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A simple computational algorithm with inertial extrapolation for generalized split common fixed point problems

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ABSTRACT

In the present paper, we explore operator norm independent inertial type accelerated iterative algorithm solving generalized split common fixed point problem, which is the problem of finding a point that belongs to the intersection of a finite family of fixed point sets of demimetric mappings such that its image under a finite number of linear transformations belongs to the intersection of another finite family of fixed point sets of demimetric mappings in the image space. We adopt rules for selecting the step size such that the implementation of our proposed algorithm does not need any prior information about the operator norms. The strong convergence result is analyzed and some applications of our proposed algorithm are demonstrated. Our result in this paper will improve and generalize many results in the literature. Numerical experiments show that our iteration method is very effective for approximating the solution point of problem under consideration.

1. Introduction

Censor and Segal [1] introduced the following split common fixed-point problem (SCFP):

$$\bar{x} \in F(U) \text{ such that } A\bar{x} \in F(T), \tag{1}$$

where $A : H_1 \to H_2$ is nonzero bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are directed operators, H_1 and H_2 are real Hilbert spaces, and $F(U)$ and $F(T)$ stand for the fixed point sets T and U, respectively. In particular, if T and U are projection operators, then the SCFP is reduced to the well-known split feasibility problem [2, 3], which is the problem of finding $\bar{x} \in C$ such that $A\bar{x} \in Q$, where C and Q are nonempty closed convex subsets in H_1 and H_2, respectively. Censor and Segal [1] took the following iterative scheme, in finite dimensional spaces, solving SCFP (1):

$$x_{n+1} = U(x_n - \gamma_n A^*(I - T)Ax_n), \tag{2}$$

where $\gamma_n = \gamma \in (0, \frac{2}{\lambda})$ and λ is the largest eigenvalue of the matrix $A^t A$. Subsequently, after the work of Censor and Segal [1], this result was extended to different class of operators, see for example [4, 5, 6, 7, 8].

For a real Hilbert space H, the self-mapping $T : H \to H$ is called κ-demimetric [9] if $F(T) \neq \emptyset$ and there is $\kappa \in (-\infty, 1)$ such that

$$\langle x - \bar{x}, x - Tx \rangle \geq \frac{1-\kappa}{2} \|x - Tx\|^2, \quad \forall (x, \bar{x}) \in H \times F(T). \tag{3}$$

The class of κ-demimetric mappings in Hilbert space contains the classes of κ-strict pseudocontractions mappings, firmly nonexpansive mappings, nonexpansive mappings, 2-generalized hybrid mappings [10], firmly quasi-nonexpansive mappings, quasi-nonexpansive mappings, demicontractive mappings and directed mappings, see [9, 11, 12]. Moreover, several well known types of mapping arising in optimization belong to the class of demimetric mappings, see for example [9, 13] and references therein.

Generalized Split Inverse Problem (GSIP) [14] is formulated as a problem of

$$\begin{cases} \text{find } x^* \in X \text{ that solves IP1} \\ \text{such that} \\ A_k(x^*) = y_k^* \in Y \text{ and } y_k^* \text{ solves IP2}, \quad \forall k \in \Lambda \end{cases} \tag{4}$$

where $\Lambda \subset \mathbb{R}$ is an index set, IP1 and IP2 are two inverse problems installed in space X and Y, respectively, $A_k : X \to Y$ for each $k \in \Lambda$. GSIP will be reduced to Split Inverse Problem (SIP) [15] if $A = A_k$ for all $k \in \Lambda$. Many models of inverse problems can be cast in this framework by choosing different inverse problems for IP1 and IP2. There is a considerable investigation in the framework of SIP, see for example [4, 16, 17, 18] and the many references therein. In this paper, we consider the type of GSIP called the *generalized split common fixed point problem* (in short, GSCFP) [14], formulated as finding

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$$\bar{x} \in \bigcap_{i=1}^{N} F(U_i) \text{ such that } A_k(\bar{x}) \in \bigcap_{j=1}^{M} F(T_j), \quad \forall k \in \{1, \dots, R\}, \tag{5}$$

where $A_k : H_1 \to H_2$ is a linear operator for all $k \in \{1, \dots, R\}$ ($R \in \mathbb{N}$), $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$ are nonlinear operators for all $i \in \{1, \dots, N\}$ ($N \in \mathbb{N}$), $j \in \{1, \dots, M\}$ ($M \in \mathbb{N}$), H_1 and H_2 are two real Hilbert spaces. If $A_k = A$ for all $k \in \{1, \dots, R\}$, GSCFP (5) will be reduced to the problem of finding

$$\bar{x} \in \bigcap_{i=1}^{N} F(U_i) \text{ such that } A(\bar{x}) \in \bigcap_{j=1}^{M} F(T_j). \tag{6}$$

Moreover, if $A_k = A$ for all $k \in \{1, \dots, R\}$, $U = U_i$ for all $i \in \{1, \dots, N\}$ and $T = T_j$ for all $j \in \{1, \dots, M\}$, GSCFP (5) will be reduced to SCFP (1).

Some authors proposed methods solving (6) for a different class of mappings; for example, for directed operators [1, 19], demicontractive mappings [20], asymptotically quasi-nonexpansive mappings [21], quasi-nonexpansive operators [5, 22]. However, the implementation of the proposed algorithm requires the estimate of the operator norm $\|A\|$, and operator norm is global invariant and is often difficult to estimate, see Theorem of Hendrickx and Olshevsky in [23]. The turning point of avoiding the estimate of $\|A\|$ came when López et al. [24] introduced a variable step size that does not depend on the operator norm $\|A\|$ solving SFP (SCFP (1) where $U = P_C$ and $T = P_Q$). Initialed by López et al. [24], some authors modified (2) and established a modified step size for SCFP (1) in general so that its choice does not need any priori knowledge of $\|A\|$, see for example in [25, 26, 27, 28].

The purpose of this paper is twofold. Firstly, to present a computationally simple algorithm to approximate the solution point of GSCFP (5) for demimetric mappings U_i and T_j. To be precise, the algorithm is formulated in a parallel computing platform and reduced to a simpler structure. Moreover, we constructed an extended variable step-sizes generated by the algorithms at each iteration, based on previously evaluated iterations so that the implementation of our algorithm does not need any prior information about the operator norms $\|A_k\|$ ($k \in \{1, \dots, R\}$). Note that iterative algorithms with a step size that does not require any prior knowledge of the operator norm are more desirable and efficient in practice. Secondly, to present inertial-type algorithm to solve the GSCFP (5) for demimetric mappings U_i and T_j, and provide the strong convergence theorem for the proposed algorithm. An inertial-type algorithm is algorithm incorporating inertial extrapolation term $\alpha_n(x_n - x_{n-1})$ by making use of the previous two iterates x_n and x_{n-1}. The inertial extrapolation term $\alpha_n(x_n - x_{n-1})$ is employed in algorithm for the purpose of speeding up the rate of convergence of the algorithm. The vector $(x_n - x_{n-1})$ is acting as an impulsion term and α_n is acting as a speed regulator, see [29]. Many inertial type accelerated iterative methods are proposed for different kind of problems, see for example [22, 30, 31] and references therein.

The outline of this paper is as follows. In Section 2, we present some necessary definitions, interesting properties and results. The main result of the paper is contained in Section 3 and Section 4, where in Section 3, we introduce our proposed algorithm and discuss its structure, and in Section 4, the strong convergence analysis of our proposed method is investigated. In Section 5, we give some applications that follow from our main result. Finally, in Section 6, we end the paper with numerical results about the new iteration method.

2. Preliminary

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Then, the metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Given $x \in H$ and $z \in C$, then $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \le 0, \quad \forall y \in C$.

Definition 1. The mapping $T : H \to H$ is called

(a) firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H;$$

(b) κ-strict pseudocontraction [32] if there exists a $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa||(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H;$$

(c) directed [1] if $F(T) \neq \emptyset$ and

$$\|Tx - \bar{x}\|^2 \le \|x - \bar{x}\|^2 - ||x - Tx\|^2, \quad \forall (x, \bar{x}) \in H \times F(T).$$

It is clear that (3) is equivalent to the following:

$$\|Tx - \bar{x}\|^2 \le \|x - \bar{x}\|^2 + \kappa\|x - Tx\|^2, \quad \forall (x, \bar{x}) \in H \times F(T).$$

Definition 2. Let $T : H \to H$ be a mapping, $I - T$ is called demiclosed at 0 if for a sequence $\{x_n\}$ in H such that $x_n \rightharpoonup \bar{x}$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $T\bar{x} = \bar{x}$ holds.

Lemma 1. *[9] The set of all fixed points $F(T)$ of κ-demimetric mapping $T : H \to H$ is closed and convex.*

Lemma 2. *[33] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_t}\}_{t\ge1}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_t} < \Gamma_{n_t+1}$ for all $t \ge 1$. Also consider the sequence of integers $\{\varphi(n)\}_{n\ge n_0}$ defined by $\varphi(n) = \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}$. Then $\{\varphi(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \varphi(n) = \infty$, and for all $n \ge n_0$, the following estimates hold: $\Gamma_{\varphi(n)} \le \Gamma_{\varphi(n)+1}$ and $\Gamma_n \le \Gamma_{\varphi(n)+1}$.*

Lemma 3. *[34] Let $\{c_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers, $\{\beta_n\}$ be sequences of real numbers such that*

$$c_{n+1} \le (1 - \alpha_n)c_n + \beta_n + \gamma_n, \quad n \ge 1,$$

where $0 < \alpha_n < 1$ and $\sum \gamma_n < \infty$.

(i) *If $\beta_n \le \alpha_n M$ for some $M \ge 0$, then $\{c_n\}$ is a bounded sequence.*
(ii) *If $\sum \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \le 0$, then $c_n \to 0$ as $n \to \infty$.*

3. Proposed method

Lemma 4. *Suppose $U_i : H_1 \to H_1$ is η_i-demimetric mapping for all $i \in \{1, \dots, N\}$ and $T_j : H_2 \to H_2$ is β_j-demimetric mapping for all $j \in \{1, \dots, M\}$. For $x \in H_1$, define $i_x \in \arg\max\{\|(I - U_i)x\| : i \in \{1, \dots, N\}\}$ and*

$$(j_x, k_x) \in \arg\max\{\|(I - T_j)A_k x\| : (j, k) \in \{1, \dots, M\} \times \{1, \dots, R\}\}.$$

Then, $\|A^*_{k_x}(I - T_{j_x})A_{k_x}x + (I - U_{i_x})x\| = 0$ iff x solves GSCFP (5).

Proof. Suppose $\|A^*_{k_x}(I - T_{j_x})A_{k_x}x + (I - U_{i_x})x\| = 0$. For $p \in \Gamma$, we have

$$\begin{aligned}
0 &= \|A^*_{k_x}(I - T_{j_x})A_{k_x}x + (I - U_{i_x})x\|\|x - p\| \\
&\ge \langle A^*_{k_x}(I - T_{j_x})A_{k_x}x + (I - U_{i_x})x, x - p\rangle \\
&= \langle A^*_{k_x}(I - T_{j_x})A_{k_n}x, x - p\rangle + \langle (I - U_{i_x})x, x - p\rangle \\
&= \langle (I - T_{j_x})A_{k_x}x, A_{k_x}x - A_{k_x}p\rangle + \langle (I - U_{i_x})x, x - p\rangle \\
&\ge \frac{1 - \eta_{j_x}}{2}\|(I - T_{j_x})A_{k_x}x\|^2 + \frac{1 - \beta_{i_x}}{2}\|(I - U_{i_x})x\|^2.
\end{aligned}$$

This implies $\|(I - T_{j_x})A_{k_x}x\| = \|(I - U_{i_x})x\| = 0$. Using the definition of i_x and (j_x, k_x), we have $\|(I - T_j)A_k x\| = \|(I - U_i)x\| = 0$ for all $(i, j, k) \in \{1, \dots, N\} \times \{1, \dots, M\} \times \{1, \dots, R\}$. Therefore, x solves GSCFP (5).

The converse is straightforward. □

We now establish a self-adaptive iterative method with inertial term for solving the GSCFP (5) by making use of the following three assumptions:

(A1) The mapping U_i is η_i-demimetric and $I-U_i$ is demiclosed at 0 for all $i \in \{1,\dots,N\}$;
(A2) The mapping T_j is β_j-demimetric and $I-T_j$ is demiclosed at 0 for all $j \in \{1,\dots,M\}$;
(A3) The mapping A_k is nonzero bounded linear operator and A_k^* denotes the adjoint of A_k for $k \in \{1,\dots,R\}$;
(A4) Γ denotes the solution set of the GSCFP (5) and Γ is nonempty.

Algorithm 1 Choose $u, x_0, x_1 \in H_1$. Let $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ be a non-negative real sequences.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.
STEP 2. Compute $t_n = (I-U_{i_n})z_n$ and $y_n = (I-T_{j_n})A_{k_n}(z_n)$, where

$$i_n \in \arg\max\{\|(I-U_i)z_n\| : i \in \{1,\dots,N\}\} \quad \text{and}$$

$$(j_n,k_n) \in \arg\max\{\|(I-T_j)A_k(z_n)\| : (j,k) \in \{1,\dots,M\}\times\{1,\dots,R\}\}.$$

If $\|A_{k_n}^*(y_n)+t_n\| = 0$, Stop. Otherwise, go to **STEP 3**.
STEP 3. Evaluate

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(z_n - \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\|A_{k_n}^*(y_n)+t_n\|^2}\big(A_{k_n}^*(y_n)+t_n\big)\Big). \tag{7}$$

STEP 4. Set $n := n+1$ and go to **STEP 1**.

Lemma 5. *The stopping condition (in STEP 2) of Algorithm 1 is satisfied ($\|A_{k_n}^*(y_n)+t_n\| = 0$ for some $n \in \mathbb{N}$) iff z_n solves GSCFP (5).*

Proof. Straightforward using Lemma 4. □

In the next section, we analyze the strong convergence of the infinite sequence $\{x_n\}$ generated by Algorithm 1 to the solution point of GSCFP (5) assuming that Algorithm 1 does not terminate (the stopping condition (in STEP 2) is not satisfied, i.e., $\mu(z_n) = \|A_{k_n}^* y_n + t_n\| \neq 0$ for all $n \in \mathbb{N}$), or by ignoring the stopping condition (in STEP 2) and setting x_{n+1} in (7) (STEP 3) by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(z_n - \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big)\Big),$$

where $\mu(z_n) = \|A_{k_n}^*(y_n)+t_n\|^2$ if $\|A_{k_n}^*(y_n)+t_n\| \neq 0$, $\mu(z_n) = 1$ otherwise.

4. Convergence analysis

Here in this section, we analysis the convergence of our proposed iterative method, Algorithm 1, for solving GSCFP (5).

Theorem 1. *If the real parameters $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ in Algorithm 1 satisfy the following conditions:*

(C1) $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 \le \theta_n \le \theta < 1$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|$;
(C3) $0 < \rho_n < \delta = \min\{1-\eta_1,\dots,1-\eta_N,1-\beta_1,\dots,1-\beta_M\}$ *and* $\liminf_{n\to\infty} \rho_n(\delta-\rho_n) > 0$;

then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\bar{x} \in \Gamma$ where $\bar{x} = P_\Gamma(u)$.

Proof. Let $s_n = z_n - \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big)$ and let $\bar{x} = P_\Gamma(u)$.

Now, using the definition of s_n, we have

$$\begin{aligned}
\|s_n - \bar{x}\|^2 &= \Big\|z_n - \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big) - \bar{x}\Big\|^2 \\
&\le \|z_n - \bar{x}\|^2 + \Big\|\rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big)\Big\|^2 \\
&\quad - 2\Big\langle \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big), z_n - \bar{x}\Big\rangle.
\end{aligned} \tag{8}$$

The result (8) in view of

$$\begin{aligned}
\Big\|\rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big)\Big\|^2 &= \Big(\rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\Big)^2 \|A_{k_n}^*(y_n)+t_n\|^2 \\
&\le \rho_n^2 \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\langle \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n)+t_n\big), z_n - \bar{x}\Big\rangle \\
&= \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big\langle A_{k_n}^*(y_n)+t_n, z_n - \bar{x}\big\rangle \\
&= \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(\langle y_n, A_{k_n}x_n - A_{k_n}\bar{x}\rangle + \langle t_n, x_n - \bar{x}\rangle\big) \\
&\ge \rho_n \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\Big(\frac{1-\beta_{j_n}}{2}\|y_n\|^2 + \frac{1-\eta_{i_n}}{2}\|t_n\|^2\Big) \\
&= \rho_n \frac{\delta}{2} \frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(\|y_n\|^2+\|t_n\|^2\big) \\
&= \rho_n \frac{\delta}{2} \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)},
\end{aligned}$$

gives

$$\begin{aligned}
\|s_n - \bar{x}\|^2 &\le \|z_n - \bar{x}\|^2 + \rho_n^2 \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)} - \rho_n\delta \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)} \\
&\le \|z_n - \bar{x}\|^2 + \rho_n(\rho_n - \delta)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}.
\end{aligned} \tag{9}$$

From condition (C3) and (9), we have that

$$\|s_n - \bar{x}\| \le \|z_n - \bar{x}\|. \tag{10}$$

By (10) and the definition of x_{n+1}, we get

$$\begin{aligned}
\|x_{n+1} - \bar{x}\| &= \|(1-\alpha_n)s_n + \alpha_n u - \bar{x}\| \\
&= \|(1-\alpha_n)(s_n - \bar{x}) + \alpha_n(u - \bar{x})\| \\
&\le (1-\alpha_n)\|s_n - \bar{x}\| + \alpha_n\|u - \bar{x}\| \\
&\le (1-\alpha_n)\|z_n - \bar{x}\| + \alpha_n\|u - \bar{x}\| \\
&= (1-\alpha_n)\|x_n + \beta_n(x_n - x_{n-1}) - \bar{x}\| + \alpha_n\|u - \bar{x}\| \\
&\le (1-\alpha_n)\|x_n - \bar{x}\| + \beta_n(1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n\|u - \bar{x}\| \\
&= (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n\Big((1-\alpha_n)\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|u - \bar{x}\|\Big) \\
&\le (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n v_n,
\end{aligned} \tag{11}$$

where $v_n = (1-\alpha_n)\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|u - \bar{x}\|$. From condition (C1) and (C2), we see that $\lim_{n\to\infty} v_n = \|u - \bar{x}\|$. Hence, $\{v_n\}$ is bounded. Let $d > 0$ such that $v_n \le d$ for all $n \in \mathbb{N}$. Then, (11) becomes

$$\|x_{n+1} - \bar{x}\| \le (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n d.$$

Thus, by Lemma 3 (i) the sequence $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, there exists $L > 0$ such that $\|x_n - \bar{x}\| \le L$ for all $n \in \mathbb{N}$. Now using the definition of x_{n+1}, z_n and (9), we get

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &= \|(1-\alpha_n)(s_n-\bar{x})+\alpha_n(u-\bar{x})\|^2\\ &\le (1-\alpha_n)\|s_n-\bar{x}\|^2+\alpha_n\|u-\bar{x}\|^2\\ &\le (1-\alpha_n)\|z_n-\bar{x}\|^2+\alpha_n\|u-\bar{x}\|^2\\ &\quad +(1-\alpha_n)\rho_n(\rho_n-\delta)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}\\ &=(1-\alpha_n)\|x_n+\theta_n(x_n-x_{n-1})-\bar{x}\|^2+\alpha_n\|u-\bar{x}\|^2\\ &\quad +(1-\alpha_n)\rho_n(\rho_n-\delta)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}\\ &\le (1-\alpha_n)\big(\|x_n-\bar{x}\|+\theta_n\|x_n-x_{n-1}\|\big)^2+\alpha_n\|u-\bar{x}\|^2\\ &\quad +(1-\alpha_n)\rho_n(\rho_n-\delta)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}\\ &\le \|x_n-\bar{x}\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n L\|x_n-x_{n-1}\|+\alpha_n\|u-\bar{x}\|^2\\ &\quad +(1-\alpha_n)\rho_n(\rho_n-\delta)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}. \end{aligned} \tag{12}$$

Let us distinguish the following two cases related to the behavior of the sequence $\{\Gamma_n\}$ where $\Gamma_n=\|x_n-\bar{x}\|^2$.

Case 1. Suppose the sequence $\{\Gamma_n\}$ decrease at infinity. Thus, there exists $n_0\in\mathbb{N}$ such that $\Gamma_{n+1}\le\Gamma_n$ for $n\ge n_0$. Then, $\{\Gamma_n\}$ converges and $\Gamma_n-\Gamma_{n+1}\to 0$ as $n\to 0$.

From (12), we have

$$\begin{aligned}(1-\alpha_n)\rho_n(\delta-\rho_n)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)} &\le (\Gamma_n-\Gamma_{n+1})+\alpha_n\|u-\bar{x}\|^2\\ &\quad +\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n L\|x_n-x_{n-1}\|.\end{aligned} \tag{13}$$

Since $\Gamma_n-\Gamma_{n+1}\to 0$ and using (C1) and (C2), we have from (13) that

$$(1-\alpha_n)\rho_n(\delta-\rho_n)\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}\to 0,\quad n\to\infty. \tag{14}$$

The conditions (C1) and (C3) together with (14) yields

$$\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}\to 0,\quad n\to\infty. \tag{15}$$

Now, if $\|A^*_{k_n}(y_n)+t_n\|\neq 0$, we have

$$\begin{aligned}\frac{\|y_n\|^2+\|t_n\|^2}{\max\{\|A_{k_n}\|^2,1\}} &= \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\max\{\|A_{k_n}\|^2,1\}(\|y_n\|^2+\|t_n\|^2)}\\ &\le \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\|A_{k_n}\|^2\|y_n\|^2+\|t_n\|^2}\\ &= \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\|A^*_{k_n}\|^2\|y_n\|^2+\|t_n\|^2}\\ &\le \frac{(\|y_n\|^2+\|t_n\|^2)^2}{\|A^*_{k_n}(y_n)\|^2+\|t_n\|^2}\\ &\le \frac{2(\|y_n\|^2+\|t_n\|^2)^2}{\|A^*_{k_n}(y_n)+t_n\|^2}=\frac{2(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}.\end{aligned} \tag{16}$$

Moreover, if $\|A^*_{k_n}(y_n)+t_n\|=0$, then $\|y_n\|^2+\|t_n\|^2=0$ and thus

$$\frac{\|y_n\|^2+\|t_n\|^2}{\max\{\|A_{k_n}\|^2,1\}}=0=\frac{0}{\mu(z_n)}=\frac{2(\|y_n\|^2+\|t_n\|^2)}{\mu(z_n)}. \tag{17}$$

Hence, by (16) and (17), for all $n\in\mathbb{N}$ we get

$$\frac{\|y_n\|^2+\|t_n\|^2}{\max\{\|A_{k_n}\|^2,1\}}\le 2\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)},$$

and so this together with (15) gives

$$\lim_{n\to\infty}(\|y_n\|^2+\|t_n\|^2)=0\Leftrightarrow \lim_{n\to\infty}\|y_n\|=\lim_{n\to\infty}\|t_n\|=0.$$

By definition of t_n and y_n, we have $\|(I-U_i)z_n\|\le\|(I-U_{i_n})z_n\|=\|t_n\|$ for all $j\in\{1,\dots,M\}$, and $\|(I-T_j)A_kz_n\|\le\|((I-T_{j_n})A_{k_n}z_n\|=\|y_n\|$ for all $(j,k)\in\{1,\dots,M\}\times\{1,\dots,R\}$. Therefore,

$$\lim_{n\to\infty}\|(I-T_j)A_kz_n\|=\lim_{n\to\infty}\|(I-U_i)z_n\|=0, \tag{18}$$

for all $(i,j,k)\in\{1,\dots,N\}\times\{1,\dots,M\}\times\{1,\dots,R\}$. Now, using the definition of s_n, we obtain

$$\begin{aligned}\|s_n-z_n\|^2 &= \left\|\rho_n\frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A^*_{k_n}(y_n)+t_n\big)\right\|^2\\ &\le \rho_n^2\Big(\frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\Big)^2\left\|A^*_{k_n}(y_n)+t_n\right\|^2\\ &\le \delta^2\frac{(\|y_n\|^2+\|t_n\|^2)^2}{\mu(z_n)}.\end{aligned} \tag{19}$$

Thus, (19) together with (15) gives

$$\|s_n-z_n\|\to 0,\quad n\to\infty. \tag{20}$$

Moreover, using the definition of z_n and (C2), we have

$$\|x_n-z_n\|=\|x_n-x_n-\beta_n(x_n-x_{n-1})\|=\beta_n\|x_n-x_{n-1}\|\to 0,\quad n\to\infty. \tag{21}$$

By (20) and (21), we get

$$\|s_n-x_n\|\le\|x_n-z_n\|+\|s_n-z_n\|\to 0,\quad n\to\infty. \tag{22}$$

Using (22), definition of x_{n+1}, (C1) and noting that $\{x_n\}$ is bounded, we have

$$\begin{aligned}\|x_{n+1}-s_n\| &= \|\alpha_nu-\alpha_ns_n\|\\ &= \|(1-\alpha_n)(s_n-x_n)+\alpha_n(u-x_n)+(x_n-s_n)\|\\ &\le (1-\alpha_n)\|s_n-x_n\|+\alpha_n\|u-x_n\|+\|s_n-x_n\|\\ &\le 2\|s_n-x_n\|+\alpha_n\|u-x_n\|\to 0,\quad n\to\infty.\end{aligned} \tag{23}$$

Results from (22) and (23) give

$$\|x_{n+1}-x_n\|\le\|x_{n+1}-s_n\|+\|s_n-x_n\|\to 0,\quad n\to\infty. \tag{24}$$

Let p be a weak cluster point of $\{x_n\}$, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l}\rightharpoonup p$ as $l\to\infty$. Using (21), we have $z_{n_l}\rightharpoonup p$ as $l\to\infty$, and hence $A_kz_{n_l}\rightharpoonup A_kp$ as $l\to\infty$ for all $k\in\{1,\dots,R\}$. Hence, by demiclosedness assumptions of $I-U_i$ and $I-T_j$ together with (18), we obtain $p\in\Gamma$.

Next, we show that $\limsup_{n\to\infty}\langle u-\bar{x},x_n-\bar{x}\rangle\le 0$. Indeed, since $\bar{x}=P_\Gamma(u)$ and $p\in\Gamma$, we obtain that

$$\limsup_{n\to\infty}\langle u-\bar{x},x_n-\bar{x}\rangle=\lim_{l\to\infty}\langle u-\bar{x},x_{n_l}-\bar{x}\rangle=\langle u-\bar{x},p-\bar{x}\rangle\le 0. \tag{25}$$

Since $\|x_{n+1}-x_n\|\to 0$ from (24), by (25), we have

$$\limsup_{n\to\infty}\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\le 0. \tag{26}$$

The definition of x_{n+1} and z_n together with (10) yields

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &= \|(1-\alpha_n)s_n+\alpha_n(u-\bar{x})\|^2\\ &\le (1-\alpha_n)\|s_n-\bar{x}\|^2+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\ &\le (1-\alpha_n)\|z_n-\bar{x}\|^2+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\ &= (1-\alpha_n)\|x_n+\theta_n(x_n-x_{n-1})-\bar{x}\|^2+2\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\ &\le (1-\alpha_n)(\|x_n-\bar{x}\|+\theta_n\|x_n-x_{n-1}\|)^2+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\ &= (1-\alpha_n)(\|x_n-\bar{x}\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\\ &\qquad +2\theta_n\|x_n-\bar{x}\|\|x_n-x_{n-1}\|)^2+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\ &\le (1-\alpha_n)\|x_n-\bar{x}\|^2+\theta_n^2\|x_n-x_{n-1}\|^2\end{aligned}$$

$$
\begin{aligned}
&+2\theta_n\|x_n-\bar{x}\|\|x_n-x_{n-1}\|+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\le (1-\alpha_n)\|x_n-\bar{x}\|^2+\frac{\theta_n^2}{\alpha_n}\|x_n-x_{n-1}\|^2\\
&+2\theta_n L\|x_n-x_{n-1}\|+2\alpha_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle.
\end{aligned}
\tag{27}
$$

Therefore, in view of (27), we have

$$
\Gamma_{n+1}\le \left(1-\alpha_n\right)\Gamma_n+\alpha_n\nu_n, \tag{28}
$$

where $\nu_n=\frac{\theta_n^2}{\alpha_n^2}\|x_n-x_{n-1}\|^2+2\frac{\theta_n}{\alpha_n}L\|x_n-x_{n-1}\|+2\langle u-\bar{x},x_{n+1}-\bar{x}\rangle$. From (26) and (C3), we have $\limsup_{n\to\infty}\nu_n\le 0$. By Lemma 3 *(ii)* and (28), we get $\Gamma_n\to 0$ as $n\to\infty$. Hence, $x_n\to\bar{x}$ as $n\to\infty$.

Case 2. Assume that $\{\Gamma_n\}$ does not decrease at infinity. Let $\varphi:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) defined by

$$
\varphi(n)=\max\{l\in\mathbb{N}: l\le n,\Gamma_l\le\Gamma_{l+1}\}.
$$

By Lemma 2, $\{\varphi(n)\}_{n=n_0}^{\infty}$ is a nondecreasing sequence, $\varphi(n)\to\infty$ as $n\to\infty$ and

$$
\Gamma_{\varphi(n)}\le\Gamma_{\varphi(n)+1}\text{ and }\Gamma_n\le\Gamma_{\varphi(n)+1},\quad \forall n\ge n_0. \tag{29}
$$

In view of $\|x_{\varphi(n)}-\bar{x}\|^2-\|x_{\varphi(n)+1}-\bar{x}\|^2=\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1}\le 0$ for all $n\ge n_0$ and (27), we have for all $n\ge n_0$

$$
\begin{aligned}
&(1-\alpha_{\varphi(n)})\rho_n(\delta-\rho_{\varphi(n)})\frac{(\|y_{\varphi(n)}\|^2+\|t_{\varphi(n)}\|^2)^2}{\mu(z_{\varphi(n)})}\\
&\le(\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1})+\frac{\theta_{\varphi(n)}^2}{\alpha_{\varphi(n)}}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2\\
&+2\theta_{\varphi(n)}L\|x_{\varphi(n)}-x_{\varphi(n)-1}\|+2\alpha_{\varphi(n)}\langle-\bar{x},x_{\varphi(n)+1}-\bar{x}\rangle.
\end{aligned}
\tag{30}
$$

Thus, (30) together with (C1) and (C2), we have

$$
\frac{(\|y_{\varphi(n)}\|^2+\|t_{\varphi(n)}\|^2)^2}{\mu(z_{\varphi(n)})}\to 0,\quad n\to\infty. \tag{31}
$$

Using similar procedure as above in Case 1, we have

$$
\lim_{n\to\infty}\|x_{\varphi(n)}-y_{\varphi(n)}\|=\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\|=0.
$$

Since $\{x_{\varphi(n)}\}$ is bounded, there exists a subsequence of $\{x_{\varphi(n)}\}$, still denoted by $\{x_{\varphi(n)}\}$ which converges weakly to p. Now repeating the argument of the proof as in Case 1, we have $\limsup_{n\to\infty}\langle u-\bar{x},x_{\varphi(n)+1}-\bar{x}\rangle\le 0$. From (28), we have

$$
\Gamma_{\varphi(n)+1}\le\left(1-\alpha_{\varphi(n)}\right)\Gamma_{\varphi(n)}+\alpha_{\varphi(n)}\nu_{\varphi(n)}, \tag{32}
$$

where

$$
\begin{aligned}
\nu_{\varphi(n)}=&\frac{\theta_{\varphi(n)}^2}{\alpha_{\varphi(n)}^2}\|x_{\varphi(n)}-x_{\varphi(n)-1}\|^2+\frac{\theta_{\varphi(n)}}{\alpha_{\varphi(n)}}2L\|x_{\varphi(n)}-x_{\varphi(n)-1}\|\\
&+2\langle u-\bar{x},x_{\varphi(n)+1}-\bar{x}\rangle.
\end{aligned}
$$

Using $\Gamma_{\varphi(n)}-\Gamma_{\varphi(n)+1}\le 0$ for all $n\ge n_0$, the inequality (32) gives

$$
0\le-\alpha_{\varphi(n)}\Gamma_{\varphi(n)}+\alpha_{\varphi(n)}\nu_{\varphi(n)}.
$$

Since $\alpha_{\varphi(n)}>0$, we obtain $\|x_{\varphi(n)}-\bar{x}\|^2=\Gamma_{\varphi(n)}\le\nu_{\varphi(n)}$. Moreover, since $\limsup_{n\to\infty}\nu_{\varphi(n)}\le 0$, we have $\lim_{n\to\infty}\|x_{\varphi(n)}-\bar{x}\|=0$. Thus, $\lim_{n\to\infty}\|x_{\varphi(n)}-\bar{x}\|=0$ together with $\lim_{n\to\infty}\|x_{\varphi(n)+1}-x_{\varphi(n)}\|=0$, gives $\lim_{n\to\infty}\Gamma_{\varphi(n)+1}=0$. Hence, from (29), we get $\lim_{n\to\infty}\Gamma_n=0$, that is, $x_n\to\bar{x}$ as $n\to\infty$.

This ends the proof. □

It should be noted that our iterative method, Algorithm 1, also works for approximation of solution of (6) under the assumptions **(A1), (A2)** and **(A3).**

Algorithm 2 Choose $u,x_0,x_1\in H_1$. Let $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ be a non-negative real sequences.

STEP 1. Evaluate $z_n=x_n+\theta_n(x_n-x_{n-1})$.

STEP 2. Compute $t_n=(I-U_{i_n})z_n$ and $y_n=(I-T_{j_n})Az_n$, where

$$
i_n\in\arg\max\{\|(I-U_i)z_n\|:i\in\{1,\dots,N\}\}
$$

and $j_n\in\arg\max\{\|(I-T_j)Az_n\|:j\in\{1,\dots,N\}\}$.

STEP 3. Evaluate

$$
x_{n+1}=\alpha_n u+(1-\alpha_n)\Big(z_n-\rho_n\frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A^*(y_n)+t_n\big)\Big),
$$

where $\mu(z_n)=\|A^*(y_n)+t_n\|^2$ if $\|A^*(y_n)+t_n\|\ne 0$, $\mu(z_n)=1$ otherwise.

STEP 4. Set $n:=n+1$ and go to **STEP 1.**

Corollary 1. *If the real parameters $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ in Algorithm 2 satisfy the following conditions:*

(C1) $0<\alpha_n<1$, $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C2) $0\le\theta_n\le\theta<1$ *and* $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|$;

(C3) $0<\rho_n<\delta=\min\{1-\eta_1,\dots,1-\eta_N,1-\beta_1,\dots,1-\beta_M\}$ *and* $\liminf_{n\to\infty}\rho_n(\delta-\rho_n)>0$;

then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to the solution point $\bar{x}$ the problem (6) where $\bar{x}=P_\Omega(u)$ and Ω the solution set of (6).

Corollary 2. *Suppose H_1 and H_2 be real Hilbert spaces, $U:H_1\to H_1$ is η-demimetric mapping, $T:H_2\to H_2$ is β-demimetric mapping, and $I-U$ and $I-T$ are demiclosed. Then the sequence $\{x_n\}$ generated by iterative algorithm*

$$
\begin{cases}
u,x_0,x_1\in H_1\ \text{chosen arbitrarily},\\
z_n=x_n+\theta_n(x_n-x_{n-1}),\\
t_n=(I-U)z_n,\quad y_n=(I-T)Az_n,\\
\mu(z_n)=\begin{cases}\|A^*(y_n)+t_n\|^2, & \text{if } \|A^*(y_n)+t_n\|\ne 0,\\ 1, & \text{otherwise},\end{cases}\\
x_{n+1}=\alpha_n u+(1-\alpha_n)\big(z_n-\rho_n\frac{\|y_n\|^2+\|t_n\|^2}{\mu(z_n)}\big(A^*(y_n)+t_n\big)\big),
\end{cases}
\tag{33}
$$

converges strongly to the solution point $\bar{x}$ the SCFP, i.e., $\bar{x}\in\Theta=\{\bar{x}\in F(U): A\bar{x}\in F(T)\}$ where $\bar{x}=P_\Theta(u)$, if the real parameters $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ in (33) satisfy the following conditions:

(C1) $0<\alpha_n<1$, $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C2) $0\le\theta_n\le\theta<1$ *and* $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|$;

(C3) $0<\rho_n<\delta=\min\{1-\eta,1-\beta\}$ *and* $\liminf_{n\to\infty}\rho_n(\delta-\rho_n)>0$.

Remark 1. In view of all studies done on (6), Algorithm 2 is a new approach and improvement of an existing result in solving the problem (6). Moreover, the algorithm (33) (Corollary 2) in also an improvement of the existing results concerning SCFP.

Remark 2. Condition (C2) of Theorem 1 can easily be satisfied because the value of $\|x_n-x_{n-1}\|$ is known before choosing θ_n. For instance, we can choose θ_n such that $0\le\theta_n\le\bar{\theta}_n$ where

$$
\bar{\theta}_n:=\begin{cases}\min\left\{\theta,\frac{\epsilon_n}{\|x_{n-1}-x_n\|}\right\}, & \text{if } x_{n-1}\ne x_n\\ \theta, & \text{otherwise,}\end{cases}
\tag{34}
$$

where $\theta\in[0,1)$, $\epsilon_n>0$ and $\epsilon_n=o(\alpha_n)$.

5. Applications

5.1. Theoretical applications

As a direct consequence of our algorithm, we can have several new algorithms for different class of mappings; for example, for κ-strict pseudo-contractions mappings, firmly-nonexpansive mappings and directed mappings. Moreover, as applications, we can obtain several new algorithms to solve problems that can be converted to the fixed point problem of demimetric mappings. Note that κ-strict pseudocontractive mapping is κ-demimetric mapping, and firmly nonexpansive mapping is -1-demimetric mapping. Moreover, both κ-strict pseudocontractive and firmly nonexpansive mapping are demiclosed [35].

For a real Hilbert space H and maximal monotone set-valued mapping $T : H \to 2^H$, the resolvent operator J_λ^T associated with T and $\lambda > 0$ is

$$J_\lambda^T(x) = (I + \lambda T)^{-1}(x), \quad x \in H. \tag{35}$$

The resolvent operator J_λ^T is single-valued and firmly nonexpansive. Moreover, $0 \in T(\bar{x})$ if and only if $\bar{x}$ is a fixed point of J_λ^T for all $\lambda > 0$; see [20].

Let H_1 and H_2 be real Hilbert spaces, $A_k : H_1 \to H_2$ is a nonzero bounded linear operator for each $k \in \{1, \dots, R\}$, $T_i : H_1 \to 2^{H_1}$ and $U_j : H_2 \to 2^{H_2}$ be maximal monotone mappings for all $i \in \{1, \dots, N\}$, $j \in \{1, \dots, M\}$. The *split system of inclusion problem* is to find $\bar{x} \in H_1$ such that

$$\begin{cases} 0 \in T_i(\bar{x}), & \forall i \in \{1, \dots, N\}, \\ 0 \in U_j(A_k(\bar{x})), & \forall j \in \{1, \dots, M\}, \quad \forall k \in \{1, \dots, R\}. \end{cases} \tag{36}$$

Let X be the solution set of (36).

Now replacing U_i and T_j of Algorithm 1 by the resolvent operators $J_\lambda^{T_i}$ and $J_\lambda^{U_j}$, we obtain the following algorithm, Algorithm 3, for solving (36).

Algorithm 3 Choose $u, x_0, x_1 \in H_1$. Let $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ be nonnegative real sequences.

STEP 1. Evaluate $z_n = x_n + \theta_n(x_n - x_{n-1})$.
STEP 2. Compute $t_n = (I - J_\lambda^{T_i})z_n$ and $y_n = (I - J_\lambda^{T_i})A_{k_n} z_n$ where

$$i_n \in \arg\max\{\|(I - J_\lambda^{T_i})z_n\| : i \in \{1, \dots, N\}\}$$

and $(j_n, k_n) \in \arg\max\{\|(I - J_\lambda^{U_j})A_k z_n\| : (j, k) \in \{1, \dots, M\} \times \{1, \dots, R\}\}$.
STEP 3. Evaluate

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\Big(z_n - \rho_n \frac{\|y_n\|^2 + \|t_n\|^2}{\mu(z_n)}\big(A_{k_n}^*(y_n) + t_n\big)\Big),$$

where $\mu(z_n) = \|A_{k_n}^*(y_n) + t_n\|^2$ if $\|A_{k_n}^*(y_n) + t_n\| \neq 0$, $\mu(z_n) = 1$ otherwise.
STEP 4. Set $n := n + 1$ and go to **STEP 1**.

Corollary 3. *If the real parameters $\{\alpha_n\}$, $\{\theta_n\}$ and $\{\rho_n\}$ in Algorithm 3 satisfy the following conditions:*

(C1) $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 \le \theta_n \le \theta < 1$ *and* $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|$;
(C3) $0 < \rho_n < 2$ *and* $\liminf_{n\to\infty} \rho_n(2 - \rho_n) > 0$;

then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to the solution point $\bar{x}$ the problem (36) where $\bar{x} = P_X(u)$.

Let H be a real Hilbert space. Note that for a proper, lower semicontinuous convex function $f : H \to \mathbb{R} \cup \{+\infty\}$ the subdifferential of f (denoted by ∂f) is a maximal monotone operator; f at $x \in H$, we denote by $\partial f(x)$, is given by $\partial f(x) = \{y \in H : f(z) \ge f(x) + \langle y, z - x\rangle, \quad \forall z \in H\}$. Notice that a point $\bar{x} \in H$ minimizes f if and only if $0 \in \partial f(\bar{x})$. On top of that, $\mathrm{prox}_{\lambda f} = (I + \lambda\partial f)^{-1}$, i.e., the point $\bar{x}$ minimizes f if and only if $\mathrm{prox}_{\lambda f}(\bar{x}) = \bar{x}$, see [36].

If $f_i : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g_j : H_2 \to \mathbb{R} \cup \{+\infty\}$ are proper, lower semicontinuous convex functions for $i \in \{1, \dots, N\}$, $j \in \{1, \dots, M\}$, then taking $T_i = \mathrm{prox}_{\lambda f_i}$ and $U_j = \mathrm{prox}_{\lambda g_j}$ in (5), Algorithm 3 yields strong convergence result solving

$$\bar{x} \in \bigcap_{i=1}^N (\arg\min f_i) \text{ such that } A_k(\bar{x}) \in \bigcap_{j=1}^M (\arg\min g_j), \quad \forall k \in \{1, \dots, R\}. \tag{37}$$

The multiple-set split feasibility problem (MSSFP) is the problem of finding

$$\bar{x} \in \bigcap_{i=1}^N C_i \text{ such that } A_k(\bar{x}) \in \bigcap_{j=1}^M Q_j, \quad \forall k \in \{1, \dots, R\}, \tag{38}$$

where C_i $(i \in \{1, \dots, N\})$ and Q_j $(j \in \{1, \dots, M\})$ nonempty closed convex subsets of H_1 and H_2, respectively. The MSSFP (38) is a special case of (37), i.e., take $f_i = \delta_{C_i}$ and $g_j = \delta_{Q_i}$ (the indicator functions) in (37).

Remark 3. The results mentioned as an application extend and improve the results in literature, for example [5, 8, 16, 17, 18, 19, 20, 21, 37] and the reference therein.

5.2. Real world application

Several problems in the fields of applications that can be cast into a MSSFP (38) and as we illustrated above MSSFP (38) is particular case of GSCFP (5). The following are exhibits of the potential application of our proposed method presented as particular case of GSCFP (5).

i. Sparse Binary Tomography: Digital image processing plays an important role in medical and astronomical imaging, file restoration, image and video coding, and other applications. Some approaches have been suggested for the image reconstruction processing analysis (see, Gibali and Petra [38]) are presented, for example l_0-superiorization [38] and $l_1 - l_2$ minimization, see [39] and reference therein. Consider l_1 constraint approach of the image reconstruction by Gibali and Petra [38]:

$$\min \frac{1}{2}\|Ax - b\|_2^2 \quad \text{subject to,} \quad x \in [0,1]^p \text{ and } \|x\|_1 \le t, \tag{39}$$

where $b \in \mathbb{R}^q$, A in $q \times p$ matrix, $[0,1]^p$ is the $[0,1]$ cube in $\mathbb{R}^p$. The problem (39) takes MSSFP (38) given by

$$\text{find } \bar{x} \in C_1 \cap C_2 \text{ such that } \bar{x} \in Q_1 \tag{40}$$

where $b \in \mathbb{R}^q$, A in $q \times p$ matrix, C_1 is cube in $\mathbb{R}^q$ $(C_1 = [0,1]^p)$, $C_2 = \{x \in \mathbb{R}^p : \|x\|_1 \le t\}$ and $Q = \{b\}$.

ii. Radiotherapy treatment is very important in the field of medicine, specially in the dose calculation process in Radiation therapy treatment planning (RTTP) [40] and intensity-modulated radiation therapy (IMRT) [41, 42]. The RTTP considered in [40] and IMRT considered in [41, 42] are examples that can be translated into MSSFP (38).

6. Numerical result

In this section, we will present two preliminary numerical experiments to show the performance of our proposed iterative algorithm, Algorithm 1, and to show our algorithm converges faster than the algorithm in [1, 19]. All the code is written in MATLAB and is performed on HP laptop with Intel(R) Core(TM) i5-7200U CPU @ 250 GHz 2.70 GHz and RAM 4.00 GB.

Table 1. For $\rho_n = 1$, $\theta_n = (10n + 10)^{-\frac{5}{2}}$, $p = 20$, $R = 5$ and for randomly generated starting points $x_0 = (1, \sqrt{d}, \sqrt{d^2}, \sqrt{d^3}, \dots)$, $x_1 = (1, \sqrt{e}, \sqrt{e^2}, \sqrt{e^3}, \dots)$ where $d, e \in [0, 1)$.

N	M		$\alpha_n = \frac{1}{100n+100}$	$\alpha_n = \frac{1}{10n+10}$	$\alpha_n = \frac{1}{n+1}$	$\alpha_n = \frac{1}{\sqrt{n+1}}$
2	3	Iter(n)	15	15	13	11
		cput	0.008303	0.008199	0.008152	0.008118
6	7	Iter(n)	13	13	14	13
		cput	0.009847	0.008701	0.008940	0.007977
19	8	Iter(n)	16	17	13	12
		cput	0.008590	0.008743	0.007932	0.008051
30	30	Iter(n)	21	21	19	17
		cput	0.011459	0.080036	0.010311	0.010291
35	35	Iter(n)	39	41	36	33
		cput	0.235961	0.015459	0.016082	0.017052

Table 2. For $\alpha_n = \frac{1}{n+1}$, $\theta_n = (n+1)^{-2}$, $p = 8$, $R = 3$ and for randomly generated starting points $x_0 = (1, \sqrt{d}, \sqrt{d^2}, \sqrt{d^3}, \dots)$, $x_1 = (1, \sqrt{e}, \sqrt{e^2}, \sqrt{e^3}, \dots)$ where $d, e \in [0, 1)$.

N	M		$\rho_n = 0.1$	$\rho_n = 0.5$	$\rho_n = 1$	$\rho_n = 1.5$	$\rho_n = 1.8$
2	2	Ite(n)	10	10	11	9	9
		cput	0.007978	0.007465	0.007998	0.008004	0.007228
4	6	Iter(n)	13	10	9	11	10
		cput	0.007896	0.007852	0.008262	0.006437	0.005454
9	9	Iter(n)	11	13	10	11	12
		cput	0.007790	0.006858	0.006925	0.007476	0.006107
21	16	Iter(n)	18	14	15	13	13
		cput	0.010914	0.007995	0.010343	0.008111	0.009817
29	30	Iter(n)	32	36	29	27	27
		cput	0.012258	0.017639	0.016162	0.010873	0.010642

Problem 1. Consider GSCFP (5) for $H_1 = H_2 = l_2$ with usual norm, where

$$U_i : x = (x^{(1)}, x^{(2)}, x^{(3)}, \dots) \mapsto \left(a_{(i,1)}x^{(1)}, a_{(i,2)}x^{(2)}, \dots, a_{(i,p)}x^{(p)}, 0, 0, 0, \dots\right),$$

$$T_j : y = (y^{(1)}, y^{(2)}, y^{(3)}, \dots) \mapsto \left(b_j y^{(1)}, b_j y^{(2)}, b_j y^{(3)}, \dots\right)$$

and

$$A_k : x = (x^{(1)}, x^{(2)}, x^{(3)}, \dots) \mapsto \left(z^{(1)}, z^{(2)}, z^{(3)}, \dots\right),$$

where $a_{(i,t)} \leq -1$ for all $(i,t) \in \{1, \dots, N\} \times \{1, \dots, p\}$, and $b_j \leq -1$ for all $j \in \{1, \dots, M\}$, $z^{(s)}$ $(s = 1, 2, 3, \dots)$ is given by

$$z^{(s)} = \begin{cases} 0, & \text{if } s \leq k, \\ x^{(m)}, & \text{if } s = k + m, \end{cases}$$

for $k \in \{1, \dots, R\}$. It is not hard to show that U_i is η_i-strict pseudo-contraction mapping (η_i-demimetric mapping) where $0 \leq \xi_i \leq \eta_i < 1$ for $\xi_i = \max\left\{\frac{a^2_{(i,t)}-1}{(a_{(i,t)}-1)^2} : t \in \{1, \dots, p\}\right\}$, and T_j is β_j-strict pseudocontraction mapping (β_j-demimetric mapping) where $0 \leq \zeta_j \leq \beta_j < 1$ for $\zeta_j = \frac{b_j^2-1}{(b_j-1)^2}$. This implies, $I - U_i$ and $I - T_j$ are demiclosed.

We study the numerical results of our algorithm for different step-sizes, $u = 0$ and for different starting points x_0 and x_1, and different N, M and p. The step size θ_n is set $\theta_n = \bar{\theta}_n$ where $\bar{\theta}_n$ is defined as in Remark 2, i.e., by taking $\theta = 0.8$ and ϵ_n with $\epsilon_n = o(\alpha_n)$ and (34). The numerical results are shown in Table 1, 2 and 3. The tables illustrate the numerical behavior of our algorithm in terms of number of iterations (Iter(n)) and CPU time in seconds (cput), where $\frac{\|(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(p)})\|_\infty}{\|x_1 - x_2\|} = \frac{1}{\|x_1 - x_2\|} \max_{t \in \{1, \dots, p\}} |x_n^{(t)}| < 0.001$ as the stopping criterion. Based on the numerical results in Table 1 and 2 we see that the CPU time and a number of iterations of the algorithm to decrease linearly for the choice of the term α_n close to 1 or ρ_n close to 1. In addition, from Table 1, 2 and 3, we observe that the number of iterations and CPU time of Algorithm 1 depends on initial points and N and M, and our iteration method can successively find high accuracy approximations to the solution GSCFP (5).

Table 3. For $p = 100$ and $R = 2$.

x_0, x_1, respectively		$N = 3 = M$	$N = 9$, $M = 6$	$N = 20 = M$
$(1, \frac{\sqrt{2}}{\sqrt{3}}, \frac{2}{3}, \frac{\sqrt{8}}{\sqrt{27}}, \dots)$	Iter(n)	18	22	25
$(1, \frac{1}{\sqrt{3}}, \frac{1}{3}, \frac{1}{\sqrt{27}}, \dots)$	cput	0.010708	0.012163	0.014925
$(1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}, \dots)$	Iter(n)	19	24	30
$(1, 2, 3, 4, 5, 0, 0, \dots)$	cput	0.012252	0.016246	0.023767
$(1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{\sqrt{8}}, \dots)$	Iter(n)	24	23	23
$(1, \frac{\sqrt{3}}{2}, \frac{3}{4}, \frac{\sqrt{27}}{8}, \dots)$	cput	0.012206	0.009742	0.019825
$(-20, 19, 21, 0, 0, \dots)$	Iter(n)	13	13	15
$(6, 20, -5, 32, 0, 0, \dots)$	cput	0.008294	0.014257	0.026192

We next consider an example of GSCFP (5) in a finite-dimensional Hilbert space to compare our method, with the results in [1, 19].

Problem 2. Consider GSCFP (5) for $H_1 = \mathbb{R}^p$ and $H_2 = \mathbb{R}^q$, with $A_k = G^{(k)}_{q \times p}$,

$$U_i : x = (x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(p)}) \mapsto \left(a_{(i,1)}x^{(2)}, a_{(i,2)}x^{(2)}, \dots, a_{(i,p)}x^{(p)}\right)$$

and

$$T_j : y = (y^{(1)}, y^{(2)}, y^{(3)}, \dots, y^{(q)}) \mapsto \left(b_{(j,1)}y^{(1)}, b_{(j,2)}y^{(2)}, \dots, b_{(j,q)}y^{(q)}\right),$$

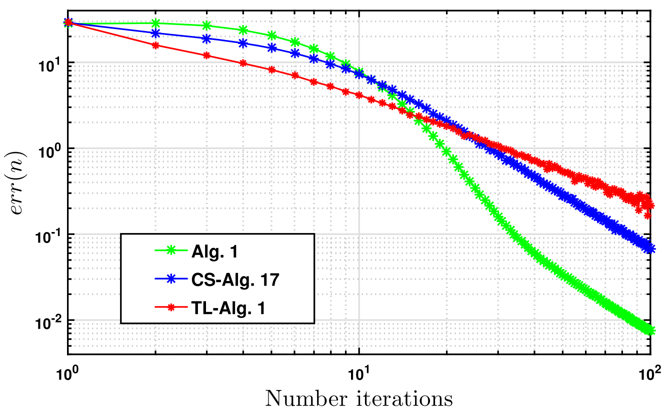

Fig. 1. Comparison of Alg. 1, CS-Alg. 17 and TL-Alg. 1 for $p = q = 100$, $u = 0$, and for randomly generated starting points $x_0, x_1 \in [-1000, 1000]$.

Table 4. Comparison of Alg. 1, CS-Alg. 17 and TL-Alg. 1 for different dimensions where $u = 0$ and x_0, x_1 are randomly generated starting points with $x_0, x_1 \in [-1000, 1000]$.

Method		$p = 2 = q$	$p = 5$, $q = 4$	$p = 8$, $q = 5$	$p = 20 = q$
Alg. 1	Iter(n)	15	15	17	19
	cput	0.042293	0.048064	0.054138	0.083525
CS-Alg. 17	Iter(n)	17	18	18	21
	cput	0.063185	0.067504	0.068096	0.093564
TL-Alg. 1	Iter(n)	15	16	17	22
	cput	0.043785	0.047113	0.070805	0.084524

where $0 \le a_{(i,t)} < 1$ for all $(i,t) \in \{1,\dots,N\} \times \{1,\dots,p\}$, $0 \le b_{(j,r)} < 1$ for all $((i,t) \in \{1,\dots,M\} \times \{1,\dots,q\})$ and $G^{(k)}_{q\times p}$ is $q \times p$ matrix for $k \in \{1,\dots,R\}$. Obviously, for this case U_i is η_i-demimetric mapping where $\xi_i \le \eta_i \le -1$ for $\xi_i = \max\left\{\frac{a^2_{(i,t)}-1}{(a_{(i,t)}-1)^2} : t \in \{1,\dots,p\}\right\}$, and T_j is β_j-demimetric where $\zeta_j \le \beta_j \le -1$ for $\zeta_j = \max\left\{\frac{b^2_{(j,r)}-1}{(b_{(j,r)}-1)^2} : r \in \{1,\dots,q\}\right\}$. This implies, each U_i and T_j are directed mapping. The mappings $I - U_i$ and $I - T_j$ are demiclosed from the fact that U_i and T_j are also nonexpansive mappings.

We take $a_{(i,t)} = \frac{t}{i+t}$ $((i,t) \in \{1,\dots,N\} \times \{1,\dots,p\})$ and $b_{(j,r)} = \frac{j}{j+r}$ $((j,r) \in \{1,\dots,M\} \times \{1,\dots,q\})$. By setting $A_k = G^{(k)}_{q\times p} = A = G_{q\times p}$ for all $k \in \{1,\dots,R\}$, we compare our iterative algorithm, Algorithm 1 (Alg. 1), with Algorithm 17 of Censor and Segal [1] (CS-Alg. 17) and Algorithm 1 of Tang and Liua [19] (TL-Alg. 1) to solve Problem 2 for randomly generated $q \times p$ matrix $G_{q\times p}$. We take the following step sizes:

Alg. 1: $\alpha_n = \frac{1}{n+1}$, $\rho_n = 1$, θ_n defined as in Remark 2 taking $\epsilon_n = \frac{1}{(n+1)^2}$;

CS-Alg. 17: $\gamma = \frac{1}{10}$, $\alpha_i = \frac{i}{2(1+\ldots+N)}$, $\beta_j = \frac{j}{2(1+\ldots+M)}$ $(i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\})$, see [1];

TL-Alg. 1: $\gamma_n = \frac{1}{10}$, $\omega_i = \frac{i}{1+\ldots+N}$, $\eta_j = \frac{j}{1+\ldots+M}$ $(i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\})$, see [19];

In Table 3 and 4, and Fig. 1 present the corresponding numerical results, where Fig. 1 demonstrates error ($err(n) = \|x_n - \bar{x}\|$) versus the number of iterations, and Table 3 shows the CPU time exclusion (cput) and a number of iterations (Iter(n)) of Alg. 1, CS-Alg. 17 and TL-Alg. 1 for the stopping criteria $\frac{\|x_n\|_\infty}{\|x_1-x_2\|} = \frac{1}{\|x_1-x_2\|} \max_{t\in\{1,\dots,p\}} |x_n^{(t)}| < 0.001$.

From Fig. 1, and Table 4 and 5, we see that our proposed iterative method outperform Algorithm 17 of Censor and Segal [1] and Algorithm 1 of Tang and Liua [19]. Our method requires most likely fewer iterations and CPU time than that of the two compared algorithms as seen from Table 4 and 5. Furthermore, the numerical results further confirm the effectiveness of our proposed method.

Table 5. Comparison of Alg. 1, CS-Alg. 17 and TL-Alg. 1 for $u = 0$ and for different starting points x_0, x_1 where $p = 6$, $q = 100$.

Initial points x_0, x_1		Alg. 1	CS-Alg. 17	TL-Alg. 1
$x_0 = (30, -22, 50, 6, 70, 100)$	Iter(n)	29	33	36
$x_1 = (37, 100, 9, -60, 80, 25)$	cput	0.010412	0.022883	0.010546
$x_0 = (13, 9, 16, 60, -7, -10)$	Iter(n)	17	26	17
$x_1 = (-29, 76, -29, 80, 970, 10)$	cput	0.009903	0.011066	0.037172
$x_0 = (-9, 8, -6, 5, -4, 3)$	Iter(n)	14	16	15
$x_1 = (45, -5, 7, 9, -4, 1)$	cput	0.007256	0.009011	0.008949
$x_0 = (1, 2, 3, 4, 5, 6)$	Iter(n)	10	15	14
$x_1 = (-5, -4, -3, -2, -1, 0)$	cput	0.006734	0.006819	0.007673
$x_0 = (-11, 2, 0, 0, -5, 0.5)$	Iter(n)	10	16	14
$x_1 = (0.5, 1, 3.5, -2, -1, 10)$	cput	0.006039	0.006199	0.007031

7. Conclusions

In this paper, we introduced a novel algorithm involving an inertial term and a step size independent of the operator norm for approximating a solution of a generalized split common fixed point problem (5) for demimetric mappings in a real Hilbert space. Under mild assumptions, we proved the convergence of the proposed iterative methods. We applied our result to solve problems that can be studied as fixed points of demimetric mapping. The numerical experiments reported show the efficiency of the proposed iterative method.

Declarations

Author contribution statement

Anteneh Getachew Gebrie, Dejene Shewakena Bedane: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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