



Research article

Dynamical behaviour of single photobioreactor with variable yield coefficient

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ABSTRACT

Scholars studied chemostat model with variable yield coefficient and a growth rate in Monod expression for the existence of natural oscillations in a bioreactor. This article explores dynamical properties of a similar simple model, analytically and numerically, in which the growth rate is a modified Haldane expression. Study includes determination of analytic conditions for existence of steady-state washout and no washout solutions, optimization of the performance of the bioreactor when no washout solution occurs, stability of the optimized steady state solution, and the ranges of the parameter values for which natural oscillations (Hopf Bifurcation) take place. Investigation shows that it is possible to gain natural oscillations for much smaller values of the substrate concentration compared to Monod-based earlier works.

1. Introduction

Recently, researchers have been attracted to the development and study of mathematical models to meet the demand for microbial products such as fermented food products, biofuel, cosmetics, medicines, vitamins, antibiotics, and food supplements [1, 2, 3]. A continuous closed stirred photobioreactor, also known as a chemostat, is an experimental tool that is used to study for better understanding the mechanism of microbial growth of some microorganisms. The growth rate in the closed photobioreactor is controlled and may be described by growth models. Chemostat systems can be effectively used for microorganism growth.

In 1950, Monod [4] and Novick et al. [5] investigated the growth of microorganisms in chemostats known as a CSTR (continuous stir tank reactor), continuous controlled laboratory instruments. These types of models are commonly called constant yield models [6]. The specific growth rate of biomass in photobioreactors can be predicted using a number of well-known kinetic models such as Andrews [7], Contois [8], Logistics [9], Moser [10, 11] and Teissier [12]. Cho et al. [13, 14] cultivate microalgae in a photobioreactor (PBR) under a chemostat environment, yielding more than a ten-fold cumulative increase in productivities. This type of model has been studied both qualitatively and numerically in the past few decades for wastewater purification systems [15, 16, 17]. Improving product yield in bioreactors or chemical reactors

has been under extensive research efforts in the past few decades [18]. Oscillatory external forcing is an important engineering instrument to improve biomass/chemical mass production to the desired scale. However, the cost of logistics associated with the implementation of external forcing has limited industrial uptake of this scheme [19, 20, 21]. Many authors have investigated the possibility of combining oscillatory operations using two reactors arranged in series [22, 23, 24]. In a natural oscillation, parameter values are determined so that a steady input of the reactants (substrate concentration) into the first bioreactor stimulates self-sustained oscillation in its output that externally forces the second reactor in a periodic fashion without requiring any extra energy.

For constant yield the chemostat system (1) and (2) has no periodic solutions [25] for any specific growth rate function. If the yield coefficient is a function of the substrate concentration, limit cycles can be seen in specific situations [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Teissier [12], Nelson and Sidhu [24] studied the chemostat model with variable yield coefficient and specific growth rate function to find natural oscillations in the bioreactor. Motivated by their work, we propose and study a chemostat model with a modified growth rate function that has not been investigated by any scholar, as far as we know.

Researchers investigate the two-component system with constant or variable yield coefficients [17, 37]. We study the case of a single bioreactor because two reactors in a series studied by [38, 39] assume no

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refluxing; therefore, conditions for natural oscillations that occur in the first reactor are solely determined by the self-dynamics of the reactor. Since the oscillations in the first reactor are used to stimulate the second reactor, it is more practical to determine the necessary conditions that ensure oscillation in the first reactor. Nelson and Sidhu [24] believed that it is better to determine a benchmark for the maximum output from a single reactor before investigating a two-reactor system. Balakrishnan and Yang [40] examined a single bio-reactor model by direct integration for parameter values in a very limited range. On the other hand, Nelson and Sidhu [24] studied the Monod-based model analytically by exploiting the theory of bifurcation and path following methods.

The contents of the manuscript are organized as follows. Chemostat model is described with our proposed specific choice of growth function in section 2.1, which is nondimensionalized in section 2.2. Dynamical properties of the proposed model has been analyzed in section 3. Finally, achievement of our analysis has been discussed in section 4.

2. Model equations

2.1. Dimensional model description

We study the following microbial system where a cell mass concentration (X) grows by consuming some substrate species (S) [31],

$$V \frac{dS}{dt} = F(S_0 - S) - VX \frac{\mu(S)}{Y(S)} \tag{1}$$

$$V \frac{dX}{dt} = F(X_0 - X) + VX\mu(S) \tag{2}$$

with the specific growth rate $\mu(S)$, like the Haldane [27] type models presented in

$$\mu(S) = \frac{\mu_m S(t)}{k_s + S + \frac{S^2}{k_i}} \left(1 - \frac{k_i}{q} \right), \tag{3}$$

and a variable yield coefficient

$$Y(S) = \alpha + \beta S, (\alpha, \beta > 0), \tag{4}$$

where F represents the flow rate (l/hr), k_s indicates the half-saturation constant (g/l), k_i represents the inhibition constant (g/l), V is the volume (l), X_0 denotes the initial cell mass concentration (g/l), S_0 denotes the initial substrate concentration (g/l), μ_m represents the maximum specific growth rate (/h), α (-) and β (l/g) are constant yield coefficients.

The objective of the study is to optimize the cell mass concentration as a function of the residence time (τ) in a reactor. The study of this kind of microbial system can be found in [19, 21, 24]. The bio-reactor model which we are studying is a very simple one, while more complex models are available in the literature. Nevertheless, we are studying the simple model because of its importance in bioreactor engineering and food industries.

2.2. Dimensionless model

We introduce the transformations $S^* = \frac{S}{k_s}$, $X^* = \frac{X}{\alpha k_s}$, and $t^* = t\mu_m$ to make the system of differential Eqs. (1) and (2) dimensionless by using Eqs. (3) and (4) as follows

$$\frac{dS^*}{dt^*} = \frac{S_0^* - S^*}{\tau^*} - \frac{S^* X^* (1 - \delta^*)}{(1 + S^* + \gamma^* S^{*2})(1 + \beta^* S^*)} \tag{5}$$

$$\frac{dX^*}{dt^*} = \frac{X_0^* - X^*}{\tau^*} + \frac{S^* X^*}{(1 + S^* + \gamma^* S^{*2})} (1 - \delta^*) \tag{6}$$

where $\tau^* = \frac{V\mu_m}{F}$, $\beta^* = \frac{\beta k_s}{\alpha}$, $\delta^* = \frac{k_i}{q}$ and $\gamma^* = \frac{k_s}{k_i}$.

The non-dimensional model contains the parameter S_0^* , X_0^* , τ^* , β^* , δ^* and γ^* . We assume $X_0^* = 0$ that represents a sterile feed condition and τ^* (residence time) as our primary bifurcation parameter, and S_0^* (substrate concentration in the feed), β^* (dimensionless yield coefficient), δ^* , and γ^* are dimensionless inhibition parameters as secondary bifurcation parameters. Specific microbial system determines the value of β^* . Therefore, it is not very flexible. Since there is one-to-one correspondence between the dimensional and dimensionless variables, we will denote them interchangeably by the same names and symbols. For example, 'the residence time' and 'the dimensionless residence time' will be denoted by the same notation τ .

3. Results and discussion

3.1. Steady-state solutions and their stability

3.1.1. Steady-state solutions

The system of differential Eqs. (5) and (6) has one washout steady state solution

$$(S, X) = (S_0, 0) \tag{7}$$

and two no washout steady-state solutions

$$NW1 = (S, X) = \left(\begin{array}{c} \frac{1 + (-1 + \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{2\gamma}, \\ \frac{1}{2\gamma^2} \left(\begin{array}{c} \gamma \left(1 + 2S_0\gamma - \tau + \delta\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2} \right) \\ -\beta \left((1 + (-1 + \delta)\tau) \left(\frac{1 + (-1 + \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{\sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}} \right) \right) \\ +\gamma \left(-2 + S_0 \left(\frac{1 + (-1 + \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{\sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}} \right) \right) \end{array} \right) \end{array} \right) \tag{8}$$

and

$$NW2 = (S, X) = \left(\begin{array}{c} \frac{-1 + (1 - \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{2\gamma}, \\ \frac{1}{2\gamma^2} \left(\begin{array}{c} \gamma \left(1 + 2S_0\gamma - \tau + \delta\tau - \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2} \right) \\ +\beta \left((1 + (-1 + \delta)\tau) \left(\frac{-1 + (1 - \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{\sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}} \right) \right) \\ +\gamma \left(2 + S_0 \left(\frac{-1 + (1 - \delta)\tau + \sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}}{\sqrt{-4\gamma + (1 + (-1 + \delta)\tau)^2}} \right) \right) \end{array} \right) \end{array} \right) \tag{9}$$

The substrate component in (8)(8) and (9)(9) are real when $0 < \delta < 1$ and $\geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$. The cell mass concentration component in (8) is nonnegative when $\gamma > \frac{1}{S_0^2}$, $0 < \delta < 1$, and $\tau \geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$, and cell mass concentration in (9) is nonnegative when $\gamma > \frac{1}{S_0^2}$, $0 < \delta < 1$, and $\frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}} \leq \tau < \frac{-1-S_0-S_0^2\gamma}{-S_0+S_0\delta}$. Both no washout solutions are physically feasible whenever $\gamma > \frac{1}{S_0^2}$, $0 < \delta < 1$, and $\frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}} \leq \tau < \frac{-1-S_0-S_0^2\gamma}{-S_0+S_0\delta}$. It is important and instructive to study the properties of X in the no

washout steady-state solutions of the system (reactor's performance) as a function of the residence time τ . In particular, we are interested in optimum values of the cell mass concentration.

In Eq. (8), it can be shown that $\frac{dX}{d\tau} = 0$ at

$$\tau_{\max} = \frac{2\beta - 4\beta^2 - 2S_0\beta^2 - \gamma + 2S_0\beta\gamma - S_0^2\beta^2\gamma}{2\beta(-1 + S_0\beta)(-1 + \delta)} \tag{10}$$

subject to the restriction that

$$S_0 > \frac{1}{\beta}, 0 < \delta < 1 \tag{11}$$

and

$$\frac{1}{S_0^2} < \gamma < \frac{4\beta^2}{1 - 2S_0\beta + S_0^2\beta^2} \tag{12}$$

At the critical point in Eq. (10), $X''(\tau)$ is negative as long as the critical point τ_{\max} is in the following interval:

$$\begin{aligned} & \frac{1}{1 - \delta} + 2\sqrt{\frac{\gamma}{(-1 + \delta)^2}} < \tau_{\max} < \text{Root} \left[-\beta + 3\beta^2 + S_0\beta^2 + 6\beta\gamma - 16\beta^2\gamma - 6S_0\beta^2\gamma - \gamma^2 \right. \\ & + 2S_0\beta\gamma^2 - S_0^2\beta^2\gamma^2 + (3\beta - 6\beta^2 - 3S_0\beta^2 - 6\beta\gamma + 6S_0\beta^2\gamma - 3\beta\delta + 6\beta^2\delta + 3S_0\beta^2\delta + \\ & 6\beta\gamma\delta - 6S_0\beta^2\gamma\delta)\#\mathbb{1} + (-3\beta + 3\beta^2 + 3S_0\beta^2 + 6\beta\delta - 6\beta^2\delta - 6S_0\beta^2\delta - 3\beta\delta^2 + 3\beta^2\delta^2 \\ & \left. + 3S_0\beta^2\delta^2)\#\mathbb{1}^2 + (\beta - S_0\beta^2 - 3\beta\delta + 3S_0\beta^2\delta + 3\beta\delta^2 - 3S_0\beta^2\delta^2 - \beta\delta^3 + S_0\beta^2\delta^3)\#\mathbb{1}^3 \& , 3 \right] \end{aligned}$$

where the upper bound of τ_{\max} depends on $\beta, \gamma, \delta,$ and S_0 . For example, $\beta = 2, \gamma = 0.9, \delta = 0.5,$ and $S_0 = 2$ gives an upper bound 6.62694 of the critical point τ_{\max} . The τ_{\max} does not exceed the upper bound when

$$\frac{1}{S_0^2} < \gamma \leq \frac{4}{S_0^2} \tag{13}$$

or

$$\left(\gamma > \frac{4}{S_0^2} \text{ and } \beta < 2\sqrt{\frac{\gamma}{(-4 + S_0^2\gamma)^2} + \frac{S_0\gamma}{-4 + S_0^2\gamma}} \right) \tag{14}$$

Therefore, X has a local maximum value at τ_{\max} .

In Eq. (9), it can be shown that $\frac{dX}{d\tau} = 0$ at τ_{\max} , subject to the restriction that $S_0 > \frac{1}{\beta}, \gamma > \frac{4\beta^2}{1 - 2S_0\beta + S_0^2\beta^2},$ and $0 < \delta < 1.$ $X''(\tau)$ is negative as long as critical point τ_{\max} is satisfied the following conditions.

$$\left(\gamma > \frac{4}{S_0^2}, 0 < \delta < 1, \text{ and } \beta > 2\sqrt{\frac{\gamma}{(-4 + S_0^2\gamma)^2} + \frac{S_0\gamma}{-4 + S_0^2\gamma}} \right) \tag{15}$$

Therefore, X has a local maximum value at τ_{\max} .

3.1.2. Stability of the washout steady state solution (7)

Jacobian Matrix: Jacobian Matrix of the system (5) and (6) at the washout steady state solution (7) is

$$J = \begin{pmatrix} -\frac{1}{\tau} & \frac{-S_0(1 - \delta)}{(1 + S_0\beta)(1 + S_0 + S_0^2\gamma)} \\ 0 & \frac{S_0(1 - \delta)}{1 + S_0 + S_0^2\gamma} - \frac{1}{\tau} \end{pmatrix} \tag{16}$$

The Jacobian matrix (16) has eigenvalues $-\frac{1}{\tau}$ and $\frac{(1 - \delta)S_0}{1 + S_0 + S_0^2\gamma} - \frac{1}{\tau}.$ Therefore, the washout steady state solution is stable if $\tau < \frac{1 + S_0 + S_0^2\gamma}{(1 - \delta)S_0}.$

3.1.3. Stability of no washout steady state solution (8)

Jacobian Matrix: Jacobian Matrix of the system (5) and (6) at the no washout steady state solution (8) is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}$$

where

$$J_{11} = (-1/\tau^2) \left(\begin{aligned} & \tau + \left(\frac{2\beta\tau}{\beta(1 + (-1 + \delta)\tau)} \right) / \left(\left(\frac{-2\gamma + \beta(1 + (-1 + \delta)\tau + \omega)}{\beta(1 + (-1 + \delta)\tau + \omega)} \right) \right)^2 \\ & - \left(2\tau \left(\frac{\beta(1 + (-1 + \delta)\tau)}{(1 + (-1 + \delta)\tau + \omega) - \gamma \left(\frac{1 + 2S_0\gamma + (-1 + \delta)\tau + \omega + \beta\gamma(-2 + S_0(1 + (-1 + \delta)\tau + \omega))}{(1 + (-1 + \delta)\tau + \omega)} \right)} \right) \right) / \left(\left(\frac{(1 + (-1 + \delta)\tau + \omega)}{(-2\gamma + \beta(1 + (-1 + \delta)\tau + \omega))} \right) \right) \right. \\ & \left. + \left(2((-1 + \delta)\tau + \omega) \left(\beta(1 + (-1 + \delta)\tau)(1 + (-1 + \delta)\tau + \omega) - \gamma \left(\frac{1 + 2S_0\gamma + (-1 + \delta)\tau + \omega + \beta\gamma(-2 + S_0(1 + (-1 + \delta)\tau + \omega))}{(-1 + \delta)\tau + \omega + \beta\gamma(-2 + S_0(1 + (-1 + \delta)\tau + \omega))} \right) \right) \right) / \right. \\ & \left. \left((-1 + \delta)(1 + (-1 + \delta)\tau + \omega)(-2\gamma + \beta(1 + (-1 + \delta)\tau + \omega)) \right) \right) \end{aligned} \right) \tag{17}$$

$$J_{12} = \frac{2\gamma}{\tau(-2\gamma + \beta(1 + (-1 + \delta)\tau + \omega))} \tag{18}$$

$$J_{21} = \frac{(-1/(\gamma(-1 + \delta)\tau^2(1 + (-1 + \delta)\tau + \omega)^2))(((4\gamma - (1 + (-1 + \delta)\tau)(1 + (-1 + \delta)\tau + \omega)(\gamma(1 + 2S_0\gamma + (-1 + \delta)\tau + \omega) - \beta((1 + (-1 + \delta)\tau)(1 + (-1 + \delta)\tau + \omega) + \gamma(-2 + S_0(1 + (-1 + \delta)\tau + \omega))))))}{(1 + (-1 + \delta)\tau + \omega)} \tag{19}$$

and. At the critical point τ_{max} , it can be shown under the restrictions (11) (12) (13), and (14), that the trace of the Jacobian matrix is negative, and the determinant of the Jacobian matrix is positive. We conclude that the steady-state solution (8) at τ_{max} is stable and it is very important for practical applications.

If $Det(J) = J_{12}J_{21} = 0$, the Jacobian matrix has a zero eigenvalue. This happens whenever $\tau = \frac{1}{1-\delta} + \frac{2\sqrt{\gamma}}{\sqrt{(-1+\delta)^2}}$, $\tau = \frac{1}{1-\delta} - \frac{2\sqrt{\gamma}}{\sqrt{(-1+\delta)^2}}$, or $\tau = \frac{1+S_0+S_0^2\gamma}{S_0}$. The conditions for the double zero eigenvalues of the Jacobian matrix are

It turns out that these conditions are satisfied from Eqs. (17), (18), and (19) when

$$S_0 = \frac{1}{\beta} - \frac{2(-1 + \delta)}{\sqrt{\gamma(-1 + \delta)^2}}, \tau = \frac{1 + 2\sqrt{\gamma(-1 + \delta)^2} - \delta}{(-1 + \delta)^2}$$

$$\gamma + 2\beta\gamma + \beta\sqrt{\gamma(-1 + \delta)^2} + 2\gamma\sqrt{\gamma(-1 + \delta)^2} \neq (1 + 2\beta)\gamma\delta$$

The positivity of S_0 in the feed immediately implies that double zero eigenvalues can arise.

3.1.4. Stability of no washout steady state solution (9)

Jacobian Matrix: Jacobian Matrix of the system (5) and (6) at the no washout steady state solution (9) is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & 0 \end{bmatrix}$$

where

$$J_{12} = \frac{-2\gamma}{\tau(2\gamma + \beta(-1 + \tau - \delta\tau + \omega))} \tag{21}$$

$$\text{and } \omega = \sqrt{-4\gamma + (-1 + (-1 + \delta)\tau)^2}$$

Under condition (15), the trace of the Jacobian matrix at the critical point is always nonnegative and the determinant of the Jacobian matrix is always nonpositive. Therefore, the maximum steady state solution (9) is not stable.

If $Det(J) = J_{12}J_{21} = 0$, the Jacobian matrix has a zero eigenvalue. This happens whenever $\tau = \frac{1}{1-\delta} + \frac{2\sqrt{\gamma}}{\sqrt{(-1+\delta)^2}}$, $\tau = \frac{1}{1-\delta} - \frac{2\sqrt{\gamma}}{\sqrt{(-1+\delta)^2}}$, or $\tau = \frac{1+S_0+S_0^2\gamma}{S_0(1-\delta)}$. But the later value of τ is not relevant. The conditions for the double zero eigenvalues of the Jacobian matrix are

$$J_{21} = \frac{(-1/(\gamma(-1 + \delta)\tau^2(-1 + \tau - \delta\tau + \omega)^2))(((4\gamma + (1 + (-1 + \delta)\tau)(-1 + (1 - \delta)\tau + \omega)(\gamma(1 + 2S_0\gamma + (-1 + \delta)\tau - \omega) + \beta(1 + (-1 + \delta)\tau)(-1 + (1 - \delta)\tau + \omega) + \beta\gamma(2 + S_0(1 + (-1 + \delta)\tau + \omega))))))}{(1 + (-1 + \delta)\tau + \omega)} \tag{22}$$

It turns out that these conditions are satisfied from Eqs. (20), (20), and (22) when

$$S_0 = \frac{1}{\beta} - \frac{2(-1 + \delta)}{\sqrt{\gamma(-1 + \delta)^2}}, \tau = \frac{1 + 2\sqrt{\gamma(-1 + \delta)^2} - \delta}{(-1 + \delta)^2}$$

$$\text{and } \gamma + 2\beta\gamma + \beta\sqrt{\gamma(-1 + \delta)^2} + 2\gamma\sqrt{\gamma(-1 + \delta)^2} \neq (1 + 2\beta)\gamma\delta$$

The positivity of S_0 in the feed immediately implies that double zero eigenvalues can arise.

3.1.5. Hopf bifurcation on the no washout steady state solution (8)

The condition for the Hopf bifurcation is $J_{11} = 0$ with $J_{12}J_{21} < 0$ [42]. The latter condition occurs when

$$0 < \delta < 1, \tau > \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}, \text{ and}$$

$$\gamma \left(1 + 2S_0\gamma - \tau + \delta\tau + \gamma\sqrt{\frac{-4\gamma + (1 + (-1 + \delta)\tau)^2}{\gamma^2}} \right) > 0$$

which is equivalent to the following cases. In each case, Hopf bifurcations occur at the values of τ that correspond to the root of $J_{11} = 0$.

$$J_{11} = (1/\tau^2)(-\tau + 2\tau(-\beta(1 + (-1 + \delta)\tau)(-1 + (1 - \delta)\tau + \omega) + \gamma(-1 - 2S_0\gamma + (1 - \delta)\tau + \omega) + \beta\gamma(-2 + S_0(1 + (-1 + \delta)\tau - \omega))) /$$

$$((2\gamma + \beta(-1 + (1 - \delta)\tau + \omega))(-1 + \tau - \delta\tau + \omega) + (2\beta\tau(\gamma(1 + 2S_0\gamma + (-1 + \delta)\tau - \omega) + \beta(1 + (-1 + \delta)\tau)(-1 + \tau - \delta\tau + \omega) + \beta\gamma(2 + S_0(-1 + \tau - \delta\tau + \omega)))) / (2\gamma + \beta(-1 + \tau - \delta\tau + \omega))^2 + (2((1 - \delta)\tau + \omega)(\gamma(1 + 2S_0\gamma + (-1 + \delta)\tau - \omega) + \beta(1 + (-1 + \delta)\tau)(-1 + (1 - \delta)\tau + \omega) + \beta\gamma(2 + S_0(-1 + (1 - \delta)\tau + \omega)))) / (((-1 + \delta)(1 + (-1 + \delta)\tau - \omega)(2\gamma + \beta(-1 + (1 - \delta)\tau + \omega)))) \tag{20}$$

Case I. $0 < S_0 < 2$, $0 < \delta < 1$, $\gamma > \frac{1}{\delta^2}$, and $\tau \geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$. It is possible to show that, at $S_0 = 1.5$, $\gamma = 3$, $\delta = 0.5$, and $\beta = 5.25$, $J_{11} = 0$ at $\tau = 9.63746$ (other values are possible) (Figure 1).

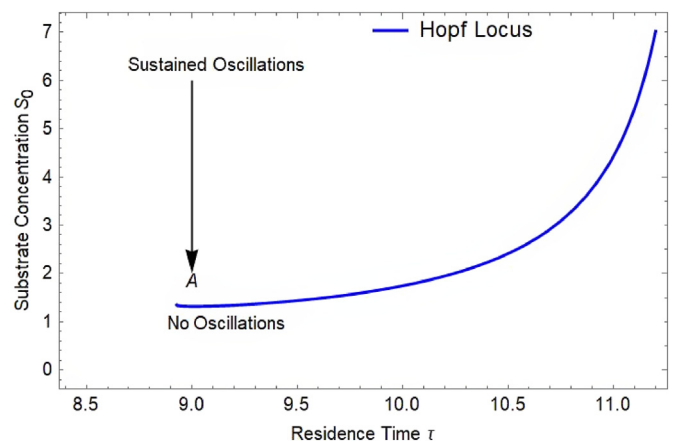


Figure 1. Hopf locus shows the emersion of the two Hopf points at an H21 degeneracy (A). Values of parameters: $\gamma = 3$, $\delta = 0.5$ and $\beta = 5.25$.

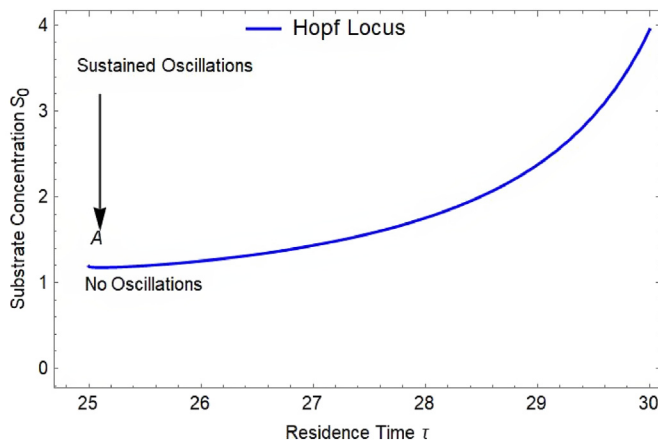


Figure 2. Hopf locus shows the emersion of the two Hopf points at an $H2_1$ degeneracy (A). Values of parameters: $\gamma = 4$, $\delta = 0.8$, and $\beta = 5.25$.

Degenerate Hopf bifurcation occurs at a parameter value (in our case, it is the residence time τ) where two Hopf points annihilate each other resulting in a single point. This is known as $H2_1$ degeneracy, which occurs when $J_{11} = 0$ and $\frac{dJ_{11}}{d\tau} = 0$ [23].

In this case, when $J_{11} = 0$, it can be shown that $\frac{dJ_{11}}{d\tau}$ vanishes for suitable values of γ , β , and δ . For example, if $\gamma = 3$, $\delta = 0.5$, $\beta = 5.25$, then $\frac{dJ_{11}}{d\tau}$ vanishes at $(S_0, \tau) \approx (1.31544, 9.01184)$. Therefore, if S_0 is small enough ($S_0 < 1.31544$) or ($S_0 < 2.30202$ g/l) natural oscillations are not possible for $\gamma = 3$, $\delta = 0.5$, $\beta = 5.25$.

Case II. $S_0 = 2, 0 < \delta < 1, 4\gamma > 1$, and $\tau \geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$. Let, $\gamma = 4$, $\delta = 0.8$, and $\beta = 5.25$, then $J_{11} = 0$, at $\tau = 28.4861$ (other values are possible) (Figure 2).

In this case, when $J_{11} = 0$ it can be shown that $\frac{dJ_{11}}{d\tau}$ vanishes for suitable values of γ , β , and δ . For example, if $\gamma = 4$, $\delta = 0.8$, $\beta = 5.25$, then $\frac{dJ_{11}}{d\tau}$ vanishes at $(S_0, \tau) \approx (1.17753, 25.1095)$. Therefore, if S_0 is small enough ($S_0 < 1.17753$) or ($S_0 < 2.06068$ g/l) natural oscillations are not possible for $\gamma = 4$, $\delta = 0.8$, $\beta = 5.25$.

Case III. $S_0 > 2, 0 < \delta < 1, 4\gamma > 1$, and $\tau \geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$. Let $S_0 = 4$, $\delta = 0.2$, $\gamma = 0.25$, and $\beta = 5.25$, then $J_{11} = 0$ at $\tau = 3.7858$ and $\tau = 2.50315$ (other values are possible) (Figure 3).

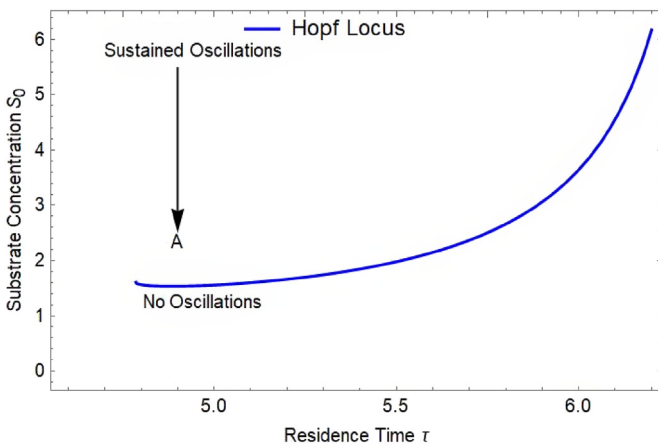


Figure 3. Hopf locus showing the emersion of the two Hopf points at a $H2_1$ degeneracy (A). Values of parameters: $\gamma = 0.25$, $\delta = 0.2$, and $\beta = 5.25$.

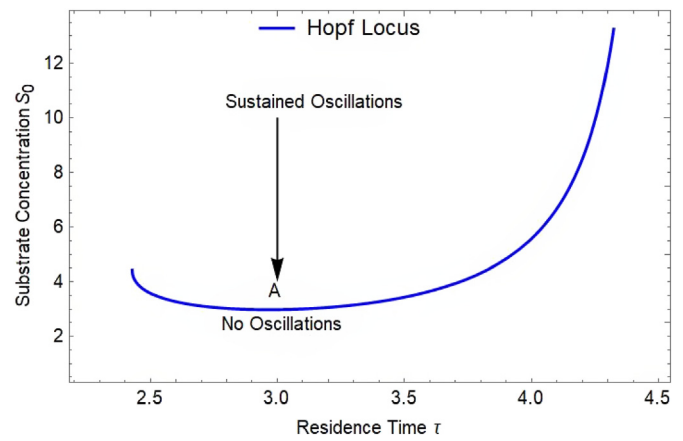


Figure 4. Hopf locus shows the emersion of the two Hopf points at an $H2_1$ degeneracy (A). Values of parameters: $\gamma = 2/9$, $\delta = 0.2$, and $\beta = 5.25$.

In this case, when $J_{11} = 0$ it can be shown that $\frac{dJ_{11}}{d\tau}$ vanishes for suitable values of γ , β , and δ . For example, if $\gamma = 0.25$, $\delta = 0.2$, $\beta = 5.25$, then $\frac{dJ_{11}}{d\tau}$ vanishes at $(S_0, \tau) \approx (1.53265, 4.88347)$. Therefore, if S_0 is small enough ($S_0 < 1.53265$) or ($S_0 < 2.68214$ g/l) natural oscillations are not possible for $\gamma = 0.25$, $\delta = 0.2$, $\beta = 5.25$.

Case IV. $S_0 > 2, 0 < \delta < 1, \frac{1}{S_0^2} < \gamma < 1/4$, and $\tau \geq \frac{1}{1-\delta} + 2\sqrt{\frac{\gamma}{(-1+\delta)^2}}$. Let $S_0 = 4$, $\delta = 0.2$, $\gamma = 2/9$, and $\beta = 5.25$, then $J_{11} = 0$ at $\tau = 3.71754$ and $\tau = 2.4431$ (other values are possible) (Figure 4).

In this case, when $J_{11} = 0$, it can be shown that $\frac{dJ_{11}}{d\tau}$ vanishes for suitable values of γ , β , and δ . For example, if $\gamma = \frac{2}{9}$, $\delta = 0.2$, and $\beta = 5.25$, then $\frac{dJ_{11}}{d\tau}$ vanishes at $(S_0, \tau) \approx (2.97799, 2.96539)$. Therefore, if the substrate concentration S_0 is sufficiently small ($S_0 < 2.97799$) or ($S_0 < 5.21148$ g/l), natural oscillations are impossible for $\gamma = \frac{2}{9}$, $\delta = 0.2$, and $\beta = 5.25$.

This work reported in [22, 25, 30] used $S_0 \geq 10$ g/l without confirming conditions under which natural oscillations are possible. The work in [20, 25] reported that natural oscillations are impossible for sufficiently small substrate concentration $S_0 < 6.84203$ g/l, for $\beta = 5.25$. Both cases above were observed for the specific growth rate equation $\mu(S) = \frac{\mu_m S}{k_s + S}$. In this work, we have used a modified Haldane equation $\mu(S) = \frac{\mu_m S}{k_s + S + \frac{S^2}{k_i}} \left(1 - \frac{k_i}{q}\right)$. It is possible to gain natural oscillations for even smaller values of S_0 .

4. Conclusions

We have studied, analytically, a simple chemostat model in a flow reactor with a variable yield coefficient in which the growth rate is taken to be a modified Haldane expression. In this study, four steady-state solutions have been discussed, which characterize no washout and washout circumstances in the closed photobioreactor. Under suitable parameter values, a stable steady-state solution attains its maximum value. We also corroborated the parameter ranges for the model with variable yield coefficient and growth rate, which describes natural oscillations in the chemostat. In all four cases of Hopf bifurcation analysis, we have found that natural oscillations can be achieved at significantly lower values of the substrate concentrations. In case I, it is $S_0 > 2.30202$ g/l. In case II, it is $S_0 > 2.06068$ g/l. In case III, it is $S_0 < 2.68214$ g/l, and in case IV, it is $S_0 > 5.21148$ g/l. A previous study reported that natural oscillation is achieved at a substrate concentration $S_0 > 6.84203$ g/l and results claim that natural oscillations are possible for sufficiently large values of substrate concentration ($S_0 > 10$ g/l) but the conditions of natural oscillations were not reported.

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