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Fitted computational method for solving singularly perturbed small time lag problem

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Abstract

Objectives: An accurate exponentially fitted numerical method is developed to solve the singularly perturbed time lag problem. The solution to the problem exhibits a boundary layer as the perturbation parameter approaches zero. A priori bounds and properties of the continuous solution are discussed.

Result: The backward-Euler method is applied in the time direction and the higher order finite difference method is employed for the spatial derivative approximation. An exponential fitting factor is induced on the difference scheme for stabilizing the computed solution. Using the comparison principle, the stability of the method is examined and analyzed. It is proved that the method converges uniformly with linear order of convergence. To validate the theoretical findings and analysis, two test examples are given. Comparison is made with the results available in the literature. The proposed method has better accuracy than the schemes in the literature.

Keywords: Accurate numerical method, Exponentially fitted method, Stability and uniform convergence

Introduction

A differential equation in which its highest order derivative term is multiplied by a small parameter is known as a singularly perturbed differential equation (SPDE). It commonly occurs in the modeling of chemical processes, fluid flows, water quality problems in river networks, mechanical systems and simulation of oil extraction from underground reservoirs [1]. The solution of such type of equation possesses a multi-scale character that varies quickly in the boundary layer region and it slows in the outer layer regions.

Due to the multi-scale character of the solution, the classical numerical methods fail to give an accurate result. Currently, it becomes interesting to develop a numerical method which gives accurate results; and its convergence does not depend on the perturbation

parameter. For solving the considered problem in Kumar et al. [2] proposed an adaptive mesh method using the concept of entropy function. Hybrid scheme of the mid-point upwind in the outer region and the central difference method in the boundary layer region are used in [3, 4]. Gowrisankar and Natesan [5] used the upwind finite difference method on a piecewise uniform mesh. The upwind finite difference method on Shishkin mesh is used in [6]. Podila and Kumar [7] used a stable finite difference scheme, which works on a uniform and an adaptive mesh. An exponentially fitted scheme is discussed in [8, 9]. The non-standard finite difference method is used in [10]. The numerical schemes developed in [11–16] works for both the large and small delay cases.

In this paper, we proposed a numerical scheme using higher order finite difference scheme fitted by the exponential fitting factor. Moreover, the main aim of this study is to develop a more accurate, stable and uniformly convergent numerical scheme for solving singularly perturbed convection-diffusion problem having small time lag.

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In this paper, C is a generic positive constant, which does not depend on the mesh parameters and the perturbation parameter. The norm $\|\cdot\|$, defined by $\|g\| = \max_{(s,t) \in \Omega} |g(s,t)|$ is the maximum norm.

Continuous problem

Consider a class of singularly perturbed convection-diffusion problem of the form

$$\begin{cases} z_t(s,t) + \mathcal{L}z(s,t) = -c(s,t)z(s,t-\tau) + g(s,t), & (s,t) \in \Omega = (0,1) \times (0,T], \\ z(s,t) = \psi_b(s,t) \text{ on } \eta_b := [0,1] \times [-\tau,0], \\ z(0,t) = \psi_l(t) \text{ on } \eta_l := \{(0,t) : 0 \leq t \leq T\}, \\ z(1,t) = \psi_r(t) \text{ on } \eta_r := \{(1,t) : 0 \leq t \leq T\}, \end{cases} \tag{1}$$

where $\mathcal{L}z(s,t) = -\varepsilon z_{ss}(s,t) + b(s,t)z_s(s,t) + a(s)z(s,t)$, $0 < \varepsilon \ll 1$ is the perturbation parameter and $\tau > 0$ is the delay parameter. The coefficients $b(s,t)$, $a(s)$, $c(s,t)$, $g(s,t)$ on $\bar{\Omega}$ and $\psi_b(s,t)$, $\psi_l(t)$, $\psi_r(t)$ on $\eta = \eta_l \cup \eta_r \cup \eta_b$ are assumed to be smooth and bounded functions that satisfy the conditions $a(s) \geq \alpha > 0$, $c(s,t) \geq \gamma > 0$, $b(s,t) \leq \beta < 0$, $a(s) + c(s,t) \geq 0$ on $\bar{\Omega}$. Under these conditions, the problem exhibits a boundary layer along $s = 0$ [2].

Properties of the analytical solution

In this part, we present the analytical aspects of the solution and its derivatives. The existence and uniqueness of a solution of (1).

Lemma 2.1 [17, 18] *The solution $z(s, t)$ of (1) satisfies $|z(s, t) - \psi_b(s, 0)| \leq Ct$, $(s, t) \in \bar{\Omega} = [0, 1] \times [0, T]$, where $C > 0$ is a constant that does not depends on ε .*

The operator $L(s, t) = z_t(s, t) + \mathcal{L}z(s, t)$ of (1) satisfies the following minimum principle.

Lemma 2.2 [19] *Let $v(s, t) \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfies $v(s, t) \geq 0$ $(s, t) \in \partial\Omega = \bar{\Omega} - \Omega$. If $Lv(s, t) \leq 0$, $(s, t) \in \Omega$, then $v(s, t) \geq 0$, $(s, t) \in \bar{\Omega}$.*

Lemma 2.3 [18, 19] *[Stability result] The solution of (1) satisfies the bound*

$$|z(s, t)| \leq \alpha^{-1} \|g\| + \max\{|\psi_l(t)|, |\psi_b(s, t)|, |\phi_r(t)|\}, \tag{2}$$

where α is the lower bound of $a(s)$ and $\|g\| = \max_{s,t \in \Omega} |g(s, t)|$.

Lemma 2.4 [20, 21] *The following bounds are satisfied for the derivative of the solution $z(s, t)$ of (1) with respect to s and t*

$$\left| \frac{\partial^n z(s, t)}{\partial s^n} \right| \leq C \left(1 + \varepsilon^{-n} \exp\left(-\frac{\beta}{\varepsilon}s\right) \right), \tag{3}$$

$(s, t) \in \bar{\Omega}$, $n = 0, 1, 2, 3, 4$,

and

$$\left| \frac{\partial^l z(s, t)}{\partial t^l} \right| \leq C, \quad (s, t) \in \bar{\Omega}, \quad l = 0, 1, 2, \tag{4}$$

where β is lower bound of $b(s, t)$.

Main text: numerical scheme

Temporal semi-discretization

Using Taylor’s series expansion for the delay term; using a uniform mesh in t -direction with step size $\Delta t = T/M$ given by $\Omega_t^M = \{t_n = n\Delta t, n = 0, 1, 2, \dots, M, t_M = T\}$, where M is the number of mesh points in $[0, T]$. Note that $T = r\tau$ for some positive integer r . Using the backward-Euler formula, we get

$$(1 - \tau c(s, t_n)) \frac{z^n(s) - z^{n-1}(s)}{\Delta t} - \varepsilon \frac{d^2 z^n(s)}{ds^2} + b(s, t_n) \frac{dz^n(s)}{ds} + (a(s) + c(s, t_n)) z^n(s) = g(s, t_n). \tag{5}$$

Equivalently, we write

$$-\varepsilon \frac{d^2 Z^n(s)}{ds^2} + b(s, t_n) \frac{dZ^n(s)}{ds} + P(s, t_n) Z^n(s) = R(s, t_n), \tag{6}$$

Spatial discretization

The spatial domain $[0, 1]$ into N equal number of sub-intervals with the length of h is given by $0 = s_0, s_1, \dots, s_N = 1$ and $s_i = ih, i = 0, 1, 2, \dots, N$. Assume smooth function $Z(s) = Z^n(s) = Z(s, t_n)$ in the interval $[0, 1]$. From Taylor’s series expansion, we have

$$\begin{aligned} Z_{i+1} &\approx Z_i + hZ'_i + \frac{h^2}{2!}Z''_i + \frac{h^3}{3!}Z'''_i + \frac{h^4}{4!}Z^{(4)}_i + \frac{h^5}{5!}Z^{(5)}_i + \frac{h^6}{6!}Z^{(6)}_i + \frac{h^7}{7!}Z^{(7)}_i + \frac{h^8}{8!}Z^{(8)}_i, \\ Z_{i-1} &\approx Z_i - hZ'_i + \frac{h^2}{2!}Z''_i - \frac{h^3}{3!}Z'''_i + \frac{h^4}{4!}Z^{(4)}_i - \frac{h^5}{5!}Z^{(5)}_i + \frac{h^6}{6!}Z^{(6)}_i - \frac{h^7}{7!}Z^{(7)}_i + \frac{h^8}{8!}Z^{(8)}_i. \end{aligned} \tag{11}$$

Combining the result in (11), we arrive at

$$\begin{aligned} Z_{i-1} - 2Z_i + Z_{i+1} &= \frac{2h^2}{2!}Z''_i + \frac{2h^4}{4!}Z^{(4)}_i + \frac{2h^6}{6!}Z^{(6)}_i + \frac{2h^8}{8!}Z^{(8)}_i + O(h^{10}) \\ Z''_{i-1} - 2Z''_i + Z''_{i+1} &= \frac{2h^2}{2!}Z^{(4)}_i + \frac{2h^4}{4!}Z^{(6)}_i + \frac{2h^6}{6!}Z^{(8)}_i + \frac{2h^8}{8!}Z^{(10)}_i + O(h^{12}). \end{aligned} \tag{12}$$

where $P(s, t_n) = a(s) + c(s, t_n) + \frac{1-\tau c(s, t_n)}{\Delta t}$, and $R(s, t_n) = g(s, t_n) + \frac{1-\tau c(s, t_n)}{\Delta t} Z^{n-1}(s)$ with the boundary $Z^n(0) = \psi_l(t_n), Z^n(1) = \psi_r(t_n), 0 \leq n \leq M$.

Now, we rewrite (6) as

$$-\varepsilon \frac{d^2 Z(s)}{ds^2} + b(s, t_n) \frac{dZ(s)}{ds} + P(s, t_n) Z(s) = R(s, t_n), \tag{8}$$

where $Z(s) = Z^n(s) = Z(s, t_n)$. At each time step the local error is defined as $e_n(s) := z(s, t_n) - Z^n(s), 0 \leq n \leq M$.

Lemma 3.1 *The local error estimate in the temporal direction satisfies the bound*

$$\|e_n\| \leq C_1(\Delta t)^2,$$

and the global error at n th time level satisfies the bound

$$\|E_n\| \leq C(\Delta t). \tag{9}$$

Proof Refer from the Appendix section. □

Lemma 3.2 *For $0 \leq n \leq M - 1$, the solution of (6)–(7), satisfies the bound*

$$\begin{aligned} \left| \frac{d^r Z^n(s)}{ds^r} \right| &\leq C \left(1 + \varepsilon^{-r} \exp \left(-\frac{\beta}{\varepsilon}(s) \right) \right), \\ s &\in \bar{\Omega}_s, 0 \leq r \leq 4. \end{aligned} \tag{10}$$

Proof For the proof see [22]. □

Using (12), we get $\frac{h^4}{12}Z^{(6)}_i$ and, we obtain

$$Z_{i-1} - 2Z_i + Z_{i+1} = \frac{h^2}{30}(Z''_{i-1} + 28Z''_i + Z''_{i+1}) + \zeta, \tag{13}$$

where $\zeta = \frac{h^4}{20}Z^{(4)}_i - \frac{13h^8}{302,400}Z^{(8)}_i + O(h^{10})$. Using (8), we get

$$\begin{aligned} -\varepsilon Z''_{i+1} &= -b(s_{i+1}, t_n)Z'_{i+1} - P(s_{i+1}, t_n)Z_{i+1} + R(s_{i+1}, t_n), \\ -\varepsilon Z''_i &= -b(s_i, t_n)Z'_i - P(s_i, t_n)Z_i + R(s_i, t_n), \\ -\varepsilon Z''_{i-1} &= -b(s_{i-1}, t_n)Z'_{i-1} - P(s_{i-1}, t_n)Z_{i-1} + R(s_{i-1}, t_n). \end{aligned} \tag{14}$$

Next, using the non-symmetric finite difference schemes we approximate Z'_{i+1}, Z'_i and Z'_{i-1} as

$$\begin{aligned} Z'_i &= \frac{Z_{i+1} - Z_{i-1}}{2h} + O(h^2), \\ Z'_{i+1} &= \frac{3Z_{i+1} - 4Z_i + Z_{i-1}}{2h} - hZ''_i + O(h^2), \\ Z'_{i-1} &= \frac{-Z_{i+1} + 4Z_i - 3Z_{i-1}}{2h} + hZ''_i + O(h^2). \end{aligned} \tag{15}$$

Substituting (15) into (14) and substituting the resulting equation into (13) after simplifying, we obtain

$$\begin{aligned}
 & - \left(\varepsilon - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30} \right) \left(\frac{Z_{i-1} - 2Z_i + Z_{i+1}}{h^2} \right) \\
 & + \frac{b(s_{i-1}, t_n)}{60h} (-3Z_{i-1} + 4Z_i - Z_{i+1}) \\
 & + \frac{28b(s_i, t_n)}{60h} (Z_{i+1} - Z_{i-1}) \\
 & + \frac{b(s_{i+1}, t_n)}{60h} (Z_{i-1} - 4Z_i + 3Z_{i+1}) + \frac{P(s_{i-1}, t_n)}{30} Z_{i-1} \\
 & + \frac{28P(s_i, t_n)}{30} Z_i + \frac{P(s_{i+1}, t_n)}{30} Z_{i+1} \\
 & = \frac{1}{30} (R(s_{i-1}, t_n) + 28R(s_i, t_n) + R(s_{i+1}, t_n)).
 \end{aligned} \tag{16}$$

Computing the exponential fitting factor

In this part, we introduce the fitting factor σ and for the obtained scheme of (6)–(7) at $(i, n)th$ level. As the theory of singular perturbation given in [9, 23], the zero order asymptotic solution of the problem of the form

$$\begin{cases} -\varepsilon Z''(s) + b(s)Z' + p(s)Z(s) = q(s), & s \in \Omega_s = (0, 1), \\ Z(0) = \alpha_l, & Z(1) = \alpha_r, \end{cases} \tag{17}$$

is given by

$$Z(s) \approx Z_0(s) + \frac{b(0)}{b(s)} (\alpha_l - Z_0(0)) \exp \left(- \int_0^s \left(\frac{b(s)}{\varepsilon} - \frac{p(s)}{b(s)} \right) ds \right) + O(\varepsilon). \tag{18}$$

From Taylor’s series expansion for $b(s)$ and $p(s)$ restricting to their first terms about $s = 0$ and the simplified form gives

$$Z(s) = Z_0(s) + (\alpha_l - Z_0(0)) \exp \left(- \frac{b(0)}{\varepsilon} s \right), \tag{19}$$

where Z_0 is the reducible problem solution. Considering h is fairly small and solving the result in (19) at s_i which gives

$$Z(s_i) = Z(ih) = Z_0(0) + (\alpha_l - Z_0(0)) \exp(-b(0)i\rho),$$

where $\rho = \frac{h}{\varepsilon}$. We present an exponentially fitting factor σ of the scheme (16)

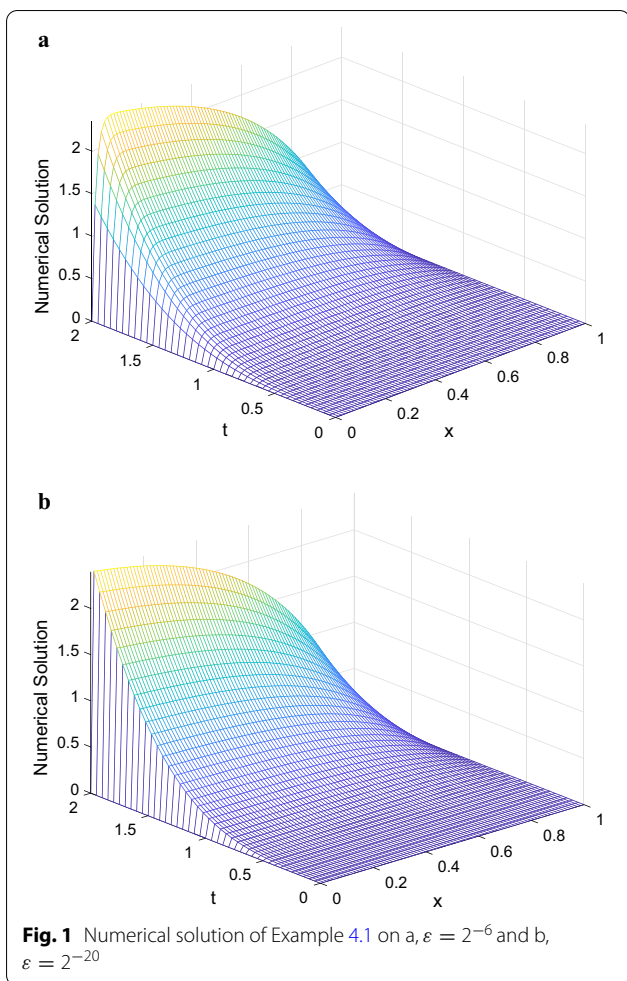
$$\begin{aligned}
 & - \left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30} \right) \left(\frac{Z_{i-1} - 2Z_i + Z_{i+1}}{h^2} \right) \\
 & + \frac{b(s_{i-1}, t_n)}{60h} (-3Z_{i-1} + 4Z_i - Z_{i+1}) \\
 & + \frac{28b(s_i, t_n)}{60h} (Z_{i+1} - Z_{i-1}) \\
 & + \frac{b(s_{i+1}, t_n)}{60h} (Z_{i-1} - 4Z_i + 3Z_{i+1}) + \frac{P(s_{i-1}, t_n)}{30} Z_{i-1} \\
 & + \frac{28P(s_i, t_n)}{30} Z_i + \frac{P(s_{i+1}, t_n)}{30} Z_{i+1} = \frac{1}{30} (R(s_{i-1}, t_n) \\
 & + 28R(s_i, t_n) + R(s_{i+1}, t_n)).
 \end{aligned} \tag{20}$$

Putting $\rho = \frac{h}{\varepsilon}$ and after multiplying both side by h and letting $h \rightarrow 0$ which gives

$$\begin{aligned}
 & - \frac{\sigma}{\rho} \lim_{h \rightarrow 0} (Z_{i-1} - 2Z_i + Z_{i+1}) \\
 & + \frac{b_0(t_n)}{60} \lim_{h \rightarrow 0} (-3Z_{i-1} + 4Z_i - Z_{i+1}) \\
 & + \frac{28b_0(t_n)}{60} \lim_{h \rightarrow 0} (Z_{i+1} - Z_{i-1}) \\
 & + \frac{b_0(t_n)}{60} \lim_{h \rightarrow 0} (Z_{i-1} - 4Z_i + 3Z_{i+1}) = 0.
 \end{aligned} \tag{21}$$

Using the results in (19) and simplifying, we obtain

$$\frac{\sigma}{\rho} (e^{b(0)\rho} + e^{-b(0)\rho} - 2) = \frac{b_0(t_n)}{60} (-30e^{b(0)\rho} + 30e^{-b(0)\rho}). \tag{22}$$



Solving for σ in (22), the exponential fitting factor σ is obtained as

$$\sigma = \frac{\rho b_0(t_n)}{2} \coth\left(\frac{\rho b(0)}{2}\right). \tag{23}$$

The discrete scheme

Using the higher order finite difference scheme of (16) and inducing the exponential fitting factor (23) for $1 \leq i \leq N - 1$ and $0 \leq n \leq M - 1$, then the fully discrete scheme is given as

$$L^{\Delta t, h} Z_i^n = \frac{1}{30} (R(s_{i-1}, t_n) + 28R(s_i, t_n) + R(s_{i+1}, t_n)), \tag{24}$$

where

$$\begin{aligned} L^{\Delta t, h} Z_i^n = & -\left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30}\right) \\ & \left(\frac{Z_{i-1}^n - 2Z_i^n + Z_{i+1}^n}{h^2}\right) \\ & + \frac{b(s_{i-1}, t_n)}{60h} (-3Z_{i-1}^n + 4Z_i^n - Z_{i+1}^n) \\ & + \frac{28b(s_i, t_n)}{60h} (Z_{i+1}^n - Z_{i-1}^n) \\ & + \frac{b(s_{i+1}, t_n)}{60h} (Z_{i-1}^n - 4Z_i^n + 3Z_{i+1}^n) + \frac{P(s_{i-1}, t_n)}{30} Z_{i-1}^n \\ & + \frac{28P(s_i, t_n)}{30} Z_i^n + \frac{P(s_{i+1}, t_n)}{30} Z_{i+1}^n. \end{aligned}$$

In the explicit form, we write

$$r_i^- Z_{i-1}^n + r_i^0 Z_i^n + r_i^+ Z_{i+1}^n = H_i^n, \tag{25}$$

where

$$\begin{aligned} r_i^- = & -\frac{1}{h^2} \left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30}\right) \\ & - \frac{3b(s_{i-1}, t_n)}{60h} + \frac{P(s_{i-1}, t_n)}{30} \\ & - \frac{28b(s_i, t_n)}{60h} + \frac{b(s_{i+1}, t_n)}{60h}, \\ r_i^0 = & \frac{2}{h^2} \left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30}\right) + \frac{4b(s_{i-1}, t_n)}{60h} \\ & - \frac{4b(s_{i+1}, t_n)}{60h} + \frac{28P(s_i, t_n)}{30}, \\ r_i^+ = & -\frac{1}{h^2} \left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30}\right) \\ & - \frac{b(s_{i-1}, t_n)}{60h} + \frac{28b(s_i, t_n)}{60h} \\ & + \frac{3b(s_{i+1}, t_n)}{60h} + \frac{P(s_{i+1}, t_n)}{30}, \\ H_i^n = & \frac{1}{30} (R(s_{i-1}, t_n) + 28R(s_i, t_n) + R(s_{i+1}, t_n)). \end{aligned} \tag{26}$$

Stability and convergence analysis

In this part, for the developed scheme of (24) we take to prove the discrete comparison principle.

Lemma 3.3 *There is a comparison function v_i^n such that $\mathcal{L}^{\Delta t, h} Z_i^n \leq \mathcal{L}^{\Delta t, h} v_i^n$ for $1 \leq i \leq N - 1$ and if $Z_0^n \leq v_0^n$ and $Z_N^n \leq v_N^n$, then $Z_i^n \leq v_i^n$ for $1 \leq i \leq N$.*

Proof The discrete operator $\mathcal{L}^{\Delta t, h} Z_i^n$ is matrix of size $(N + 1) \times (N + 1)$ with its entries for $1 \leq i \leq N - 1$ are r_i^- , r_i^0 , and r_i^+ . We observe that

$|r_i^-| > 0$, $|r_i^0| > 0$, $|r_i^+| > 0$ and $|r_i^0| \geq |r_i^-| + |r_i^+|$, giving that the matrix is diagonally dominant. Then, it satisfies the property of M matrix. Thus, the non-negative inverse of the matrix exists. So, it guarantees the existence of unique discrete solution [22, 24]. \square

Lemma 3.4 (Stability result) *If the solution of (24) be Z_i^n , then it satisfies*

$$|Z_i^n| \leq \frac{\|\mathcal{L}^{\Delta t, h} Z_i^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\},$$

From the discrete comparison principle, we get $\vartheta_{i,n}^\pm \geq 0$, $i = 0, 1, 2, \dots, N$. Hence, the necessary bound is satisfied. \square

Lemma 3.5 *If v_i^n be any mesh function such that $v_0^n = v_N^n = 0$. Then it satisfies*

$$|v_i^n| \leq \frac{1}{\zeta} \max_m |\mathcal{L}^{\Delta t, h} v_m^n|. \tag{28}$$

Using Taylor’s series approximation, we have the bounds

$$\begin{aligned} \left| -\left(\frac{d}{ds^2} - \delta_s^2\right)Z^n(s_i) \right| &\leq Ch^2 \left\| \frac{d^4 Z^n(s_i)}{ds^4} \right\|, \\ \left| \frac{dZ^n(s_{i-1})}{ds} - \left(\frac{-Z_{i+1}^n + 4Z_i^n - 3Z_{i-1}^n}{2h} + h\frac{d^2 Z^n(s_i)}{ds^2}\right) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_i)}{ds^3} \right\|, \\ \left| \frac{dZ^n(s_{i+1})}{ds} - \left(\frac{3Z_{i+1}^n - 4Z_i^n + Z_{i-1}^n}{2h} - h\frac{d^2 Z^n(s_i)}{ds^2}\right) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_i)}{ds^3} \right\|, \\ \left| \left(\frac{d}{ds} - \delta_s^0\right)Z^n(s_i) \right| &\leq Ch^2 \left\| \frac{d^3 Z^n(s_i)}{ds^3} \right\|, \quad \left| \delta_s^2 Z^n(s_i) \right| \leq C \left\| \frac{d^2 Z^n(s_i)}{ds^2} \right\|, \end{aligned} \tag{29}$$

where $P(s_i, t_n) \geq \zeta > 0$.

Proof let $\Pi = \frac{\|\mathcal{L}^{\Delta t, h} Z_i^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\}$ and set the barrier functions $\vartheta_{i,n}^\pm$ by $\vartheta_{i,n}^\pm = \Pi \pm Z_i^n$. On the boundaries, we obtain $\vartheta_{0,n}^\pm = \Pi \pm Z_0^n = \frac{\|\mathcal{L}^{\Delta t, h} Z_i^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_l(t_n) \geq 0$,

$\vartheta_{N,n}^\pm = \Pi \pm Z_N^n = \frac{\|\mathcal{L}^{\Delta t, h} Z_i^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\} \pm \psi_r(t_n) \geq 0$. On the discretized spatial domain s_i , $1 < i < N - 1$, we have

where $\|Z^{(k)}(s_i)\| = \max_{s_i} |Z_n^{(k)}(s_i)|$, $k = 2, 3, 4$.

In the next theorem we bound the truncation error in space direction discretization.

Theorem 3.1 *Consider the sufficiently smooth functions $a(s)$, $b(s, t_n)$ and $c(s, t_n)$ of (6–7) so that $Z^n(s) \in C^4[0, 1]$. Then, the solution Z_i^n of (24) satisfies the bound*

$$|\mathcal{L}^{\Delta t, h}(Z^n(s_i) - Z_i^n)| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-3} \exp\left(-\frac{\beta}{\varepsilon} s_i\right) \right). \tag{30}$$

$$\begin{aligned} \mathcal{L}^{\Delta t, h} \vartheta_{i,n} &= -\left(\varepsilon\sigma - \frac{hb(s_{i-1}, t_n)}{30} + \frac{hb(s_{i+1}, t_n)}{30}\right) \left(\frac{\Pi \pm Z_{i-1}^n - 2(\Pi \pm Z_i^n) + \Pi \pm Z_{i+1}^n}{h^2}\right) \\ &\quad + \frac{b(s_{i-1}, t_n)}{60h} (-3(\Pi \pm Z_{i-1}^n) + 4(\Pi \pm Z_i^n) - (\Pi \pm Z_{i+1}^n)) \\ &\quad + \frac{28b(s_i, t_n)}{60h} (\Pi \pm Z_{i+1}^n - (\Pi \pm Z_{i-1}^n)) \\ &\quad + \frac{b(s_{i+1}, t_n)}{60h} (\Pi \pm Z_{i-1}^n - 4(\Pi \pm Z_i^n) + 3(\Pi \pm Z_{i+1}^n)) \\ &\quad + \frac{P(s_{i-1}, t_n)}{30} (\Pi \pm Z_{i-1}^n) + \frac{28P(s_i, t_n)}{30} (\Pi \pm Z_i^n) + \frac{P(s_{i+1}, t_n)}{30} (\Pi \pm Z_{i+1}^n) \\ &= \left(\frac{P(s_{i-1}, t_n)}{30} + \frac{28P(s_i, t_n)}{30} + \frac{P(s_{i+1}, t_n)}{30}\right) \Pi \pm \mathcal{L}^{\Delta t, h} Z_i^n \\ &= \left(\frac{P(s_{i-1}, t_n)}{30} + \frac{28P(s_i, t_n)}{30} + \frac{P(s_{i+1}, t_n)}{30}\right) \left(\frac{\|\mathcal{L}^{\Delta t, h} Z_i^n\|}{\zeta} + \max\{|\psi_l(t_n)|, |\psi_r(t_n)|\}\right) \pm \frac{1}{30} (R_{i-1}^n + 28R_i^n + R_{i+1}^n) \geq 0. \end{aligned} \tag{27}$$

Table 1 Example 4.1, maximum absolute errors for the case $\tau = 0.5\varepsilon$

$\varepsilon \downarrow$	N = M = 16	32	64	128	256
2^{-0}	1.3260e-03	7.4390e-04	3.9431e-04	2.0293e-04	1.0294e-04
2^{-2}	1.3097e-02	7.2784e-03	3.8340e-03	1.9676e-03	9.9661e-04
2^{-4}	2.7648e-02	1.5726e-02	8.3824e-03	4.3310e-03	2.2011e-03
2^{-6}	3.7674e-02	1.9780e-02	1.0209e-02	5.3769e-03	2.7671e-03
2^{-8}	3.7154e-02	2.2029e-02	1.2040e-02	5.6488e-03	2.8289e-03
2^{-10}	3.7206e-02	2.1942e-02	1.1903e-02	6.2248e-03	3.1986e-03
2^{-12}	3.7221e-02	2.1951e-02	1.1906e-02	6.1986e-03	3.1627e-03
2^{-14}	3.7224e-02	2.1953e-02	1.1907e-02	6.1992e-03	3.1627e-03
2^{-16}	3.7225e-02	2.1953e-02	1.1907e-02	6.1993e-03	3.1627e-03
2^{-18}	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
2^{-20}	3.7225e-02	2.1953e-02	1.1908e-02	6.1994e-03	3.1627e-03
Proposed scheme					
$E^{N,M}$	3.7225e-02	2.2029e-02	1.2040e-02	6.2248e-03	3.1986e-03
$r^{N,M}$	0.77417	0.87156	0.95174	0.96059	-
Method in [14]					
$E^{N,M}$	8.3951e-02	4.9224e-02	2.6666e-02	1.3880e-02	7.0816e-03
$r^{N,M}$	0.77019	0.88436	0.94199	0.97086	-

Proof In the spatial direction the local truncation error is given by

$$\begin{aligned}
 & \left| \mathcal{L}^{\Delta t, h}(Z^n(s_i) - Z_i^n) \right| = \left| -\varepsilon \left(\frac{d}{ds^2} - \sigma \delta_s^2 \right) Z^n(s_i) + \frac{b(s_{i-1}, t_n)}{60h} \right. \\
 & \quad \left(\frac{dZ^n(s_{i-1})}{ds} - \left(\frac{-Z_{i+1}^n + 4Z_i^n - 3Z_{i-1}^n}{2h} + h \frac{d^2 Z^n(s_i)}{ds^2} \right) \right) \\
 & \quad + \frac{28b(s_i, t_n)}{60h} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_i) + \frac{b(s_{i+1}, t_n)}{60h} \\
 & \quad \left. + \left(\frac{dZ^n(s_{i+1})}{ds} - \left(\frac{3Z_{i+1}^n - 4Z_i^n + Z_{i-1}^n}{2h} - h \frac{d^2 Z^n(s_i)}{ds^2} \right) \right) \right| \\
 & \leq \left| \left(\varepsilon (b(s_i, t_n) \frac{\rho}{2} \coth(b(0) \frac{\rho}{2}) - 1) \delta_s^2 Z^n(s_i) \right) \right| + \left| \varepsilon \left(\frac{d^2}{ds^2} - \delta_s^2 \right) Z^n(s_i) \right| \\
 & \quad \left| \frac{b(s_{i-1}, t_n)}{60h} \left(\frac{dZ^n(s_{i-1})}{ds} - \left(\frac{-Z_{i+1}^n + 4Z_i^n - 3Z_{i-1}^n}{2h} + h \frac{d^2 Z^n(s_i)}{ds^2} \right) \right) \right| \\
 & \quad + \left| \frac{28b(s_i, t_n)}{60h} \left(\frac{d}{ds} - \delta_s^0 \right) Z^n(s_i) \right| + \left| \frac{b(s_{i+1}, t_n)}{60h} \right. \\
 & \quad \left. \left(\frac{dZ^n(s_{i+1})}{ds} - \left(\frac{3Z_{i+1}^n - 4Z_i^n + Z_{i-1}^n}{2h} - h \frac{d^2 Z^n(s_i)}{ds^2} \right) \right) \right|,
 \end{aligned}$$

where $\sigma = b(s_i, t_n) \frac{\rho}{2} \coth(b(0) \frac{\rho}{2})$ and $\rho = \frac{h}{\varepsilon}$.

Generally, $\forall \rho > 0$, we put as

For the constants C_1 and C_2 we have $|\rho \coth(\rho) - 1| \leq C_1 \rho^2$, for $\rho \leq 1$. For $\rho \rightarrow \infty$, since $\lim_{\rho \rightarrow \infty} \coth(\rho) = 1$ which gives $|\rho \coth(\rho) - 1| \leq C_1 \rho$.

$$C_1 \frac{\rho^2}{\rho + 1} \leq \rho \coth(\rho) - 1 \leq C_2 \frac{\rho^2}{\rho + 1}, \tag{31}$$

Table 2 Example 4.2, maximum absolute errors for the case $\tau = 0.5\varepsilon$

$\varepsilon \downarrow$	N = M = 16	32	64	128	256
2^{-0}	2.4094e-04	4.5191e-05	5.3745e-06	4.6776e-06	3.4511e-06
2^{-2}	5.8848e-04	1.7058e-04	1.1039e-04	6.2476e-05	3.3191e-05
2^{-4}	2.1117e-03	7.2963e-04	3.0344e-04	1.3611e-04	6.4377e-05
2^{-6}	4.6754e-03	2.2578e-03	8.2105e-04	2.8430e-04	1.0734e-04
2^{-8}	4.8645e-03	2.9411e-03	1.5511e-03	6.7433e-04	2.3656e-04
2^{-10}	4.8682e-03	2.9448e-03	1.6029e-03	8.4404e-04	4.2558e-04
2^{-12}	4.8691e-03	2.9452e-03	1.6032e-03	8.4463e-04	4.3880e-04
2^{-14}	4.8693e-03	2.9453e-03	1.6032e-03	8.4466e-04	4.3881e-04
2^{-16}	4.8693e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
2^{-18}	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
2^{-20}	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
$E^{N,M}$	4.8694e-03	2.9454e-03	1.6032e-03	8.4467e-04	4.3881e-04
$r^{N,M}$	0.7253	0.8775	0.9245	0.9448	-

we obtain

$$\varepsilon [b(s_i, t_n) \frac{\rho}{2} \coth \left(b(0) \frac{\rho}{2} \right) - 1] \leq \varepsilon \frac{(h/\varepsilon)^2}{h/\varepsilon + 1} = \frac{h^2}{h + \varepsilon}. \tag{32}$$

Using the bound for the difference of the derivatives in (29) and (32), we obtain

$$\begin{aligned} \left| \mathcal{L}^{\Delta t, h} (Z^n(s_i) - Z_i^n) \right| &\leq \frac{Ch^2}{h + \varepsilon} \left\| \frac{d^2 Z^n(s_i)}{ds^2} \right\| \\ &+ Ch^2 \left\| \frac{d^3 Z^n(s_i)}{ds^3} \right\| + C\varepsilon h^2 \left\| \frac{d^4 Z^n(s_i)}{ds^4} \right\|. \end{aligned}$$

From Lemma 3.2, we obtain the bound for the derivatives

$$\left| \mathcal{L}^{\Delta t, h} (Z^n(s_i) - Z_i^n) \right| \leq \frac{Ch^2}{h + \varepsilon} \left(1 + \varepsilon^{-2} \exp \left(-\frac{\beta}{\varepsilon} s_i \right) \right) + Ch^2 \left[\left(1 + \varepsilon^{-3} \exp \left(-\frac{\beta}{\varepsilon} s_i \right) \right) + \left(\varepsilon + \varepsilon^{-3} \exp \left(-\frac{\beta}{\varepsilon} s_i \right) \right) \right]. \tag{33}$$

Evidently, $\varepsilon^{-2} \leq \varepsilon^{-3}$, then we obtain

$\left| \mathcal{L}^{\Delta t, h} (Z^n(s_i) - Z_i^n) \right| \leq \frac{Ch^2}{h + \varepsilon} (1 + \varepsilon^{-3} \exp(-\frac{\beta}{\varepsilon} s_i))$ thus, it gives the desired bound. \square

Lemma 3.6 For a fixed mesh and as $\varepsilon \rightarrow 0$, it gives

$$\lim_{\varepsilon \rightarrow 0} \max_i \frac{\exp(-\beta s_i / \varepsilon)}{\varepsilon^n} = 0, \quad n = 1, 2, 3, \dots, \tag{34}$$

where $s_i = ih, 1 \leq i \leq N - 1$.

Proof The proof is considered in [9]. \square

Theorem 3.2 Let Z_i^n be the solution of (24), then we have the following uniform error bound

$$\sup_{\varepsilon \in (0,1]} \max_i |Z^n(s_i) - Z_i^n| \leq Ch, \quad i = 0, 1, 2, \dots, N. \tag{35}$$

Proof Plugging the result in Lemma 3.6 into (30), we arrive at

$$\left| \mathcal{L}^{\Delta t, h} (Z^n(s_i) - Z_i^n) \right| \leq \frac{Ch^2}{h + \varepsilon}. \tag{36}$$

Hence, the result leads $|Z^n(s_i) - Z_i^n| \leq \frac{Ch^2}{h + \varepsilon}$. Using the sup over all $\varepsilon \in (0, 1]$, we get

$$\sup_{\varepsilon \in (0,1]} \max_i |Z^n(s_i) - Z_i^n| \leq Ch. \tag{37}$$

From the preceding theorem for the case when $\varepsilon > h$, the obtained scheme gives second order uniformly convergent. For the case when $\varepsilon \ll h$, the scheme is first order uniformly convergent in spatial direction. \square

Theorem 3.3 Let z and Z are the solutions of (1) and (24) respectively, then we have the following uniform error bound

$$\sup_{\varepsilon \in (0,1]} |z - Z| \leq C(h + (\Delta t)). \tag{38}$$

Proof The proof can be done by the combination of Lemma 3.1 and Theorem 3.2. \square

Numerical results and discussions

In this part, we are considering two model examples to validate the theoretical results obtained by the proposed method. If the exact solutions of the considered examples are not known the maximum pointwise error is estimated by using the double mesh principle. So, the maximum pointwise error is calculated by $E_\epsilon^{N,M} = \max_{i,n} |Z_{i,n}^{N,M} - Z_{i,n}^{2N,2M}|$, and the ϵ -uniform error is estimated by $E_\epsilon^{N,M} = \max_{i,n} (E_\epsilon^{N,M})$. The rate of convergence is calculated by $r_\epsilon^{N,M} = \log 2(E_\epsilon^{N,M}/E_\epsilon^{2N,2M})$, and the ϵ -uniform rate of convergence is computed by $r^{N,M} = \log 2(E^{N,M}/E^{2N,2M})$.

Example 4.1 Consider the problem $\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + (\frac{1+x^2}{2})u = t^3 - u(x, t - \tau)$, $(x, t) \in (0, 1) \times (0, 2]$ with interval condition $u(x,t) = 0$, on $(x, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$, $t \in (0, 2]$.

Example 4.2 Consider the problem

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} - (2 + x^2) \frac{\partial u}{\partial x} + xu(x, t) + u(x, t - \tau) = 10t^2 \exp(-t)x(1 - x),$$

$(x, t) \in (0, 1) \times (0, 2]$ with interval condition $u(x, t) = 0$, on $(x, t) \in [0, 1] \times [-\tau, 0]$ and the boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$, $t \in (0, 2]$.

For different values of ϵ and an equal number of mesh points the maximum pointwise error, ϵ -uniform error and ϵ -uniform rate of convergence of the proposed method are displayed in Tables 1 and 2. We observe from these Tables, as $\epsilon \rightarrow 0$, the maximum pointwise error after showing increment remains uniform. This shows that the scheme is stable and uniformly convergent irrespective of the values of ϵ . The ϵ -uniform error and ϵ -uniform rate of convergence of the method are indicated in the last rows of each Tables and it confirms that the numerical results agree with the theoretical result.

In Fig. 1, we show the numerical solution of the scheme for Example 4.1 for different values of ϵ . From these Figures, it can be seen that a strong boundary layer is created on the left side of the spatial domain for small ϵ .

In the last section of Table 1, the comparison of the proposed scheme with the results of the existing published work of [14] is given. As one observes, the developed scheme gives more accurate result than the scheme in [14].

Conclusion

We developed a numerical scheme to solve a singularly perturbed parabolic convection-diffusion equation that exhibits a boundary layer. The proposed scheme consists of the backward-Euler method in the time direction and an exponentially fitted finite difference scheme for the spatial direction. Using the comparison principle, the stability of the discrete scheme is examined and analysed. The uniform convergence of the scheme is discussed theoretically. To validate the theoretical finding of the scheme, we considered two model examples and the numerical results are given by applying maximum pointwise absolute error, ϵ -uniform error and ϵ -uniform rate of convergence in Tables. The proposed method contributes more accurate, stable and ϵ -uniform numerical result with linear order of convergence.

Limitations

- The proposed scheme is not layer resolving method since there is no sufficient number of mesh points in the boundary layer region.

Appendix

Proof of Lemma 3.1 Since the function $Z^n(s)$ satisfies

$$\begin{aligned} (1 - \tau c(s, t_n) + \Delta t \mathcal{L})Z^n(s) - \Delta g(s, t_n) &= Z^{n-1}(s) \\ Z(t_{n-1}) &= (1 - \tau c(t_n) + \Delta t \mathcal{L})Z(t_n) - \Delta g(t_n) \\ &+ \int_{t_{n-1}}^{t_n} (t_{n-1} - s) \frac{\partial^2 Z(s)}{\partial t^2} ds \\ &= (1 - \tau c(t_n) + \Delta t \mathcal{L})Z(t_n) \\ &- \Delta g(t_n) + O(\Delta t^2). \end{aligned}$$

Then $e_n(s)$ is the solution of boundary value problem of type

$$((1 - \tau c(s, t_n) + \Delta t \mathcal{L})e_n(s) = O(\Delta t^2).$$

$e_n(0) = 0 = e_n(1)$ by manimum principle, we obtain $\|e_n\| \leq C_1(\Delta t^2)$.

Taking the summation of all local error estimate up to n^{th} time step the global error estimate is given by

$$\begin{aligned} \|E_n\| &= \left\| \sum_{l=1}^n e_l \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \dots + \|e_n\| \\ &\leq C_1 T(\Delta t), \text{ since } (n)\Delta t \leq T \\ &= C(\Delta t), \text{ where } C_1 T = C, \end{aligned}$$

where C is a constant not depends on ϵ and Δt .

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Authors contributions

SKT and MMW carried out the numerical scheme development, algorithms writing, MATLAB code writing and the numerical experimentation. TGD and GFD formulated the problem, design, and write draft of the manuscript. All authors read, commented and approved the final version of the manuscript.

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