Heliyon 10 (2024) e24001

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

A new Topp-Leone Kumaraswamy Marshall-Olkin generated family of distributions with applications

Mintodê Nicodème Atchadé [a,b], Melchior A.G. N'bouké [a], Aliou Moussa Djibril [a], Aned Al Mutairi [e], Manahil SidAhmed Mustafa [c], Eslam Hussam [d,*], Hassan Alsuhabi [f], Said G. Nassr [g]

[a] National Higher School of Mathematics Genius and Modelization, National University of Sciences, Technologies, Engineering and Mathematics, Abomey, Republic of Benin
[b] University of Abomey-Calavi/International Chair in Mathematical Physics and Applications (ICMPA: UNESCO-Chair), 072 BP 50 Cotonou, Republic of Benin
[c] Department of Statistics, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia
[d] Department of Accounting, College of Business Administration in Hawtat bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia
[e] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[f] Department of Mathematics, Al-Qunfudah University College, Umm Al-Qura University, Mecca, Saudi Arabia
[g] Department of Statistics and Insurance, Faculty of Commerce, Arish University, Al-Arish 45511, Egypt

ARTICLE INFO

Keywords:
Topp-Leone distribution
Kumaraswamy-G family
Marshall-Olkin-G family
Maximum Likelihood Estimation
Statistical modeling

ABSTRACT

We aim in this paper to propose a novel class of distributions that was created by merging the Topp-Leone distribution and the Generated families of Kumaraswamy and Marshall-Olkin. Its cumulative distribution function characterizes it and includes rational and polynomial functions. In particular, the following desirable properties of the new family are presented: Shannon entropy, order statistics, the quantile power series, and several associated measures and functions. Then, using a specific family member identified before, we create a parametric statistical model with the basic distribution being the inverse exponential distribution. Finally, a thorough investigation has been made to implement this new distribution with three data sets: the glass fibers data set, the glass Alumina data set and the hailing times data set. In comparison to six prominent competitors, the new model performs favorably on all statistical tests and criteria that were examined.

1. Introduction

There is an increasing drive to create robust and adaptable statistical models in the context of describing, analyzing, and explaining occurrences from various spheres of life. Such models can be constructed from a family of distributions, and their degree of flexibility can be increased by modifying a basic distribution by one or more shape parameters. In statistics, parameter addition is an essential method for improving model accuracy and better representing real data. This approach consists in including new parameters in a statistical model to better fit it to the specific characteristics of the observed data. By enabling the model to capture more complex relationships between variables and to explain more variations, parameter addition enables more reliable results to be obtained,

* Corresponding author.
E-mail addresses: aoalmutairi@pnu.edu.sa (A.A. Mutairi), msida@ut.edu.sa (M.S. Mustafa), eslmhussam1986@gmail.com (E. Hussam), dr.saidstat@gmail.com (S.G. Nassr).

https://doi.org/10.1016/j.heliyon.2024.e24001
Received 8 July 2023; Received in revised form 9 December 2023; Accepted 2 January 2024

Available online 8 January 2024
2405-8440/

taking into account factors often overlooked by simpler models. However, it is crucial to carry out this procedure with care, avoiding overloading the model, to ensure relevant interpretation of results and appropriate generalization to new data. In this article, we will discuss a newly developed broad family of distributions that was produced by mixing the Topp-Leone distribution [1], with other distributions as the Generated families of Kumaraswamy [2], Kumaraswamy censored model [3] and Marshall-Olkin [4]. Typical instances of this family of distributions include the Marshall-Olkin Topp Leone-G [5], [6], type II half logistic [7], exponentiated generalized Topp-Leone-G [8], Garhy-G [9], half-logistic odd Weibull-Topp-Leone-G [10], truncated inverted Kumaraswamy-G [11], Topp-Leone Kumaraswamy-G [12], odd log-logistic Poisson-G [13], Kumaraswamy-G [14], type II power Topp-Leone-G [15], Fréchet Topp-Leone-G [16], Topp-Leone Gompertz-G [17], Topp-Leone odd Lindley-G [18], type II Topp-Leone-G [19], Topp–Leone modified Weibull [20], Marshall-Olkin extended Gompertz Makeham [21], Marshall Olkin alpha power extended Weibull [22], Marshall-Olkin alpha power inverse Weibull [23], Marshall-Olkin alpha power Lomax [24,25], a generalized Birnbaum-Saunders distribution [26], Topp–Leone modified Weibull model [20], a new version of Topp–Leone distribution with engineering applications [27], reliability analysis of exponential distributions [28], Marshall-Olkin improved Rayleigh distribution [29,30]. Some authors worked on Bayesian inferences such as [31] and a comprehensive study of lognormality tests [32]. In these papers, the authors have explored a large family of statistical distributions by combining the Topp-Leone distribution with other distributions and have developed specific models and data-fit tests for these distributions. This research contributes to a better understanding and use of these distributions in different fields of statistical application.
As it relates to this topic, let us get in to more detail about the new distributions family.
The Topp-Leone distribution is one of the most practical established distributions depending on one parameter α. Its cumulative distribution function (CDF) is defined in (1):

$$F_1(x;\alpha) = x^{\alpha}(2-x)^{\alpha}, x \in [0,1], \alpha > 0. \tag{1}$$

We build the new family of distributions using the well known Kumaraswamy and Marshall-Olkin families of distributions, respectively, as defined by (2) and (3):

$$F_2(x;a,b,\sigma) = 1 - \{1 - R^a(x;\sigma)\}^b, x \in \mathbb{R}, \tag{2}$$

$$F_3(x;p,\sigma) = \frac{R(x;\sigma)}{1 - q\bar{R}(x;\sigma)}, \tag{3}$$

where $a > 0, b > 0, q \in (0;1)$ and $R(x;\sigma)$ is the CDF of a fundamental continuous distribution depending on $\sigma = (\sigma_1, \sigma_2, \dots)$ and $\bar{R}(x;\sigma)$ the survival function of $R(x;\sigma)$.

F_1, F_2 and F_3 are differentiable and monotonically non decreasing; and $\mathbb{D}_{F_3} \subset \mathbb{D}_{F_2} \subset \mathbb{D}_{F_1}$, where $\mathbb{D}_{F_1}, \mathbb{D}_{F_2}$ and $\mathbb{D}_{F_3}$ are the respective definition domains of F_1, F_2 and F_3.

So, we have the CDF of the new TLKMO-G family written as:

$$F(x;a,b,\alpha,q,\sigma) = F_1\left(F_2\left(F_3(x;q,\sigma),a,b,v\right),\alpha\right), i.e,$$

$$F(x;\phi) = \left(1 - \left\{1 - \left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^b\right)^{\alpha}\left(2 - \left(1 - \left\{1 - \left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^b\right)\right)^{\alpha},$$

where $\phi = (a,b,\alpha,q,\sigma)$.

After some algebra, we lead to:

$$F(x;\phi) = \left(1 - \left\{1 - \left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right)^{\alpha}, \tag{4}$$

$x \in \mathbb{D}_R$, where $\mathbb{D}_R$ is the domain to which x belongs to R; $a, b, \alpha > 0,\ 0 < q < 1$.
This family of distributions is designated as the Topp-Leone Kumaraswamy Marshall-Olkin family (TLKMO).

The new family's primary drives are:

1. to develop models with concise and simple expressions;
2. to add one or more parameters to existing templates to make them more flexible;
3. to derive good estimates of parameters.

The Topp-Leone Kumaraswamy Marshall-Olkin Inverse Exponential (TLKMOIEx), a new 5-parameters lifetime distribution with a significant applicability potential, is a special member of the (TLKMO) family that is characterized by the exponential distribution as the base distribution.

2. The fundamentals of the TLKMO-G family

This section outlines the fundamentals of the TLKMO-G family with an emphasis on its key purposes.

2.1. Probability density function (pdf)

In order to obtain the pdf for the TLKMO-G family, differentiation must first be performed so we will differentiate equation (4) as a function of x and is represented as (5):

$$f(x;\phi) = \frac{2ab\alpha(1-q)r(x;\sigma)}{\left(1-q\bar{R}(x;\sigma)\right)^2}\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}^{a-1}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b-1} \times\left\{1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right\}^{\alpha-1}, \tag{5}$$

where $r(x;\sigma)$ is the pdf corresponding to $R(x;\sigma)$. Some asymptotic findings on $f(x;\phi)$ can be deduced from this expression. When $R(x;\sigma)\to 0$, we have (6):

$$f(x;\phi) \sim \frac{2^\alpha ab^\alpha \alpha r(x;\sigma)}{(1-q)^{a\alpha}}\{R(x;\sigma)\}^{a\alpha-1}. \tag{6}$$

Moreover, for $R(x;\sigma)\to 1$, we obtain (7):

$$f(x;\phi) \sim 2ab\alpha(1-q)r(x;\sigma)\left\{1-\{R(x;\sigma)\}^a\right\}^{2b-1}. \tag{7}$$

Beginning with the critical points provided by the solutions of the nonlinear equation according to x, it is possible to study the variations of $f(x;\phi)$ as follows:
$\{\ln[f(x;\phi)]\}'=0$, one has (8):

Let's put $\Lambda = \{\ln[f(x;\phi)]\}'$,

$$\Lambda = \frac{r(x;\sigma)'}{r(x;\sigma)} - \frac{2qr(x;\sigma)}{1-q\bar{R}(x;\sigma)} + \frac{(a-1)(1-q)r(x;\sigma)}{R(x;\sigma)\left(1-q\bar{R}(x;\sigma)\right)} - \frac{a(2b-1)(1-q)r(x;\sigma)\{R(x;\sigma)\}^{a-1}}{\left\{1-q\bar{R}(x;\sigma)\right\}^{a+1}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}} + \frac{2ab(\alpha-1)(1-q)r(x;\sigma)\{R(x;\sigma)\}^{a-1}}{\left\{1-q\bar{R}(x;\sigma)\right\}^{a+1}\left\{1-\left(1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right)^{2b}\right\}}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b-1}. \tag{8}$$

The nature of the critical point x_c, is revealed by the sign of x : $\{\ln[f(x;\phi)]\}''\,|_{x=x_c}$.

2.2. Hazard rate function

The equation (9) gives the TLKMO-G family's hazard rate function (hrf):

$$\begin{aligned} h(x;\phi) &= \frac{f(x;\phi)}{1-F(x;\phi)} \\ &= \frac{2ab\alpha(1-q)r(x;\sigma)}{\left(1-q\bar{R}(x;\sigma)\right)^2}\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}^{a-1}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b-1} \\ &\times \frac{\left\{1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right\}^{\alpha-1}}{1-\left(1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right)^\alpha}, \end{aligned} \tag{9}$$

indeed, for $R(x;\sigma)\to 0$, we get (10):

$$h(x;\phi) \sim \frac{2^\alpha ab^\alpha \alpha r(x;\sigma)}{(1-q)^{a\alpha}}\{R(x;\sigma)\}^{a\alpha-1}. \tag{10}$$

Additionally, if $R(x;\sigma)\to 1$ we get (11):

$$h(x;\phi) \sim \frac{2ab(1-q)r(x;\sigma)}{\left[1-\{R(x;\sigma)\}^a\right]}. \tag{11}$$

Consequently, when $R(x;\sigma)\to 0$, but not when $R(x;\sigma)\to 1$, the parameter α significantly affects the asymptotes. Likewise to how $f(x;\phi)$, variations can be investigated, $h(x;\phi)$ variations can be as well using (12).

$$\{\ln[h(x;\phi)]\}' = \{\ln[f(x;\phi)]\}' + h(x;\phi). \tag{12}$$

2.3. A particular member: the TLKMOIEx distribution

In this work, we select the inverse exponential with $\theta > 0$ as the fundamental distribution to characterize the TLKMOIEx. The resulting CDF is in (13):

$$R(x;\theta) = e^{-\frac{\theta}{x}}, \tag{13}$$

the inverse exponential's pdf is in (14):

$$r(x;\theta) = \frac{\theta}{x^2} e^{-\frac{\theta}{x}}, \tag{14}$$

and the related hrf is provided by (15):

$$h(x;\theta) = \frac{\theta e^{-\frac{\theta}{x}}}{x^2\left(1 - e^{-\frac{\theta}{x}}\right)}. \tag{15}$$

Additionally, it has shown to be quite adaptable for modeling data with an underlying nonmonotonic hrf. Accordingly, the following CDF in (16) defined the TLKMOIEx distribution:

$$F(x;\phi) = \left(1 - \left\{1 - \left(\frac{e^{-\frac{\theta}{x}}}{1 - q\left(1 - e^{-\frac{\theta}{x}}\right)}\right)^a\right\}^{2b}\right)^\alpha, \tag{16}$$

where $x \in \mathbb{D}_R; a, b, \alpha > 0,\ 0 < q < 1$.
The associated pdf and hrf are available by the equations (17) and (18), respectively:

$$\begin{aligned} f(x;\phi) = {} & \frac{2ab\alpha\theta(1-q)e^{-\frac{\theta}{x}}}{x^2\left(1 - q\left(1 - e^{-\frac{\theta}{x}}\right)\right)^2} \left(\frac{e^{-\frac{\theta}{x}}}{1 - q\left(1 - e^{-\frac{\theta}{x}}\right)}\right)^{a-1} \left(1 - \left\{\frac{e^{-\frac{\theta}{x}}}{1 - q\left(1 - e^{-\frac{\theta}{x}}\right)}\right\}^a\right)^{2b-1} \\ & \times \left(1 - \left\{1 - \left(\frac{e^{-\frac{\theta}{x}}}{1 - q\left(1 - e^{-\frac{\theta}{x}}\right)}\right)^a\right\}^{2b}\right)^{\alpha-1}, \end{aligned} \tag{17}$$

and

$$h(x;\phi) = \frac{f(x;\phi)}{1 - \left(1 - \left\{1 - \left(\frac{e^{-\frac{\theta}{x}}}{1 - q\left(1 - e^{-\frac{\theta}{x}}\right)}\right)^a\right\}^{2b}\right)^\alpha}. \tag{18}$$

The potential forms of (17) and (18) are shown in Fig. 1 and Fig. 2 and Fig. 3 and Fig. 4, respectively. Fig. 1 and Fig. 2 in particular show that the pdf remains pretty close to the normal distribution even when the settings are altered. As seen in Fig. 3 and Fig. 4, the TLKMOIEx distribution presents a hrf with a trend that can be rising, falling, turning upside down, or displaying a bathtub-shaped pattern. All of these curvature features are excellent for creating flexible statistical models.

3. Some TLKMO-G family mathematical properties

3.1. Order statistics

Proposition 1. *In numerous fields of statistical theory and practical applications, order statistics play a pivotal role. In this context, we present the fundamental principles of order statistics within the scope of the TLKMO-G family of distributions.*

Consider a set of n independent random variables, denoted as $X_1, X_2, \ldots, X_n$, each following a distribution from the TLKMO-G family. The pdf for the ith order statistic ($X_{i:n}$) is thus expressed by (19):

$$f_{i:n}(x;\phi) = \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{k=0}^{m} \sum_{l=0}^{u} \sum_{s=0}^{\infty} \tau K(x), \tag{19}$$

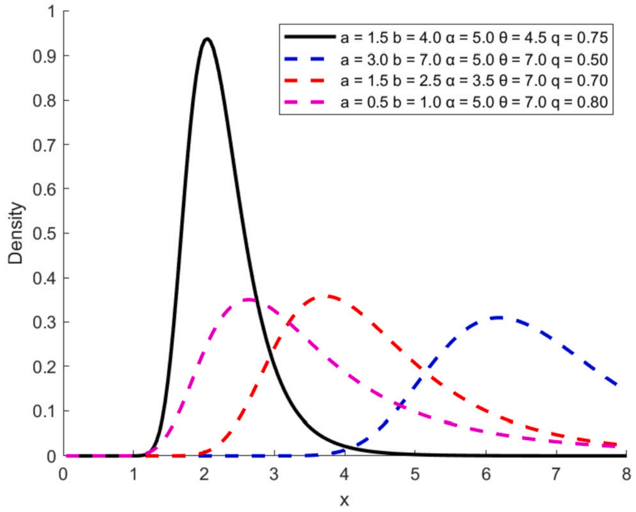


Fig. 1. pdf of the new TLKMOIEx for various sets of parameters.

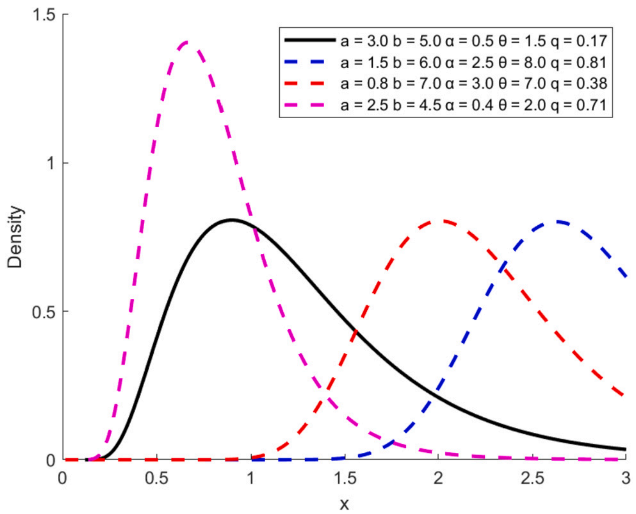


Fig. 2. pdf of the new TLKMOIEx for other sets of parameters.

where

$$m = \alpha(i+j) - 1, u = 2b(k+1) - 1; \quad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt,$$

$$\tau = (-1)^{j+k+l} 2ab\alpha(1-q) \binom{n-i}{j} \binom{\alpha(i+j)-1}{k} \binom{2b(k+1)-1}{l} \gamma_s,$$

$$K(x) = r(x;\sigma)\{R(x;\sigma)\}^s.$$

Proof.

$$f_{i:n}(x;\phi) = \frac{f(x;\phi)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \{F(x;\phi)\}^{i+j-1}.$$

Let $H(x) = f(x;\phi)\{F(x;\phi)\}^{i+j-1}$,

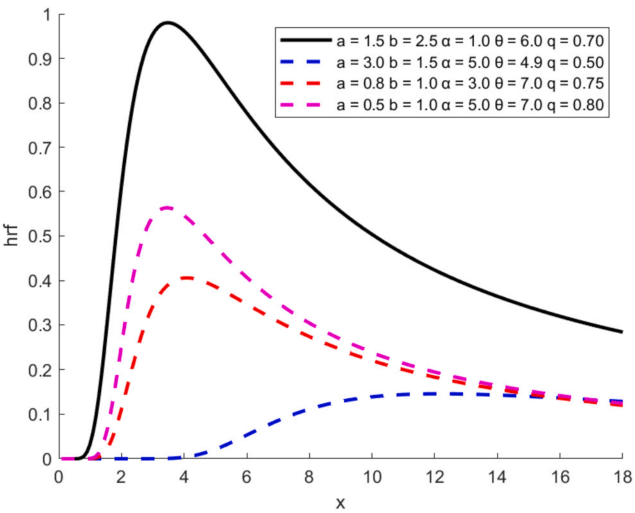


Fig. 3. New TLKMOIEx's hazard rate functions for various sets of parameters.

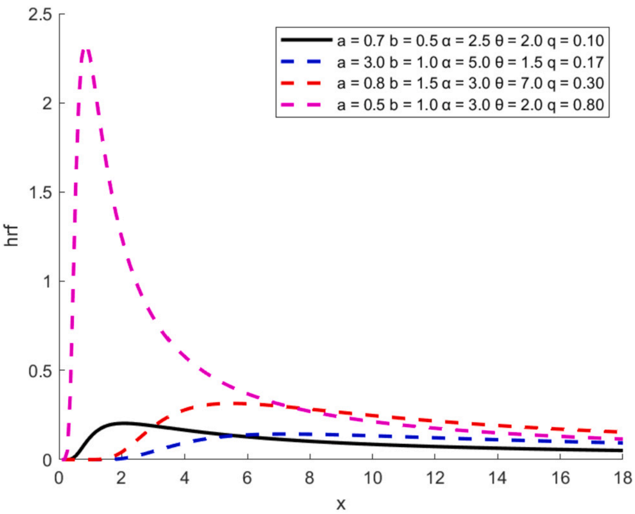


Fig. 4. New TLKMOIEx's hazard rate functions for other sets of parameters.

$$H(x) = \frac{2ab\alpha(1-q)r(x;\sigma)}{\left(1-q\bar{R}(x;\sigma)\right)^2}\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}^{a-1}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b-1}$$
$$\times\left\{1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right\}^{\alpha(i+j)-1}.$$

The generalized binomial formula yields the following result:

$$\left\{1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b}\right\}^{\alpha(i+j)-1} = \sum_{k=0}^{m}(-1)^k\binom{\alpha(i+j)-1}{k}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2kb},$$

where $m = \alpha(i+j)-1$,

$$H(x) = \sum_{k=0}^{m}(-1)^k\frac{2ab\alpha(1-q)r(x;\sigma)}{\left(1-q\bar{R}(x;\sigma)\right)^2}\binom{\alpha(i+j)-1}{k}\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}^{a-1}\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b(k+1)-1},$$

$$\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right\}^{2b(k+1)-1} = \sum_{l=0}^{u}(-1)^l\binom{2b(k+1)-1}{l}\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}^{al},$$

where $u = 2b(k+1)-1$.

Then,

$$H(x)=\sum_{k=0}^{m}\sum_{l=0}^{u}(-1)^{k+l}\,2ab\alpha(1-q)\binom{\alpha(i+j)-1}{k}\binom{2b(k+1)-1}{l}\frac{r(x;\sigma)\{R(x;\sigma)\}^{a(l+1)-1}}{\left(1-q\bar{R}(x;\sigma)\right)^{a(l+1)+1}},\tag{20}$$

and

$$\begin{aligned}f_{i:n}(x;\phi)=&\frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\sum_{k=0}^{m}\sum_{l=0}^{u}(-1)^{j+k+l}\,2ab\alpha(1-q)\binom{n-i}{j}\binom{\alpha(i+j)-1}{k}\\&\times\binom{2b(k+1)-1}{l}\frac{r(x;\sigma)\{R(x;\sigma)\}^{a(l+1)-1}}{\left(1-q\bar{R}(x;\sigma)\right)^{a(l+1)+1}}.\end{aligned}\tag{21}$$

An expansion for $\{R(x;\sigma)\}^{\lambda}$ ($\lambda>0$ real non-integer), can be written out as follows:

$$\{R(x;\sigma)\}^{\lambda}=\sum_{\nu=0}^{\infty}S_{\nu}(\lambda)\{R(x;\sigma)\}^{\nu},$$

where

$$S_{\nu}(\lambda)=\sum_{j=\nu}^{\infty}(-1)^{\nu+j}\binom{\lambda}{j}\binom{j}{\nu}.\tag{22}$$

Then, by using (22), we get:

$$\begin{aligned}\frac{\{R(x;\sigma)\}^{a(l+1)-1}}{\left(1-q\bar{R}(x;\sigma)\right)^{a(l+1)+1}}&=\frac{\sum_{s=0}^{\infty}\alpha_s\{R(x;\sigma)\}^{s}}{\sum_{s=0}^{\infty}\beta_s\{R(x;\sigma)\}^{s}}\\&=\sum_{s=0}^{\infty}\gamma_s\{R(x;\sigma)\}^{s},\end{aligned}\tag{23}$$

where:

$$\begin{aligned}&\alpha_s=\alpha_s(a,i)=\sum_{w=s}^{\infty}(-1)^{w+s}\binom{a(l+1)-1}{w}\binom{w}{s}\\&\beta_s=\beta_s(a,q,i)=\sum_{w=s}^{\infty}(-1)^{w+s}q^{w}\binom{a(l+1)+1}{w}\binom{w}{s}\\&\text{and for } k\geq 1\,\gamma_s=\gamma_s(a,q,i)=\frac{1}{\beta_0}\left(\alpha_s-\frac{1}{\beta_0}\sum_{t=1}^{s}\alpha_t\beta_{s-t}\right)\\&\gamma_0=\frac{\alpha_0}{\beta_0}.\end{aligned}\tag{24}$$

By taking (24) in (23) and the result in (21), we complete the proof of proposition 1.

3.2. Quantile power series

Proposition 2. *The TLKMO-G family's quantile power series is provided by (25):*

$$Q(u)=\sum_{k=0}^{\infty}e_k u^k,\tag{25}$$

where $e_k=\sum_{k=0}^{\infty}a_i\gamma_{i,k}\gamma_{i,0}=C_0^i$ and (for $k>1$),

$$\gamma_{i,k}=\left(kC_0\right)^{-1}\sum_{m=1}^{k}[m(i+1)-k]\,C_m\gamma_{i,k-m}.$$

Proof. For more clarity, by setting $x_u=Q(u;\phi)$ for $u\in(0;1)$. Consequently, x_u satisfies the equation $u=F(x;\phi)$ according to the definition of a qf, implying that

$$u=\left(1-\left\{1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^{a}\right\}^{2b}\right)^{\alpha}.$$

Following some algebraic manipulations, we obtain the following result:

$$R(x;\sigma) = \frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}},$$

and

$$x = Q_R\left(\frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}\right). \tag{26}$$

This expression lacks a clear form, therefore let's write it as a series expansion instead:

$$Q(u) = \sum_{i=0}^{\infty} a_i u^i. \tag{27}$$

Using the findings in [33], we have:

$$(Q(u))^n = \left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i} u^i, \tag{28}$$

where n is an integer ≥ 1, and the coefficients are calculated as follows:

$$c_{n,i} = \left(i a_0\right)^{-1} \sum_{m=1}^{i} \left[m(n+1)-i\right] a_m c_{n,i-m}, \tag{29}$$

where $c_{n,0} = a_0^n$.
We gain the following by deriving an expansion for Q_R's argument in (26):

$$\begin{aligned} V &= \frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}} \\ &= \frac{\sum_{k=0}^{\infty} A_k u^k}{\sum_{k=0}^{\infty} B_k u^k} \\ &= \sum_{k=0}^{\infty} C_k u^k, \end{aligned} \tag{30}$$

where

$$A_k = (1-q) \sum_{i,k,l=0}^{\infty} (-1)^{i+k+l} \binom{\frac{1}{a}}{i} \binom{\frac{i}{2b}}{k} \binom{\frac{k}{\alpha}}{l}$$

$$B_k = q \sum_{i,k,l=0}^{\infty} (-1)^{i+k+l+1} \binom{\frac{1}{a}}{i} \binom{\frac{i}{2b}}{k} \binom{\frac{k}{\alpha}}{l}$$

$$C_k = \frac{1}{B_0}\left(A_k - \frac{1}{B_0}\sum_{t=1}^{k} B_t C_{k-t}\right)$$

$$C_0 = A_0 / B_0.$$

Consequently, by combining (27) and (30) the qf of X follows from (26):

$$Q(u) = Q_R\left(\sum_{k=0}^{\infty} C_k u^k\right) = \sum_{i=0}^{\infty} a_i \left(\sum_{k=0}^{\infty} C_k u^k\right)^i.$$

Using (28) and (29) further puts an end to the demonstration of the proposition 2.

$$x = Q_R\left(\frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}\right).$$

The TLKMO-G's first, second and third quartiles are:

$$Q_1 = Q\left(\frac{1}{4};\phi\right),$$

$$Q_2 = Q\left(\frac{1}{2};\phi\right),$$

$$Q_3 = Q\left(\frac{3}{4};\phi\right),$$

The median, the inter-quartile range, the skewness and the kurtosis are provided by, respectively:

$$M = Q_R\left(\frac{(1-q)\left[1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right]^{\frac{1}{a}}}{1-q\left[1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right]^{\frac{1}{a}}}\right)$$

$$IQR = Q_3 - Q_1$$

$$S = \frac{Q_3 + Q_1 - 2M}{IQR}$$

$$K = \frac{Q\left(\frac{7}{8};\phi\right) - Q\left(\frac{5}{8};\phi\right) + Q\left(\frac{3}{8};\phi\right) - Q\left(\frac{1}{8};\phi\right)}{IQR}.$$

The quantile function of the special member $R(x;\theta) = e^{-\frac{\theta}{x}}$ can be found as follows:

$$e^{-\frac{\theta}{x}} = \frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}.$$

Then,

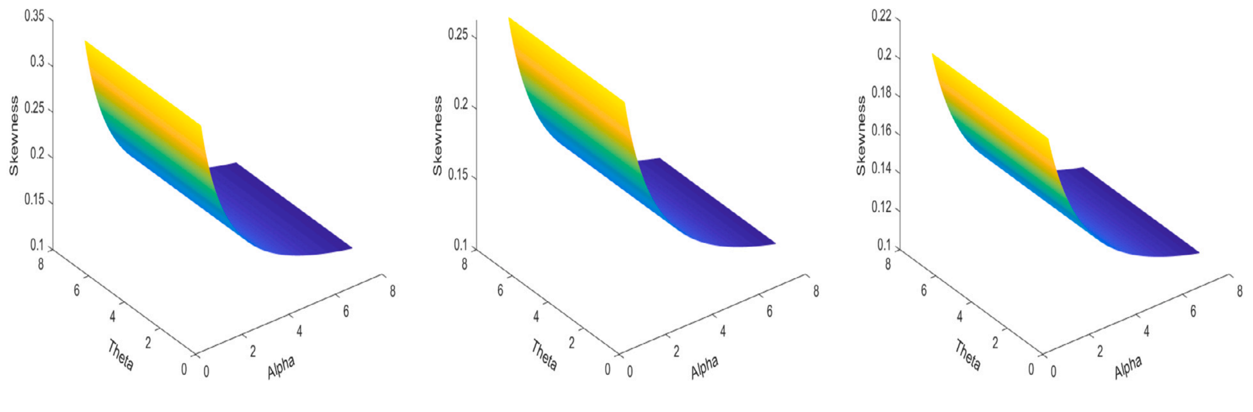
$$x = -\theta\left\{\ln\left(\frac{(1-q)\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}{1-q\left(1-\left(1-u^{\frac{1}{\alpha}}\right)^{\frac{1}{2b}}\right)^{\frac{1}{a}}}\right)\right\}^{-1}.$$

Table 1 provides values for Q_1, M, Q_3, S, and K of the TLKMOIEx distribution for various parameters.

Fig. 5a, Fig. 5b, and Fig. 5c show the 3D graphical representations of skewness for different values of q. Analogously, Fig. 6a, Fig. 6b, and Fig. 6c exhibit the kurtosis for different values of q. From these graphical representations, we see that for small values of q, the measures of skewness and kurtosis are high, whereas for slightly larger values, these measures are lower and the corresponding curves have almost the same shape.

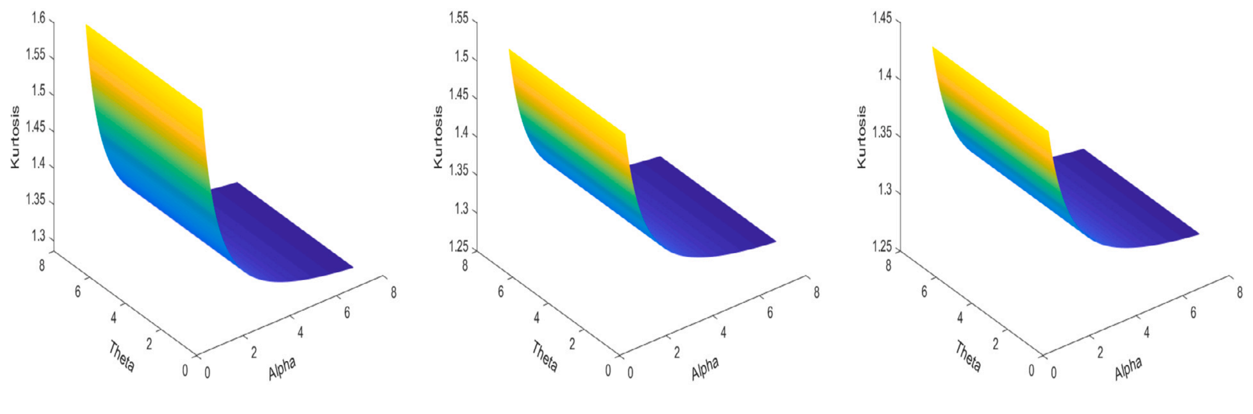
Table 1
Q_1, M, Q_3, S and K for TLKMOIEx.

$(a, b, \alpha, q, \theta)$	Q_1	M	Q_3	S	K
(0.5, 0.5, 0.5, 0.5, 0.5)	0.0802	0.1456	0.2991	0.4023	1.9206
(2.5, 2.0, 1.5, 0.52, 0.5)	0.3744	0.4972	0.6854	0.2104	1.3850
(4.5, 4.0, 3.5, 0.57, 1.5)	1.7719	2.1246	2.6006	0.1487	1.3118
(5.0, 4.5, 4.0, 0.60, 2.0)	2.4344	2.8760	3.4625	0.1409	1.3047
(5.5, 5, 4.5, 0.67, 2.5)	2.8600	3.3233	3.9294	0.1336	1.2987
(6.0, 5.5, 5.0, 0.72, 3.0)	3.2641	3.7430	4.3622	0.1276	1.2939
(6.5, 6.0, 5.5, 0.77, 3.5)	3.5297	3.9975	4.5956	0.1223	1.2898
(7.0, 6.5, 5.0, 0.77, 4.0)	4.0807	4.6171	5.2968	0.1178	1.2860
(7.5, 7.0, 5.5, 0.78, 4.5)	4.6728	5.2543	5.9866	0.1149	1.2837
(8.0, 7.5, 6.0, 0.79, 5.0)	4.6728	5.2543	5.9866	0.1149	1.2837
(8.5, 8.0, 6.5, 0.80, 5.5)	5.8234	6.4791	7.2968	0.1100	1.2800
(12.0, 11.0, 9.0, 0.82, 7.0)	8.5432	9.3502	10.3387	0.1011	1.2733
(17.0, 14.0, 12.0, 0.84, 9.0)	12.7772	13.8395	15.1290	0.0966	1.2698
(57.0, 47.0, 37.0, 0.86, 27.0)	74.8293	78.5800	83.0079	0.0828	1.2593



(a) (b) (c)

Fig. 5. Skewness plots of the TLKMOIEx model for (a) $q = 0.1$, (b) $q = 0.7$ and (c) $q = 0.9$.



(a) (b) (c)

Fig. 6. Kurtosis plots of the TLKMOIEx model for (a) $q = 0.1$, (b) $q = 0.7$ and (c) $q = 0.9$.

3.3. Renyi entropy

Equation (31) defines the Renyi entropy for the TLKMO-G family.

$$v_{\mathbb{R}}(X) = \frac{1}{1-\rho} \log\left(\sum_{k=0}^{n(\alpha-1)} \sum_{l=0}^{n} \sum_{s=0}^{\infty} \tau_{k,l,s} I_s(\rho, \sigma) \right), \quad \rho \neq 1, \quad \rho > 0, \tag{31}$$

$$\text{where } \tau_{k,l,s} = (-1)^{k+l}\,\gamma_s\,(2ab\alpha(1-q))^{\rho}\binom{\rho(\alpha-1)}{k}\binom{2b(k+\rho)-\rho}{l}$$

$$I_s(\rho,\sigma) = \int_{\mathbb{R}} [r(x;\sigma)]^{\rho}\,[R(x;\sigma)]^{s}\,dx.$$

Proof. As a measure of uncertainty, the Renyi entropy associated to a continuous random variable on $\mathbb{R}$ is defined as:

$$v_R(X) = \frac{1}{1-\rho} log\left\{\int_R f(x,\eta)^{\rho}\,dx\right\}, \rho \neq 1,$$

$$\begin{aligned}
f(x;\phi) &= [2ab\alpha(1-q)\,r(x;\sigma)]^{\rho}\,\frac{[R(x;\sigma)]^{\rho(a-1)}}{\left[1-q\bar{R}(x;\sigma)\right]^{\rho(a+1)}}\left[1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^{a}\right]^{\rho(2b-1)} \\
&\quad\times\left[1-\left[1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^{a}\right]^{2b}\right]^{\rho(\alpha-1)}, \\
&= \sum_{k=0}^{\rho(\alpha-1)}\sum_{l=0}^{n}(-1)^{k+l}\,(2ab\alpha(1-q))^{\rho}\,[r(x;\sigma)]^{\rho}\binom{\rho(\alpha-1)}{k} \\
&\quad\times\binom{2b(k+\rho)-\rho}{l}\frac{[R(x;\sigma)]^{a(l+\rho)-\rho}}{\left(1-q\bar{R}(x;\sigma)\right)^{a(l+\rho)+\rho}}, \\
&= \sum_{k=0}^{\rho(\alpha-1)}\sum_{l=0}^{n}\sum_{s=0}^{\infty}(-1)^{k+l}\,(2ab\alpha(1-q))^{\rho}\,[r(x;\sigma)]^{\rho}\binom{\rho(\alpha-1)}{k} \\
&\quad\times\binom{2b(k+\rho)-\rho}{l}[r(x;\sigma)]^{\rho}\,\gamma_s\,[R(x;\sigma)]^{s},
\end{aligned}$$

$$\begin{aligned}
\text{where }\ \alpha_s &= \alpha_s(a,i) = \sum_{r=s}^{\infty}(-1)^{r+s}\binom{a(l+\rho)-\rho}{r}\binom{r}{s} \\
\beta_s &= \beta_s(a,q,i) = \sum_{r=s}^{\infty}(-1)^{r+s}\,q^{r}\binom{a(l+\rho)+\rho}{r}\binom{r}{s}, \text{ and, for } k\geq 1 \\
\gamma_s &= \gamma_s(a,q,i) = \frac{1}{\beta_0}\left(\alpha_s - \frac{1}{\beta_0}\sum_{t=1}^{s}\alpha_t\beta_{s-t}\right) \\
\gamma_0 &= \frac{\alpha_0}{\beta_0}.
\end{aligned}$$

3.4. Shannon entropy

In order to determine the amount of information held in the data, we examine the Shannon entropy of the TLKMO-G family.

Proposition 3. *We can write the Shannon entropy equation as shown in* (32)*:*

$$\begin{aligned}
\gamma_x &= -\log(2ab\alpha(1-q)) + 2\mathbb{E}\left\{\log\left(1-q\bar{R}(x;\sigma)\right)\right\} - \mathbb{E}\{\ln[r(x;\sigma)]\} - (a-1)\mathbb{E}\left\{\log\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)\right\} \\
&\quad+(2b-1)\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i}\gamma_{k,1}\mathbb{E}\left(\{R(x;\sigma)\}^{k}\right) + (\alpha-1)\sum_{i=1}^{\infty}\sum_{j=0}^{2bi}\sum_{k=0}^{\infty}\frac{(-1)^{j}}{i}\binom{2bi}{j}\gamma_{k,2}\mathbb{E}\left(\{R(x;\sigma)\}^{k}\right),
\end{aligned} \tag{32}$$

where
for $k \geq 1$

$$\begin{aligned}
&\alpha_{k,1} = \alpha_{k,1}(a,i) = \sum_{j=k}^{\infty}(-1)^{j+k}\binom{ai}{j}\binom{j}{k};\ \beta_{k,1} = \beta_{k,1}(a,q,i) = \sum_{j=k}^{\infty}(-1)^{j+k}\,q^{j}\binom{ai}{j}\binom{j}{k} \\
&\textit{and}\ \ \gamma_{k,1} = \frac{1}{\beta_{0,1}}\left(\alpha_{k,1} - \frac{1}{\beta_{0,1}}\sum_{s=1}^{k}\alpha_{s,1}\beta_{k-s,1}\right) \textit{ for } k\geq 1,
\end{aligned} \tag{33}$$

$$\alpha_{k,2}=\alpha_{k,2}(a,i)=\sum_{t=k}^{\infty}(-1)^{t+k}\binom{aj}{t}\binom{t}{k};\ \beta_{k,2}=\beta_{k,2}(a,q,i)=\sum_{t=k}^{\infty}(-1)^{t+k}q^{t}\binom{aj}{t}\binom{t}{k}$$
$$\text{and}\ \ \gamma_{k,2}=\frac{1}{\beta_{0,2}}\left(\alpha_{k,2}-\frac{1}{\beta_{0,2}}\sum_{s=1}^{k}\alpha_{s,2}\beta_{k-s,2}\right)\ \text{for}\ k\geq 1;\ \gamma_{0,i}=\frac{\alpha_{0,i}}{\beta_{0,i}}, i=\{1,2\}. \tag{34}$$

Proof. We can write the Shannon entropy equation as shown below:

$$\chi=-\mathbb{E}\{\ln[f(x;\phi)]\}.$$

By using logarithm of (5), we have:

$$\ln[f(x;\phi)]=\ln\{2ab\alpha(1-q)\}+\ln\{r(x;\sigma)\}-2\ln\left\{1-q\bar{R}(x;\sigma)\right\}+(a-1)\ln\left\{\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right\}$$
$$+(2b-1)\ln\left\{1-\left[\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right]^{a}\right\}+(\alpha-1)\ln\left\{1-\left(1-\left[\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right]^{a}\right)^{2b}\right\}. \tag{35}$$

In the equation (35), using $\ln(1-z)=-\sum_{i=1}^{\infty}\frac{z^i}{i}$, and after minor adjustments, we obtain:

$$\ln\left\{1-\left[\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right]^{a}\right\}=-\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{i}\gamma_{k,1}\{R(x;\sigma)\}^{k}, \tag{36}$$

and

$$\ln\left\{1-\left(1-\left[\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right]^{a}\right)^{2b}\right\}=-\sum_{i=1}^{\infty}\sum_{j=0}^{2bi}\sum_{k=0}^{\infty}\frac{(-1)^{j}}{i}\binom{2bi}{j}\gamma_{k,2}\{R(x;\sigma)\}^{k}, \tag{37}$$

where $\gamma_{k,1}$, and $\gamma_{k,2}$ are respectively given by (33) and (34).

By replacing (36) and (37) in (35), and by taking the opposite of the mean, we complete the proof of the proposition 3.

3.5. Moments

Numerous crucial distributional characteristics, including dispersion, tendency, asymmetry and kurtosis are studied using the moments function. For any function $\psi(x)$, under one condition the existences and convergence of all quantities, we have:

$$\mathbb{E}\{\psi(x)\}=\int_{\mathbb{R}}\psi(x)f(x;\phi)\,dx. \tag{38}$$

By doing some algebra on (5), we get:

$$f(x;\phi)=\sum_{k=0}^{\alpha-1}\sum_{l=0}^{u}\sum_{s=0}^{\infty}(-1)^{k+l}\,2ab\alpha(1-q)\binom{\alpha-1}{k}\binom{2b(k+1)-1}{l}r(x;\sigma)\gamma_s\{R(x;\sigma)\}^{s},$$

where γ_s is given by (24).

By using (17) in (38), we have:

$$\mathbb{E}\{\psi(x)\}=\int_{\mathbb{R}}\psi(x)\frac{2ab\alpha\theta(1-p)e^{-\frac{\theta}{x}}}{x^2\left(1-p\left(1-e^{-\frac{\theta}{x}}\right)\right)^2}\left(\frac{e^{-\frac{\theta}{x}}}{1-p\left(1-e^{-\frac{\theta}{x}}\right)}\right)^{a-1}$$
$$\times\left(1-\left\{\frac{e^{-\frac{\theta}{x}}}{1-p\left(1-e^{-\frac{\theta}{x}}\right)}\right\}^{a}\right)^{2b-1}\left(1-\left\{1-\left(\frac{e^{-\frac{\theta}{x}}}{1-p\left(1-e^{-\frac{\theta}{x}}\right)}\right)^{a}\right\}^{2b}\right)^{\alpha-1}dx.$$

3.6. Moment-weighted probability (MWP)

Proposition 4. *The $(u+v)^{th}$ MWP of X denoted $M_{r,v}$ is given by (39):*

$$M_{u,v}=\sum_{k=0}^{\alpha-1}\sum_{l=0}^{n}\sum_{s=0}^{\infty}\Gamma_{k,l,s}I_{u,v}, \tag{39}$$

where $\Gamma_{k,l,s} = (-1)^{k+l}\, 2ab\alpha\,(1-q)\binom{\alpha+\upsilon-1}{k}\binom{2b(k+1)-1}{l}$

$$I_{u,\upsilon} = \int_{\mathbb{R}} x^u r(x;\sigma)\,[R(x;\sigma)]^s\,dx.$$

Proof. By definition, the $(u+\upsilon)^{th}$, MWP is defined by:

$$M_{u,\upsilon} = E\left(X^u F^\upsilon(x)\right) = \int_0^{+\infty} x^u f F^\upsilon dx, \tag{40}$$

$$\begin{aligned} f(x;\phi)\,[F(x;\phi)]^\upsilon &= \frac{2ab\alpha(1-q)\,r(x;\sigma)}{\left(1-q\bar{R}(x;\sigma)\right)^2}\left[\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right]^{a-1}\left[1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right]^{2b-1} \\ &\times\left[1-\left[1-\left(\frac{R(x;\sigma)}{1-q\bar{R}(x;\sigma)}\right)^a\right]^{2b}\right]^{\alpha+\upsilon-1}, \\ &= \sum_{k=0}^{\alpha-1}\sum_{l=0}^{n}\sum_{s=0}^{\infty}(-1)^{k+l}\,2ab\alpha(1-q)\binom{\alpha+\upsilon-1}{k}\binom{2b(k+1)-1}{l}r(x;\sigma) \\ &\times\gamma_s\,[R(x;\sigma)]^s, \end{aligned} \tag{41}$$

where γ_s is given by (24).
By introducing (41) into (40), we thus complete the demonstration.

4. Maximum Likelihood Estimation (MLE) technique

In this section, we delve into the TLKMO-G family, accompanied by its CDF as presented in equation (4). To estimate a, b, α, q, and σ, the MLE method was chosen for its noteworthy theoretical and practical advantages. Considering a n random sample drawn from the distribution of X, the log-likelihood function, utilizing the pdf detailed in equation (5), can be formulated as:

$$\begin{aligned} L(x;\phi) &= \prod_{i=1}^{n} f\left(x_i;\phi\right) \\ &= \prod_{i=1}^{n}\frac{2ab\alpha(1-q)\,r(x;\sigma)}{\left(1-q\bar{R}\left(x_i;\sigma\right)\right)^2}\left\{\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right\}^{a-1}\left\{1-\left(\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right)^a\right\}^{2b-1} \\ &\times\left\{1-\left\{1-\left(\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right)^a\right\}^{2b}\right\}^{\alpha-1}, \end{aligned}$$

$$\begin{aligned} l &= \ln\left[L(x;\phi)\right] \\ &= n\ln(2)+n\ln(a)+n\ln(b)+n\ln(\alpha)+n\ln(1-q)+\sum_{i=1}^{n}\ln\left\{r\left(x_i;\sigma\right)\right\} \\ &-2\sum_{i=1}^{n}\ln\left\{1-q\bar{R}\left(x_i;\sigma\right)\right\}+(a-1)\sum_{i=1}^{n}\ln\left\{\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right\} \\ &+(2b-1)\sum_{i=1}^{n}\ln\left\{1-\left[\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right]^a\right\} \\ &+(\alpha-1)\sum_{i=1}^{n}\ln\left\{1-\left(1-\left[\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right]^a\right)^{2b}\right\}. \end{aligned} \tag{42}$$

As is typically done, we equate all partial derivatives of l to zero, as illustrated below:

$$\frac{\partial l}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left\{\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right\}-(2b-1)\sum_{i=1}^{n}\frac{\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\ln\left\{\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right\}}{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a}$$
$$+2b(\alpha-1)\sum_{i=1}^{n}\frac{\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\ln\left\{\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right\}\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b-1}}{1-\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b}},$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+2\sum_{i=1}^{n}\ln\left\{1-\left(\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right)^a\right\}-2(\alpha-1)\sum_{i=1}^{n}\frac{\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b}\ln\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}}{1-\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b}},$$

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left\{1-\left\{1-\left(\frac{R\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right)^a\right\}^{2b}\right\},$$

$$\frac{\partial l}{\partial q}=-\frac{n}{1-q}+(a+1)\sum_{i=1}^{n}\left(\frac{\bar{R}\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}\right)-a(2b-1)\sum_{i=1}^{n}\frac{\left[\frac{R(x_i;\sigma)\bar{R}(x_i;\sigma)}{(1-q\bar{R}(x_i;\sigma))^2}\right]\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^{a-1}}{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a}$$
$$+2ab(\alpha-1)\sum_{i=1}^{n}\frac{\left[\frac{R(x;\sigma)\bar{R}(x_i;\sigma)}{(1-q\bar{R}(x_i;\sigma))^2}\right]\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^{a-1}\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b-1}}{1-\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b}},$$

$$\frac{\partial l}{\partial^{(\sigma)}}=\sum_{i=1}^{n}\frac{r^{(\sigma)}\left(x_i;\sigma\right)}{r\left(x_i;\sigma\right)}+(a-1)\sum_{i=1}^{n}\frac{R^{(\sigma)}\left(x_i;\sigma\right)}{R\left(x_i;\sigma\right)}-q(a+1)\sum_{i=1}^{n}\frac{R^{(\sigma)}\left(x_i;\sigma\right)}{1-q\bar{R}\left(x_i;\sigma\right)}$$
$$-a(2b-1)(1-q)\sum_{i=1}^{n}\frac{R^{(\sigma)}\left(x_i;\sigma\right)\left\{R\left(x_i;\sigma\right)\right\}^{a-1}}{\left(1-q\bar{R}\left(x_i;\sigma\right)\right)\left\{\left(1-q\bar{R}\left(x_i;\sigma\right)\right)^a-\left\{R\left(x_i;\sigma\right)\right\}^a\right\}}$$
$$+2ab(\alpha-1)(1-q)\sum_{i=1}^{n}\frac{\left\{\frac{R^{(\sigma)}(x_i;\sigma)}{(1-q\bar{R}(x_i;\sigma))^2}\right\}\left\{\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right\}^{a-1}\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b-1}}{1-\left\{1-\left(\frac{R(x_i;\sigma)}{1-q\bar{R}(x_i;\sigma)}\right)^a\right\}^{2b}}.$$

$w^{(\sigma)}(.)$ is the derivative of w with respect to σ. Given the large number of parameters, solving the system of equations corresponding to each derivative equal to zero is very tedious. Consequently, the basic distribution $R(x;\theta)=e^{-\frac{\theta}{x}}$ has been introduced in (42) to obtain the maximum likelihood related to the TLKMOIEx distribution. For the calculations, we employed the Mathematica software and thanks to the Maximize syntax of the latter followed by the positivity constraints of each parameter, this syntax returns us the values of the estimates $\hat{l},\hat{a},\hat{b},\hat{\alpha},\hat{q}$ and $\hat{\sigma}$.

5. Real data exploration

In this particular section, we underscore the practical utility of the TLKMOIEx distribution by showcasing its application in three real-world scenarios. Specifically, we compare the goodness-of-fit statistics and maximum likelihood estimates (MLEs) of model parameters between the TLKMOIEx model and its competitors. The competing models under consideration are as follows:

1. Exponentiated Marshall–Olkin Inverse Exponential Exponential (EMIEE) [34]
2. Logistic Inverse Exponential (LIEx) [35]
3. Alpha-power Inverse Weibull (AIW) [36]
4. Inverse Weibull Inverse Exponential (IWIEx) [37]
5. Topp-Leone Inverse Rayleigh (TIR) [38]
6. Standard Inverse Exponential.

Table 2
Maximum likelihood parameters estimates for the first data set.

Model	a	b	α	β	θ	λ	γ	q
TLKMOIEx	2.3373	52.4794	0.2674	-	9.3309	-	-	0.9582
EMIEEx	-	-	4.8290	2.5445	-	-	29.621	-
TIR	-	-	0.6500	-	5.7973	-	5.3420	-
IWIEx	-	-	0.0845	2.7673	14.3801	-	-	-
AIW	-	-	193.0620	3.8768	-	0.6365	-	-
LIEx	8.9077	0.7228	-	-	-	-	-	-
IEx	-	-	-	-	1.4083	-	-	-

Table 3
The metrics for the glass fibers data set.

Model	$-\hat{l}$	AIC	CAIC	BIC	HQIC	AD	W*	KS
TLKMOIEx	16.9078	43.8156	44.8682	54.5313	48.0301	1.8699	0.3702	0.1888
EMIEEx	30.9446	67.8892	68.2960	74.3186	70.4179	4.2573	0.7819	0.2271
TIR	33.6669	73.3338	73.7406	79.7632	75.8625	4.8228	0.8919	0.2601
IWIEx	47.5901	101.1802	101.5870	107.6096	103.7089	6.8063	1.2956	0.2474
AIW	37.8862	81.7724	82.1792	88.2018	84.3011	4.9893	0.8678	0.2163
LIEx	89.7887	183.5774	184.0218	186.3798	184.4739	4.1526	0.5305	0.1969
IEx	89.4392	180.8784	180.9440	183.0215	181.7213	17.5378	3.7685	0.4879

Table 4
Maximum likelihood parameters estimates for the second data set.

Model	a	b	α	β	θ	λ	γ	q
TLKMOIEx	24.7853	18126.7	0.1752	-	13.074	-	-	0.9786
EMIEEx	-	-	998.516	2.2749	-	-	1966.94	-
TIR	-	-	0.3486	-	126.045	-	271.928	-
IWIEx	-	-	0.1990	7.1618	16.0931	-	-	-
AIW	-	-	22.6256	9.2064	-	24673.3	-	-
LIEx	17.2519	1.6576	-	-	-	-	-	-
IEx	-	-	-	-	3.5004	-	-	-

Table 5
The metrics for the glass Alumina data set.

Model	$-\hat{l}$	AIC	CAIC	BIC	HQIC	AD	W*	KS
TLKMOIEx	12.6123	35.2246	37.7246	42.2306	37.4659	0.9481	0.1433	0.1510
EMIEEx	20.3752	46.7504	47.6735	50.9540	48.0952	1.3923	0.2172	0.1807
TIR	16.2199	38.4398	39.3629	42.6434	39.7846	1.1462	0.1709	0.1692
IWIEx	22.4816	50.9632	51.8863	55.1668	52.3080	1.6686	0.2639	0.1998
AIW	20.6126	47.2252	48.1483	51.4288	48.5700	1.3539	0.1977	0.1816
LIEx	48.5039	101.0078	101.4522	103.8102	101.9043	1.2756	0.1536	0.1821
IEx	68.0602	138.1204	138.2633	139.5216	138.5600	10.7104	2.3164	0.5787

To ascertain the most efficient model, we employed MATLAB and Mathematica software to calculate various statistical measures, including the Anderson-Darling (AD) statistic, Cramer-von Mises (W*), Kolmogorov-Smirnov (KS) statistic, and the Akaike Information Criterion (AIC), along with its adjusted version and the negative log-likelihood.

Dataset I: This data refers to [39] and describes the strength of 1.5 cm glass fibers: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

Dataset II: This data is Alumina (Al2O3) taken from [40] and is consisted of 30 observations: 2.68, 2.70, 2.77, 3.00, 3.12, 3.13, 3.20, 3.20, 3.28, 3.30, 3.40, 3.50, 3.51, 3.60, 3.70, 3.71, 3.73, 3.75, 3.80, 3.80, 3.90, 3.90, 3.95, 3.95, 4.00, 4.00, 4.00, 4.00, 4.00, 4.00.

Dataset III: The third data set comprises of hailing times and comes from the field of civil engineering. [41]: 3.200, 3.400, 3.500, 3.500, 3.600, 3.600, 3.900, 4.150, 4.300, 4.400, 4.400, 4.400, 4.540, 4.600, 4.700, 4.700, 4.730, 4.750, 4.790, 4.800, 4.800, 4.820, 4.900, 4.950, 5.100, 5.100, 5.150, 5.200, 5.200, 5.300, 5.400, 5.400, 5.400, 5.400, 5.400, 5.600, 5.600, 5.700, 5.700, 5.700, 5.800, 5.800, 5.800, 5.800, 5.900, 5.900, 5.900, 5.900, 5.900, 5.900, 5.900, 5.900, 5.900, 5.900, 6.000, 6.000, 6.000, 6.000, 6.000, 6.000, 6.000, 6.000, 6.200, 6.300, 6.400, 6.500, 6.500, 6.500, 6.600, 6.700, 6.800, 6.800, 6.900, 7.000, 7.000, 7.000, 7.100, 7.300, 7.400, 7.500, 7.900, 8.000, 8.200, 8.500, 8.600.

Tables 3, 5 and 7 show the comparison criteria obtained for the three datasets I, II and III respectively.

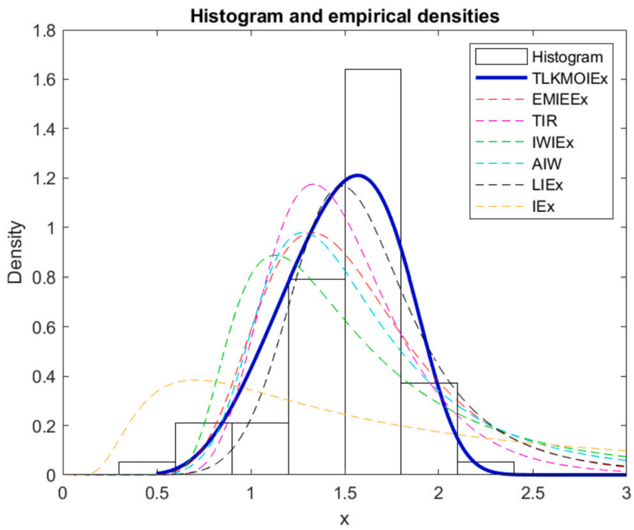


Fig. 7. Empirical pdfs (dataset I).

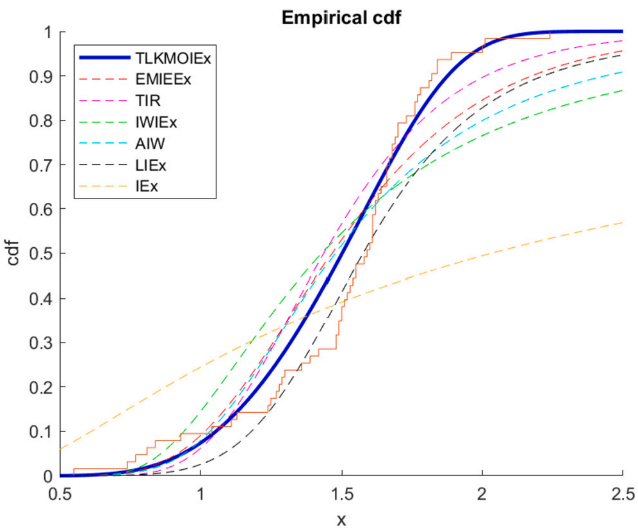


Fig. 8. Empirical CDFs (dataset I).

Table 6
Maximum likelihood parameters estimates for the third data set.

Model	a	b	α	β	θ	λ	γ	q
TLKMOIEx	23.4145	29.1664	1.4990	-	1.4889	-	-	0.3753
EMIEEx	-	-	65.0531	0.90607	-	-	102.3930	-
TIR	-	-	1.5238	-	10.2640	-	18.1467	-
IWIEx	-	-	0.0054	4.4055	918.9540	-	-	-
AIW	-	-	45.0628	5.6082	-	2804.7800	-	-
LIEx	10.2843	2.6568	-	-	-	-	-	-
IEx	-	-	-	-	5.4430	-	-	-

We see that the information criteria obtained for the TLKMOIEx model are less than the values observed for the six competing models EMIEEx, TIR, IWIEx, AIW, LIEx and IEx for the three datasets.

Fig. 7 and Fig. 8 present the pdf and the CDF for the dataset I. It shows that the TLKMOIEx model approaches the dataset I better than the EMIEEx, TIR, IWIEx, AIW, LIEx and IEx models.

Similarly, Fig. 9 and Fig. 10 present the pdf and the CDF for the dataset II. It reveals that the TLKMOIEx model approaches the dataset II better than the EMIEEx, TIR, IWIEx, AIW, LIEx and IEx models.

And similarly, Fig. 11 and Fig. 12 present the pdf and the CDF for the dataset III. It reveals that the TLKMOIEx model approaches the dataset III better than the EMIEEx, TIR, IWIEx, AIW, LIEx and IEx models.

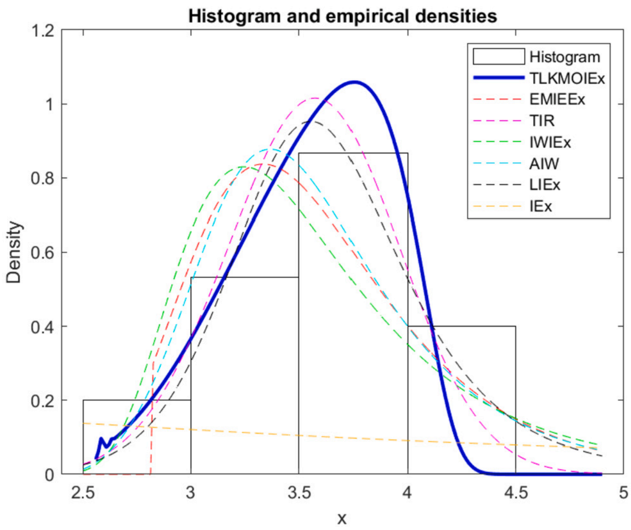


Fig. 9. Empirical pdfs (dataset II).

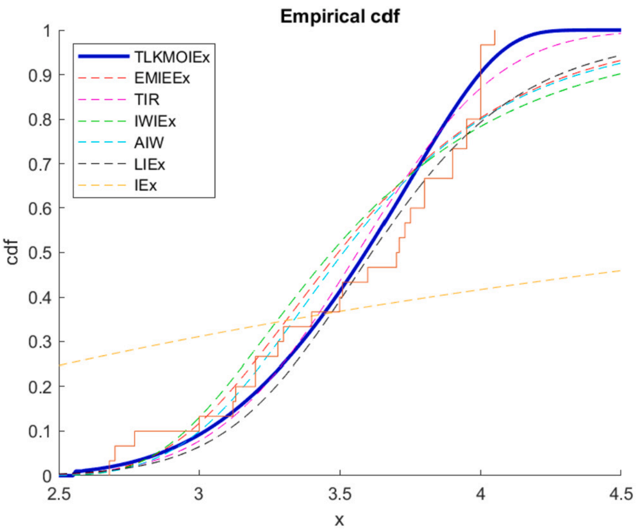


Fig. 10. Empirical CDFs for Dataset II.

Table 7
The metrics for the hailing times data set.

Model	$-\hat{l}$	AIC	CAIC	BIC	HQIC	AD	W*	KS
TLKMOIEx	133.4750	270.9500	271.0963	275.8353	272.9150	0.7428	0.1418	0.0995
EMIEEx	137.3370	278.6740	278.8203	283.5593	280.6390	1.4706	0.2543	0.1208
TIR	134.6910	273.3820	273.5283	278.2673	275.3470	0.9727	0.1673	0.1141
IWIEx	146.0150	296.0300	296.1763	300.9153	297.9950	3.0708	0.5074	0.1407
AIW	139.9020	283.8040	283.9503	288.6893	285.7690	1.7361	0.2611	0.1212
LIEx	219.9150	443.8300	443.9763	448.7153	445.7950	2.0942	0.2981	0.1592
IEx	232.9120	469.8240	469.9703	474.7093	471.7890	24.8972	5.2915	0.4690

Tables 2, 4 and 6 contain the values of the estimates. From the analysis of the tables (3, 5 and 7) and figures (Fig. 7, Fig. 8, Fig. 9, Fig. 10, Fig. 11 and Fig. 12), we deduce that the TLKMOIEx model is a better fit for the three datasets than the EMIEEx, TIR, IWIEx, AIW, LIEx and IEx models. For more reading about distributions and statistical inferences and modeling see [42–47].

6. Conclusion

In this study, a new family of distributions called TLKMO-G was developed. This family is nothing else than the combination of the distribution of Topp-Leone, the Kumaraswamy-G and the Marshall-Olkin families. By doing so, we obtain a family whose

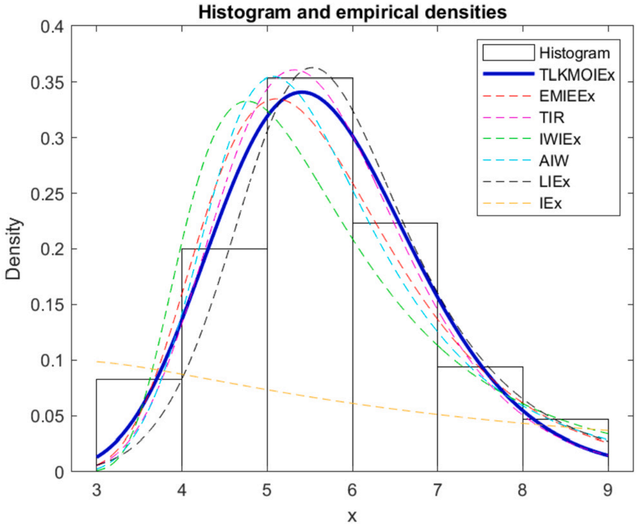


Fig. 11. Graphical representation of the empirical pdfs (dataset III).

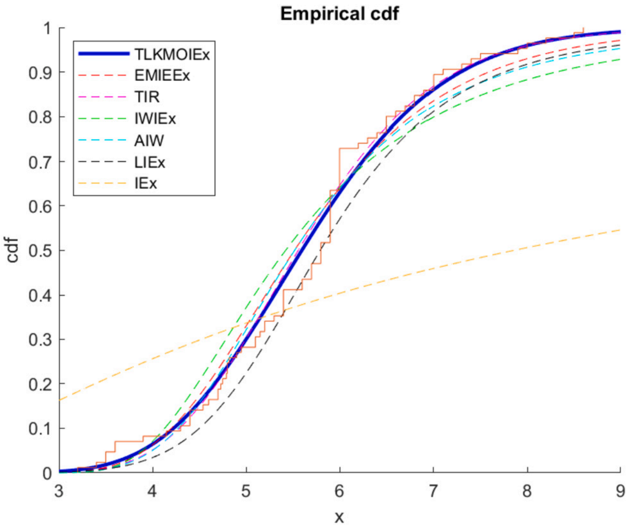


Fig. 12. Empirical CDFs for Dataset III.

resulting distributions contain at least five parameters, which led to much more flexibility, even when changing the set of parameters each time. We then performed a simulation study after taking the inverse exponential distribution as a special member and we find that the pdfs have much more normality which allows the application of the new TLKMOIEx model in many real-life domains. We derived the Shannon entropy and generalized moments and some other properties. Subsequently, we employed the MLE technique to estimate the parameters of the newly introduced TLKMOIEx model. Three practical datasets were used to study this new model. Six competing models were considered, but none of them outperformed the TLKMOIEx model in terms of comparison criteria and statistical tests.

Among the interesting perspectives, we aim to take a base distribution with at least two parameters in order to increase the number of parameters of the TLKMO-G family to investigate if we will get more flexibility.

7. Future work

Working on bivariate distribution opens the way to many applications that single variate distribution blocks. So in this context we are intending to extend our work on this distribution to make a bivariate distribution with many applications. These applications may be in medical fields as COVID-19 data or cancer data, or any other medical applications. Another application that appears on the horizon is engineering applications like accelerated life tests and their kinds, such as progressive or constant or the step stress life test.

CRediT authorship contribution statement

Mintodê Nicodème Atchadé: Data curation, Conceptualization. **Melchior A.G. N'bouké:** Data curation, Conceptualization. **Aliou Moussa Djibril:** Formal analysis, Data curation. **Aned Al Mutairi:** Writing – review & editing, Methodology, Investigation. **Manahil SidAhmed Mustafa:** Writing – review & editing, Writing – original draft, Visualization. **Eslam Hussam:** Data curation, Conceptualization. **Hassan Alsuhabi:** Resources, Project administration, Methodology. **Said G. Nassr:** Writing – review & editing, Writing – original draft, Visualization.

Declaration of competing interest

There is no conflict of interest regarding the submitted paper.

Data availability

All data underlying the results are available in the article with related references.

Acknowledgements

Princess Nourah Bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R368), Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia. This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445).

References

[1] A. Al-Shomrani, O. Arif, A. Shawky, S. Hanif, M.Q. Shahbaz, Topp-Leone family of distributions: some properties and application, Pak. J. Stat. Oper. Res. 12 (2016) 443–451.
[2] Z.M. Nofal, E. Altun, A.Z. Afify, M. Ahsanullah, The generalized Kumaraswamy-G family of distributions, J. Stat. Theory Appl. 18 (2019) 329.
[3] M.M. Yousef, E.M. Almetwally, Multi stress-strength reliability based on progressive first failure for Kumaraswamy model: bayesian and non-bayesian estimation, Symmetry 13 (2021) 2120.
[4] M. Alizadeh, M. Tahir, G.M. Cordeiro, M. Mansoor, M. Zubair, G. Hamedani, The Kumaraswamy Marshal-Olkin family of distributions, J. Egypt. Math. Soc. 23 (2015) 546–557.
[5] M.A. Khaleel, P.E. Oguntunde, J.N. Al, N.A. Ibrahim, M.H.A. Abujarad, The Marshall-Olkin Topp Leone-G family of distributions: a family for generalizing probability models, Sci. Afr. 8 (2020) e00470.
[6] H.M. Yousof, M. Alizadeh, S. Jahanshahi, T.G.R.I. Ghosh, G. Hamedani, The transmuted Topp-Leone G family of distributions: theory, characterizations and applications, J. Data Sci. 15 (2017) 723–740.
[7] A.H. Soliman, M.A.E. Elgarhy, M. Shakil, Type ii half logistic family of distributions with applications, Pak. J. Stat. Oper. Res. (2017) 245–264.
[8] H.M. Reyad, M. Alizadeh, F. Jamal, S. Othman, G.G. Hamedani, The exponentiated generalized Topp Leone G family of distributions: properties and applications, Pak. J. Stat. Oper. Res. (2019) 1–24.
[9] A.S. Hassan, M. Elgarhy, Z. Ahmad, Type ii generalized Topp-Leone family of distributions: properties and applications, J. Data Sci. 17 (2019).
[10] F. Chipepa, D. Wanduku, B. Oluyede, Half-logistic odd Weibull-Topp-Leone-G family of distributions: model, properties and applications, Afr. Stat. 15 (2020) 2481–2510.
[11] R.A. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, Truncated inverted Kumaraswamy generated family of distributions with applications, Entropy 21 (2019) 1089.
[12] S. Ibrahim, S.I. Doguwa, A. Isah, J.M. Haruna, The Topp Leone Kumaraswamy-G family of distributions with applications to cancer disease data, 2020.
[13] M. Alizadeh, H.M. Yousof, M. Rasekhi, E. Altun, The odd log-logistic Poisson-G family of distributions, J. Math. Ext. 12 (2018) 81–104.
[14] M.H. Tahir, M.A. Hussain, G.M. Cordeiro, M. El-Morshedy, M.S. Eliwa, A new Kumaraswamy generalized family of distributions with properties, applications, and bivariate extension, Mathematics 8 (2020) 1989.
[15] R.A. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, Type ii power Topp-Leone generated family of distributions with statistical inference and applications, Symmetry 12 (2020) 75.
[16] H. Reyad, M.Ç. Korkmaz, A.Z. Afify, G. Hamedani, S. Othman, The Fréchet Topp Leone-G family of distributions: properties, characterizations and applications, Ann. Data Sci. 8 (2021) 345–366.
[17] B. Oluyede, S. Chamunorwa, F. Chipepa, M. Alizadeh, The Topp-Leone Gompertz-G family of distributions with applications, J. Stat. Manag. Syst. (2022) 1–25.
[18] H. Reyad, M. Alizadeh, F. Jamal, S. Othman, The Topp Leone odd Lindley-G family of distributions: properties and applications, J. Stat. Manag. Syst. 21 (2018) 1273–1297.
[19] M. Elgarhy, M. Arslan Nasir, F. Jamal, G. Ozel, The type ii Topp-Leone generated family of distributions: properties and applications, J. Stat. Manag. Syst. 21 (2018) 1529–1551.
[20] S.A. Alyami, I. Elbatal, N. Alotaibi, E.M. Almetwally, H.M. Okasha, M. Elgarhy, Topp–Leone modified Weibull model: theory and applications to medical and engineering data, Appl. Sci. 12 (2022) 10431.
[21] R.A. Mohamed, A.A. Al-Babtain, I. Elbatal, E.M. Almetwally, H.M. Almongy, Classical and bayesian inference of Marshall-Olkin extended Gompertz Makeham model with modeling of physics data, J. Math. 2022 (2022).
[22] E.M. Almetwally, Marshall Olkin alpha power extended Weibull distribution: different methods of estimation based on type i and type ii censoring, Gazi Univ. J. Sci. 35 (2021) 293–312.
[23] A.M. Basheer, E.M. Almetwally, H.M. Okasha, Marshall-Olkin alpha power inverse Weibull distribution: non bayesian and bayesian estimations, J. Stat. Appl. Probab. 10 (2021) 327–345.
[24] H.M. Almongy, E.M. Almetwally, A.E. Mubarak, Marshall–Olkin alpha power Lomax distribution: estimation methods, applications on physics and economics, Pak. J. Stat. Oper. Res. 17 (2021) 137–153.
[25] M. Mahdizadeh, E. Zamanzade, New insights on goodness-of-fit tests for ranked set samples, Stat. Pap. (2022) 1–23.
[26] M. Tamandi, A. Jamalizadeh, M. Mahdizadeh, A generalized Birnbaum-Saunders distribution with application to the air pollution data, Electron. J. Appl. Stat. Anal. 12 (2019) 26–43.
[27] M.N. Atchadé, M. N'bouké, A.M. Djibril, S. Shahzadi, E. Hussam, R. Aldallal, H.M. Alshanbari, A.M. Gemeay, A.-A.H. El-Bagoury, A new power Topp–Leone distribution with applications to engineering and industry data, PLoS ONE 18 (2023) e0278225.

[28] M. Mahdizadeh, E. Zamanzade, Efficient reliability estimation in two-parameter exponential distributions, Filomat 32 (2018) 1455–1463.

[29] S. MirMostafaee, M. Mahdizadeh, A.J. Lemonte, The Marshall–Olkin extended generalized Rayleigh distribution: properties and applications, Commun. Stat., Theory Methods 46 (2017) 653–671.

[30] E. Zamanzade, M. Mahdizadeh, Goodness of fit tests for Rayleigh distribution based on phi-divergence, Rev. Colomb. Estad. 40 (2017) 279–290.

[31] S. MirMostafaee, M. Mahdizadeh, M. Aminzadeh, Bayesian inference for the Topp–Leone distribution based on lower k-record values, Jpn. J. Ind. Appl. Math. 33 (2016) 637–669.

[32] M. Mahdizadeh, E. Zamanzade, A comprehensive study of lognormality tests, Electron. J. Appl. Stat. Anal. 10 (2017) 349–373.

[33] D. Zwillinger, A. Jeffrey, Table of Integrals, Series, and Products, Elsevier, 2007.

[34] R.A. Bantan, C. Chesneau, F. Jamal, M. Elgarhy, On the analysis of new covid-19 cases in Pakistan using an exponentiated version of the m family of distributions, Mathematics 8 (2020) 953.

[35] P.E. Oguntunde, A.O. Adejumo, M.A. Khaleel, H.I. Okagbue, O.A. Odetunmibi, The logistic inverse exponential distribution: basic structural properties and application, in: Proceedings of the World Congress on Engineering and Computer Science, vol. 2, 2018.

[36] D.A. Ramadan, W. Magdy, On the alpha-power inverse Weibull distribution, Int. J. Comput. Appl. 181 (2018) 6–12.

[37] M.A. Aldahlan, The inverse Weibull inverse exponential distribution with application, Int. J. Contemp. Math. Sci. 14 (2019) 17–30.

[38] N. Yahia, H. Mohammed, The type ii Topp-Leone generalized inverse Rayleigh distribution, Int. J. Contemp. Math. Sci. 14 (2019) 113–122.

[39] F. Chipepa, B. Oluyede, B. Makubate, et al., The Topp-Leone-Marshall-Olkin-G family of distributions with applications, Int. J. Stat. Probab. 9 (2020) 15–32.

[40] S. Ibrahim, S.I. Doguwa, A. Isah, M.J. Haruna, Some properties and applications of Topp Leone Kumaraswamy Lomax distribution, J. Stat. Model. Anal. 3 (2021).

[41] M.A. Khaleel, P.E. Oguntunde, J.N. Al Abbasi, N.A. Ibrahim, M.H. AbuJarad, La famille de distributions Marshall-Olkin Topp Leone-G: une famille pour généraliser les modèles de probabilité, Sci. Afr. 8 (2020) e00470.

[42] M. Siddiqa, A. Zubair, A. Kamal, M. Ijaz, T. Abushal, Prevalence and associated factors of stunting, wasting and underweight of children below five using quintile regression analysis (pdhs 2017–2018), Sci. Rep. 12 (2022) 1–8.

[43] A. Ahmad, N. Alsadat, M.N. Atchade, S.Q. ul Ain, A.M. Gemeay, M.A. Meraou, E.M. Almetwally, M.M. Hossain, E. Hussam, New hyperbolic sine-generator with an example of Rayleigh distribution: simulation and data analysis in industry, Alex. Eng. J. 73 (2023) 415–426.

[44] T.A. Abushal, J. Kumar, A.H. Muse, A.H. Tolba, Estimation for Akshaya failure model with competing risks under progressive censoring scheme with analyzing of thymic lymphoma of mice application, Complexity (2022) 2022.

[45] M. Amin, S. Afzal, M.N. Akram, A.H. Muse, A.H. Tolba, T.A. Abushal, Outlier detection in gamma regression using Pearson residuals: simulation and an application, AIMS Math. 7 (2022) 15331–15347.

[46] M. Kilai, G.A. Waititu, W.A. Kibira, M. Abd El-Raouf, T.A. Abushal, A new versatile modification of the Rayleigh distribution for modeling covid-19 mortality rates, Results Phys. 35 (2022) 105260.

[47] O. Agossou, M.N. Atchadé, A.M. Djibril, Modeling the effects of preventive measures and vaccination on the covid-19 spread in Benin Republic with optimal control, Results Phys. 31 (2021) 104969.