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A Stronger Multi-observable Uncertainty Relation

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Uncertainty relation lies at the heart of quantum mechanics, characterizing the incompatibility of non-commuting observables in the preparation of quantum states. An important question is how to improve the lower bound of uncertainty relation. Here we present a variance-based sum uncertainty relation for *N* incompatible observables stronger than the simple generalization of an existing uncertainty relation for two observables. Further comparisons of our uncertainty relation with other related ones for spin-$\frac{1}{2}$ and spin-1 particles indicate that the obtained uncertainty relation gives a better lower bound.

Uncertainty relation is one of the fundamental building blocks of quantum theory, and now plays an important role in quantum mechanics and quantum information[1–4]. It is introduced by Heisenberg[5] in understanding how precisely the simultaneous values of conjugate observables could be in microspace, i.e., the position X and momentum P of an electron. Kennard[6] and Weyl[7] proved the uncertainty relation

$$\Delta X \Delta P \geq \frac{\hbar}{2}, \tag{1}$$

where the standard deviation of an operator X is defined by $\Delta X = \sqrt{\langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2}$. Later, Robertson proposed the well-known formula of uncertainty relation[8]

$$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{1}{2}\langle\psi|[A, B]|\psi\rangle\right|^2, \tag{2}$$

which is applicable to arbitrary incompatible observables, and the commutator is defined by $[A, B] = AB - BA$. The uncertainty relation was further strengthed by Schrödinger[9] with the following form

$$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{1}{2}\langle[A, B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle\right|^2. \tag{3}$$

Here the commutator defined as $\{A, B\} \equiv AB + BA$.

It is realized that the traditional uncertainty relations may not fully capture the concept of incompatible observables as the lower bound could be trivially zero while the variances are not. An important question in uncertainty relation is how to improve the lower bound and immune from triviality problem[10,11]. Various attempts have been made to find stronger uncertainty relations. One typical kind of relation is that of Maccone and Pati, who derived two stronger uncertainty relations

$$(\Delta A)^2 + (\Delta B)^2 \geq \pm i\langle\psi|[A, B]|\psi\rangle + |\langle\psi|A \pm iB|\psi^{\perp}\rangle|^2, \tag{4}$$

$$(\Delta A)^2 + (\Delta B)^2 \geq \frac{1}{2}|\langle\psi^{\perp}_{A+B}|A + B|\psi\rangle|^2 = \frac{1}{2}[\Delta(A + B)]^2, \tag{5}$$

where $\langle\psi|\psi^{\perp}\rangle = 0, |\psi^{\perp}_{A+B}\rangle \propto (A + B - \langle A + B\rangle)|\psi\rangle$, and the sign on the right-hand side of the inequality takes $+(-)$ while $i\langle[A, B]\rangle$ is positive (negative). The basic idea behind these two relations is adding additional terms to improve the lower bound. Along this line, more terms[12–14] and weighted form of different terms[15,16] have been put into the uncertainty relations. It is worth mentioning that state-independent uncertainty relations can

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immune from triviality problem[17–20]. Recent experiments have also been performed to verify the various uncertainty relations[21–24].

Besides the conjugate observables of position and momentum, multiple observables also exist, e.g., three component vectors of spin and angular momentum. Hence, it is important to find uncertainty relation for multiple incompatible observables. Recently, some three observables uncertainty relations were studied, such as Heisenberg uncertainty relation for three canonical observables[25], uncertainty relations for three angular momentum components[26], uncertainty relation for three arbitrary observables[14]. Furthermore, some multiple observables uncertainty relations were proposed, which include multi-observable uncertainty relation in product[27,28] and sum[29,30] form of variances. It is worth noting that Chen and Fei derived an variance-based uncertainty relation[30]

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{N-2}\left\{\sum_{1\leq i<j\leq N}\left[\Delta(A_i + A_j)\right]^2 - \frac{1}{(N-1)^2}\left[\sum_{1\leq i<j\leq N}\Delta(A_i + A_j)\right]^2\right\}, \tag{6}$$

for arbitrary N incompatible observables, which is stronger than the one such as derived from the uncertainty inequality for two observables[10].

In this paper, we investigate variance-based uncertainty relation for multiple incompatible observables. We present a new variance-based sum uncertainty relation for multiple incompatible observables, which is stronger than an uncertainty relation from summing over all the inequalities for pairs of observables[10]. Furthermore, we compare the uncertainty relation with existing ones for a spin-$\frac{1}{2}$ and spin-1 particle, which shows our uncertainty relation can give a tighter bound than other ones.

Results

Theorem 1. For arbitrary N observables $A_1, A_2, \ldots, A_N$, the following variance-based uncertainty relation holds

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{N}\left[\Delta\left(\sum_{i=1}^{N}A_i\right)\right]^2 + \frac{1}{N^2}\left[\sum_{1\leq i<j\leq N}\Delta\left(A_i - A_j\right)\right]^2. \tag{7}$$

The bound becomes nontrivial as long as the state is not common eigenstate of all the N observables.

Proof: To derive (7), start from the equality

$$\sum_{1\leq i<j\leq N}\left[\Delta(A_i - A_j)\right]^2 = N\sum_{i=1}^{N}(\Delta A_i)^2 - \left[\Delta\left(\sum_{i=1}^{N}A_i\right)\right]^2, \tag{8}$$

then using the inequality

$$N\sum_{1\leq i<j\leq N}\left[\Delta(A_i - A_j)\right]^2 \geq \left[\sum_{1\leq i<j\leq N}\Delta(A_i - A_j)\right]^2, \tag{9}$$

we obtain the uncertainty relation (7) QED.

When $N=2$ we have the following corollary

Corollary 1.1. *For two incompatible observables A and B, we have*

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}[\Delta(A+B)]^2 + \frac{1}{4}[\Delta(A-B)]^2 \geq \frac{1}{2}[\Delta(A+B)]^2, \tag{10}$$

which is derived from Theorem 1 *for* $N=2$, *and stronger than uncertainty relation* (5).

To show that our relation (7) has a stronger bound, we consider the result in ref. 10, the relation (5) is derived from the uncertainty equality

$$\Delta A^2 + \Delta B^2 = \frac{1}{2}[\Delta(A+B)]^2 + \frac{1}{2}[\Delta(A-B)]^2. \tag{11}$$

Using the above uncertainty equality, one can obtain two inequalities for arbitrary N observables, namely

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{2(N-1)}\sum_{1\leq i<j\leq N}\left[\Delta(A_i + A_j)\right]^2 \tag{12}$$

and

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{2(N-1)}\sum_{1\leq i<j\leq N}\left[\Delta(A_i - A_j)\right]^2. \tag{13}$$

The bound in (6) is tighter than the one in (12)[30]. However, the lower bound in (6) is not always tighter than the one in (13) (see Fig. 1).

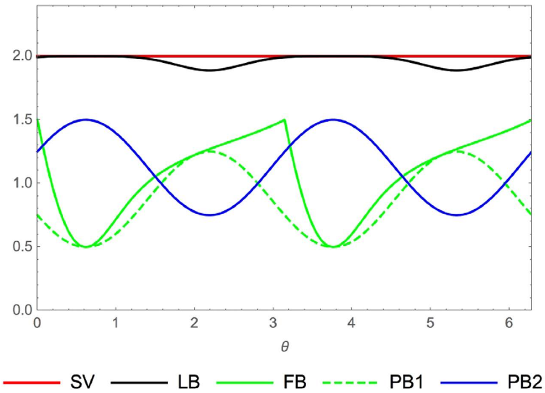

Figure 1. Example of comparison between our relation (7) and the ones (6), (12), (13). The upper line is the sum of the variances (SV) $(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2$. The black line is the lower bound (LB) given by our relation (7). The solid green line is the bound (6) (FB). The dashed green line is the bound (12) (PB1). The blue line is the bound (13) (PB2).

Example 1. To give an overview that the relation (7) has a better lower bound than the relations (6), (12), (13), we consider a family of qubit pure states given by the Bloch vector $\vec{r} = (\frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\cos\theta, \sin\theta)$, and choose the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have $(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 = 2$, $[\Delta(\sigma_x + \sigma_y)]^2 = 2(\sin\theta)^2$, and $[\Delta(\sigma_y + \sigma_z)]^2 = [\Delta(\sigma_x + \sigma_z)]^2 = \frac{5}{4} + \frac{1}{4}\cos 2\theta - \frac{\sqrt{2}}{2}\sin 2\theta$. Similarly, $[\Delta(\sigma_x - \sigma_y)]^2 = 2$, and $[\Delta(\sigma_y - \sigma_z)]^2 = [\Delta(\sigma_x - \sigma_z)]^2 = \frac{5}{4} + \frac{1}{4}\cos 2\theta + \frac{\sqrt{2}}{2}\sin 2\theta$. The comparison between the lower bounds (6), (12), (13) and (7) is given in Fig. 1. Apparently, our bound is tighter than (6), (12) and (13). We shall show with detailed proofs and examples that our uncertainty relation (7) has better lower bound than that of (6), (12), (13) in the following sections.

Comparison between the lower bound of our uncertainty relation (7) with that of inequality (12).

First, we compare our relation (7) with the one (12). Note that $\Delta(A_i + A_j)^2 = \Delta A_i^2 + \Delta A_j^2 + \langle\{A_i, A_j\}\rangle - 2\langle A_i\rangle\langle A_j\rangle$, the relation (12) becomes

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{2(N-1)}\left\{(N-1)\sum_{i=1}^{N}(\Delta A_i)^2 + \sum_{1\leq i<j\leq N}\left[\left\langle\left\{A_i, A_j\right\}\right\rangle - 2\langle A_i\rangle\left\langle A_j\right\rangle\right]\right\}. \tag{14}$$

Simplify the above inequality, we obtain

$$\sum_{i=1}^{N}(\Delta A_i)^2 \geq \frac{1}{(N-1)}\sum_{1\leq i<j\leq N}\left[\left\langle\left\{A_i, A_j\right\}\right\rangle - 2\langle A_i\rangle\left\langle A_j\right\rangle\right], \tag{15}$$

which is equal to the relation (12).

Similarly, by using $\Delta(A_i - A_j)^2 = \Delta A_i^2 + \Delta A_j^2 - \left\langle\left\{A_i, A_j\right\}\right\rangle + 2\langle A_i\rangle\left\langle A_j\right\rangle$, our relation (7) becomes

$$\begin{aligned}\sum_{i=1}^{N}(\Delta A_i)^2 \geq\ & \frac{1}{N}\left[\sum_{i=1}^{N}(\Delta A_i)^2 + \sum_{1\leq i<j\leq N}\left[\left\langle\left\{A_i, A_j\right\}\right\rangle - 2\langle A_i\rangle\left\langle A_j\right\rangle\right]\right] \\ & + \frac{1}{N^2}\left\{(N-1)\sum_{i=1}^{N}(\Delta A_i)^2 - \sum_{1\leq i<j\leq N}\left[\left\langle\left\{A_i, A_j\right\}\right\rangle\right.\right. \\ & \left.\left. - 2\langle A_i\rangle\left\langle A_j\right\rangle\right] + \sum_{1\leq i<j\leq N 1\leq i'<j'\leq N}^{i\neq i' or j\neq j'}\Delta(A_i - A_j)\Delta(A_{i'} - A_{j'})\right\}.\end{aligned} \tag{16}$$

Simplify the above inequality, we get

$$\sum_{i=1}^{N}(\Delta A_i)^2 \ \geq \ \frac{1}{N-1}\sum_{1\leq i<j\leq N}\left[\left\langle\left\{A_i, A_j\right\}\right\rangle - 2\langle A_i\rangle\left\langle A_j\right\rangle\right] + \frac{1}{(N-1)^2}\sum_{1\leq i<j\leq N 1\leq i'<j'\leq N}^{i\neq i' \text{ or } j\neq j'}\Delta(A_i - A_j)\Delta(A_{i'} - A_{j'}), \tag{17}$$

which is equal to the relation (7). It is easy to see that the right-hand side of (17) is greater than the right-hand side of (15). Hence, the relation (7) is stronger than the relation (12).

Comparison between the lower bound of our uncertainty relation (7) with that of inequalities (6) and (13). Here, we will show the uncertainty relation (7) is stronger than inequalities (13) and (6) for a spin-$\frac{1}{2}$ particle and measurement of Pauli-spin operators $\sigma_x, \sigma_y, \sigma_z$. Then the uncertainty relation (7) has the form

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 \geq \frac{1}{3}\left[\Delta(\sigma_x + \sigma_y + \sigma_z)\right]^2 + \frac{1}{9}\left(\sum_{1\leq i<j\leq 3}\Delta(\sigma_i - \sigma_j)\right)^2, \tag{18}$$

the relation (13) is given by

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 \geq \frac{1}{4}\sum_{1\leq i<j\leq 3}\left[\Delta(\sigma_i - \sigma_j)\right]^2, \tag{19}$$

and the relation (6) says that

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 \geq \sum_{1\leq i<j\leq 3}\left[\Delta(\sigma_i + \sigma_j)\right]^2 - \frac{1}{4}\left(\sum_{1\leq i<j\leq 3}\Delta(\sigma_i + \sigma_j)\right)^2. \tag{20}$$

We consider a qubit state and its Bloch sphere representation

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}), \tag{21}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices and the Bloch vector $\vec{r} = (x, y, z)$ is real three-dimensional vector such that $\|\vec{r}\| \leq 1$. Then we have $(\Delta\sigma_x)^2 = \mathrm{Tr}[\rho\sigma_x\sigma_x] - \mathrm{Tr}[\rho\sigma_x]^2 = 1 - x^2$, $(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 = 3 - (x^2 + y^2 + z^2)$. The relation (18) has the form

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta_\rho\sigma_z)^2 \geq \frac{1}{9}\alpha^2 + \frac{1}{3}(3 - (x + y + z)^2), \tag{22}$$

where we define $\alpha = \sqrt{2 - (x - y)^2} + \sqrt{2 - (x - z)^2} + \sqrt{2 - (y - z)^2}$. And the relation (19) becomes

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 \geq \frac{1}{2}(3 - (x^2 + y^2 + z^2) + xy + xz + yz). \tag{23}$$

Let us compare the lower bound of (22) with that of (23). The difference of these two bounds is

$$\begin{aligned}&\frac{1}{9}\alpha^2 + \frac{1}{6}(x^2 + y^2 + z^2) - \frac{7}{6}(xy + xz + zy) - \frac{1}{2}\\ &\geq (\sqrt{2 - (x - y)^2}\sqrt{2 - (x - z)^2}\sqrt{2 - (y - z)^2})^{\frac{2}{3}} - (x^2 + y^2 + z^2) - \frac{1}{2}\\ &\geq (\sqrt{2 - (x - y)^2}\sqrt{2 - (x - z)^2}\sqrt{2 - (y - z)^2})^{\frac{2}{3}} - \frac{3}{2},\end{aligned} \tag{24}$$

for all $x, y, z \in [-1, 1]$. When $x = y = z = \pm 1/\sqrt{3}$, the above inequality becomes equality, then the Eq. (24) has the minimum value $1/2 > 0$. This illustrates that the uncertainty relation (7) is stronger that the one (13) for a spin-$\frac{1}{2}$ particle and measurement of Pauli-spin operators $\sigma_x, \sigma_y, \sigma_z$.

Let us compare the uncertainty relation (18) with (20). The relation (20) has the form

$$(\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + (\Delta\sigma_z)^2 \geq 6 - 2(xy + xz + yz) - 2(x^2 + y^2 + z^2) - \frac{1}{4}\beta^2, \tag{25}$$

where we define $\beta = \sqrt{2 - (x + y)^2} + \sqrt{2 - (x + z)^2} + \sqrt{2 - (y + z)^2}$. Then the difference of these two bounds of relation (22) and (25) becomes

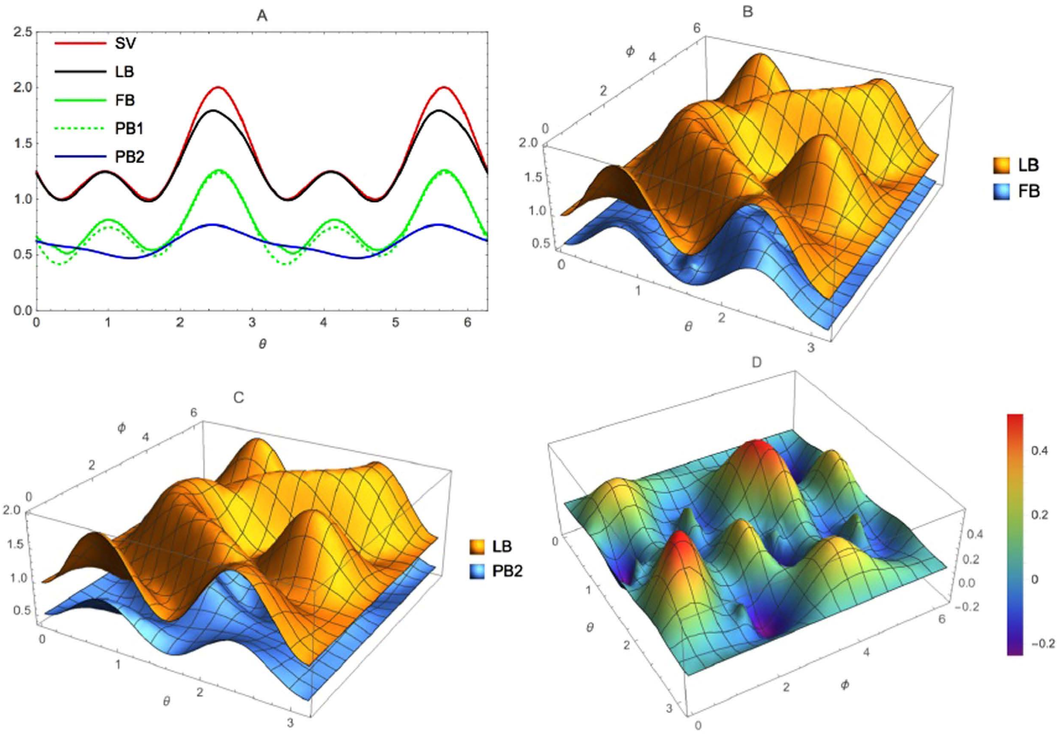

Figure 2. Example of comparison between our relation (7) and ones (6), (12), (13). We choose $A_1 = L_x$, $A_2 = L_y$ and $A_3 = L_y$ three components of the angular momentum for spin-1 particle, and a family of states parametrized by θ and ϕ as $|\psi\rangle = \sin\theta\cos\phi|1\rangle + \sin\theta\sin\phi|0\rangle + \cos\theta|-1\rangle$. (**A**) For $\phi = \pi/4$, the comparison between our relation (7) and ones (6), (12), (13). The upper line is the sum of the variances (SV) $(\Delta L_x)^2 + (\Delta L_y)^2 + (\Delta L_z)^2$. The black line is the lower bound (LB) given by our relation (7). The solid green line is the bound (6) (FB). The dashed green line is the bound (12) (PB1). The blue line is the bound (13) (PB2). (**B**) The comparison between our relation (7) and (6), which shows that our relation (7) (LB) has stronger bound than (6) (FB). (**C**) The comparison between our relation (7) and (13), which shows that our relation (7) (LB) has stronger bound than (13) (PB2). (**D**) The lower bound of the relation (6) minus the lower bound of the relation (13).

$$\frac{1}{9}\alpha^2 + \frac{1}{4}\beta^2 + \frac{2}{3}(x+y+z)^2 + (x^2+y^2+z^2) - 5$$
$$\geq \left(\frac{\sqrt{6}}{9}\alpha + \frac{\sqrt{2}+\sqrt{6}}{4}\beta\right)^2\left(\frac{3}{8+3\sqrt{3}}\right) + \frac{1}{2}(x-z)^2 - 5, \tag{26}$$

where we have twice used Cauchy's inequality. When $\alpha = \frac{\sqrt{6}}{3}$, $\beta = \frac{\sqrt{2}+\sqrt{6}}{2}$, $x + y + z = 0$ and $x = -y = \pm 1/\sqrt{2}$, $z = 0$ *or* $x = -z = \pm 1/\sqrt{2}$, $y = 0$ *or* $y = -z = \pm 1/\sqrt{2}$, $x = 0$, the above inequality becomes equality, then the Eq. (26) has the minimum value $\sqrt{3} - \frac{4}{3} > 0$. This illustrates that the uncertainty relation (7) is stronger that the one (6) for a spin-$\frac{1}{2}$ particle and measurement of Pauli-spin operators σ_x, σ_y, σ_z.

Example 2. For spin-1 systems, we consider the following quantum state characterized by θ and ϕ

$$|\psi\rangle = \sin\theta\cos\phi|1\rangle + \sin\theta\sin\phi|0\rangle + \cos\theta|-1\rangle, \tag{27}$$

with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. By choosing the three angular momentum operators ($\hbar = 1$)

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the comparison between the lower bounds (6), (12), (13) and (7) is shown by Fig. 2. The results suggest that the relation (7) can give tighter bounds than other ones (6), (12), (13) for a spin-1 particle and measurement of angular momentum operators L_x, L_y, L_z.

Conclusion

We have provided a variance-based sum uncertainty relation for N incompatible observables, which is stronger than the simple generalizations of the uncertainty relation for two observables derived by Maccone and Pati [Phys. Rev. Lett. **113**, 260401 (2014)]. Furthermore, our uncertainty relation gives a tighter bound than the others by comparison for a spin-$\frac{1}{2}$ particle with the measurements of spin observables σ_x, σ_y, σ_z. And also, in the case of spin-1 with measurement of angular momentum operators L_x, L_y, L_z, our uncertainty relation predicts a tighter bound than other ones.

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Author Contributions

Q.-C.S. and J.-L.L. and G.-X.P. and C.-F.Q. contribute equally to this work, and agree with the manuscript submitted.

Additional Information

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