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L^p bounds for rough parabolic maximal operators

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ABSTRACT

In this article, we establish the L^p estimates for a certain class of rough parabolic maximal functions related to surfaces of revolution. The obtained estimates allow us to apply an extrapolation argument to extend and improve some previously known results.

1. Preliminaries and statement of results

Throughout this article, we assume that $n \geq 2$ and $\mathbf{S}^{n-1}$ is the unit sphere in the Euclidean space $\mathbf{R}^n$, which is equipped with the normalized Lebesgue surface measure $d\mu = d_n\mu(\cdot)$.

For $j \in \{1, 2, \cdots, n\}$, let α_j be fixed real numbers in the interval $[1, \infty)$, and let $\Psi : \mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}$ be a function given by $\Psi(v, \lambda) = \sum_{j=1}^{n} \frac{v_j^2}{\lambda^{2\alpha_i}}$ with $v = (v_1, v_2, \cdots, v_n) \in \mathbf{R}^n$. For each fixed $v \in \mathbf{R}^n$, the unique solution of the equation $\Psi(v, \lambda) = 1$ is denoted by $\lambda(v)$. The authors of [1] proved that $(\mathbf{R}^n, \lambda)$ is a metric space which is frequently called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$. Let D_λ (with $\lambda > 0$) be the diagonal $n \times n$ matrix

$$D_\lambda = \begin{bmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{bmatrix}.$$

The polar coordinates transform in $(\mathbf{R}^n, \lambda)$ is given by the following:

$$\begin{aligned} v_1 &= \lambda^{\alpha_1} \cos \vartheta_1 \cdots \cos \vartheta_{n-2} \cos \vartheta_{n-1}, \\ v_2 &= \lambda^{\alpha_2} \cos \vartheta_1 \cdots \cos \vartheta_{n-2} \sin \vartheta_{n-1}, \\ &\vdots \\ v_{n-1} &= \lambda^{\alpha_{n-1}} \cos \vartheta_1 \sin \vartheta_2, \\ v_n &= \lambda^{\alpha_n} \sin \vartheta_1, \end{aligned}$$

$v \in \mathbf{R}^n$. Hence, $dv = \lambda^{\alpha-1} J(v')d\lambda d\mu(v')$, where

$$\alpha = \sum_{j=1}^{n} \alpha_j, \quad J(v') = \sum_{j=1}^{n} \alpha_j (v'_j)^2, \quad v' = D_{\lambda(v)^{-1}} v \in \mathbf{S}^{n-1},$$

and $\lambda^{\alpha-1} J(v')$ refers to the Jacobian of the above transforms.

The authors of [1] proved that there is a real constant C satisfying $1 \leq J(v') \leq C$ for all $v' \in \mathbf{S}^{n-1}$, and $J(v') \in C^\infty(\mathbf{S}^{n-1})$. Let $K_{\Omega,g}$ be the kernel on $\mathbf{R}^n$ given by

$$K_{\Omega,g}(v) = \frac{g(\lambda(v))\Omega(v)}{\lambda(v)^\alpha},$$

where g is a real measurable function on $\mathbf{R}^+$, and Ω is an integrable function over $\mathbf{S}^{n-1}$ that satisfies the conditions

$$\Omega(D_\lambda v) = \Omega(v), \quad \forall \lambda > 0, \tag{1.1}$$

and

$$\int_{\mathbf{S}^{n-1}} \Omega(v')J(v')d\mu(v') = 0. \tag{1.2}$$

For a suitable mapping $\psi : \mathbf{R}^+ \to \mathbf{R}$, we define the maximal operator $\mathcal{M}^{(\tau)}_{\Omega,\psi}$, for $f \in \mathcal{S}(\mathbf{R}^{n+1})$, by

$$\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)(x, x_{n+1}) = \sup_{g \in \Gamma_\tau(\mathbf{R}^+, \frac{d\lambda}{\lambda})} \left| T_{\Omega,h,\psi}(f)(x, x_{n+1}) \right|, \tag{1.3}$$

where

$$T_{\Omega,g,\psi}(f)(x, x_{n+1}) = p.v \int_{\mathbf{R}^n} f(x - v, x_{n+1} - \psi(\lambda(v)))K_{\Omega,g}(v)dv \tag{1.4}$$

and $\Gamma_\tau(\mathbf{R}^+, \frac{d\lambda}{\lambda})$ $(\tau \geq 1)$ is denoted to the class of all measurable functions $g : \mathbf{R}^+ \to \mathbf{R}$ such that

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$$\|g\|_{\Gamma_\tau(\mathbf{R}^+,\frac{d\lambda}{\lambda})}=\left(\int_0^\infty |g(\lambda)|^\tau \frac{d\lambda}{\lambda}\right)^{1/\tau}\leq 1.$$

The parabolic singular operator $T_{\Omega,1,\lambda}$ ($g\equiv 1$ and $\psi(\lambda)=\lambda$) was introduced by Fabes and Riviére in [1] in which the authors established the L^p ($1<p<\infty$) boundedness of $T_{\Omega,1,\lambda}$ whenever $\Omega\in C^1(\mathbf{S}^{n-1})$. Later on, the authors of [2] improved the above result. Precisely, they proved that $T_{\Omega,1,\lambda}$ is bounded for any $p\in(1,\infty)$ under the condition that $\Omega\in L(\log L)(\mathbf{S}^{n-1})$. Recently, a considerable amount of research has been done to obtain the L^p boundedness of the operator $T_{\Omega,g,\psi}$, the readers are refereed (for instance to [3, 4] and the references therein).

When $\alpha_1=\cdots=\alpha_n=1$, then $\alpha=n$, $\lambda(x)=|x|$ and $(\mathbf{R}^n,\lambda)=(\mathbf{R}^n,|\cdot|)$. In this case, we denote $T_{\Omega,g,\psi}$ by $T^c_{\Omega,g,\psi}$ and $\mathcal{M}^{(\tau)}_{\Omega,\psi}$ by $\mathcal{M}^{(\tau),c}_{\Omega,\psi}$. Also, when $\psi(\lambda)=\lambda$ and $g\equiv 1$, then the operator $T^c_{\Omega,g,\psi}$ becomes the classical Calderón-Zygmund singular integral operator T^c_Ω given by

$$T^c_\Omega(f)(x)=p.v\int_{\mathbf{R}^n} f(x-v)\frac{\Omega(v')}{|v|^n}dv.$$

Historically, the study of the singular operator T^c_Ω started by Calderón and Zygmund in [5] in which they showed that T^c_Ω is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$ provided that $\Omega\in L(\log L)(\mathbf{S}^{n-1})$. Moreover, they found that the condition $\Omega\in L(\log L)(\mathbf{S}^{n-1})$ is optimal in the sense that T^c_Ω may lose the L^p boundedness for any p if Ω is assumed to be in the space $L(\log L)^{1-\gamma}(\mathbf{S}^{n-1})$ for some $\gamma\in(0,1)$. Subsequently, the study of the L^p boundedness of $T^c_{\Omega,g,\psi}$ under various conditions on the kernels has attracted the attention of many mathematicians. For more information about the importance of such operators and their developments, the readers are refereed to [6, 7, 8, 9, 10, 11, 12], among numerous references.

Again, when $\psi(\lambda)=\lambda$, then the operator $\mathcal{M}^{(\tau),c}_{\Omega,\psi}$ is just the classical maximal operator which is denoted by $\mathcal{M}^{(\tau),c}_{\Omega}$. The operator $\mathcal{M}^{(\tau),c}_{\Omega}$ was first introduced by the authors of [13], who proved that when $\Omega\in C(\mathbf{S}^{n-1})$ and $g\in\Gamma_\tau(\mathbf{R}^+,\frac{dr}{r})$ for some $1\leq\tau\leq 2$, then $\mathcal{M}^{(\tau),c}_{\Omega}$ is bounded on $L^p(\mathbf{R}^n)$ for all $p\in((n\tau)',\infty)$. Afterward, Al-Salman improved this result in [14]. Precisely, he established the $L^p(\mathbf{R}^n)$ ($p\geq 2$) boundedness of $\mathcal{M}^{(2),c}_{\Omega}$ under the condition $\Omega\in L(\log L)^{1/2}(\mathbf{S}^{n-1})$. Furthermore, he found that the condition $\Omega\in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is optimal in the sense that the $L^2(\mathbf{R}^n)$ boundedness of $\mathcal{M}^{(2),c}_{\Omega}$ is not true when the exponent $1/2$ in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is replaced by any number $\kappa\in(0,1/2)$. Recently, Al-Qassem in [6] improved the result in [14]. In fact, he obtained that if $g\in\Gamma_\tau(\mathbf{R}^+,\frac{dr}{r})$ for some $1\leq\tau\leq 2$, $\Omega\in L(\log L)^{1/\tau'}(\mathbf{S}^{n-1})$ and the function ψ is $C^2([0,\infty))$, increasing and convex with $\psi(0)=0$, then $\mathcal{M}^{(\tau),c}_{\Omega,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for any $\tau'\leq p<\infty$ with $1<\tau\leq 2$, and also it is bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\tau=1$.

In this paper, we shall get certain estimates for $\mathcal{M}^{(\tau)}_{\Omega,\psi}$ under weak conditions on the kernels, and then we use these estimates in an extrapolation argument to establish some new extended and improved results in parabolic maximal functions. Also, we shall derive and present several applications of our main result. The main result of this paper is described in the following theorem.

Theorem 1.1. *Let $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $1<q\leq 2$ with $\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\leq 1$ and satisfy the conditions (1.1)-(1.2). Assume that ψ is a real-valued polynomial and $\mathcal{M}^{(\tau)}_{\Omega,\psi}$ is given by (1.3) for some $\tau\in[1,2]$. Then there is a positive constant C_p such that*

$$\left\|\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)\right\|_{L^p(\mathbf{R}^{n+1})}\leq C_p\left(1+\log(e+\|\Omega\|_{L^q(\mathbf{S}^{n-1})})\right)^{1/\tau'}\|f\|_{L^p(\mathbf{R}^{n+1})}\tag{1.5}$$

for $\tau'\leq p<\infty$ with $1<\tau\leq 2$, and

$$\left\|\mathcal{M}^{(1)}_{\Omega,\psi}(f)\right\|_{L^\infty(\mathbf{R}^{n+1})}\leq C\|f\|_{L^\infty(\mathbf{R}^{n+1})}.\tag{1.6}$$

By using the inequalities (1.5)-(1.6) and applying an extrapolation argument (see [15, 16, 17, 18]), we obtain the following result:

Theorem 1.2. *Let ψ be given as in Theorem 1.1 and Ω belong to the space $B_q^{(0,-1/\tau)}(\mathbf{S}^{n-1})\cup L(\log L)^{1/\tau'}(\mathbf{S}^{n-1})$ for some $q>1$. Then $\mathcal{M}^{(\tau)}_{\Omega,\psi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for $\tau'\leq p<\infty$ with $1<\tau\leq 2$, and it is bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\tau=1$.*

As a direct consequence of Theorem 1.2 and the observation that

$$\left|T_{\Omega,g,\psi}(f)(x,x_{n+1})\right|\leq\|g\|_{\Gamma_\tau(\mathbf{R}^+,\frac{d\lambda}{\lambda})}\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)(x,x_{n+1})$$

for any $1\leq\tau\leq 2$, we deduce the following:

Corollary 1.3. *Assume that ψ and Ω are given as in Theorem 1.2. Let $g\in\Gamma_\tau(\mathbf{R}^+,\frac{d\lambda}{\lambda})$ for some $1\leq\tau\leq 2$. Then the singular operator $T_{\Omega,g,\psi}$ given by (1.4) is bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\tau=1$; and it bounded on $L^p(\mathbf{R}^{n+1})$ for $\tau'\leq p<\infty$ with $1<\tau\leq 2$.*

In fact, conclusion from Corollary 1.3 and using a standard duality argument (see [[6], Theorem 1.3]), one can easily satisfy the L^p boundedness of $T_{\Omega,g,\psi}$ for any $1<p<\infty$ with $1<\tau\leq 2$ whenever $\Omega\in B_q^{(0,-1/\tau)}(\mathbf{S}^{n-1})\cup L(\log L)^{1/\tau'}(\mathbf{S}^{n-1})$.

The generalized parabolic Marcinkiewicz operator related to the maximal operator $\mathcal{M}^{(\tau)}_{\Omega,\psi}$ is given by

$$\mu^{(\tau)}_{\Omega,\psi}(f)(x,x_{n+1})$$
$$=\left(\int_{\mathbf{R}^+}\left|\frac{1}{t}\int_{\lambda(v)\leq t} f(x-v,x_{n+1}-\psi(\lambda(v)))\Omega(y)(\lambda(v))^{-\alpha+1}dv\right|^{\tau'}dt\right)^{1/\tau'}.\tag{1.7}$$

It is clear that for any $1\leq\tau\leq 2$, we have

$$\mu^{(\tau)}_{\Omega,\psi}(f)(\cdot)\leq C\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)(\cdot).$$

Therefore, we can derive the following result:

Corollary 1.4. *Let ψ and Ω be given as in Theorem 1.2. Suppose that the generalized parabolic Marcinkiewicz operator $\mu^{(\tau)}_{\Omega,\psi}$ is given by (1.7) for some $1\leq\tau\leq 2$. Then the operator $\mu^{(\tau)}_{\Omega,\psi}$ is bounded on $L^\infty(\mathbf{R}^{n+1})$ for $\tau=1$; and it is of type (p,p) for all $p\in[\tau',\infty)$ with $1<\tau\leq 2$.*

It is worth mentioning that the authors of [19] established the L^p boundedness of $\mu^{(2)}_{\Omega,\psi}$ for all $1<p<\infty$ whenever ψ is a real-valued polynomial and $\Omega\in L(\log L)^{1/2}(\mathbf{S}^{n-1})\cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q>1$. On the other side, under other constraints different form the above corollary, the operator $\mu^{(2)}_{\Omega,\psi}$ was studied in [20, 21, 22, 23].

Throughout the rest of this article, whenever the letter C appears, it refers to a bounded positive constant that may vary at each occurrence but independent of the essential variables. Also, whenever $\Omega\in L^q(\mathbf{S}^{n-1})$) for some $q>1$, we let $\beta_\Omega=\log(e+\|\Omega\|_{L^q(\mathbf{S}^{n-1})})$.

2. Preliminary lemmas

In this section, we give some auxiliary. Let us start with the following lemma.

Lemma 2.1. *Let $\Omega\in L^1(\mathbf{S}^{n-1})$ satisfy the conditions (1.1)-(1.2), and let ψ be a real-valued polynomial. Define the maximal function $M^\lambda_{\Omega,\psi}$ by*

$$M^\lambda_{\Omega,\psi}f(x)=\sup_{\mathbf{j}\in\mathbf{Z}}\int_{2^j\leq\lambda(y)\leq 2^{j+1}}|f(x-v,x_{n+1}-\psi(\lambda(v)))|\frac{|\Omega(v)|}{\lambda(v)^\alpha}dv.$$

Then, for $1<p\leq\infty$, we have

$$\left\|M^\lambda_{\Omega,\psi}(f)\right\|_{L^p(\mathbf{R}^{n+1})}\leq C_p\|f\|_{L^p(\mathbf{R}^{n+1})}\|\Omega\|_{L^1(\mathbf{S}^{n-1})}.$$

Proof. By a simple change of variables, it is easy to see that

$$M_{\Omega,\psi}^{\lambda} f(x) = \sup_{\mathbf{j}\in\mathbf{Z}} \int_{\mathbf{S}^{n-1}} \int_{2^j}^{2^{j+1}} \left| f(x - D_\lambda u, x_{n+1} - \psi(\lambda))\Omega(u)J(u) \right| \frac{d\lambda}{\lambda} d\mu(u)$$

$$\leq C \sup_{\mathbf{j}\in\mathbf{Z}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \left(\int_{2^j}^{2^{j+1}} \left| f(x - D_\lambda u, x_{n+1} - \psi(\lambda)) \right| \frac{d\lambda}{\lambda} \right) d\mu(u).$$

Since the maximal function $M = \sup_{\mathbf{j}\in\mathbf{Z}} \int_{2^j}^{2^{j+1}} \left| f(x - D_\lambda u, x_{n+1} - \psi(\lambda)) \right| \frac{d\lambda}{\lambda}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $1 < p \leq \infty$ (see [24]), then we directly get

$$\left\| M_{\Omega,\psi}^{\lambda}(f) \right\|_{L^p(\mathbf{R}^{n+1})} \leq C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^{n+1})}. \quad \square$$

The next lemma can be derived by following the same approaches (with only minor modifications) found in [6, 25, 26].

Lemma 2.2. *Let $1 < q \leq 2$, and let $\Omega \in L^q(\mathbf{S}^{n-1})$ satisfy the conditions (1.1)-(1.2) with $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Assume that $\psi(.)$ is an arbitrary function on $\mathbf{R}^+$. For $k \in \mathbf{Z}$, define $\mathcal{J}_{k,\Omega,\psi} : \mathbf{R}^{n+1} \to \mathbf{R}$ by*

$$\mathcal{J}_{k,\Omega,\psi}(\xi,\eta) = \int_1^{2^{2\beta_\Omega}} \left| \int_{\mathbf{S}^{n-1}} \Omega(u) \mathcal{A}_{k,\Omega,\psi}(\lambda,u) J(u) d\mu(u) \right|^2 \frac{d\lambda}{\lambda}, \tag{2.1}$$

where

$$\mathcal{A}_{k,\Omega,\psi}(\lambda,u) = e^{-i\left[2^{-(k+1)\beta_\Omega} D_\lambda u\cdot\xi + \psi(2^{-(k+1)\beta_\Omega}\lambda)\eta\right]} \tag{2.2}$$

Then, there exist constants $C > 0$ and $0 \leq \epsilon \leq 1$ so that

$$\mathcal{J}_{k,\Omega,\psi}(\xi,\eta) \leq C\beta_\Omega \min\{ \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{-\frac{\epsilon}{4m\beta_\Omega}}, \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{\frac{\epsilon}{4m\beta_\Omega}} \},$$

where m is denoted to the distinct numbers of $\{\alpha_j\}$.

Proof. It is easy to check that

$$\mathcal{J}_{k,\Omega,\psi}(\xi,\eta) \leq C \int_1^{2^{2\beta_\Omega}} \left(\int_{\mathbf{S}^{n-1}} |\Omega(u)| \left| \mathcal{A}_{k,\Omega,\psi}(\lambda,u) \right| d\mu(u) \right)^2 \frac{d\lambda}{\lambda}$$

$$\leq C \int_1^{2^{2\beta_\Omega}} \left(\int_{\mathbf{S}^{n-1}} |\Omega(u)| \, d\mu(u) \right)^2 \frac{d\lambda}{\lambda} \leq C\beta_\Omega \|\Omega\|^2_{L^1(\mathbf{S}^{n-1})} \leq C\beta_\Omega. \tag{2.3}$$

On one hand, by using [[27], Lemma 2.2], we obtain that

$$\left| \int_1^{2^{2\beta_\Omega}} \mathcal{A}_{k,\Omega,\psi}(\lambda,u) \overline{\mathcal{A}_{k,\Omega,\psi}(\lambda,v)} \frac{d\lambda}{\lambda} \right| \leq C \left| \left\{ D_{2^{-(k+1)\beta_\Omega}} (u-v)\cdot\xi \right\} \right|^{-\frac{1}{4m}}$$

$$\leq C \left(|(u-v)\cdot\zeta| \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right| \right)^{-\frac{1}{4m}}, \tag{2.4}$$

where $\zeta = \frac{D_{2^{-(k+1)\beta_\Omega}}\xi}{\left|D_{2^{-(k+1)\beta_\Omega}}\xi\right|}$. Combining (2.4) with the trivial estimate

$$\left| \int_1^{2^{2\beta_\Omega}} \mathcal{A}_{k,\Omega,\psi}(\lambda,u) \overline{\mathcal{A}_{k,\Omega,\psi}(\lambda,v)} \frac{d\lambda}{\lambda} \right| \leq C\beta_\Omega \tag{2.5}$$

leads to

$$\left| \int_1^{2^{2\beta_\Omega}} \mathcal{A}_{k,\Omega,\psi}(\lambda,u) \overline{\mathcal{A}_{k,\Omega,\psi}(\lambda,v)} \frac{d\lambda}{\lambda} \right| \leq C \left(|(u-v)\cdot\zeta| \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right| \right)^{-\frac{\epsilon}{4m}} \beta_\Omega^{1-\epsilon}$$

for any $\epsilon \in [0,1]$. Now, by Hölder's inequality, we obtain

$$\left(\mathcal{J}_{k,\Omega,\psi}(\zeta)\right)^{q'}$$

$$\leq C \|\Omega\|^{2q'}_{L^q(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left| \int_1^{2^{2\beta_\Omega}} \mathcal{A}_{k,\Omega,\psi}(\lambda,u) \overline{\mathcal{A}_{k,\Omega,\psi}(\lambda,v)} \frac{d\lambda}{\lambda} \right|^{q'} d\mu(u) d\mu(v).$$

Hence, as $1 < q \leq 2$, we choose ϵ so that $\frac{q'\epsilon}{2m} < 1$. So, we deduce

$$\mathcal{J}_{k,\Omega,\psi}(\zeta) \leq C \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{-\frac{\epsilon}{4m}} \|\Omega\|^2_{L^1(\mathbf{S}^{n-1})} \beta_\Omega^{1-\epsilon},$$

which when combined with the trivial estimate (2.3) gives

$$\mathcal{J}_{k,\Omega,\psi}(\zeta) \leq C \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{-\frac{\epsilon}{4m\beta_\Omega}} \beta_\Omega^{1-\frac{\epsilon}{\beta_\Omega}} \leq C\beta_\Omega \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{-\frac{\epsilon}{4m\beta_\Omega}}. \tag{2.6}$$

On the other hand, by using the cancellation condition (1.2), we have

$$\left| \int_{\mathbf{S}^{n-1}} \Omega(u) \mathcal{A}_{k,\Omega,\psi}(\lambda,u) J(u) d\mu(u) \right| \leq C \int_{\mathbf{S}^{n-1}} |\Omega(u)| \left| e^{-i2^{-(k+1)\beta_\Omega} D_\lambda u\cdot\xi} - 1 \right| d\mu(u)$$

$$\leq C \left| D_{2^{-(k+1)\beta_\Omega}\lambda} \xi \right| \|\Omega\|_{L^1(\mathbf{S}^{n-1})}.$$

Thus, when the last estimate is combined with the trivial estimate

$$\left| \int_{\mathbf{S}^{n-1}} \Omega(u) \mathcal{A}_{k,\Omega,\psi}(\lambda,u) J(u) d\mu(u) \right| \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})},$$

we get

$$\left| \int_{\mathbf{S}^{n-1}} \Omega(u) \mathcal{A}_{k,\Omega,\psi}(\lambda,u) J(u) d\mu(u) \right| \leq C \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \left| D_{2^{-(k+1)\beta_\Omega}\lambda} \xi \right|^{\frac{\epsilon}{8m\beta_\Omega}}.$$

Therefore,

$$\mathcal{J}_{k,\Omega,\psi}(\zeta) \leq C\beta_\Omega \left| D_{2^{-(k+1)\beta_\Omega}} \xi \right|^{\frac{\epsilon}{4m\beta_\Omega}}. \tag{2.7}$$

Consequently, by (2.6) and (2.7), the proof is complete. $\square$

3. Proof of Theorem 1.1

To prove this theorem, we employ similar arguments used in the proof of [Theorem 1.6, [6]] and [Theorem 1.1, [17]]. By the duality,

$$\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)(x, x+1)$$

$$= \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x - D_\lambda v, x_{n+1} - \psi(\lambda))\Omega(v)J(v)d\mu(v) \right|^{\tau'} \frac{d\lambda}{\lambda} \right)^{1/\tau'},$$

which gives

$$\left\| \mathcal{M}^{(\tau)}_{\Omega,\psi}(f) \right\|_{L^p(\mathbf{R}^{n+1})} = \left\| \mathcal{B}_\lambda(f) \right\|_{L^p(L^{\tau'}(\mathbf{R}^+, \frac{d\lambda}{\lambda}), \mathbf{R}^{n+1})}, \tag{3.1}$$

where $\mathcal{B}_\lambda : L^p(\mathbf{R}^{n+1}) \to L^p(L^{\tau'}(\mathbf{R}^+, \frac{d\lambda}{\lambda}), \mathbf{R}^{n+1})$ is the linear operator given by

$$\mathcal{B}_\lambda(f)(x, x_{n+1}) = \int_{\mathbf{S}^{n-1}} f(x - D_\lambda v, x_{n+1} - \psi(\lambda))\Omega(v)J(v)d\mu(v).$$

In order to handle our main result, it is enough to show that

$$\left\| \mathcal{B}_\lambda(f) \right\|_{L^\infty(L^\infty(\mathbf{R}^+, \frac{d\lambda}{\lambda}), \mathbf{R}^{n+1})} \leq C_p \|f\|_{L^\infty(\mathbf{R}^{n+1})}, \tag{3.2}$$

and

$$\left\| \mathcal{B}_\lambda(f) \right\|_{L^p(L^2(\mathbf{R}^+, \frac{d\lambda}{\lambda}), \mathbf{R}^{n+1})} \leq C_p (1+\beta_\Omega)^{1/2} \|f\|_{L^p(\mathbf{R}^{n+1})} \tag{3.3}$$

for $2 \le p < \infty$; and then apply the interpolation theorem to the inequalities (3.2)-(3.3) to get

$$\left\|\mathcal{M}^{(\tau)}_{\Omega,\psi}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C_p(1+\beta_\Omega)^{1/\tau'}\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{3.4}$$

for $\tau' \le p < \infty$ with $1 < \tau < 2$.

Let us first prove (3.2). In this case we consider $\tau = 1$; assume that $g \in L^1(\mathbf{R}^+, \frac{d\lambda}{\lambda})$ and $f \in L^\infty(\mathbf{R}^{n+1})$. Then for all $(x, x_{n+1}) \in \mathbf{R}^n \times \mathbf{R}$, we have

$$\left|\int_0^\infty h(\lambda)\int_{\mathbf{S}^{n-1}} f(x - D_\lambda v, x_{n+1} - \psi(\lambda))\Omega(v)J(v)d\mu(v)\frac{d\lambda}{\lambda}\right| \le C\|f\|_{L^\infty(\mathbf{R}^{n+1})}\|g\|_{L^1(\mathbf{R}^+,\frac{d\lambda}{\lambda})}.$$

Hence, for any g with $\|g\|_{\Gamma_1(\mathbf{R}^+,\frac{d\lambda}{\lambda})} \le 1$, we reach

$$\mathcal{M}^{(1)}_{\Omega,\psi} f(x, x_{n+1}) \le C\|f\|_{L^\infty(\mathbf{R}^{n+1})}$$

for almost every where $(x, x_{n+1}) \in \mathbf{R}^{n+1}$, which implies

$$\|\mathcal{B}_\lambda(f)\|_{L^\infty(L^\infty(\mathbf{R}^+,\frac{d\lambda}{\lambda}),\mathbf{R}^{n+1})} = \left\|\mathcal{M}^{(1)}_{\Omega,\psi} f\right\|_{L^\infty(\mathbf{R}^{n+1})} \le C\|f\|_{L^\infty(\mathbf{R}^{n+1})}.$$

Now consider the case $\tau = 2$. Let $\{\varphi_k\}_{k\in\mathbf{Z}}$ be a collection of C^∞ functions on $(0, \infty)$ that satisfy the following:

$$\operatorname{supp}\varphi_k \subseteq \mathcal{I}_{k,\beta_\Omega} = \left[2^{-(k+1)\beta_\Omega}, 2^{-(k-1)\beta_\Omega}\right];\ \ \varphi_k(u) = \varphi_k(\lambda(u)))$$

$$0 \le \varphi_k \le 1;\quad \sum_{k\in\mathbf{Z}}\varphi_k(\lambda) = 1;\quad \textit{and}\quad \left|\frac{d^k\varphi_k(\lambda)}{d\lambda^k}\right| \le \frac{C_k}{\lambda^k}.$$

Let Φ_k be the multiplier operators in $\mathbf{R}^{n+1}$ given by

$$\widehat{(\Phi_k f)}(\xi,\eta) = \varphi_k(\lambda(\xi))\hat{f}(\xi,\eta)\quad for\ (\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}.$$

Then by Minkowski's inequality, we have for any $f \in \mathcal{S}(\mathbf{R}^{n+1})$,

$$\mathcal{M}^{(2)}_{\Omega,\psi}(f)(x, x_{n+1}) \le \sum_{j\in\mathbf{Z}} E^{(2)}_{\Omega,\psi,j}(f)(x, x_{n+1}), \tag{3.5}$$

where

$$E^{(2)}_{\Omega,\psi,j}(f)(x, x_{n+1}) = \left(\sum_{k\in\mathbf{Z}}\int_{\mathcal{I}_{k,\beta_\Omega}}\left|\mathcal{H}_{k+j,\lambda}f(x, x_{n+1})\right|^2\frac{d\lambda}{\lambda}\right)^{1/2},$$

and

$$\mathcal{H}_{j,\lambda}f(x, x_{n+1}) = \int_{\mathbf{S}^{n-1}}(\Phi_j f)(x - D_\lambda v, x_{n+1} - \psi(\lambda))\Omega(v)J(v)d\mu(v).$$

Therefore, to satisfy (3.3), it suffices to show

$$\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C_p\left(1+\beta_\Omega\right)^{1/2}2^{\kappa|j|}\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{3.6}$$

for some $C_p, \kappa > 0$ and for all $p \ge 2$. The L^2-norm of $E^{(2)}_{\Omega,\psi,j}(f)$ is estimated as follows:

$$\begin{aligned}\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|^2_{L^2(\mathbf{R}^{n+1})} &\le \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}}\int_{\Delta_{k+j}}\left|\hat{f}(\xi,\eta)\right|^2\mathcal{J}_{k,\Omega,\psi}(\xi,\eta)d\xi d\eta\\ &\le C\left(1+\beta_\Omega\right)2^{\frac{-\epsilon\delta|j|}{4m}}\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}}\int_{\Delta_{k+j}}\left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta\\ &\le C\left(1+\beta_\Omega\right)2^{\frac{-\epsilon\delta|j|}{4m}}\|f\|^2_{L^2(\mathbf{R}^{n+1})},\end{aligned} \tag{3.7}$$

where $\delta = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\Delta_k = \{\xi \in \mathbf{R}^n : \lambda(\xi) \in \mathcal{I}_{k,\beta_\Omega}\}$. The last inequality is obtained by using Fubini's theorem, Plancherel's theorem and Lemma 2.2. Hence, when we choose ϵ small enough, then the inequality (3.6) holds for $p = 2$.

On the other side, if $p > 2$, then by the duality, there is $h \in L^{(p/2)'}(\mathbf{R}^{n+1})$ with $\|h\|_{L^{(p/2)'}(\mathbf{R}^{n+1})} = 1$ such that

$$\begin{aligned}\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|^2_{L^p(\mathbf{R}^{n+1})} = \int_{\mathbf{R}^{n+1}}\int_1^{2^{2\beta_\Omega}}\Bigg|\int_{\mathbf{S}^{n-1}}&\mathcal{A}_{k+j,\Omega,\psi}(\lambda,v)\\ &\times f(x - D_{2^{-(k+j+1)\beta_\Omega}\lambda}v, x_{n+1} - \psi(2^{-(k+j+1)\beta_\Omega}\lambda))d\mu(v)\Bigg|^2\\ &\times\frac{d\lambda}{\lambda}\left|h(x, x_{n+1})\right|dxdx_{n+1}.\end{aligned}$$

Thus, by Hölder's inequality and Lemma 2.1, we conclude that

$$\begin{aligned}&\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|^2_{L^p(\mathbf{R}^{n+1})}\\ &\le \int_{\mathbf{R}^{n+1}}\left|f(z, z_{n+1})\right|^2\int_1^{2^{2\beta_\Omega}}\int_{\mathbf{S}^{n-1}}|\Omega(v)|\\ &\quad\times\left|h(z + D_{2^{-(k+j+1)\beta_\Omega}\lambda}v, z_{n+1} + \psi(2^{-(k+j+1)\beta_\Omega}\lambda))\right|d\mu(v)\frac{d\lambda}{\lambda}dzdz_{n+1}\\ &\le C\beta_\Omega\left\|\sum_{k\in\mathbf{Z}}\left|\Phi_{k+j}f\right|^2\right\|_{L^{(p/2)}(\mathbf{R}^{n+1})}\left\|M^\lambda_{\Omega,\psi}\tilde{h}(z)\right\|_{L^{(p/2)'}(\mathbf{R}^{n+1})}\\ &\le C_p\beta_\Omega\|f\|^2_{L^p(\mathbf{R}^{n+1})}\|h\|_{L^{(p/2)'}(\mathbf{R}^{n+1})}\|\Omega\|_{L^1(\mathbf{S}^{n-1})},\end{aligned}$$

where $\tilde{h}(z, z_{n+1}) = h(-z, -z_{n+1})$. Hence,

$$\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C_p\left(1+\beta_\Omega\right)^{1/2}\|f\|_{L^p(\mathbf{R}^{n+1})},$$

which when Combined with (3.7) gives that there is $0 < \kappa < 1$ so that

$$\left\|E^{(2)}_{\Omega,\psi,j}(f)\right\|_{L^p(\mathbf{R}^{n+1})} \le C2^{-\kappa|j|}\left(1+\beta_\Omega\right)^{1/2}\|f\|_{L^p(\mathbf{R}^{n+1})} \tag{3.8}$$

for all $p \ge 2$. Therefore, by (3.5) and (3.8), we satisfy the inequality (3.3) for $\tau = 2$. Consequently, the proof of the main result is complete.

4. Conclusions

The appropriate L^p estimates for the parabolic maximal operator given by (1.4) are established whenever ψ is a real-valued polynomial and Ω is in $L^q(\mathbf{R}^n)$ for some $1 < q \le 2$. These obtained estimates are employed in an extrapolation argument, similar to that used in [17], to prove the L^p boundedness of the aforementioned operator when Ω belongs to the block space $B_q^{(0,-1/\tau)}(\mathbf{S}^{n-1})$ or to the space $L(\log L)^{1/\tau'}(\mathbf{S}^{n-1})$, which are considered as improvements and extensions to the results in [13, 14]. Moreover, some applications of our results are presented. Precisely, the boundedness of the parabolic singular operator related to our maximal operator is given. In fact, this obtained result generalizes the results in [1, 2, 5]. Furthermore, the boundedness of the generalized parabolic Marcinkiewicz operator associated to the such operator is achieved, which extends the results in [19].

Declarations

Author contribution statement

M. Ali, Q. Katatbeh: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data.

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