Dissimilarity Based Choquet Integrals

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Abstract. In this paper, in order to generalize the Choquet integral,
we replace the difference between inputs in its definition by a restricted
dissimilarity function and refer to the obtained function as d-Choquet
integral. For some particular restricted dissimilarity function the cor-
responding d-Choquet integral with respect to a fuzzy measure is just
the 'standard' Choquet integral with respect to the same fuzzy mea-
sure. Hence, the class of all d-Choquet integrals encompasses the class
of all 'standard' Choquet integrals. This approach allows us to construct
a wide class of new functions, d-Choquet integrals, that are possibly,
unlike the 'standard' Choquet integral, outside of the scope of aggrega-
tion functions since the monotonicity is, for some restricted dissimilarity
function, violated and also the range of such functions can be wider than
$[0,1]$, in particular it can be $[0,n]$.

Keywords: Choquet integral · d-Choquet integral · Dissimilarity ·
Pre-aggregation function · Aggregation function · Monotonicity ·
Directional monotonicity

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1   Introduction

The Choquet integral [4] can be regarded as a generalization of additive aggregation functions replacing the requirement of additivity by that of comonotone additivity. In recent years it was shown that, in some cases, additive aggregation functions are not appropriate to model even quite simple situations, which, on the other hand, can be treated with Choquet integrals [1,5,8,9,14,15]. In the literature, some generalizations of the Choquet integral appeared: in [11,12,16] the product operator was replaced by a more general function; in the same pattern, using the distributivity of the product operator and then replacing its two instances by two different functions under some constraints, in [6,17], generalizations of the Choquet integral were obtained; some Choquet-like integrals defined in terms of pseudo-addition and pseudo-multiplication are studied in [18]; a fuzzy t-conorm integral that is a generalization of Choquet integral is introduced in [19]; a non-linear integral that need not be increasing is introduced in [20]; a concave integral generalizing the Choquet integral is introduced in [13]; and a level dependent Choquet integral was also introduced in [10]. An overview of some recent extensions of the Choquet integral can be found in [7].

   Our aim is to replace the difference between the inputs in the definition of the Choquet integral by a restricted dissimilarity function [2,3] in order to generalize the Choquet integral. We refer to the obtained function as d-Choquet integral. This approach allows us to construct a wide class of new functions, d-Choquet integrals, which, unlike the "standard" Choquet integral, may be possibly outside of the scope of aggregation functions, since the monotonicity may be violated for some restricted dissimilarity function, and also the range of such functions can be wider than $[0, 1]$. Our work can be seen as the first step to the generalization of the Choquet integral to various settings where the difference causes problems (for example, intervals).

   The structure of the paper is as follows. First, we present some preliminary concepts. In Sect. 3, we introduce the notion of d-Choquet integral, describe its construction in terms of automorphisms and study its monotonicity and directional monotonicity. Conclusions and future research are described in Sect. 4.

2   Preliminaries

The necessary basic notions and terminology are recalled in this section.

   A function $\delta : [0, 1]^2 \to [0, 1]$ is called a restricted dissimilarity function on $[0, 1]$ if it satisfies, for all $x, y, z \in [0, 1]$, the following conditions:

1. $\delta(x, y) = \delta(y, x)$;
2. $\delta(x, y) = 1$ if and only if $\{x, y\} = \{0, 1\}$;
3. $\delta(x, y) = 0$ if and only if $x = y$;
4. if $x \leq y \leq z$, then $\delta(x, y) \leq \delta(x, z)$ and $\delta(y, z) \leq \delta(x, z)$.

   An n-ary aggregation function is a mapping $A : [0, 1]^n \to [0, 1]$ satisfying the following properties:

(A1) A is increasing in each argument;

(A2) $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

An automorphism of $[0, 1]$ is a continuous, strictly increasing function $\varphi :$ $[0, 1] \rightarrow [0, 1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover, the identity on $[0, 1]$ is denoted by Id.

It is well-known that a function $f : [0, 1]^n \rightarrow [0, 1]$ is additive if

$$f(x_1 + y_1, \ldots, x_n + y_n) = f(x_1, \ldots, x_n) + f(y_1, \ldots, y_n) \tag{1}$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $(x_1 + y_1, \ldots, x_n + y_n) \in [0, 1]^n$. From now on, $[n]$ denotes the set $\{1, \ldots, n\}$. Vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ are comonotone if there exists a permutation $\sigma : [n] \rightarrow [n]$ such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$ and $y_{\sigma(1)} \leq \ldots \leq y_{\sigma(n)}$. A function $f : [0, 1]^n \rightarrow [0, 1]$ is called comonotone additive if Equality (1) holds for all comonotone vectors $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $(x_1 + y_1, \ldots, x_n + y_n) \in [0, 1]^n$.

A function $\mu : 2^{[n]} \rightarrow [0, 1]$ is called a fuzzy measure on $[n]$ if $\mu(\emptyset) = 0$, $\mu([n]) = 1$ and $\mu(A) \leq \mu(B)$ for all $A \subseteq B \subseteq [n]$.

Let $\boldsymbol{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector such that $\boldsymbol{r} \neq \boldsymbol{0}$. A function $f : [0, 1]^n \rightarrow [0, 1]$ is $\boldsymbol{r}$-increasing if, for all $(x_1, \ldots, x_n) \in [0, 1]^n$ and for all $c \in]0, 1]$ such that $(x_1 + cr_1, \ldots, x_n + cr_n) \in [0, 1]^n$, it holds

$$f(x_1 + cr_1, \ldots, x_n + cr_n) \geq f(x_1, \ldots, x_n).$$

A function $f : [0, 1]^n \rightarrow [0, 1]$ is said to be an n-ary pre-aggregation function if $f(0, \ldots, 0) = 0$, $f(1, \ldots, 1) = 1$ and f is $\boldsymbol{r}$-increasing for some real n-dimensional vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ such that $\boldsymbol{r} \neq \boldsymbol{0}$ and $r_i \geq 0$ for every $i = 1, \ldots, n$. In this case, we say that f is an $\boldsymbol{r}$-pre-aggregation function.

3 d-Choquet Integral

A new approach to generalization of Choquet integral based on dissimilarity functions is introduced in this section and its monotonicity is studied.

The discrete Choquet integral on the unit interval with respect to a fuzzy measure $\mu : 2^{[n]} \rightarrow [0, 1]$ is defined as a mapping $C_\mu : [0, 1]^n \rightarrow [0, 1]$ such that

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^n (x_{\sigma(i)} - x_{\sigma(i-1)}) \mu \left(A_{\sigma(i)} \right) \tag{2}$$

where σ is a permutation on $[n]$ satisfying $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$.

In order to generalize the Choquet integral, we replace the difference $x_{\sigma(i)} - x_{\sigma(i-1)}$ by a restricted dissimilarity function $\delta : [0, 1]^2 \rightarrow [0, 1]$ and refer to the obtained function as d-Choquet integral.

Definition 1. *Let n be a positive integer and $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$. Let $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function. An n-ary discrete d-Choquet integral on $[0,1]$ with respect to μ and δ is defined as a mapping $C_{\mu,\delta} : [0,1]^n \to [0,n]$ such that*

$$C_{\mu,\delta}(x_1,\ldots,x_n) = \sum_{i=1}^{n} \delta(x_{\sigma(i)}, x_{\sigma(i-1)}) \mu\left(A_{\sigma(i)}\right) \tag{3}$$

where σ is a permutation on $[n]$ satisfying $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$, with the convention $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} = \{\sigma(i),\ldots,\sigma(n)\}$.

It is easy to check that, in general, the range of $C_{\mu,\delta}$ need not be a subset of $[0,1]$, but it is a subset of $[0,n]$. The following condition assures that the outputs of d-Choquet integral $C_{\mu,\delta}$ would be from $[0,1]$, which is often a desired property for some applications:

(P1) $\delta(0,x_1) + \delta(x_1,x_2) + \ldots + \delta(x_{n-1},x_n) \leq 1$ for all $x_1,\ldots,x_n \in [0,1]$ where $x_1 \leq \ldots \leq x_n$.

Under the condition we obtain $C_{\mu,\delta} : [0,1]^n \to [0,1]$ so we have the following straightforward result.

Proposition 1. *Let $C_{\mu,\delta} : [0,1]^n \to [0,n]$ be an n-ary discrete d-Choquet integral on $[0,1]$ with respect to μ and δ given by Definition 1. If δ satisfies the condition (P1), then*

$$C_{\mu,\delta}(x_1,\ldots,x_n) \in [0,1]$$

for all $x_1,\ldots,x_n \in [0,1]$ and for any measure μ.

Example 1. Let μ be a fuzzy measure on $\{1,2,3\}$ defined by $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = 0.3$, $\mu(\{1,2\}) = 0.75$, $\mu(\{2,3\}) = 0.55$ and $\mu(\{1,3\}) = 0.6$.
  (i) Then

$$C_\mu(0.2, 0.9, 0.6) = 0.2 \cdot 1 + 0.4 \cdot 0.55 + 0.3 \cdot 0.3 = 0.51.$$

It is easy to see that for $\delta(x,y) = |x-y|$ it holds $C_{\mu,\delta} = C_\mu$ for any possible inputs and any measure μ.
  (ii) However, if $\delta(x,y) = (x-y)^2$ we have

$$C_{\mu,\delta}(0.2, 0.9, 0.6) = 0.04 \cdot 1 + 0.16 \cdot 0.55 + 0.09 \cdot 0.3 = 0.155.$$

  (iii) Finally, taking

$$\delta(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{|x-y|+1}{2}, & \text{otherwise,} \end{cases}$$

we obtain

$$C_{\mu,\delta}(0.2, 0.9, 0.6) = 0.6 \cdot 1 + 0.7 \cdot 0.55 + 0.65 \cdot 0.3 = 1.18,$$

where we can see that $C_{\mu,\delta}(0.2, 0.9, 0.6) > 1$. This may happen since the condition (P1) is not satisfied as can be seen if we take, for instance, $(0, 0.5, 1)$.

The construction of d-Choquet integrals is directly connected with the construction of restricted dissimilarity functions with desired properties. In [2], a construction method for restricted dissimilarity functions in terms of automorphisms was introduced.

Proposition 2 *[2]. If φ_1, φ_2 are two automorphisms of $[0,1]$, then the function $\delta : [0,1]^2 \to [0,1]$ defined by*

$$\delta(x,y) = \varphi_1^{-1}\Big(|\varphi_2(x) - \varphi_2(y)| \Big)$$

is a restricted dissimilarity function.

We write $C_{\mu,\varphi_1,\varphi_2}$ instead of $C_{\mu,\delta}$ if the restricted dissimilarity function δ is given in terms of automorphisms φ_1, φ_2 as in the previous proposition.

Proposition 3. *Let n be a positive integer, $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 2. If $\varphi_1 \geq Id$ for all $x \in [0,1]$, then δ satisfies (P1).*

Example 2. The restricted dissimilarity functions $\delta(x,y) = (x-y)^2$, $\delta(x,y) = |\sqrt{x} - \sqrt{y}|$, $\delta(x,y) = |x^2 - y^2|$ and $\delta(x,y) = (\sqrt{x} - \sqrt{y})^2$ satisfy the condition $(P1)$ which means that the corresponding d-Choquet integrals have the ranges in $[0,1]$. However, for the restricted dissimilarity function $\delta(x,y) = \sqrt{|x-y|}$, the condition $(P1)$ is violated. For instance, $\delta(0, 0.1) + \delta(0.1, 1) = 1.2649 > 1$.

Clearly, d-Choquet integral $C_{\mu,\delta}$ for the restricted dissimilarity function $\delta(x,y) = |x-y|$ recovers the "standard" Choquet integral.

Theorem 1. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$, $\delta : [0,1]^2 \to [0,1]$ be the function $\delta(x,y) = |x-y|$, $C_{\mu,\delta} : [0,1]^n \to [0,1]$ be an n-ary discrete d-Choquet integral on $[0,1]$ with respect to μ and δ given by Definition 1 and $C_\mu : [0,1]^n \to [0,1]$ be an n-ary discrete Choquet integral on $[0,1]$ with respect to μ given by Eq. (2). Then*

$$C_{\mu,\delta}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

Corollary 1. *Let n be a positive integer, $\mu : 2^{[n]} \to [0,1]$ be a fuzzy measure on $[n]$. Then*

$$C_{\mu,Id,Id}(x_1, \ldots, x_n) = C_\mu(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in [0,1]$.

3.1   Monotonicity of d-Choquet Integrals

In general, d-Choquet integrals are not monotone, hence we study conditions under which a d-Choquet integral is increasing in each component. Note that, since the boundary conditions are satisfied, any increasing d-Choquet integral is an aggregation function.

Theorem 2. *Let n be a positive integer, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be an n-ary d-Choquet integral with respect to μ and δ. Then the following assertions are equivalent:*

(i) *For any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq \ldots \leq x_n$, $y_1 \leq \ldots \leq y_n$, $x_1 \leq y_1, \ldots, x_n \leq y_n$.*
(ii) *$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) \leq \delta(0, y_1) + \delta(y_1, y_2) + \ldots + \delta(y_{m-1}, y_m)$ for all $m \in [n]$ and $x_1, \ldots, x_m, y_1, \ldots, y_m \in [0,1]$ where $x_1 \leq \ldots \leq x_m$, $y_1 \leq \ldots \leq y_m$, $x_1 \leq y_1, \ldots, \leq x_m \leq y_m$.*

Corollary 2. *Let n be a positive integer, δ be a restricted dissimilarity function and $C_{\mu,\delta}$ be an n-ary d-Choquet integral with respect to μ and δ. If for all $m \in [n]$ there exists an increasing function $f_m : [0,1] \to [0,1]$ such that $\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) = f_m(x_m)$ for all $x_1, \ldots, x_m \in [0,1]$ where $x_1 \leq \ldots \leq x_m$, then for any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$.*

Corollary 3. *Let n be a positive integer, $\delta : [0,1]^2 \to [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 2. Let $C_{\mu,\varphi_1,\varphi_2}$ be an n-ary d-Choquet integral with respect to μ and δ. If $\varphi_1 = Id$, then for any fuzzy measure μ on $[n]$, $C_{\mu,\delta}(x_1, \ldots, x_n) \leq C_{\mu,\delta}(y_1, \ldots, y_n)$ whenever $x_1 \leq y_1, \ldots, x_n \leq y_n$.*

It is easy to see that for $\varphi_1 = Id$ we have:

$$C_{\mu,Id,\varphi_2}(x_1, \ldots, x_n) = C_\mu(\varphi_2(x_1), \ldots, \varphi_2(x_n)),$$

i.e. C_{μ,Id,φ_2} is fully determined by a "standard" Choquet integral C_μ. It also means that C_{μ,Id,φ_2} is an aggregation function.

Since the monotonicity is not always satisfied, we also study directional monotonicity. From the previous results it is clear that, in general, an n-ary d-Choquet integral is not $\boldsymbol{r}$-increasing for a vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ such that there exists $k \in \{1, \ldots, n\}$ with $r_i \neq 0$ if and only if $i = k$. In what follows we focus on the directional monotonicity with respect to the vector $\boldsymbol{r} = (1, \ldots, 1)$.

Theorem 3. *Let n be a positive integer and $C_{\mu,\delta} : [0,1]^n \to [0,n]$ be an n-ary d-Choquet integral with respect to a fuzzy measure μ and a restricted dissimilarity function δ. Then*

(i) *$C_{\mu,\delta}$ is $\boldsymbol{1}$-increasing for any fuzzy measure μ whenever*

$$\delta(x + c, y + c) \geq \delta(x, y)$$

for all $x, y, c \in [0,1]$ such that $x + c, y + c \in [0,1]$;
(ii) *$C_{\mu,\delta}$ is $\boldsymbol{1}$-increasing for any fuzzy measure μ whenever for all $m \in [n]$ there exists an increasing function $f_m : [0,1] \to [0,1]$ such that*

$$\delta(0, x_1) + \delta(x_1, x_2) + \ldots + \delta(x_{m-1}, x_m) = f_m(x_m)$$

for all $x_1, \ldots, x_m \in [0,1]$ where $x_1 \leq \ldots \leq x_m$.

Corollary 4. *Let n be a positive integer, $\delta : [0,1]^2 \rightarrow [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 2. Let $C_{\mu,\varphi_1,\varphi_2} : [0,1]^n \rightarrow [0,n]$ be an n-ary d-Choquet integral with respect to μ and δ. Then $C_{\mu,\delta}$ is **1**-increasing for any fuzzy measure μ whenever at least one of the following conditions is satisfied:*

(i) φ_2 is convex;
(ii) $\varphi_1 = Id$.

The conditions under which an n-ary d-Choquet integral is a **1**-pre-aggregation function directly follow from the previous results.

Corollary 5. *Let n be a positive integer, $\delta : [0,1]^2 \rightarrow [0,1]$ be a restricted dissimilarity function given in terms of automorphisms φ_1, φ_2 as in Proposition 2. Then the n-ary d-Choquet integral with respect to μ and δ is a **1**-pre-aggregation function for any fuzzy measure μ on $[n]$ whenever at least one of the following conditions is satisfied:*

(i) $\varphi_1 = Id$;
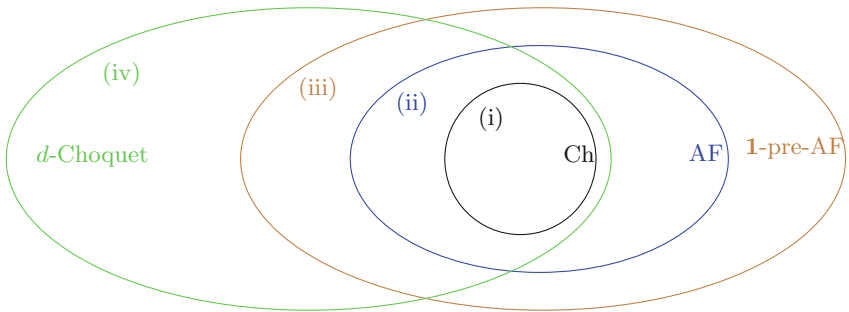*(ii) $\varphi_1 > Id$ and φ_2 is convex.*

Fig. 1. The relations among the classes of all standard Choquet integrals (Ch), aggregation functions (AF), **1**-pre-aggregation functions (**1**-pre-AF) and d-Choquet integrals (d-Choquet).

Taking the following restricted dissimilarity functions, we obtain an example of d-Choquet integral which is (see Fig. 1):

(i) a standard Choquet integral, if $\delta(x,y) = |x - y|$;
(ii) an aggregation function which is not a standard Choquet integral, if $\delta(x,y) = |\sqrt{x} - \sqrt{y}|$ or $\delta(x,y) = |x^2 - y^2|$;
(iii) an **1**-pre-aggregation function which is not an aggregation function, if $\delta(x,y) = (x - y)^2$;
(iv) a d-Choquet integral which is not an **1**-pre-aggregation function, if $\delta(x,y) = \sqrt{|\sqrt{x} - \sqrt{y}|}$.

4   Conclusions

In this paper we have introduced a generalization of the Choquet integral replacing the difference by a restricted dissimilarity function. Our approach results in a wide class of d-Choquet integrals that encompasses the class of all "standard" Choquet integrals. We have shown that, based on the choice of a restricted dissimilarity function, the d-Choquet integral is or is not an aggregation function or pre-aggregation function.

The class of d-Choquet integrals can be useful in applications where Choquet integrals have shown themselves valuable, for instance in image processing, decision making or classification. In particular, these new functions can be used in the settings where the difference causes problems, for example for intervals. In this sense, we intend to make a research of possibilities to apply our results in image processing and classification problems in the interval-valued setting in such a way that classical fuzzy algorithms which make use of the Choquet integral can be appropriately formulated.

The Choquet integral can be equivalently expressed in terms of differences of capacity weights on a nested family of sets or in terms of the Moebius transform. We intend to develop an analysis similar to this one regarding the former; nevertheless the equivalence between the two representations will be most probably lost in our more general setting. The possible extension of our generalization idea to Choquet integrals expressed in terms of Moebius transform does not seem so easily achievable, since it is not clear how the information provided by a restricted dissimilarity function can be included in such representation. We will analyze this question in future works.

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