1 - (1+x^{\lambda})^{-1}$, respectively, for $\lambda > 0$. The cdf, pdf and hazard rate function of the BIII-TL-LLoG distribution are given by

$$F(x;\alpha,\beta,b,\lambda) = -log[1-1-1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})^{-\alpha} \tag{16a}$$

$$f(x;\alpha,\beta,b,\lambda) = \frac{2\alpha\beta b\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}1+x^{\lambda})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b-1}}{[1-1+x^{\lambda})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}]}$$
$$\times -\log[1-1-1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})^{-\alpha-1}$$

and

$$h(x;\alpha,\beta,b,\lambda) = \frac{2\alpha\beta b\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}1+x^{\lambda})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b-1}}{[1-1+x^{\lambda})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}]}$$
$$\times -\log[1-1-1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})^{-\alpha-1}$$
$$\times 1-1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta-1}$$
$$\times 1+-\log[1-1-1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])$$
$$(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])(1+x^{\lambda})(1+x^{\lambda})^{-2})(1+x^{\lambda})(1+x^{\lambda})^{-2})^{b}])^{-\beta})^{-\alpha})^{-1}$$

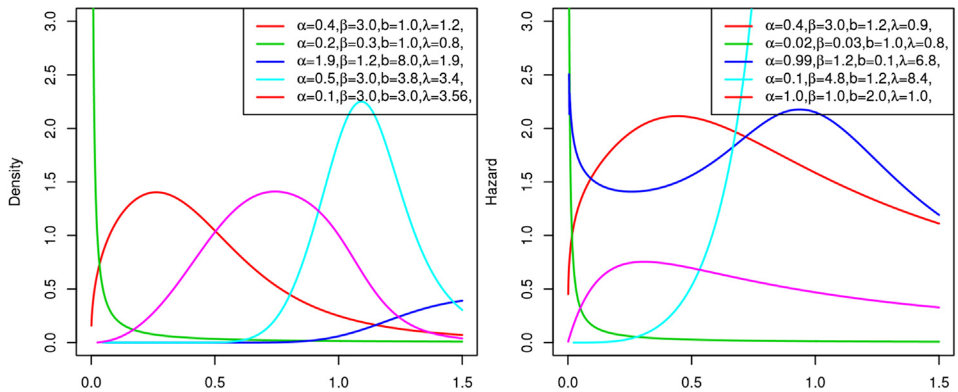

Figure 1. Plots of the pdf and hrf for the BIII-TL-LLoG distribution.

for $\alpha, \beta, b, \lambda > 0$.

Figure 1 shows the flexibility of the BIII-TL-LLoG distribution for selected parameter values. From this figure, we note that the BIII-TL-LLoG pdf takes various shapes that include reverse-J, uni-modal, left or right-skewed shapes. Furthermore, the BIII-TL-LLoG distribution exhibit decreasing, increasing, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL-LLoG distribution are shown in Figure 2. Plots of sk and ck versus the shape parameter β, clearly shows the dependence of these functions on the parameter β.

4.2. Burr III-Topp-Leone-Lindley (BIII-TL-L) distribution

Consider the Lindley distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$ and $G(x;\lambda) = 1 - \left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}$, respectively, for $\lambda > 0$. The cdf, pdf and hazard rate function of the BIII-TL-L distribution are given by

$$F(x;\alpha,\beta,b,\lambda) = \left(1 + \left(-\,log\left[1 - \left(1 - \left(\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-\lambda x}\right)^2\right)^b\right]\right)^{-\beta}\right)^{-\alpha} \tag{18}$$

$$f(x;\alpha,\beta,b,\lambda) = \frac{2\alpha\beta b\frac{\lambda^2}{(1+\lambda)}(1+x)\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\left(1-\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\right)^{b-1}}{\left[1-\left(1-\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\right)^b\right]}$$
$$\times -\log\left[1-1-1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\left)\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\right)^b\right]\Big)$$
$$\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\Big)^b\Big]\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)$$
$$\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^2 e^{-2\lambda x}\Big)^b\Big]\Big)^{-\beta}\Big)^{-\alpha-1}$$

and

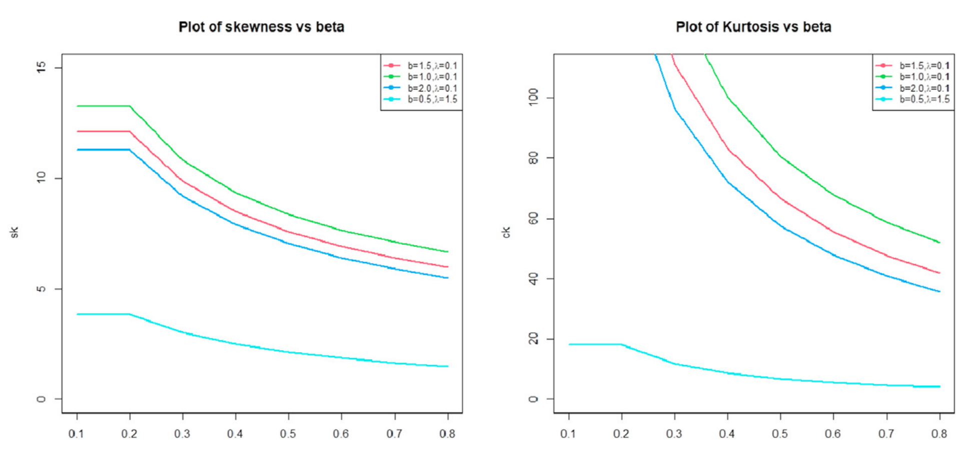

Figure 2. Plots of skewness and kurtosis for the BIII-TL-LLoG distribution.

$$h(x;\alpha,\beta,b,\lambda)=\frac{2\alpha\beta b\frac{\lambda^{2}}{(1+\lambda)}(1+x)e^{-\lambda x}\overline{G}(x;\xi)\Big(1-\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b-1}}{\Big[1-\Big(1-\left(1+\frac{\lambda x}{1+\lambda}\right)e^{-2\lambda x}\Big)^{b}\Big]}$$

$$\times -\log\Big[1-1-1+\frac{\lambda x}{1+\lambda}\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)$$

$$\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)^{-\beta}\Big)^{-\alpha-1}\quad\times 1-1$$

$$+\frac{\lambda x}{1+\lambda}\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)^{-\beta-1}\quad 1+-\log\Big[1-1-1$$

$$+\frac{\lambda x}{1+\lambda}\Big)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)^{-\beta}\Big)\Big(1+\frac{\lambda x}{1+\lambda}\Big)$$

$$\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)^{-\beta}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)$$

$$\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)\Big(\left(1+\frac{\lambda x}{1+\lambda}\right)\left(1+\frac{\lambda x}{1+\lambda}\right)^{2}e^{-2\lambda x}\Big)^{b}\Big]\Big)^{-\beta}\Big)^{-\alpha}\Big)^{-1}$$

for $\alpha,\beta,b,\lambda > 0$. Figure 3 demonstrate the applicability of the BIII-TL-L distribution for selected parameter values. The pdfs of the BIII-TL-L distribution takes different shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the BIII-TL-L distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL-L distribution are shown in Figure 4. Plots of sk and ck versus the shape parameter b, clearly shows the dependence of these functions on the parameter b.

4.3. Burr III-Topp-Leone-Weibull (BIII-TL-W) distribution

By taking the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda)=\lambda x^{\lambda-1}e^{-x^{\lambda}}$ and $G(x;\lambda)=1-e^{-x^{\lambda}}$, respectively, for $\lambda > 0$, we obtain the BIII-TL-W distribution with cdf, pdf and hazard rate function given by

$$F\Big(x;\alpha,\beta,b,\lambda\Big)=-log\Big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)\Big(-log\Big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)$$
$$\Big(-log\Big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta}\Big)^{-\alpha} \tag{20a}$$

$$f(x;\alpha,\beta,b,\lambda)=\frac{2\alpha\beta b\lambda x^{\lambda-1}e^{-2x^{\lambda}}\big(1-e^{-2x^{\lambda}}\big)^{b-1}}{\big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\big]}\Big(-\log\Big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta-1}$$

and

$$h(x;\alpha,\beta,b,\lambda)=\frac{2\alpha\beta b\lambda x^{\lambda-1}e^{-2x^{\lambda}}\big(1-e^{-2x^{\lambda}}\big)^{b-1}}{\big[1-\big(1-e^{-2x^{\lambda}}\big)^{b}\big]}1-e^{-2x^{\lambda}}\Big)\Big(1-e^{-2x^{\lambda}}\Big)$$

$$\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta-1}\times-\log\Big[1-1$$
$$-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)\Big(1-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)\Big(1-e^{-2x^{\lambda}}\Big)$$

$$\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta}\Big)^{-\alpha-1}\times 1+-\log\Big[1-1$$
$$-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)\Big(1-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta}\Big)\Big(1-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)$$

$$\Big(1-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta}\Big)\Big(1-e^{-2x^{\lambda}}\Big)\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)\Big(1-e^{-2x^{\lambda}}\Big)$$

$$\big(1-e^{-2x^{\lambda}}\big)^{b}\Big]\Big)^{-\beta}\Big)^{-\alpha}\Big)^{-1}$$

for α, β, b, $\lambda > 0$.

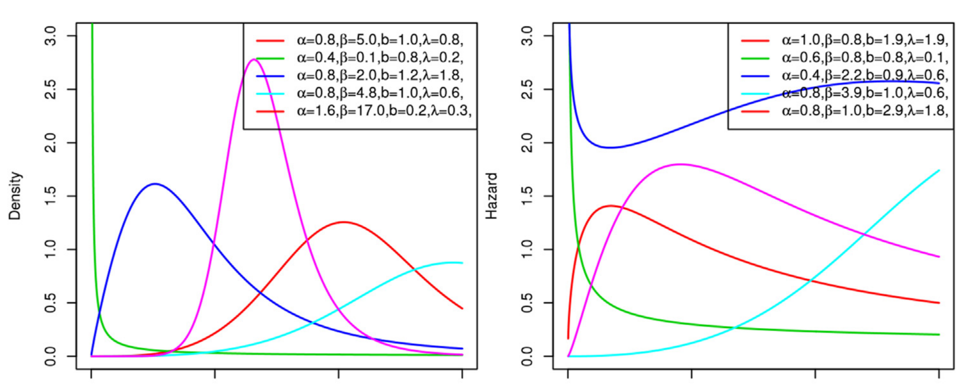

Figure 3. Plots of the pdf and hrf for the BIII-TL-L distribution.

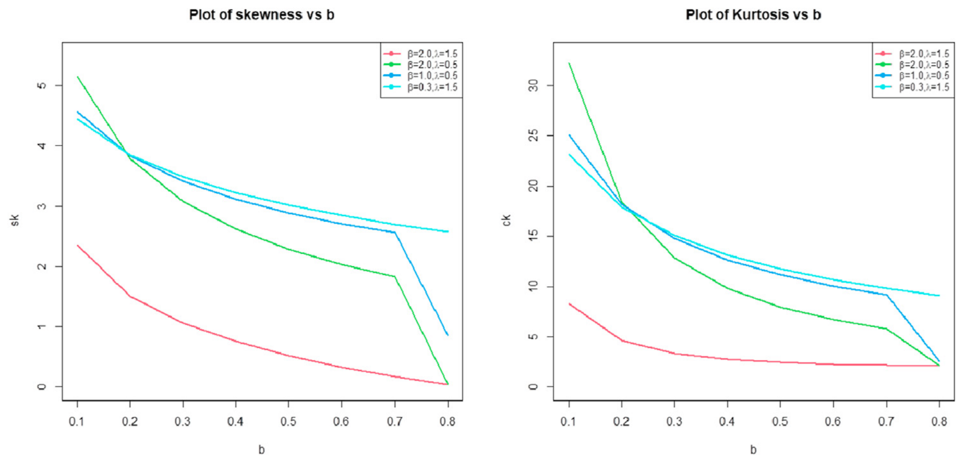

Figure 4. Plots of skewness and kurtosis for the BIII-TL-L distribution.

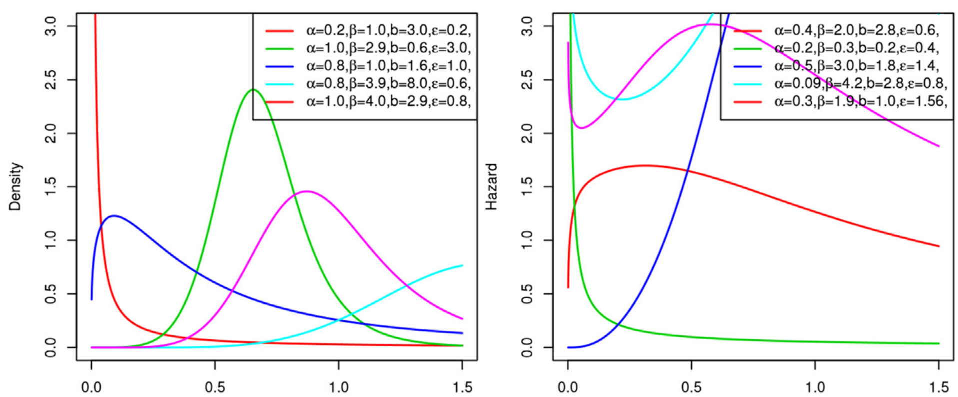

Figure 5. Plots of the pdf and hrf for the BIII-TL-W distribution.

Figure 5 illustrate the flexibility of the BIII-TL-W distribution for selected parameter values in terms of its pdfs that takes various shapes including reverse-J, uni-modal, left or right-skewed. Also, the BIII-TL-W distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL-W distribution are shown in Figure 6. Plots of sk and ck versus the shape parameter λ, clearly shows the dependence of these functions on the parameter λ.

5. Simulation study

In this section, a simulation study was conducted for the BIII-TL-W distribution to assess consistency of the maximum likelihood estimators. We generated N = 1000 samples for sample sizes n= 25, 50, 100, 200, 400, 800 and 1000. We fixed the values of $\alpha = 1$ and $\beta = 2$ for the following sets of parameters values (I: $\beta = 2.0, b = 1.1, \lambda = 1.1$), (II: $\beta = 2.0, b = 1.5, \lambda = 0.5$), (III: $\beta = 2.0, b = 1.1, \lambda = 2.0$), (IV: $\beta = 2.0, b = 1.1, \lambda = 1.5$), (V: $\beta = 2.0, b = 0.5, \lambda = 1.1$) and (VI: $\beta = 2.0, b = 1.1, \lambda = 0.5$). We estimate the mean, root mean square error (RMSE), and average bias. If the model performs better, we except the mean to approximate the true parameter values, the RMSE, and bias to decay toward zero for an increase in sample size. From the results in Tables 1, 2, and 3, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values.

6. Applications

In this section, we demonstrate the applicability and usefulness of the BIII-TL-W distribution. We apply the model to three real data examples. We compare the BIII-TL-W distribution to nested and several non-nested models. Model performance was assessed using the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cram'er-von Mises (W^*), sum of squares (SS), and Andersen-Darling (A^*) (see Chen and Balakrishnan [22] for details), Kolmogorov-Smirnov (K–S) statistic (and its p-value). The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K–S statistic is regarded as the best model.

Model parameters (standard errors in parentheses) and the goodness-of-fit statistics are presented in Tables 4, 5, and 6. Furthermore, we also provide fitted densities and probability plots (as described by Chambers et al. [23]) to demonstrate how well our model fits the selected data sets.

The non-nested models considered in this paper are: exponentiated Weibull (EW) by Pal et al. [24] Marshall-Olkin-inverse Weibull (MO-IW) by Pakungwati et al. [25] Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) by Chipepa et al. [26], Kumaraswamy-Weibull (KW) by Cordeiro et al. [27], beta odd Lindley-exponential (BOL-E) and beta odd Lindley-uniform by Chipepa et al. [28] the Topp-Leone-Weibull-Lomax (TLWLx) distributions by Jamal et al. [29] and the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [30] distributions. The pdfs of the non-nested models are as follows:

Table 1. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

		I: $\beta = 2.0, b = 1.1, \lambda = 1.1$			II: $\beta = 2.0, b = 1.5, \lambda = 0.5$		
n		Mean	RMSE	Bias	Mean	RMSE	Bias
β	25	2.8934	3.1902	0.8934	3.0162	4.2433	1.0162
	50	2.7121	1.4118	0.7121	2.4764	1.1606	0.4764
	100	2.5364	1.1547	0.5364	2.4625	1.2612	0.4625
	200	2.3398	0.8743	0.3398	2.3353	0.7139	0.3353
	400	2.3889	0.8706	0.3889	2.3225	0.6147	0.3225
	800	2.2632	0.6461	0.2632	2.3094	0.5371	0.3094
	1000	2.1550	0.4679	0.1550	2.1389	0.2971	0.1389
b	25	1.2562	0.6532	0.1562	1.3990	0.6454	-0.1010
	50	1.2749	0.5794	0.1749	1.4785	0.5050	-0.0215
	100	1.2382	0.4897	0.1382	1.5599	0.4071	0.0599
	200	1.1951	0.4217	0.0951	1.6024	0.3515	0.1024
	400	1.2612	0.3854	0.1612	1.6314	0.2572	0.1314
	800	1.2064	0.2972	0.1064	1.6289	0.2324	0.1289
	1000	1.1670	0.2154	0.0670	1.5556	0.1550	0.0556
λ	25	1.4299	1.4321	0.3299	0.7021	0.6192	0.2021
	50	1.1396	0.7701	0.0396	0.5521	0.3611	0.0521
	100	1.0846	0.5634	-0.0154	0.4772	0.2302	-0.0228
	200	1.0851	0.4523	-0.0149	0.4539	0.1851	-0.0461
	400	0.9657	0.3455	-0.1343	0.4302	0.1280	-0.0698
	800	0.9973	0.2792	-0.1027	0.4299	0.1141	-0.0701
	1000	1.0234	0.2097	-0.0766	0.4631	0.0796	-0.0369

Table 2. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

N		III: $\beta = 2.0, b = 1.1, \lambda = 2.0$			IV: $\beta = 2.0, b = 1.1, \lambda = 1.5$		
β	25	3.1823	5.8036	1.1823	3.0549	4.5594	1.0549
	50	2.7396	1.4264	0.7396	2.7110	1.4072	0.7110
	100	2.5741	1.1580	0.5741	2.5688	1.1571	0.5688
	200	2.4006	0.9213	0.4006	2.3807	0.9121	0.3807
	400	2.3717	0.8740	0.3717	2.3760	0.9041	0.3760
	800	2.2801	0.6418	0.2801	2.2690	0.6472	0.2690
	1000	2.1240	0.3861	0.1240	2.1059	0.3075	0.1059
b	25	1.2894	0.6652	0.1894	1.2554	0.6627	0.1554
	50	1.3066	0.5862	0.2066	1.2902	0.5810	0.1902
	100	1.2784	0.5065	0.1784	1.2707	0.4999	0.1707
	200	1.2371	0.4337	0.1371	1.2194	0.4243	0.1194
	400	1.2475	0.3786	0.1475	1.2403	0.3834	0.1403
	800	1.2200	0.2995	0.1200	1.2088	0.3000	0.1088
	1000	1.1435	0.2022	0.0435	1.1374	0.1802	0.0374
λ	25	2.5351	2.6951	0.5351	1.9968	2.0961	0.4968
	50	2.0203	1.3843	0.0203	1.5442	1.0649	0.0442
	100	1.9200	1.0268	-0.0800	1.4386	0.7609	-0.0614
	200	1.9141	0.8166	-0.0859	1.4470	0.5978	-0.0530
	400	1.8113	0.6200	-0.1887	1.3647	0.4788	-0.1353
	800	1.7980	0.5075	-0.2020	1.3598	0.3800	-0.1402
	1000	1.9187	0.3790	-0.0813	1.4384	0.2739	-0.0616

$$f_{EW}(x;\alpha,\beta,\delta) = \alpha\beta\delta x^{\beta-1} e^{-\alpha x^{\beta}} (1 - e^{-\alpha x^{\beta}})^{\delta},$$

for $\alpha, \beta, \delta > 0$,

$$f_{MO-IW}(x;\alpha,\theta,\lambda) = \frac{\alpha\lambda\theta^{-\lambda} x^{-\lambda-1} e^{-(\theta x)^{-\lambda}}}{[\alpha - (\alpha-1)e^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha, \theta, \lambda > 0$,

$$\begin{aligned} f_{KOL-LLoG}(x;a,b,\lambda,c) &= ab\left[\frac{\lambda^2}{(1+\lambda)}\frac{cx^{c-1}}{(1+x^c)^{-1}}\exp\left\{-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\}\right] \\ &\times \left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\}\right]^{a-1} \\ &\times \left(1-\left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\}\right]^{a}\right)^{b-1}, \end{aligned}$$

for $a, b, \lambda, c > 0$,

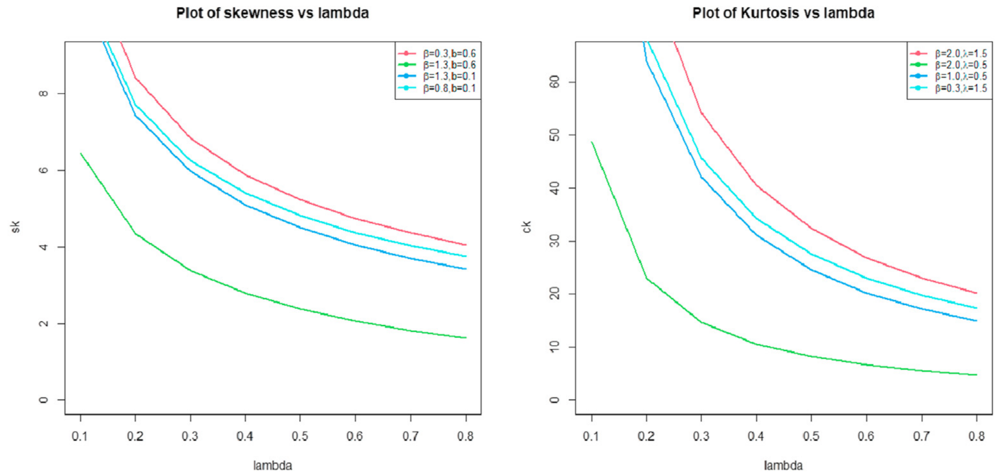

Figure 6. Plots of skewness and kurtosis for the BIII-TL-LLoG distribution.

$$f_{TL-WLx}(x;a,b,\alpha,\theta) = 2\theta\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})\alpha-1 \times \exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right) \times \left[1-\exp\left(-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)\right)\right]^{\theta-1},$$

for $a,b,\alpha,\theta>0$,

$$f_{BW}(x;a,b,\alpha,\beta)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}e^{-b(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1},$$

for $a,b,\alpha,\beta>0$,

$$f_{KwW}(x;a,b,\alpha,\beta)=ab\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}(1-e^{-(\alpha x)^{\beta}})^{a-1}(1-(1-e^{-(\alpha x)^{\beta}})^{a})^{b-1},$$

for $a,b,\alpha,\beta>0$,

$$f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda)=\frac{\alpha\theta^{2}\gamma\lambda^{\gamma}x^{\gamma-1}e^{-(\lambda x)^{\gamma}}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^{\gamma}})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}})^{\theta-1}} \times \left(1-\beta\log\left[\frac{e^{-(\lambda x)^{\gamma}}}{(1-e^{-(\lambda x)^{\gamma}})^{\alpha}+e^{-\alpha(\lambda x)^{\gamma}}}\right]\right),$$

for $\alpha,\beta,\gamma,\theta,\lambda>0$,

$$f_{BOL-U}(x;a,b,\lambda,\theta)=\frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1} \times\left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1} \times\frac{\lambda^{2}}{(1+\lambda)}\frac{\theta^{2}}{(\theta-x)^{3}}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\},$$

for $a,b,\lambda,\theta>0$,

Table 3. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

		V: $\beta=2.0, b=0.5, \lambda=1.1$			V: $\beta=2.0$, $b=1.1$, $\lambda=0.5$		
β	25	2.9072	1.9720	0.9072	2.7863	1.6889	0.7863
	50	2.5159	1.3356	0.5159	2.6559	1.3349	0.6559
	100	2.1945	0.7258	0.1945	2.5370	1.1424	0.5370
	200	2.0834	0.4820	0.0834	2.4272	0.9042	0.4272
	400	2.0276	0.1765	0.0276	2.4242	0.8620	0.4242
	800	2.0304	0.1164	0.0304	2.3088	0.6752	0.3088
	1000	2.0107	0.0947	0.0107	2.1880	0.4878	0.1880
b	25	0.7583	0.6868	0.2583	1.1707	0.6517	0.0707
	50	0.6452	0.5098	0.1452	1.2145	0.5706	0.1145
	100	0.5422	0.2993	0.0422	1.2238	0.4836	0.1238
	200	0.5120	0.2016	0.0120	1.2572	0.4445	0.1572
	400	0.5009	0.0983	0.0009	1.2865	0.3881	0.1865
	800	0.5008	0.0695	0.0008	1.2297	0.3093	0.1297
	1000	0.4935	0.0546	-0.0065	1.1827	0.2274	0.0827
λ	25	1.4116	1.2874	0.3116	0.7175	0.7672	0.2175
	50	1.2297	0.7620	0.1297	0.5373	0.3794	0.0373
	100	1.1814	0.4484	0.0814	0.4790	0.2551	-0.0210
	200	1.1619	0.3406	0.0619	0.4519	0.2023	-0.0481
	400	1.1162	0.2023	0.0162	0.4221	0.1560	-0.0779
	800	1.0924	0.1384	-0.0076	0.4404	0.1307	-0.0596
	1000	1.1058	0.1171	0.0058	0.4562	0.1004	-0.0438

Table 4. Parameter estimates and goodness-of-fit statistics for various models fitted for 50 mm carbon fibres data set.

Model	Estimates				Statistics							
	α	β	b	λ	-2 log L	AIC	AICC	BIC	W	A	K–S	p-value
BIII-TL-W	0.2810	315.7900	3.3189	0.0186	169.9	177.9	178.6	186.7	0.0436	0.2746	0.0655	0.9397
	(0.0528)	(7.3570×10^{-5})	(0.0191)	(0.0019)								
BIII-TL-W	1	1.3025×10^{3}	3.1693	0.0022	183.3	189.3	189.7	195.9	0.2594	1.3937	0.0940	0.6036
	-	(1.6745×10^{-9})	(0.0010)	(9.0894×10^{-5})								
BIII-TL-W	4.4797	1	1.7917	1.6324	246.3	252.3	252.6	258.8	0.8256	4.6362	0.3126	$4.9910\ 10^{6}$
	(1.1085)	-	(1.1008)	(0.2677)								
BIII-TL-W	1	1	30.9215	0.7252	238.0	242.0	242.2	246.4	0.8997	4.9689	0.2734	0.0001
	-	-	(4.4254)	(0.0522)								
BIII-TL-W	14.5919	1.8553	1	1	232.2	236.2	236.4	240.6	0.8337	4.6806	0.2147	0.0046
	(2.6580)	(0.1303)	-	-								
	α	β	δ									
EW	1.3186	0.9022	14.2572	-	203.3	209.3	209.7	215.8	0.3958	2.1888	0.1794	0.0286
	(0.4391)	(0.1897)	(7.3551)	-								
	α	λ	θ									
MO-IW	3.2588×10^{5}	4.8957	4.9322	-	183.3	189.3	189.7	195.9	0.2594	1.3938	0.0938	0.6076
	(4.2992×10^{-6})	(0.5157)	(1.3813)	-								
	a	b	λ	c								
KOL-LLoG	0.6471	8.2776	0.0100	2.9691	170.7	178.7	179.4	187.5	0.0686	0.4107	0.0766	0.8340
	(0.6643)	(0.0216)	(0.0435)	(2.2420)								
	a	b	λ	θ								
BOL-E	1.9173	0.7603	0.5773	0.5904	171.1	179.1	179.7	187.8	0.0715	0.4397	0.0816	0.7719
	(0.8770)	(1.4267)	(0.4193)	(0.2677)								
BOL-U	3.1542	54.3899	0.3438	8.3728	172.4	180.4	181.1	189.2	0.0946	0.5635	0.0884	0.6815
	(0.9488)	(103.7123)	(0.3682)	(1.7135)								
	β	λ	θ	γ								
ELOLLW	0.3771	0.8337	0.1368	2.7796	171.0	179.0	179.7	187.8	0.0754	0.4345	0.0735	0.8682
	(0.2975)	(0.3392)	(0.0810)	(0.4686)								
	a	b	α	β								
KW	151.4900	3.1194×10^{3}	1.1465×10^{3}	0.1332	175.3	183.3	183.9	192.0	0.1407	0.7595	0.0978	0.5533
	(1.6535×10^{-7})	(1.0314×10^{-9})	(8.9253×10^{-9})	(6.3281×10^{-4})								
	a	b	α	θ								
TL-WLx	4.1798	0.0482	2.7916	1.0206	171.7	179.7	180.4	188.5	0.0832	0.4965	0.0820	0.7671
	(6.5430)	(0.0809)	(0.9700)	(0.5160)								

Table 5. Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set.

Model	Estimates				Statistics							
	α	β	b	λ	2 log L	AIC	AICC	BIC	W	A	K–S	p-value
BIII-TL-W	0.2636	42.9734	6.0260	0.1614	335.7	343.7	344.0	354.8	0.0542	0.3273	0.0558	0.8518
	(0.0756)	(0.0626)	(0.7436)	(0.0251)								
BIII-TL-W	1	829.4500	3.2043	0.0050	356.7	362.7	362.9	371.0	0.3566	2.2409	0.0814	0.4095
	-	(4.6370×10^{9})	(0.0011)	(8.5385×10^{-5})								
BIII-TL-W	10.5571	1	4.0364	1.3668	512.0	518.0	518.2	526.4	1.0326	5.9088	0.3513	3.5330×10^{-13})
	(2.2004)	-	(1.6171)	(0.1328)								
BIII-TL-W	1	1	164.0618	0.7217	454.2	458.2	458.3	463.8	1.6979	9.3934	0.2828	1.0890×19^{8}
	-	-	(25.7737)	(0.0294)								
BIII-TL-W	397.4000	3.0250	1	1	421.5	425.5	425.6	431.0	1.4019	7.8438	0.1996	0.0002
	(135.8649)	(0.1840)	-	-								
	α	β	δ	-								
EW	0.7015	1.1441	23.9388	-	381.5	387.5	387.7	395.8	0.6594	3.8923	0.1676	0.0025
	(0.2973)	(0.2012)	(12.5289)	-								
	α	λ	θ	-								
MO-IW	2.4077×10^{3}	7.0579	0.6972	-	356.6	362.6	362.9	371.0	0.3597	2.2545	0.0805	0.4237
	(8.9459×10^{7})	(0.5495)	(0.0630)	-								
	a	b	λ	c								
KOL-LLoG	1.0957	120.67	0.0037	2.4535	337.5	345.5	345.9	356.6	0.0870	0.5406	0.0706	0.5930
	(0.4297)	(0.0018)	(0.0075)	(0.7661)								
	a	b	λ	θ								
BOL-E	2.3297	16.5497	0.1086	0.3806	337.1	345.1	345.5	356.2	0.0712	0.4408	0.0648	0.7001
	(1.0946)	(0.0092)	(0.0866)	(0.1066)								
BOL-U	6.1194	52.1616	0.5308	12.8957	339.9	347.9	348.3	359.1	0.1413	0.8678	0.1009	0.1773
	(5.6613)	(47.2416)	(0.4513)	(10.2622)								
	β	λ	θ	γ								
ELOLLW	9.0995	0.1612	5.4785	4.2260	336.4	344.4	344.8	355.6	0.0698	0.4309	0.0628	0.7363
	(2.6518)	(0.0328)	(4.4044)	(0.5822)								
	a	b	α	β								
KW	0.8261	0.5375	0.2326	5.4234	337.1	345.1	345.4	356.2	0.0812	0.4914	0.0685	0.6310
	(0.6252)	(7.3712)	(0.6203)	(1.2277)								
	a	b	α	θ								
TL-WLx	1.4492	0.1120	4.8671	0.8940	337.1	345.1	345.5	356.3	0.0829	0.5027	0.0693	0.6167
	(5.4469)	(0.5233)	(5.3307)	(0.6418)								

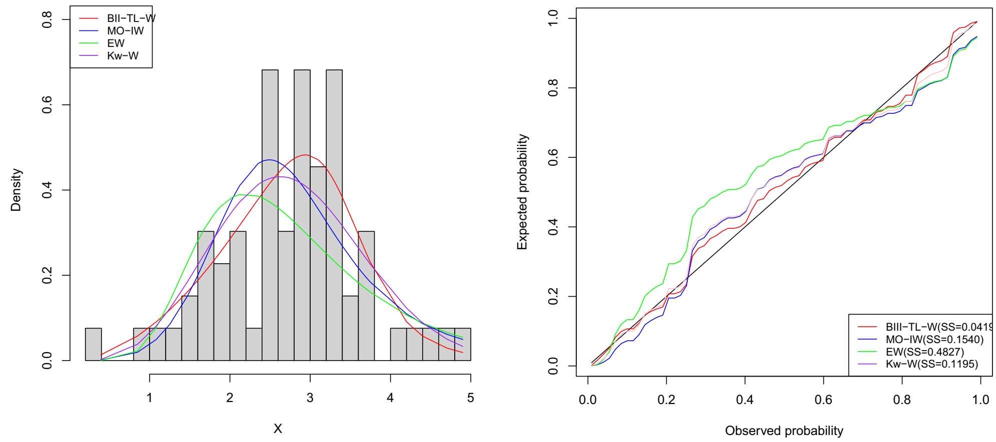

Figure 7. Fitted pdfs and probability plots for test carbon fibres data set.

$$f_{BOL-E}(x;a,b,\lambda,\theta)=\frac{1}{B(a,b)}\left[1-\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{a-1}$$
$$\times\left[\frac{\lambda+e^{-\theta x}}{(1+\lambda)e^{-\theta x}}\exp\left\{-\lambda\frac{(1-e^{-\theta x})}{e^{-\theta x}}\right\}\right]^{b-1}$$
$$\times\frac{\lambda^2}{(1+\lambda)}\frac{(\theta e^{-\theta x})}{e^{-3\theta x}}\exp\left\{-\lambda\frac{1-e^{-\theta x}}{e^{-\theta x}}\right\}$$

for $a,b,\lambda,\theta>0$,
and

$$f_{OLLEW}(x;\alpha,\beta,\gamma,\theta)=\frac{\theta\beta\gamma x^{\beta-1}e^{-(x/\alpha)^\beta}\left[1-e^{-(x/\alpha)^\beta}\right]^{\gamma\theta-1}\left(1-\left[1-e^{-(x/\alpha)^\beta}\right]^\gamma\right)^{\theta-1}}{\alpha\beta\left(\left[1-e^{-(x/\alpha)^\beta}\right]^{\theta\gamma}+\left(1-\left[1-e^{-(x/\alpha)^\beta}\right]^\gamma\right)^\theta\right)^2},$$

for $\alpha,\beta,\gamma,\theta>0$.

For the ELOLLOW distribution, we considered the case when $\alpha = 1$.

6.1. 50 mm carbon fibres data set

The data set was also analyzed by Chipepa et al. [15]. The observations represent breaking stress of carbon fibres of 50 mm length (GPa). The data are 0.39, 0.85, 1.08,1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09,3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 2.7845\times10^{-3} & 3.8799\times10^{-6} & -6.9246\times10^{-4} & -5.5296\times10^{-5}\\ 3.8799\times10^{-6} & 5.4125\times10^{-9} & -9.9981\times10^{-7} & -8.0752\times10^{-8}\\ -6.9246\times10^{-4} & -9.9981\times10^{-7} & 3.6427\times10^{-4} & 3.4091\times10^{-5}\\ -5.5296\times10^{-5} & -8.0752\times10^{-8} & 3.4091\times10^{-5} & 3.5229\times10^{-6}\end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha\in[0.2810\pm0.1034]$, $\beta\in[315.7900\pm0.0001]$, $b\in[3.3189\pm0.0374]$ and $\lambda\in[0.0186\pm0.0037]$.

We conclude from the values of the goodness-of-fit statistics A^*, W^*, K–S and the p-value of the K–S statistic that the BIII-TL-W model fits the carbon fibres data set better than the several models considered in this paper. The values of the SS also show that the BIII-TL-W model performs better than the selected models as shown in Figure 7.

6.2. Fracture toughness of silicon nitride data

We also considered the data set that was analyzed by Nadarajah and Kotz [31] and by Ali et al [32] The data are observations on fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data observations are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.0057 & -0.0043 & -0.0521 & -0.0017\\ -0.0043 & 0.0039 & 0.0465 & 0.0015\\ -0.0521 & 0.0465 & 0.5529 & 0.0186\\ -0.0017 & 0.0015 & 0.0186 & 0.0006\end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha\in[0.2636\pm0.1482]$, $\beta\in[42.9735\pm0.1227]$, $b\in[6.0260\pm1.4575]$ and $\lambda\in[0.1614\pm0.0493]$.

Furthermore, the second example also affirms that the BIII-TL-W model performs better than the selected models. The values of the goodness-of-fit statistics A^*, W^*, K–S and the p-value of the K–S statistic indicates that the BIII-TL-W model fits the fracture toughness of silicon nitride better than the several models considered in this paper. The values of the SS also show that the BIII-TL-W model performs better than the selected models as shown in Figure 8.

6.3. Turbocharger failure times data

The data set represents failure times (10^3h) of turbocharger of one type of engine as report by Xu et al. [33]. The data are 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 3.6070\times10^{-6} & 3.8199\times10^{-9} & -1.1435\times10^{-4} & -7.2012\times10^{-7}\\ 3.8199\times10^{-9} & 1.4832\times10^{-10} & -4.6320\times10^{-6} & -4.3688\times10^{-8}\\ -1.1435\times10^{-4} & -4.6320\times10^{-6} & 0.1446 & 1.3651\times10^{-3}\\ -7.2012\times10^{-7} & -4.3688\times10^{-8} & 1.3651\times10^{-3} & 1.4766\times10^{-5}\end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha\in[0.0117\pm0.0037]$, $\beta\in[266.5100\pm2.3870\times10^{-5}]$, $b\in[13.7790\pm0.7455]$ and $\lambda\in[0.2440\pm0.0075]$.

More so, from the third example, we conclude that the BIII-TL-W model performs better than the selected models. The values of the

Table 6. Parameter estimates and goodness-of-fit statistics for various models fitted for turbocharger data set.

Model	Estimates				Statistics							
	α	β	b	λ	2 log L	AIC	AICC	BIC	W	A	K–S	p-value
BIII-TL-W	0.0117	4266.5100	13.7790	0.2440	155.1	163.1	164.3	169.9	0.0144	0.1123	0.0568	0.9995
	(0.0019)	(1.2179×10^{-5})	(0.3803)	(0.0038)								
BIII-TL-W	1	1.1833×10^{3}	3.1843	0.0024	177.4	183.4	184.1	188.5	0.2143	1.4078	0.1440	0.3778
	-	(2.7152×10^{-9})	(0.0014)	(7.6008×10^{-5})								
BIII-TL-W	0.2375	1	315.6306	0.4823	195.0	201.0	201.7	206.1	0.4458	2.6669	0.2380	0.0215
	(0.0876)	-	(110.5283)	(0.0304)								
BIII-TL-W	1	1	96.4767	0.5266	212.6	216.6	216.9	220.0	0.6714	3.7950	0.3481	0.0001
	-	-	(22.5959)	(0.0393)								
BIII-TL-W	80.4692	1.9565	1	1	202.7	206.7	207.0	210.0	0.5959	3.4259	0.2451	0.0164
	(33.3439)	(0.1990)	-	-								
	α	β	δ									
EW	0.7549	0.8389	17.9156	-	187.4	193.4	194.1	198.5	0.3195	1.9997	0.1672	0.2134
	(0.5062)	(0.2517)	(14.4923)	-								
	α	λ	θ									
MO-IW	$4.8365\ 10^{3}$	4.8391	0.9275	-	177.4	183.4	184.1	188.5	0.2153	1.4131	0.1438	0.3800
	$(8.4139\ 10^{6})$	(0.6512)	(0.2303)	-								
	a	b	λ	c								
KOL-LLoG	0.7277	344.2800	0.0001	2.7055	164.9	172.9	174.0	179.6	0.0745	0.5557	0.1063	0.7564
	(0.0915)	$(6.2303 \text{ x } 10^{7})$	$(6.0421 \text{ x } 10^{5})$	(0.0249)								
	a	b	λ	θ								
BOL-E	0.2105	0.3251	0.0003	1.1911	159.0	167.0	168.1	173.7	0.0343	0.2175	0.0832	0.9449
	(0.0371)	(0.2603)	(0.0002)	(0.0088)								
BOL-U	1.2762	34.0275	0.1087	10.9050	157.5	165.5	166.6	172.2	0.0271	0.1680	0.0682	0.9923
	(0.4957)	(0.0049)	(0.0513)	(0.9765)								
	β	λ	θ	γ								
ELOLLW	3.2768	0.1209	2.9438	3.4997	163.8	171.8	172.9	178.5	0.0636	0.4815	0.1017	0.8027
	(1.7673)	(0.0284)	(1.9669)	(0.6303)								
	a	b	α	β								
KW	0.3778	21.9782	0.0630	9.9157	164.3	172.3	173.5	179.1	0.0727	0.5442	0.1091	0.7278
	(0.2433)	(0.0873)	(0.0067)	(6.3940)								
	a	b	α	θ								
TL-WLx	0.6569	0.2080	16.4716	0.1692	157.1	165.1	166.2	171.8	0.0162	0.1306	0.0839	0.9410
	(0.1754)	(0.0897)	(6.2762)	(0.0726)								

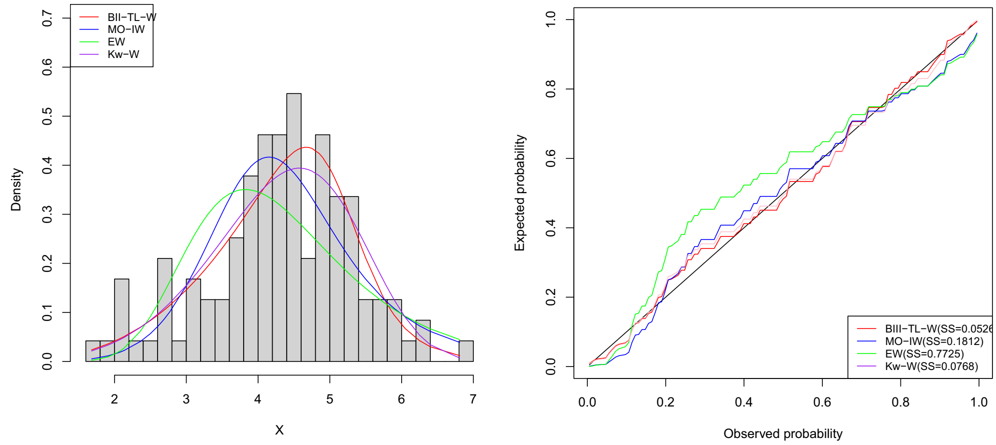

Figure 8. Fitted pdfs and probability plots for silicon nitride data set.

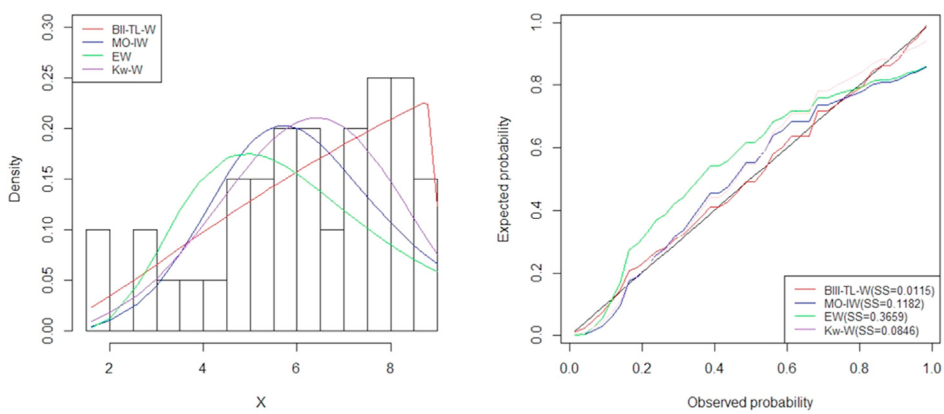

Figure 9. Fitted pdfs and probability plots for turbocharger data set.

Table 7. Likelihood ratio test results.

Model	df	Data set I	Data set II	Data set III
		$\chi2$ (p-value)	$\chi2$(p-value)	$\chi2$ (p-value)
BIII-TL-W(β, b, λ)	1	13.4 (0.0003)	21.0 ($<$ 0.00001)	22.3 ($<$ 0.00001)
BIII-TL-W(α, b, λ)	1	76.7 ($<$ 0.00001)	176.3 ($<$ 0.00001)	39.9 ($<$ 0.00001)
BIII-TL-W(b, λ)	2	68.1 ($<$ 0.00001)	118.5 ($<$ 0.00001)	57.5 ($<$ 0.00001)
BIII-TL-W(α, β)	2	62.3 ($<$ 0.00001)	85.8 ($<$ 0.00001)	47.6 ($<$ 0.00001)

goodness-of-fit statistics A^*, W^*, K–S and the p-value of the K–S statistic indicates that the BIII-TL-W model fits the turbocharger failure times data set better than the several models considered in this paper. The values of the SS also show that the BIII-TL-W model performs better than the selected models as shown in Figure 9.

6.4. Likelihood ratio test results

Likelihood ratio test results to assess if the BIII-TL-W distribution fit the given data sets better than the nested models are presented in this subsection.

From the likelihood ratio test results (see Table 7 for details), we conclude that the BIII-TL-W distribution performs better than the nested models on the three data sets at 5% significance level.

7. Concluding remarks

We present a new family of distributions called the Burr III-Topp-Leone-G (BIII-TL-G). We further study in detail its structural properties including moments, probability weighted moments, distribution of order statistics, probability weighted moments, and entropy. The new model can be expressed as an infinite linear combination of the exponentiated-G distribution. We also derive the maximum likelihood estimates of the BIII-TL-G family of distributions. Three special cases of the proposed model were also presented. A simulation study was conducted to assess consistency of the maximum likelihood estimates. Three real data examples are presented to demonstrate the applicability and usefulness of the BIII-TL-G family using the Weibull distribution as a special case of the baseline distribution. The BIII-TL-W distribution performs better than the nested and several non-nested models considered in this paper.

Declarations

Author Contribution statement

F. Chipepa, B. Oluyede: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

P. O. Peter: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Declaration of interests statement

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Additional information

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