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Research article

# Statistical analysis of the $k/n(G)$ system with dependent competing failure components influenced by Gumbel-Hougarrd Copula and progressively hybrid censored data

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ABSTRACT

Consider a $k/n(G)$ system, in which the system components are composed of multiple dependent failure mechanisms, and the dependence between the mechanisms is connected by the Gumbel-Hougarrd (GH) Copula. This paper presents a progressively hybrid censored test based on the $k/n(G)$ system. Based on the censored test, the IFM(Marginal inference) method is used to estimate the model parameters and system reliability. Meanwhile, the MH (Metropolis-Hastings) sampling mixed with the Gibbs sampling method is proposed to realize the Bayes estimation of the model parameters and system reliability. Also, under the non-informative prior conditions, the conditional posterior density of the shape parameters of the marginal Weibull distribution is proved to be log-concave. The Monte Carlo simulation results showed that the proposed Bayes method is better than the traditional IFM method. Finally, the model and method proposed in this paper are applied to real data.

## 1. Introduction

The $k/n(G)$ system is one of the classical reliability models that works if and only if at least $k$ of the $n$ components work. The $k/n(G)$ system is a series system if $k = n$, and it is a parallel system if $k = 1$. In particular, the $k/n(G)$ system is a common single component system when $n = k = 1$. The $k/n(G)$ system is widely used in engineering practice to improve product reliability [1–3]. In the study of the $k/n(G)$ system reliability, the failure mechanism of components in the system needs to be paid attention to. There have been many studies on the reliability analysis of the $k/n(G)$ system component failure caused by multiple independent mechanism competing failures [4–6]. However, due to the influence of common external loads (such as noise, vibration, etc.), the component failure is often caused by mutually dependent competition mechanisms. Therefore, it is necessary to consider the dependence between the mechanisms in the system components.

Copula function can effectively establish the dependent structure between multivariate random variables. It is characterized by discussing the edge distribution and correlation separately, which is easy to construct and has good statistical properties. Copula theory was proposed by Sklar [7] in 1959 and has been widely applied in different fields such as wind energy assessment [8], hydrology research [9], and economic analysis [10]. Based on the Copula theory, the Copula function family can be effectively applied to model

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the complex dependence structure among dependence multiple random variables. For the $k/n$ system with dependent competitive failure components, Salehi [11] considers a weighted $k/n$ system that the component dependency structures are connected by Copula functions. Xu [12] modeled the cumulative degradation of the $k/n$ system components through a heterogeneous stochastic process, and the dependence among components was characterized by a Copula function. Mahmoudi [13] discussed a weighted $k/n$ system having $m \geq 2$ types of components each with its positive integer-valued weight, in which the dependence of the system component lifetimes is modeled by a Copula function. Saberzadeh [14] considered the reliability of complex systems based on degradation data and proposed a flexible class of multivariate stochastic processes to describe the components dependence by a Copula function. Ahmadi [15] used a generic $n$-variate Farlie-Gumbel-Morgenstern (FGM) Copula function to build reliability modeling and discussed a method to maintain a load-sharing $k/n$ system subject to hidden failures. The above studies focus on the dependence between the system components, but there are few studies on the component consisting of multiple dependent mechanisms. Therefore, this paper will consider the dependence between the mechanisms. Meanwhile, the Gumbel-Hougaard (GH) Copula function is used to model the life distribution when the $k/n(G)$ system components have dependent competing failures.

In engineering, it is impossible to observe the failure time of all systems due to time, cost or other constraints. For example, the perfect units are removed from the experiment and put into application. So in many cases, the actual observed sample is censored [16–20]. The lifetime reliability under the various progressively type II censoring samples has been heavily researched when the data is derived from a single variable [21–24]. For the multivariable system, Bera [25] considered the stress−strength reliability under progressive type-II censoring and derived the parameter MLE, Bayes estimate, and bootstrap confidence interval estimate. Rostamian [26] considered the estimation reliability of the inverse Gaussian distribution under progressive type-II censoring. Under the type-II progressive censoring scheme, Farghal et al. [27] analyzed the generalized inverted exponential competing risks model, and the failure modes are independent. Most of the above studies are censored test analysis of single components. So far, little research has been done to analyze the $k/n(G)$ system based on progressive censoring data. In this paper, for the $k/n(G)$ system, we proposed a new progressively hybrid censored test, and then estimate the unknown model parameters and reliability when the data are from the new progressively hybrid censored test.

The structure of this paper is as follows. Section 2 models the reliability of mechanical dependencies in the $k/n(G)$ system Components. A type II progressively hybrid censored is presented in section 3. The likelihood function of the $k/n(G)$ system is presented. Section 4 introduces the parameter and system reliability estimation methods. In section 5, a simulation study is carried out to evaluate the performance of the two methodologies. The example data are presented to verify the proposed models in section 6. Section 7 presents conclusions.

## 2. Model description

Suppose that the system component failure is caused by $d$ $(d > 1)$ dependent competing failure mechanisms, the mechanisms failure times are $T_1, \cdots, T_d$. Assuming that every failure mechanism acting on a component can cause the component to fail. The component life $T$ is determined by the minimum value of the mechanisms failure time, i.e. $T = \min(T_1, \cdots, T_d)$. In this paper, the dependence between two $(d = 2)$ failure mechanisms is established by GH Copula function.

### *2.1. GH Copula function*

The Copula is a function used to link multiple univariate marginal distribution functions of random variables into a multivariate joint distribution. Common used Copulas are Archimedean, Elliptical and Extreme-value Copula [28,29]. Archimedean Copula function not only has excellent features such as simple construction and strong adaptability, but also can be easily extended from two-dimensional to $m$-dimensional cases. Archimedean Copula function has wide applications in the fields of economics, finance and engineering [30–32].

GH Copula function belongs to Archimedean Copula family and is widely used because it can capture upper tail dependence between variables effectively. For example, Saberzaden [14] used the GH Copula function to establish the dependence of the components in the system elements and investigate the reliability of the system. Fang [33] analyzed the reliability of the degrading system with dependent failures via GH Copula. Bai [34] used the GH Copula function to describe the dependence between stresses and strengths. It can be seen that GH copula function is superior to other copula functions in capturing complex data dependencies due to its flexibility and multi-parameter structure. Therefore, we will specifically at the GH Copula in this paper. The two-dimensional GH Copula function is given by:

$$C_\theta(u_1, u_2) = \exp\left\{ -\left((-\ln u_1)^\theta + (-\ln u_2)^\theta\right)^{\frac{1}{\theta}} \right\},\ \theta \in [1, \infty), \tag{1}$$

where, $u_1, u_2 \in [0, 1]$ is the marginal distributions, and $\theta$ denotes the strength of dependence between $u_1, u_2$. When $\theta = 1$, $u_1$ and $u_2$ are independent of each other, and when $\theta \to \infty$, $u_1$ and $u_2$ tend to be completely dependent. The GH Copula density function graph is in the form of a “T” shape, which is suitable for asymmetric changes of variables with obvious upper tail dependence.

### *2.2. Dependent competing failure model*

The model discussed in this paper is based on the following three assumptions:

(I) The $k/n(G)$ system consists of $n$ independent and identically distributed components, each component has only two states function or fail.

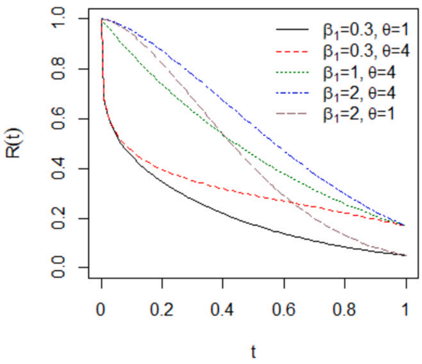


**Fig. 1.** Component reliability $R(t)$, where $\lambda = 1.5, \beta_2 = 1.5$.

(II) There are two causes of component failure in the $k/n(G)$ system: failure mechanism 1 and failure mechanism 2. And the failure of system component failure is caused by two dependent competing failure mechanisms. Let the survival time of the two mechanisms be $T_1, T_2$. And suppose that $T_l(l = 1, 2)$ follows the Weibull distribution with the same scale parameter $\lambda$ and different shape parameter $\beta_l$. The Weibull cumulative distribution function (CDF) and probability density function (PDF) are given by

$$Q_l(t|\lambda, \beta_l) = 1 - \exp\{-\lambda t^{\beta_l}\},\ t > 0,\ \lambda > 0,\ \beta_l > 0,$$
$$q_l(t|\lambda, \beta_l) = \lambda \beta_l t^{\beta_l - 1} \exp\{-\lambda t^{\beta_l}\},\ t > 0,\ \lambda > 0,\ \beta_l > 0.$$

(III) The dependence between two failure mechanisms in the system component is connected by the GH Copula function, and the expression of the GH Copula function is shown in Eq. (1).

Based on the above assumptions, the reliability function of the component for the $k/n(G)$ system is deduced by

$$\begin{aligned} R(t) &= P(\min\{T_1, T_2\} > t) = P(T_1 > t, T_2 > t) \\ &= C(u_1, u_2) = \exp\left\{ -\lambda w^{\frac{1}{\theta}}(t)\right\},\ t > 0, \end{aligned}$$

where $u_l = 1 - Q_l(t|\lambda, \beta_l) = S_l(t|\lambda, \beta_l)$ is the survival function, $l = 1, 2$, and $w(t) = t^{\beta_1\theta} + t^{\beta_2\theta}$. At the same time, the CDF and PDF of the component are derived as

$$G(t) = 1 - \exp\left\{ -\lambda w^{\frac{1}{\theta}}(t)\right\},$$
$$g(t) = \lambda\left(\beta_1 t^{\beta_1\theta - 1} + \beta_2 t^{\beta_2\theta - 1}\right) w^{\frac{1}{\theta} - 1}(t) \exp\left\{ -\lambda w^{\frac{1}{\theta}}(t)\right\}.$$

Thus, the CDF, PDF, and reliability function of the $k/n(G)$ system are derived as

$$\begin{aligned} F_{k/n}(t) &= \sum_{h=n-k+1}^{n} \binom{n}{h} \left(G(t)\right)^h \left(1 - G(t)\right)^{n-h} \\ &= \sum_{h=n-k+1}^{n} \binom{n}{h} \left(1 - \exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^h \left(\exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^{n-h}, \\ f_{k/n}(t) &= (n - k + 1)\binom{n}{n-k+1} g(t)(G(t))^{n-k}(1 - G(t))^{k-1} \\ &= (n - k + 1)\binom{n}{n-k+1} \lambda\left(\beta_1 t^{\beta_1\theta - 1} + \beta_2 t^{\beta_2\theta - 1}\right) w^{\frac{1}{\theta} - 1}(t) \\ &\quad \times \left(1 - \exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^{n-k} \left(\exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^{k}, \\ R_{k/n}(t) &= 1 - F_{k/n}(t) = \sum_{h=k}^{n} \binom{n}{h} \left(\exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^h \left(1 - \exp\{-\lambda w^{\frac{1}{\theta}}(t)\}\right)^{n-h}. \end{aligned} \tag{2}$$

Fig. 1 and Fig. 2 show the component and system reliability functions when the model parameters take different values. From Fig. 1 and Fig. 2, it can be seen that with fixed parameters $\lambda = 1.5, \beta_2 = 1.5$ and $\theta$, as the shape parameter $\beta_1$ increases, the component reliability and the system reliability both increase. When the shape parameter $\beta_1$ is fixed, as the dependence parameter $\theta$ increases, the component reliability and system reliability also increase. Therefore, it is necessary to consider the interdependence of mechanisms when modeling competing failures.

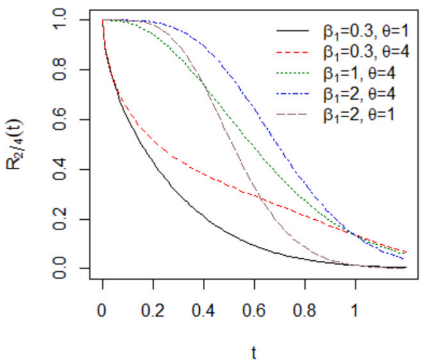


**Fig. 2.** System reliability $R_{2/4}(t)$, where $\lambda = 1.5, \beta_2 = 1.5$.

## 3. Progressively hybrid censored test

It is common in product testing that the perfect product is removed from the test for some purposes (e.g., use, sale, or purchase, etc.), even though the product has a few flaws that do not affect its use. In engineering, these problems are a class of censored problems. To deal with this kind of problem, we proposed a progressively hybrid censored test for the $k/n(G)$ system.

Suppose $w$ $k/n(G)$ systems with independent and identical failure time distribution were put to test. Assuming the component failure time and failure mechanism for $k/n(G)$ systems are unmasked. Let $\delta_{i,j}$ be an indicator function corresponding to the component failure mechanism, i.e.

$$\delta_{i,j} = \begin{cases} 1, & \text{the failure mechanism of the } i\text{th component of system } j \text{ is } 1, \\ 0, & \text{the failure mechanism of the } i\text{th component of system } j \text{ is } 2, \end{cases} \quad i = 1, \cdots, n;\ j = 1, \cdots, w.$$

Assuming the censored test scheme was predetermined as

$$\boldsymbol{r} = (r_1, \cdots, r_{m-1}, r_m, \cdots, r_v),\ r_s \geq 0,\ s = 1, \cdots, v,\ v \geq 1.$$

Specifically, the censored test was as follows.

When the first $k/n(G)$ system is observed fail (where $n-k+1$ components are observed to fail), since the system components and failure mechanisms are unmasked, the corresponding $n-k+1$ components failure times can be observed, denoted as $(\boldsymbol{t}_1, \boldsymbol{\delta}_1) = \big((t_{1,1}, \delta_{1,1}), \cdots, (t_{n-k+1,1}, \delta_{n-k+1,1})\big)$. At this time, randomly remove $r_1$ systems out of the remaining $(w-1)$ systems that have not failed and require that the components in the removed $r_1$ systems are all working. When second $k/n(G)$ system is observed fail, the observed $n-k+1$ components failure times are denoted as $(\boldsymbol{t}_2, \boldsymbol{\delta}_2) = \big((t_{1,2}, \delta_{1,2}), \cdots, (t_{n-k+1,2}, \delta_{n-k+1,2})\big)$, and randomly remove $r_2$ out of the remaining $(w-2-r_1)$ systems that have not failed, and require that the components in the removed $r_2$ systems are all working, and so on. When the $m$th $k/n(G)$ system is observed fail, the corresponding $n-k+1$ components failure times are denoted as $(\boldsymbol{t}_m, \boldsymbol{\delta}_m) = \big((t_{1,m}, \delta_{1,m}), \cdots, (t_{n-k+1,m}, \delta_{n-k+1,m})\big)$, and the number of working systems remaining is $(w - m - r_1 - \cdots - r_{m-1})$.

Two cases are considered according to the above inspection policy, and they are listed as follow.

Case I: When the $i$th system is observed fail, if the number of $k/n(G)$ systems that their components are all working (denoted as $r_m^*$) is smaller than the pre-determined $r_v (1 \leq m \leq v)$, $k/n(G)$ systems with size $r_v^*$ are removed, and the censored ends, and the test continues until all the $(w - m - r_1 - \cdots - r_{m-1} - r_m^*)$ systems have failed. In this case, the censored scheme is $r_1, \cdots, r_m^*$.

Case II: When the $v$th $k/n(G)$ system is observed fail, if the number of $k/n(G)$ systems with working components is greater than or equal to $r_v$, then the censored scheme is $r_1, \cdots, r_v$, and the test is continued until the failure of all $(w - v - r_1 - \cdots - r_v)$ systems.

The test results are given in matrix form as follows

$$\boldsymbol{d} = \begin{pmatrix} (\boldsymbol{t}_1, \boldsymbol{\delta}_1) \\ (\boldsymbol{t}_2, \boldsymbol{\delta}_2) \\ \cdots \\ (\boldsymbol{t}_{w-a}, \boldsymbol{\delta}_{w-a}) \end{pmatrix}^T = \begin{pmatrix} (t_{1,1}, \delta_{1,1}) & \cdots & (t_{1,w-a}, \delta_{1,w-a}) \\ (t_{2,1}, \delta_{2,1}) & \cdots & (t_{2,w-a}, \delta_{2,w-a}) \\ \cdots & \cdots & \cdots \\ (t_{n-k+1,1}, \delta_{n-k+1,1}) & \cdots & (t_{n-k+1,w-a}, \delta_{n-k+1,w-a}) \end{pmatrix},$$

where

$$a = \begin{cases} r_1 + \cdots + r_m^*, & \text{case I}, \\ r_1 + \cdots + r_v, & \text{case II}. \end{cases}$$

The PDF of system component failure caused by mechanism $l\ (=1,2)$ is given by

$$g_l(t) = \frac{\partial C(u_1, u_2)}{\partial u_l}\Big|_{u_1 = S_1(t), u_2 = S_2(t)} \cdot q_l(t) = \lambda \beta_l t^{\beta_l \theta - 1} w^{\frac{1}{\theta} - 1}(t) \exp\left\{ -\lambda w^{\frac{1}{\theta}}(t) \right\}. \tag{3}$$

Combined with Eq. (3), the likelihood function is given by

$$L(\boldsymbol{v}|\boldsymbol{d}) = \prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1} \left(\lambda\beta_1 t_{i,j}^{\beta_1\theta-1}\right)^{\delta_{i,j}} \left(\lambda\beta_2 t_{i,j}^{\beta_2\theta-1}\right)^{1-\delta_{i,j}} w^{\frac{1}{\theta}-1}(t_{i,j})R(t_{i,j})$$
$$\times \prod_{j=1}^{w-a}\left(R(t_{n-k+1,j})\right)^{k-1}\prod_{s=1}^{b}\left(R(t_{n-k+1,s})\right)^{nr_s}, \tag{4}$$

where $\boldsymbol{v} = (\lambda, \beta_1, \beta_2, \theta)$. In case I, $b = m$, $r_m = r_m^*$, and $b = v$ in case II.

Taking the logarithm of Eq. (4) and applying the first-order derivative to the parameters $\lambda, \beta_1, \beta_2, \theta$ to make it equal to 0 forms a four-element system of equations, which can be solved to obtain the maximum likelihood estimate (MLE) of the parameter. Nevertheless, Eq. (4) is very complicated, and the system of equations consisting of derivatives has no explicit solution. In this paper, the IFM method and the Bayes method are used to estimate the model parameters and system reliability.

## 4. Estimation method

### 4.1. IFM estimation method

The IFM estimation is a method to estimate the marginal distribution parameters and Copula parameters separately, which can effectively reduce the complexity of multi-parameter estimation [35]. The likelihood function for the marginal distribution parameters $(\lambda, \beta_1, \beta_2)$ is given by

$$\begin{aligned} L_1 &= L(\lambda, \beta_1, \beta_2|\boldsymbol{d}) \\ &= \prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1}\left(q_1(t_{i,j};\lambda,\beta_1)\overline{Q}_2(t_{i,j};\lambda,\beta_2)\right)^{\delta_{i,j}}\left(q_2(t_{i,j};\lambda,\beta_2)\overline{Q}_1(t_{i,j};\lambda,\beta_1)\right)^{1-\delta_{i,j}} \\ &\times\left(\overline{Q}_1(t_{n-k+1,j};\lambda,\beta_1)\overline{Q}_2(t_{n-k+1,j};\lambda,\beta_2)\right)^{k-1}\prod_{s=1}^{b}\left(\overline{Q}_1(t_{n-k+1,s};\lambda,\beta_1)\overline{Q}_2(t_{n-k+1,s};\lambda,\beta_2)\right)^{nr_s} \\ &\propto \prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1}\lambda(\beta_1 t_{i,j}^{\beta_1-1})^{\delta_{i,j}}(\beta_2 t_{i,j}^{\beta_2-1})^{1-\delta_{i,j}}\exp\left\{-\lambda w_1(t_{i,j})\right\} \\ &\times\exp\left\{-\lambda(k-1)w_1(t_{n-k+1,j})\right\}\prod_{s=1}^{b}\exp\left\{-n\lambda r_s w_1(t_{n-k+1,s})\right\}. \end{aligned} \tag{5}$$

where $w_1(t) = t^{\beta_1} + t^{\beta_2}$.

Taking the logarithm of Eq. (5) and partial derivatives for the parameters $\lambda, \beta_1, \beta_2$, we obtain

$$\frac{\partial \ln L_1}{\partial \lambda} = \frac{N}{\lambda} - \sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1} w_1(t_{i,j}) - (k-1)\sum_{j=1}^{w-a} w_1(t_{n-k+1,j})$$
$$- n\sum_{s=1}^{b} r_s w_1(t_{n-k+1,s}) = 0, \tag{6}$$

$$\frac{\partial \ln L_1}{\partial \beta_l} = \frac{N_l}{\beta_l} + \sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1}\left(\delta_0 - \lambda t_{i,j}^{\beta_l}\right)\ln t_{i,j} - (k-1)\lambda\sum_{j=1}^{w-a} t_{n-k+1,j}^{\beta_l}\ln t_{n-k+1,j}$$
$$- n\lambda\sum_{s=1}^{b} r_s t_{n-k+1,s}^{\beta_l}\ln t_{n-k+1,s} = 0. \tag{7}$$

When $l = 1$, $\delta_0 = \delta_{i,j}$, and when $l = 2$, $\delta_0 = 1 - \delta_{i,j}$. $N = N_1 + N_2$, $N_1 = \sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1}\delta_{i,j}$ and $N_2 = \sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1}(1-\delta_{i,j}) = (w-a)(n-k+1) - N_1$, denotes the total number of observed component failures due to failure mechanism 1 and 2, respectively.

By solving the equations consisting of Eq. (6) and Eq. (7), the parameters $\lambda, \beta_1, \beta_2$ IFM estimates are obtained and denoted as $\widehat{\lambda}, \widehat{\beta}_1, \widehat{\beta}_2$. By replacing the parameter estimates $\widehat{\lambda}, \widehat{\beta}_1, \widehat{\beta}_2$ in Eq. (4), the estimate $\widehat{\theta}$ of the dependence parameter $\theta$ can be obtained by solving the Eq. (8).

$$\frac{\partial \ln L(\theta|\widehat{\lambda},\widehat{\beta}_1,\widehat{\beta}_2,\boldsymbol{d})}{\partial\theta} = \sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1}\Big(\delta_{i,j}\widehat{\beta}_1\ln t_{i,j} + (1-\delta_{i,j})\widehat{\beta}_2\ln t_{i,j} - \frac{\ln w(t_{i,j})}{\theta^2}$$
$$+ \frac{\ln w(t_{i,j})}{\theta^2} + (\frac{1}{\theta}-1)\frac{1}{w(t_{i,j})}\frac{\partial w(t_{i,j})}{\partial\theta} - \widehat{\lambda}\frac{\partial w^{\frac{1}{\theta}}(t_{i,j})}{\partial\theta}\Big)$$

$$-(k-1)\hat{\lambda}\sum_{j=1}^{w-a}\frac{\partial w^{\frac{1}{\theta}}(t_{n-k+1,j})}{\partial\theta}-n\hat{\lambda}\sum_{s=1}^{b}r_s\frac{\partial w^{\frac{1}{\theta}}(t_{n-k+1,s})}{\partial\theta}, \tag{8}$$

where $\frac{\partial w(t)}{\partial\theta}=t^{\beta_1\theta}\ln t^{\beta_1}+t^{\beta_2\theta}\ln t^{\beta_2}$, $\frac{\partial w^{\frac{1}{\theta}}(t)}{\partial\theta}=\frac{1}{\theta}\left(\frac{1}{w(t)}\frac{\partial w(t)}{\partial\theta}-\frac{1}{\theta}\ln w(t)\right)w^{\frac{1}{\theta}}(t)$.

Given time $t$, replace $(\hat{\lambda},\hat{\beta}_1,\hat{\beta}_2,\hat{\theta})$ into Eq. (2) to compute the estimate $\hat{F}_{k/n}(t)$ of the reliability of the $k/n(G)$ system.

### 4.2. Bayes estimation method

In this section, assume that the parameters $\boldsymbol{v}=(\lambda,\beta_1,\beta_2,\theta)$ are independent of each other with non-informative uniform prior distributions. Under this assumption, the joint posterior density of the parameters $\boldsymbol{v}$ is Eq. (4). The conditional posterior density function of $\lambda,\beta_1,\beta_2$, and $\theta$, respectively, is proportional to

$$\pi_1(\lambda|\beta_1,\beta_2,\theta,\boldsymbol{d})\propto\lambda^{(w-a)(n-k+1)}\prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1}w^{\frac{1}{\theta}-1}(t_{i,j})R(t_{i,j})$$
$$\times\prod_{j=1}^{w-a}\left(R(t_{n-k+1,j})\right)^{k-1}\prod_{s=1}^{b}\left(R(t_{n-k+1,s})\right)^{nr_s}. \tag{9}$$

$$\pi_{l+1}(\beta_l|\lambda,\beta_{3-l},\theta,\boldsymbol{d})\propto\beta_l^{N_l}\prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1}t_{i,j}^{\beta_l\theta\delta_0}w^{\frac{1}{\theta}-1}(t_{i,j})R(t_{i,j})$$
$$\times\prod_{j=1}^{w-a}\left(R(t_{n-k+1,j})\right)^{k-1}\prod_{s=1}^{b}\left(R(t_{n-k+1,s})\right)^{nr_s}, l=1,2. \tag{10}$$

$$\pi_4(\theta|\lambda,\beta_1,\beta_2,\boldsymbol{d})\propto\prod_{j=1}^{w-a}\prod_{i=1}^{n-k+1}\left(t_{i,j}^{\beta_1\theta\delta_{i,j}}\right)\left(t_{i,j}^{\beta_2\theta(1-\delta_{i,j})}\right)w^{\frac{1}{\theta}-1}(t_{i,j})R(t_{i,j})$$
$$\times\prod_{j=1}^{w-a}\left(R(t_{n-k+1,j})\right)^{k-1}\prod_{s=1}^{b}\left(R(t_{n-k+1,s})\right)^{nr_s}. \tag{11}$$

Where the conditional posterior density function Eq. (10) of the parameter $\beta_l, l=1,2$ is log-concave, i.e.

$$\frac{\partial^2\log\pi_{l+1}(\beta_l|\lambda,\beta_{3-l},\theta,\boldsymbol{d})}{\partial\beta_l^2}<0.$$

The proof is presented in the Appendix.

Apparently, there are no analytical expressions for Eqs. (9)-(11). Consequently, in this paper the Metropolis-Hastings (MH) sampling mixed with the Gibbs sampling method is used to estimate the parameters [36]. Where the normal distribution is taken as the proposal distribution to sample $\lambda$ and $\theta$, the Gibbs sampling based on log-concave is used to sample $\beta_1$ and $\beta_2$. Repeated $M$ times, the Markov Chain Monte Carlo (MCMC) sample of length $M$ is obtained, that is denoted as

$$\left(\lambda^{(1)},\beta_1^{(1)},\beta_2^{(1)},\theta^{(1)},F_{k/n}^{(1)}(t)\right),\cdots,\left(\lambda^{(M)},\beta_1^{(M)},\beta_2^{(M)},\theta^{(M)}.F_{k/n}^{(M)}(t)\right)$$

Then discard the first $M_0$ sample as burn-in, the following $M-M_0$ samples are used to obtain the Bayes posterior mean and variance, where

$$E(v|\boldsymbol{d})\approx\hat{v}=\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}v^{(i)},\ Var(v|\boldsymbol{d})\approx\frac{1}{M-M_0}\sum_{i=1}^{M-M_0}(v^{(i)}-\hat{v})^2,$$

and $v=\lambda,\beta_1,\beta_2,\theta$, or $F_{k/n}(t)$.

## 5. Simulation study

A Monte Carlo technique is adopted to study the performance of the proposed approach in Sections 3 and 4 under the six-group $k/n(G)$ systems and nine-group progressively hybrid censored schemes. Where six-group $k/n(G)$ systems are $k/n=(1/4,\ 2/4,\ 4/4,\ 1/8,\ 4/8,\ 8/8)$, and nine-group progressively hybrid censored schemes are shown in Table 1. In addition, margin parameters and dependence parameter $\boldsymbol{v}=(\lambda,\beta_1,\beta_2,\theta)=(1.5,1.5,2,4)$ are selected to generate the sample. Further, six-group time $t=(0.85,0.75,0.5,0.85,0.75,0.5)$ was selected to estimate the reliability that corresponds to above six-group $k/n(G)$ systems.

Under the nine censored schemes, we repeat simulation process $N=1000$ times. The first $M_0=100$ samples have been discarded and the subsequent $M=1000$ are used to obtain the estimates of model parameters. The estimated results of parameters under each simulation process in ascending order are denoted as $\hat{v}^{[1]},\hat{v}^{[2]},\ldots,\hat{v}^{[N]}$, $v=\lambda,\beta_1,\beta_2,\theta$, or $F_{k/n}(t)$. Then a two-sided confidence

**Table 1**
Progressively censored schemes.

| $w$ | $v$ | Scheme | $\boldsymbol{r}$ |
|---|---|---|---|
| 20 | 10 | [1] | $(1,2,1,1,2,1,2,0,0,0)$ |
| | | [2] | $(\underbrace{2,\cdots,2}_{5},\underbrace{0,\cdots,0}_{5})$ |
| | | [3] | $(\underbrace{1,\cdots,1}_{10})$ |
| 40 | 20 | [4] | $(2,1,1,1,2,1,1,1,2,1,1,1,2,1,1,1,0,0,0,0)$ |
| | | [5] | $(\underbrace{2,\cdots,2}_{10},\underbrace{0,\cdots,0}_{10})$ |
| | | [6] | $(\underbrace{1,\cdots,1}_{20})$ |
| 80 | 40 | [7] | $(\underbrace{2,1,1,1,\cdots,2,1,1,1}_{32},\underbrace{0,\cdots,0}_{8})$ |
| | | [8] | $(\underbrace{2,\cdots,2}_{20},\underbrace{0,\cdots,0}_{20})$ |
| | | [9] | $(\underbrace{1,\cdots,1}_{40})$ |

**Table 2**
Parameters and system reliability estimated by IFM estimation method when $n = 4$.

| Scheme | $k$ | $\lambda$ | | | $\beta_1$ | | | $\beta_2$ | | | $\theta$ | | | $R_{k/4}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP |
| [1] | 1 | -0.619 | 0.043 | 94.8 | -0.030 | 0.140 | 94.7 | -0.182 | 0.148 | 95.6 | -0.008 | 1.264 | 99.6 | 0.152 | 0.018 | 94.4 |
| | 2 | -0.663 | 0.066 | 97.5 | -0.105 | 0.324 | 95.2 | 0.192 | 0.504 | 94.5 | 0.188 | 0.840 | 92.7 | 0.111 | 0.077 | 96.7 |
| | 4 | -0.678 | 0.132 | 95.2 | -0.495 | 0.692 | 97.6 | 2.950 | 4.428 | 98.6 | 0.142 | 1.948 | 99.8 | -0.276 | 0.385 | 99.7 |
| [2] | 1 | -0.617 | 0.042 | 95.1 | -0.035 | 0.137 | 94.8 | -0.178 | 0.158 | 95.5 | 0.037 | 1.213 | 96.9 | 0.154 | 0.018 | 93.1 |
| | 2 | -0.655 | 0.075 | 97.1 | -0.141 | 0.351 | 95.1 | 0.230 | 0.592 | 94.5 | 0.169 | 0.965 | 91.4 | 0.102 | 0.097 | 95.8 |
| | 4 | -0.686 | 0.124 | 94.6 | -0.489 | 0.663 | 98.0 | 2.946 | 4.525 | 99.5 | -0.125 | 1.915 | 99.7 | -0.255 | 0.374 | 99.7 |
| [3] | 1 | -0.623 | 0.043 | 94.1 | -0.023 | 0.137 | 93.9 | -0.188 | 0.143 | 96.0 | -0.033 | 1.279 | 99.6 | 0.151 | 0.019 | 95.1 |
| | 2 | -0.669 | 0.063 | 97.5 | -0.083 | 0.296 | 95.8 | 0.170 | 0.463 | 95.4 | 0.204 | 0.714 | 94.3 | 0.117 | 0.066 | 97.4 |
| | 4 | -0.677 | 0.123 | 95.6 | -0.559 | 0.673 | 97.4 | 2.950 | 3.667 | 99.0 | -0.202 | 1.981 | 99.7 | -0.340 | 0.394 | 99.6 |
| [4] | 1 | -0.621 | 0.031 | 95.2 | -0.029 | 0.096 | 94.6 | -0.184 | 0.105 | 94.7 | 0.093 | 1.030 | 92.2 | 0.160 | 0.013 | 91.7 |
| | 2 | -0.665 | 0.035 | 95.1 | -0.092 | 0.183 | 94.5 | 0.153 | 0.306 | 95.7 | 0.241 | 0.301 | 98.5 | 0.129 | 0.014 | 98.9 |
| | 4 | -0.683 | 0.114 | 97.9 | -0.485 | 0.548 | 99.1 | 2.463 | 3.510 | 98.7 | -0.031 | 1.790 | 99.6 | -0.195 | 0.345 | 99.1 |
| [5] | 1 | -0.619 | 0.030 | 94.4 | -0.031 | 0.097 | 95.0 | -0.176 | 0.109 | 95.2 | 0.113 | 0.989 | 90.2 | 0.161 | 0.013 | 91.1 |
| | 2 | -0.660 | 0.043 | 97.7 | -0.115 | 0.206 | 96.2 | 0.182 | 0.340 | 95.7 | 0.234 | 0.410 | 97.6 | 0.126 | 0.035 | 98.7 |
| | 4 | -0.695 | 0.085 | 95.7 | -0.411 | 0.503 | 99.1 | 2.414 | 3.815 | 98.7 | 0.057 | 1.563 | 82.6 | -0.108 | 0.299 | 88.3 |
| [6] | 1 | -0.621 | 0.029 | 95.1 | -0.029 | 0.097 | 95.6 | -0.187 | 0.103 | 95.8 | 0.075 | 1.058 | 94.4 | 0.159 | 0.014 | 93.2 |
| | 2 | -0.663 | 0.051 | 99.3 | -0.094 | 0.195 | 95.4 | 0.144 | 0.290 | 95.6 | 0.242 | 0.293 | 98.8 | 0.127 | 0.038 | 99.4 |
| | 4 | -0.686 | 0.107 | 97.1 | -0.547 | 0.548 | 99.2 | 2.801 | 3.182 | 98.8 | -0.100 | 1.895 | 99.6 | -0.259 | 0.368 | 99.6 |
| [7] | 1 | -0.621 | 0.021 | 94.4 | -0.031 | 0.067 | 94.8 | -0.183 | 0.074 | 95.0 | 0.175 | 0.716 | 90.1 | 0.166 | 0.008 | 91.0 |
| | 2 | -0.663 | 0.025 | 95.1 | -0.094 | 0.129 | 94.5 | 0.147 | 0.204 | 95.3 | 0.250 | 0.038 | 99.9 | 0.130 | 0.002 | 98.1 |
| | 4 | -0.690 | 0.082 | 97.4 | -0.447 | 0.475 | 99.9 | 2.461 | 3.489 | 94.5 | 0.044 | 1.614 | 80.1 | -0.124 | 0.310 | 91.1 |
| [8] | 1 | -0.620 | 0.021 | 93.9 | -0.032 | 0.067 | 95.3 | -0.179 | 0.075 | 96.1 | 0.192 | 0.624 | 92.3 | 0.167 | 0.007 | 93.8 |
| | 2 | -0.662 | 0.025 | 95.8 | -0.106 | 0.124 | 95.9 | 0.173 | 0.218 | 95.4 | 0.248 | 0.115 | 99.7 | 0.130 | 0.004 | 99.5 |
| | 4 | -0.697 | 0.060 | 95.5 | -0.383 | 0.410 | 88.0 | 2.122 | 3.314 | 93.1 | 0.123 | 1.321 | 87.3 | -0.048 | 0.252 | 88.2 |
| [9] | 1 | -0.621 | 0.022 | 95.3 | -0.029 | 0.069 | 94.9 | -0.186 | 0.071 | 95.4 | 0.156 | 0.777 | 90.6 | 0.165 | 0.009 | 93.0 |
| | 2 | -0.665 | 0.024 | 95.0 | -0.086 | 0.124 | 94.5 | 0.143 | 0.211 | 95.0 | 0.249 | 0.112 | 99.8 | 0.130 | 0.004 | 99.6 |
| | 4 | -0.691 | 0.067 | 94.8 | -0.557 | 0.514 | 99.7 | 2.945 | 3.642 | 98.7 | -0.092 | 1.895 | 99.7 | -0.246 | 0.362 | 99.6 |

interval for $\nu$ with a confidence level of $100(1-\gamma)\%$ is $(\hat{\nu}_L, \hat{\nu}_U) = (\hat{\nu}^{[N\gamma/2]}, \hat{\nu}^{[N(1-\gamma/2)]})$. The relative bias (RB), standard deviation (SD), and interval coverage probability (CP) with $95\%(\gamma = 0.05)$ confidence were calculated, and the calculation results were listed in Table 2–Table 5.

From Tables 2–5, it is readily observed that:

(a) For IFM estimation, as the number of $k/n(G)$ system $w$ increases, the RB and SD of the estimated parameters and system reliability decrease. In most cases, when $k = n$ (series system), the RB and SD are larger than $k = 1$ (parallel system), especially for the shape parameter $\beta_2$. With the increase of the number of $k/n(G)$ system $w$, RB and SD did not change significantly, and CP of some parameters exceeded 98.0% or even 99.9%.

(b) For Bayes estimation, RB and SD for each parameter and system reliability estimation by using Bayes estimation method are relatively small on the whole, which indicates that the degree of dispersion in the process of parameter and reliability estimation is relatively small. Moreover, when the sample size is the same, there is little difference in RB and SD results estimated by Bayes

**Table 3**
Parameters and system reliability estimated by Bayes estimation method when $n = 4$.

| Scheme | $k$ | $\lambda$ | | | $\beta_1$ | | | $\beta_2$ | | | $\theta$ | | | $R_{k/4}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP |
| [1] | 1 | 0.039 | 0.088 | 94.8 | 0.068 | 0.166 | 94.9 | 0.004 | 0.176 | 95.4 | 0.058 | 0.169 | 94.4 | 0.021 | 0.040 | 95.1 |
| | 2 | 0.041 | 0.095 | 93.7 | 0.083 | 0.182 | 95.0 | 0.023 | 0.196 | 94.8 | 0.054 | 0.169 | 94.3 | 0.033 | 0.042 | 96.1 |
| | 4 | 0.052 | 0.077 | 94.0 | 0.063 | 0.182 | 95.0 | 0.113 | 0.270 | 95.7 | 0.052 | 0.182 | 94.9 | 0.033 | 0.047 | 94.8 |
| [2] | 1 | 0.043 | 0.087 | 94.3 | 0.059 | 0.161 | 95.6 | -0.001 | 0.168 | 94.9 | 0.059 | 0.169 | 94.1 | 0.018 | 0.038 | 95.3 |
| | 2 | 0.047 | 0.093 | 94.3 | 0.069 | 0.186 | 93.9 | 0.009 | 0.204 | 95.2 | 0.051 | 0.166 | 94.6 | 0.024 | 0.045 | 96.0 |
| | 4 | 0.053 | 0.079 | 96.7 | 0.060 | 0.221 | 95.3 | 0.107 | 0.268 | 96.3 | 0.051 | 0.178 | 93.7 | 0.030 | 0.048 | 95.2 |
| [3] | 1 | 0.032 | 0.095 | 94.3 | 0.069 | 0.168 | 95.5 | 0.008 | 0.179 | 95.0 | 0.059 | 0.155 | 94.7 | 0.026 | 0.038 | 95.1 |
| | 2 | 0.036 | 0.093 | 94.4 | 0.089 | 0.194 | 95.3 | 0.026 | 0.206 | 94.7 | 0.054 | 0.159 | 94.5 | 0.035 | 0.042 | 95.6 |
| | 4 | 0.043 | 0.082 | 95.1 | 0.089 | 0.220 | 95.4 | 0.135 | 0.255 | 95.5 | 0.051 | 0.169 | 93.8 | 0.042 | 0.043 | 95.0 |
| [4] | 1 | 0.035 | 0.080 | 93.8 | 0.049 | 0.110 | 94.6 | -0.013 | 0.119 | 95.3 | 0.062 | 0.154 | 96.1 | 0.019 | 0.030 | 95.8 |
| | 2 | 0.037 | 0.089 | 94.5 | 0.061 | 0.125 | 95.2 | -0.010 | 0.139 | 94.9 | 0.055 | 0.160 | 95.1 | 0.027 | 0.033 | 96.2 |
| | 4 | 0.052 | 0.080 | 94.8 | 0.053 | 0.149 | 94.3 | 0.071 | 0.250 | 97.4 | 0.049 | 0.174 | 94.1 | 0.033 | 0.035 | 95.1 |
| [5] | 1 | 0.042 | 0.080 | 94.7 | 0.051 | 0.114 | 95.7 | -0.016 | 0.118 | 95.7 | 0.016 | 0.031 | 95.9 | 0.016 | 0.031 | 95.9 |
| | 2 | 0.043 | 0.090 | 94.2 | 0.049 | 0.131 | 95.4 | -0.018 | 0.138 | 95.3 | 0.058 | 0.167 | 94.5 | 0.023 | 0.034 | 95.1 |
| | 4 | 0.053 | 0.077 | 95.0 | 0.053 | 0.152 | 95.3 | 0.068 | 0.248 | 97.6 | 0.052 | 0.179 | 94.0 | 0.033 | 0.035 | 95.4 |
| [6] | 1 | 0.030 | 0.081 | 94.8 | 0.057 | 0.107 | 94.8 | -0.010 | 0.118 | 94.4 | 0.062 | 0.156 | 94.2 | 0.024 | 0.029 | 94.4 |
| | 2 | 0.031 | 0.093 | 93.5 | 0.069 | 0.123 | 94.6 | -0.004 | 0.139 | 94.9 | 0.056 | 0.169 | 95.0 | 0.033 | 0.032 | 95.3 |
| | 4 | 0.041 | 0.085 | 95.2 | 0.066 | 0.158 | 94.9 | 0.083 | 0.244 | 97.7 | 0.053 | 0.161 | 94.4 | 0.039 | 0.032 | 95.8 |
| [7] | 1 | 0.031 | 0.068 | 94.2 | 0.049 | 0.075 | 96.0 | -0.025 | 0.082 | 94.9 | 0.068 | 0.146 | 94.9 | 0.021 | 0.023 | 95.1 |
| | 2 | 0.036 | 0.077 | 94.6 | 0.054 | 0.092 | 95.6 | -0.023 | 0.100 | 95.1 | 0.064 | 0.156 | 94.4 | 0.029 | 0.025 | 95.2 |
| | 4 | 0.057 | 0.088 | 94.8 | 0.045 | 0.109 | 94.8 | 0.034 | 0.192 | 93.7 | 0.051 | 0.168 | 94.7 | 0.030 | 0.028 | 95.9 |
| [8] | 1 | 0.037 | 0.069 | 93.6 | 0.045 | 0.079 | 95.7 | -0.025 | 0.084 | 94.7 | 0.066 | 0.147 | 94.9 | 0.018 | 0.024 | 96.0 |
| | 2 | 0.039 | 0.083 | 93.8 | 0.049 | 0.093 | 94.8 | -0.025 | 0.103 | 95.1 | 0.067 | 0.149 | 93.0 | 0.028 | 0.025 | 95.0 |
| | 4 | 0.051 | 0.088 | 94.0 | 0.049 | 0.114 | 95.5 | 0.021 | 0.180 | 94.7 | 0.052 | 0.163 | 94.6 | 0.031 | 0.028 | 95.1 |
| [9] | 1 | 0.026 | 0.070 | 94.6 | 0.053 | 0.082 | 94.6 | -0.021 | 0.087 | 94.9 | 0.068 | 0.151 | 95.0 | 0.025 | 0.024 | 95.9 |
| | 2 | 0.029 | 0.079 | 94.1 | 0.052 | 0.085 | 95.4 | -0.026 | 0.094 | 95.5 | 0.064 | 0.150 | 94.2 | 0.031 | 0.024 | 95.7 |
| | 4 | 0.039 | 0.093 | 95.0 | 0.053 | 0.110 | 94.9 | 0.036 | 0.189 | 94.8 | 0.051 | 0.167 | 94.3 | 0.035 | 0.026 | 96.1 |

**Table 4**
Parameters and system reliability estimated by IFM estimation method when $n = 8$.

| Scheme | $k$ | $\lambda$ | | | $\beta_1$ | | | $\beta_2$ | | | $\theta$ | | | $R_{k/8}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP |
| [1] | 1 | -0.611 | 0.028 | 93.7 | -0.038 | 0.087 | 95.1 | -0.213 | 0.088 | 95.6 | -0.049 | 1.217 | 99.6 | 0.023 | 0.001 | 91.4 |
| | 4 | -0.669 | 0.068 | 95.5 | -0.248 | 0.350 | 95.1 | 0.438 | 0.528 | 93.9 | 0.146 | 1.050 | 90.0 | 0.069 | 0.124 | 95.4 |
| | 8 | -0.683 | 0.166 | 96.8 | -0.303 | 0.512 | 93.2 | 4.186 | 7.757 | 97.8 | 0.191 | 0.940 | 94.1 | 0.035 | 0.220 | 94.1 |
| [2] | 1 | -0.611 | 0.029 | 94.4 | -0.037 | 0.092 | 95.3 | -0.210 | 0.090 | 95.3 | -0.050 | 1.237 | 99.8 | 0.023 | 0.001 | 89.1 |
| | 4 | -0.665 | 0.079 | 94.7 | -0.273 | 0.369 | 94.3 | 0.464 | 0.570 | 93.9 | 0.121 | 1.149 | 89.3 | 0.059 | 0.137 | 94.5 |
| | 8 | -0.686 | 0.163 | 96.6 | -0.309 | 0.507 | 94.8 | 4.297 | 7.786 | 98.9 | 0.200 | 0.871 | 94.9 | 0.032 | 0.218 | 94.9 |
| [3] | 1 | -0.611 | 0.028 | 94.9 | -0.035 | 0.089 | 94.7 | -0.212 | 0.084 | 94.3 | -0.053 | 1.209 | 99.8 | 0.023 | 0.001 | 92.0 |
| | 4 | -0.670 | 0.084 | 95.9 | -0.245 | 0.353 | 95.1 | 0.440 | 0.549 | 94.6 | 0.145 | 1.057 | 89.6 | 0.067 | 0.130 | 95.4 |
| | 8 | -0.691 | 0.158 | 95.6 | -0.308 | 0.551 | 98.7 | 4.030 | 7.116 | 98.2 | 0.169 | 1.092 | 91.9 | 0.013 | 0.250 | 91.9 |
| [4] | 1 | -0.610 | 0.020 | 95.4 | -0.038 | 0.065 | 94.9 | -0.215 | 0.063 | 95.5 | -0.014 | 1.084 | 99.6 | 0.023 | 0.001 | 96.5 |
| | 4 | -0.673 | 0.042 | 98.5 | -0.238 | 0.229 | 96.6 | 0.425 | 0.382 | 95.6 | 0.204 | 0.699 | 94.5 | 0.093 | 0.070 | 98.2 |
| | 8 | -0.677 | 0.126 | 97.1 | -0.272 | 0.427 | 93.4 | 3.269 | 6.132 | 92.0 | 0.186 | 0.978 | 93.6 | 0.078 | 0.215 | 93.6 |
| [5] | 1 | -0.609 | 0.021 | 95.5 | -0.041 | 0.503 | 99.1 | -0.215 | 0.062 | 95.1 | -0.002 | 1.090 | 99.7 | 0.023 | 0.001 | 97.2 |
| | 4 | -0.669 | 0.056 | 98.2 | -0.255 | 0.245 | 96.6 | 0.435 | 0.403 | 95.7 | 0.196 | 0.757 | 93.6 | 0.088 | 0.087 | 98.0 |
| | 8 | -0.682 | 0.119 | 96.6 | -0.274 | 0.380 | 95.1 | 3.284 | 6.218 | 93.7 | 0.207 | 0.812 | 95.7 | 0.096 | 0.181 | 95.4 |
| [6] | 1 | -0.611 | 0.019 | 95.0 | -0.039 | 0.063 | 94.7 | -0.213 | 0.064 | 95.4 | -0.006 | 1.076 | 99.7 | 0.023 | 0.001 | 97.5 |
| | 4 | -0.674 | 0.044 | 98.7 | -0.231 | 0.229 | 96.7 | 0.411 | 0.379 | 96.0 | 0.204 | 0.686 | 93.6 | 0.094 | 0.070 | 98.5 |
| | 8 | -0.677 | 0.123 | 95.9 | -0.245 | 0.429 | 93.7 | 2.992 | 5.721 | 92.5 | 0.191 | 0.941 | 94.1 | 0.095 | 0.208 | 94.1 |
| [7] | 1 | -0.610 | 0.014 | 94.7 | -0.040 | 0.044 | 95.1 | -0.217 | 0.045 | 96.0 | 0.016 | 0.951 | 99.4 | 0.023 | 0.001 | 99.5 |
| | 4 | -0.673 | 0.018 | 95.9 | -0.233 | 0.133 | 94.6 | 0.405 | 0.244 | 95.8 | 0.242 | 0.253 | 98.3 | 0.105 | 0.008 | 99.2 |
| | 8 | -0.675 | 0.105 | 98.2 | -0.242 | 0.329 | 94.8 | 2.899 | 5.252 | 92.0 | 0.236 | 0.470 | 98.6 | 0.130 | 0.130 | 95.7 |
| [8] | 1 | -0.610 | 0.014 | 95.6 | -0.039 | 0.044 | 94.7 | -0.214 | 0.044 | 95.4 | 0.034 | 0.949 | 98.9 | 0.024 | 0.001 | 95.7 |
| | 4 | -0.674 | 0.018 | 95.6 | -0.239 | 0.137 | 95.5 | 0.421 | 0.251 | 95.4 | 0.240 | 0.294 | 98.1 | 0.105 | 0.007 | 98.7 |
| | 8 | -0.683 | 0.102 | 96.8 | -0.265 | 0.319 | 96.5 | 3.135 | 5.617 | 92.7 | 0.234 | 0.502 | 98.4 | 0.121 | 0.133 | 96.5 |
| [9] | 1 | -0.610 | 0.014 | 93.1 | -0.039 | 0.045 | 95.0 | -0.215 | 0.046 | 95.1 | 0.030 | 0.964 | 98.8 | 0.023 | 0.001 | 99.5 |
| | 4 | -0.675 | 0.018 | 94.6 | -0.227 | 0.135 | 95.1 | 0.401 | 0.243 | 95.3 | 0.242 | 0.263 | 98.1 | 0.105 | 0.006 | 98.9 |
| | 8 | -0.671 | 0.110 | 97.5 | -0.232 | 0.338 | 95.5 | 2.745 | 5.093 | 92.2 | 0.229 | 0.574 | 97.9 | 0.131 | 0.142 | 96.4 |

**Table 5**
Parameters and system reliability estimated by Bayes estimation method when $n = 8$.

| Scheme | $k$ | $\lambda$ | | | $\beta_1$ | | | $\beta_2$ | | | $\theta$ | | | $R_{k/8}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP | RB | SD | CP |
| [1] | 1 | 0.036 | 0.076 | 94.9 | 0.052 | 0.108 | 94.6 | -0.015 | 0.117 | 95.0 | 0.063 | 0.166 | 94.8 | 0.004 | 0.008 | 95.7 |
| | 4 | 0.041 | 0.091 | 94.5 | 0.061 | 0.132 | 95.2 | -0.012 | 0.140 | 95.0 | 0.057 | 0.155 | 94.9 | 0.027 | 0.038 | 95.9 |
| | 8 | 0.051 | 0.069 | 95.3 | 0.039 | 0.167 | 94.8 | 0.130 | 0.246 | 95.3 | 0.052 | 0.169 | 93.8 | 0.062 | 0.072 | 95.3 |
| [2] | 1 | 0.037 | 0.078 | 94.9 | 0.050 | 0.110 | 95.5 | -0.016 | 0.114 | 95.8 | 0.064 | 0.157 | 94.9 | 0.004 | 0.008 | 94.8 |
| | 4 | 0.043 | 0.090 | 94.3 | 0.057 | 0.124 | 95.2 | -0.017 | 0.134 | 95.5 | 0.056 | 0.162 | 94.7 | 0.024 | 0.037 | 95.9 |
| | 8 | 0.053 | 0.075 | 94.7 | 0.045 | 0.170 | 94.4 | 0.125 | 0.251 | 95.8 | 0.053 | 0.167 | 94.2 | 0.067 | 0.070 | 95.7 |
| [3] | 1 | 0.038 | 0.081 | 94.1 | 0.049 | 0.113 | 95.4 | -0.018 | 0.119 | 94.8 | 0.063 | 0.160 | 95.7 | 0.004 | 0.009 | 95.4 |
| | 4 | 0.041 | 0.090 | 93.9 | 0.072 | 0.135 | 94.1 | -0.007 | 0.143 | 95.4 | 0.055 | 0.163 | 94.2 | 0.030 | 0.038 | 95.5 |
| | 8 | 0.046 | 0.075 | 93.9 | 0.069 | 0.184 | 95.5 | 0.151 | 0.244 | 94.0 | 0.055 | 0.169 | 94.7 | 0.085 | 0.071 | 95.8 |
| [4] | 1 | 0.035 | 0.066 | 94.9 | 0.043 | 0.076 | 94.8 | -0.027 | 0.080 | 95.2 | 0.069 | 0.138 | 95.1 | 0.005 | 0.006 | 94.4 |
| | 4 | 0.034 | 0.084 | 93.4 | 0.055 | 0.095 | 94.7 | -0.026 | 0.102 | 94.9 | 0.064 | 0.157 | 95.2 | 0.031 | 0.028 | 95.9 |
| | 8 | 0.053 | 0.074 | 95.3 | 0.040 | 0.120 | 95.1 | 0.103 | 0.248 | 99.3 | 0.054 | 0.172 | 94.7 | 0.069 | 0.053 | 95.1 |
| [5] | 1 | 0.036 | 0.066 | 93.8 | 0.043 | 0.074 | 94.8 | -0.027 | 0.082 | 95.0 | 0.069 | 0.126 | 93.3 | 0.005 | 0.006 | 95.4 |
| | 4 | 0.039 | 0.083 | 93.7 | 0.049 | 0.092 | 95.2 | -0.029 | 0.099 | 94.6 | 0.064 | 0.153 | 94.4 | 0.028 | 0.027 | 96.4 |
| | 8 | 0.054 | 0.070 | 94.1 | 0.045 | 0.123 | 94.7 | 0.093 | 0.248 | 99.0 | 0.053 | 0.160 | 95.2 | 0.068 | 0.054 | 95.6 |
| [6] | 1 | 0.033 | 0.067 | 95.1 | 0.047 | 0.077 | 94.7 | -0.025 | 0.081 | 95.4 | 0.069 | 0.137 | 94.7 | 0.005 | 0.006 | 94.8 |
| | 4 | 0.028 | 0.086 | 95.3 | 0.056 | 0.095 | 94.4 | -0.025 | 0.101 | 94.8 | 0.065 | 0.156 | 94.7 | 0.033 | 0.027 | 96.2 |
| | 8 | 0.043 | 0.088 | 95.0 | 0.062 | 0.123 | 94.9 | 0.124 | 0.248 | 98.3 | 0.055 | 0.161 | 94.7 | 0.087 | 0.050 | 95.7 |
| [7] | 1 | 0.030 | 0.054 | 94.7 | 0.043 | 0.056 | 95.9 | -0.030 | 0.057 | 94.8 | 0.073 | 0.118 | 94.4 | 0.005 | 0.005 | 96.4 |
| | 4 | 0.028 | 0.071 | 95.1 | 0.048 | 0.066 | 95.6 | -0.034 | 0.073 | 94.9 | 0.074 | 0.144 | 93.8 | 0.035 | 0.019 | 94.5 |
| | 8 | 0.053 | 0.084 | 94.6 | 0.035 | 0.085 | 95.0 | 0.054 | 0.228 | 94.9 | 0.053 | 0.166 | 94.9 | 0.062 | 0.043 | 95.6 |
| [8] | 1 | 0.031 | 0.051 | 94.7 | 0.042 | 0.054 | 96.0 | -0.031 | 0.058 | 95.2 | 0.075 | 0.117 | 94.8 | 0.005 | 0.005 | 97.1 |
| | 4 | 0.031 | 0.068 | 94.8 | 0.043 | 0.065 | 94.7 | -0.037 | 0.072 | 95.2 | 0.073 | 0.137 | 93.2 | 0.033 | 0.019 | 94.9 |
| | 8 | 0.057 | 0.088 | 94.7 | 0.039 | 0.092 | 95.0 | 0.044 | 0.213 | 92.0 | 0.052 | 0.168 | 93.6 | 0.062 | 0.045 | 95.1 |
| [9] | 1 | 0.029 | 0.052 | 94.1 | 0.044 | 0.053 | 94.8 | -0.029 | 0.058 | 95.6 | 0.071 | 0.124 | 95.0 | 0.006 | 0.005 | 96.0 |
| | 4 | 0.027 | 0.072 | 95.0 | 0.048 | 0.069 | 95.8 | -0.036 | 0.073 | 95.5 | 0.073 | 0.141 | 94.0 | 0.035 | 0.020 | 95.5 |
| | 8 | 0.047 | 0.090 | 94.5 | 0.045 | 0.089 | 94.5 | 0.069 | 0.232 | 97.5 | 0.055 | 0.163 | 95.4 | 0.073 | 0.041 | 94.9 |

**Table 6**
Small appliance life data [37].

| Component failure mechanism | Component failure time |
|---|---|
| $\delta = 1$ | 0.4120, 0.4960, 0.5350, 0.5460, 0.6490, 0.7130, 0.8160, 0.8810, 1.1170, 1.2380, 1.3210, 1.3420, 1.4890, 1.5880, 1.7890, 2.1130, 2.1570, 2.4320, 2.7620, 2.8990, 3.1640, 3.3800, 3.4120, 3.8300, 4.0040, 4.0170 |
| $\delta = 0$ | 0.0045, 0.1480, 0.4370, 0.4870, 0.6870, 0.7030, 1.3120, 1.5850, 2.2030, 2.7570, 3.0080, 3.3190, 4.1290 |

method under different censored schemes. In addition to the estimation of the dependence parameter $\theta$, RB and SD are larger in most cases when $k = n$ (series system) than $k = 1$ (parallel system). As the number of $k/n(G)$ system $w$ increases, RB and SD for parameter and system reliability estimation decrease, and as the number of $k/n(G)$ system components $n$ increases, RB and SD for parameter and system reliability estimation are also decreased. Except for individual values ($\beta_2$ corresponding to rows 12, 15, 18 of Table 3), CP remains near 95.0%.

(c) Compare the estimation results of IFM and Bayes method, it is found that the RB of parameters $\lambda$ and $\beta_1$ are negative for IFM estimation, which indicates that these two parameters are underestimated, especially the RB of $\lambda$ reaches 0.6, indicating a serious underestimation trend. Both $\beta_2$ and $\theta$ were underestimated and overestimated, RB of $\beta_2$ was more than 2, and SD was volatile significantly. The Bayes method to parameter $\lambda$, $\beta_1$ and $\theta$ slightly overestimated but RB is relatively stable and almost are controlled within 10%, and SD less volatile.

Therefore, from the RB and SD results of parameter and system reliability estimation, the proposed Bayes method in this paper is superior to the traditional IFM method.

## 6. Illustrative example

To illustrate the availability of the proposed system reliability models, we consider a data set from a small appliance life test with 18 failure mechanisms [37] as an example, and the data are listed in the Table 6. For electrical appliances, the dependence between the failure mechanisms will affect the product's reliability, so the dependence of mechanisms needs to be considered. The data unit is changed to 1000 cycles for simple calculation. Since most failures are from mechanism 5, this paper mainly considers mechanism 5. In this case, the data consists of two failure causes: $\delta = 1$ (failure mechanism 5) and $\delta = 0$ (other failure mechanisms), where $N = 39$, $N_1 = 26$, and $N_2 = 13$.

**Table 7**
Parameter and system reliability estimates from IFM and Bayes methods.

| | IFM | Bayes | |
|---|---|---|---|
| parameter | estimate | estimate | CI |
| $\lambda$ | 0.3868(0.3684)* | 0.3155(0.1922)* | (0.2041, 0.4422)(0.1123, 0.2884)* |
| $\beta_1$ | 2.2570(2.2570)* | 1.5797(1.9997)* | (1.4095, 1.7508)(1.6215, 2.3587)* |
| $\beta_2$ | 1.6630(1.6630)* | 1.0922(1.0528)* | (1.0227, 1.1664)(1.0034, 1.1214)* |
| $\theta$ | 1.8590( – )* | 2.9394( – )* | (2.6550, 3.2118)( – , – )* |
| $R_{2/3}(0.8160)$ | 0.9808(0.8917)* | 0.8427(0.9394)* | (0.7421, 0.9255)(0.8741, 0.9774)* |
| MTTF | 1.1400(0.9964)* | 1.6376(1.5640)* | (1.2553,2.1877)(1.1746, 2.1050)* |

()* represents the estimate result when the failure mechanism is independent ($\theta = 1$).

Assumed that the survival time of the two failure mechanisms obeys the Weibull distribution with the same scale parameters and different shape parameters. The likelihood function of the marginal distribution parameters $(\lambda, \beta_1, \beta_2)$ is as follows:

$$L_2(\lambda, \beta_1, \beta_2|t) = \prod_{i=1}^{N} \left[q_1(t;\lambda,\beta_1)\overline{Q}_2(t;\lambda,\beta_2)\right]_{t=t_i}^{\delta_i} \left[q_2(t;\lambda,\beta_2)\overline{Q}_1(t;\lambda,\beta_1)\right]_{t=t_i}^{1-\delta_i}$$
$$= \lambda\left(\beta_1 t_i^{\beta_1-1}\right)^{\delta_i}\left(\beta_2 t_i^{\beta_2-1}\right)^{1-\delta_i} \exp\{-\lambda(t_i^{\beta_1} + t_i^{\beta_1})\}. \tag{12}$$

Based on Table 6 and Eq. (12), the MLE of $(\lambda, \beta_1, \beta_2)$ is obtained as $(0.1985, 1.6704, 0.9329)$ by the numerical method. At this time, the null hypothesis becomes that the survival time corresponding to the two failure mechanisms follows the Weibull distribution of parameters $(0.1985, 1.6704)$ and $(0.1985, 0.9329)$, respectively. The Kolmogorov-Smirnov (K-S) test is a non-parametric statistical test used to compare whether the distribution of two samples is the same, and the $P$-value is a key indicator in its results. K-S test method was used to calculate the $P$-value of the failure time of the two mechanisms as 0.9779 and 0.9985, respectively. It can be seen from the results that the $P$-value estimated by K-S test corresponding to the two mechanisms are relatively large, so there is not enough evidence to reject the null hypothesis. Therefore, it was considered that the lifetimes of the two mechanisms were subject to Weibull distributions with the same scale parameter $\lambda$ and different shape parameter $\beta$.

In order to simulate the method proposed in this paper, the data in Table 6 are randomly divided into experimental groups with capacity $w = 13$. Each group consists of 3 ($n = 3$) data, taking $k = 2$, which constitutes $w$ $k/n(G)$ system. Then take $v = 6$, and the censored scheme is $\boldsymbol{r} = (2, 1, 2, 1, 1, 0)$. The following is a progressively hybrid censored data set obtained by the simulation method in Section 3, where the actual censored scheme is $\boldsymbol{r}^* = (2, 1, 1)$.

$$\begin{pmatrix} 0.4370 & 0 \\ 0.6490 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.5350 & 1 \\ 0.8160 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.1480 & 0 \\ 0.8810 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.4120 & 1 \\ 1.1170 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.4960 & 1 \\ 1.2380 & 1 \\ - & - \end{pmatrix}$$
$$\begin{pmatrix} 0.7130 & 1 \\ 1.3120 & 0 \\ - & - \end{pmatrix}\begin{pmatrix} 0.4870 & 0 \\ 1.3210 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.5460 & 1 \\ 1.7890 & 1 \\ - & - \end{pmatrix}\begin{pmatrix} 0.0045 & 0 \\ 2.4320 & 1 \\ - & - \end{pmatrix}$$

Under the assumption that the failure mechanism is independent ($\theta = 1$) and dependent ($\theta$ is unknown), the IFM and Bayes methods proposed in section 4 are used to estimate the parameters. Take the time as $t = 0.8160$, calculate the system reliability $R_{2/3}(0.8160)$. For the Bayes estimation method, the chain length of a MCMC is taken as 4000, the burn-in value $M_0$ is taken as 2000, and the credibility $\alpha$ is chosen as 0.05 to compute two-sided confidence intervals (CI) based on the order statistic of the parameter and the reliability of the system. Confidence intervals can reflect the uncertainty of parameter estimates. At the same time, the mean time to fail of appliance (MTTF= $E(T) = \int_0^\infty t f_{k/n}(t)dt$) is calculated based on the parameter estimation results. The MTTF is used to measure the average amount of time a system can operate normally before experiencing a failure. The calculation results of the two calculation methods are listed in Table 7. Fig. 3 shows the MCMC chain trajectory diagram of each parameter. Fig. 4 shows the $k/n(G)$ system density function estimated by IFM and Bayes under the assumption of failure mechanism dependence and independence.

As seen from Table 7, the IFM and the Bayes method have apparent differences in the estimation of shape parameter and dependence parameter, and such differences significantly impact the system reliability estimation. For example, under the IFM and the Bayes methods, $R_{2/3}(0.8160)$ is 0.9808 and 0.8427, respectively, with a difference of about 0.15. While MTTF is 1.1400 and 1.6375, the difference between the results is about 0.50. When the dependence between failure mechanisms is ignored, although the difference in MTTF is not very large, there is also a significant difference in reliability, indicating that it is necessary to consider the dependence between failure mechanisms. From Fig. 4, it is found that the differences in parameter estimation between the two methods lead to significant differences in the kurtosis and skewness of the system density function, which in turn leads to substantial differences in the reliability and MTTF. According to Fig. 3, the MCMC chain is converged, which indicates that the Bayes estimation is feasible.

Combined with the analysis of the simulation results in Section 5, it can be concluded that the Bayes method in this example is more stable and reliable. The estimated result of $\theta$ is $2.9394 > 1$, indicating that the dependence between failure mechanisms is vital and should not be ignored. It can be seen from Table 7 and Fig. 4 that when the dependence between failure mechanisms is ignored, both reliability and MTTF are affected. Therefore, it is necessary to consider the dependence between the mechanisms.

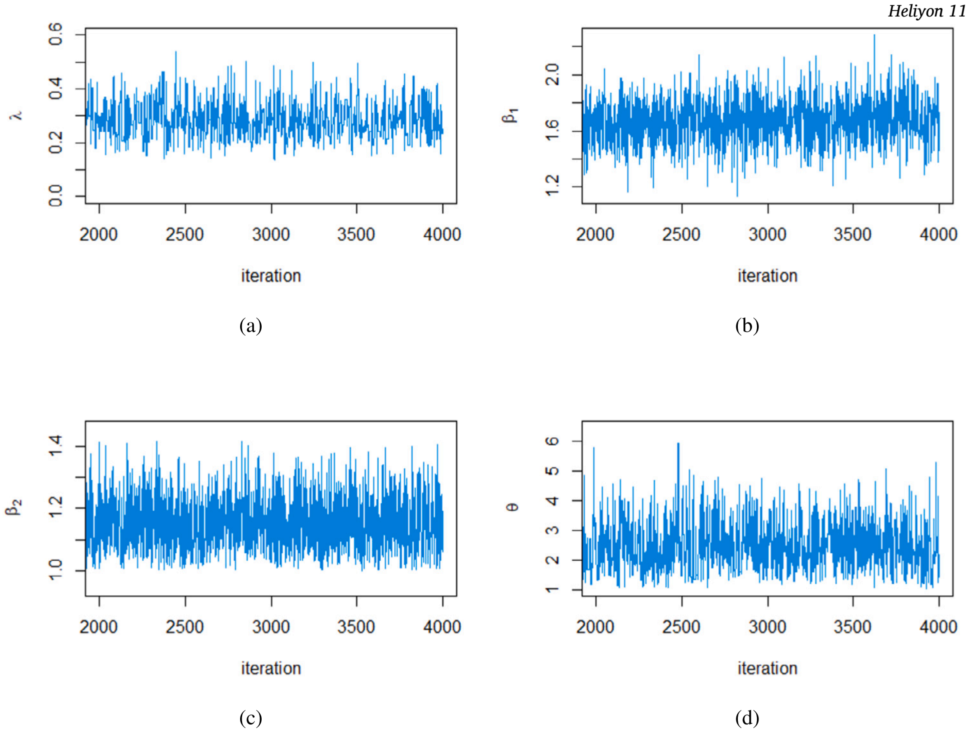


**Fig. 3.** (a), (b), (c), and (d) are trace plots of the MCMC samples.

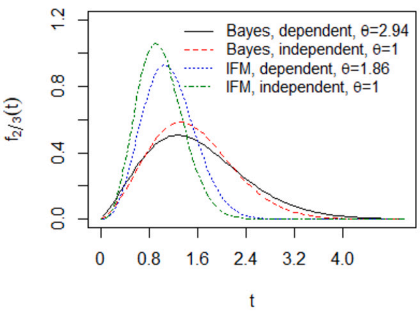


**Fig. 4.** Density function $f_{2/3}(t)$.

## 7. Conclusion

The reliability analysis of the $k/n(G)$ system based on a progressively hybrid censored life test was discussed in this paper. Based on the traditional $k/n(G)$ system, the system components with interdependent competing failure mechanism was considered. The dependencies between the mechanisms were connected by Gumbel-Hougaard (GH) Copula function, and the mechanism lifetime followed the Weibull distribution with different shape parameters but the same scale parameters. A progressively hybrid censored for the $k/n(G)$ system was presented. The maximum likelihood estimation of parameters based on the IFM (Inference for the margins) method was considered. It is proved that the conditional posterior density of the marginal distribution shape parameter without information prior was log-concave. MH (Metropolis-Hastings) sampling mixed with Gibbs sampling was proposed for the Bayes estimation of model parameters and system reliability.

Simulation experiments by the Monte Carlo method showed that the relative error (RB) and standard deviation(SD) of parameters estimated by the IFM method fluctuate significantly. In particular, the estimated RB of marginal scale parameters exceeds 50%. Both RB and SD of parameter estimation by the Bayes method are stable. RB was almost within 10% of them. According to the RB and SD parameter and system reliability estimation results, the Bayes method was superior to the IFM method. Finally, the model and the proposed method were applied to a practical problem. The MC simulation and example also show that the progressively hybrid censored test based on the $k/n(G)$ system and the method proposed in this paper are effective. Therefore, the model and method can solve the $k/n(G)$ system reliability problem in engineering.

On the basis of this paper, the future work can be extended to the following aspects: (1) Consider the dependence between parameters, and establish the corresponding Bayes estimation method. (2) Consider the system components degrade with time. Wiener process, Gamma process and Inverse Gaussian process are used to describe the degradation failure process of system components. (3) Consider the reliability of the weight $k/n(G)$ system, that is, the system components have different effects on the $k/n(G)$ system reliability.

## 8. Limitations

The following are some limitations of the proposed approach.

• The Bayes estimation method chooses the uniform distribution as the non-information prior, and the results are not guaranteed valid when the non-information prior chooses other distributions.

• In Bayes estimation, the model parameters are assumed to be independent of each other.

## CRediT authorship contribution statement

**Yanjie Shi:** Writing – review & editing, Writing – original draft. **Zaizai Yan:** Supervision, Software. **Xiuyun Peng:** Methodology, Formal analysis.

## Ethics statement

Review and/or approval by an ethics committee was not needed for this study because the study is based entirely on reliability analysis and does not involve human subjects or animal experimentation.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A

Logarithm of Eq. (10),

$$\begin{aligned}
&\ln \pi_{l+1}(\beta_l|\lambda, \beta_{3-l}, \theta, \boldsymbol{d}) \\
&= N_l \ln \beta_l + \sum_{j=1}^{w-a} \sum_{i=1}^{n-k+1} \left( \beta_l \theta \delta_0 \ln t_{i,j} + (\frac{1}{\theta} - 1) \ln w(t_{i,j}) - \lambda w^{\frac{1}{\theta}}(t_{i,j}) \right) \\
&- \sum_{j=1}^{w-a} \lambda(k-1) w^{\frac{1}{\theta}}(t_{n-k+1,j}) - \sum_{s=1}^{b} \lambda n r_s w^{\frac{1}{\theta}}(t_{n-k+1,s}).
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\frac{\partial \ln \pi_{l+1}(\beta_l|\lambda, \beta_{3-l}, \theta, \boldsymbol{d})}{\partial \beta_l} \\
&= \frac{N_l}{\beta_l} + \sum_{j=1}^{w-a} \sum_{i=1}^{n-k+1} \left( \theta \delta_0 \ln(t_{i,j}) + (\frac{1}{\theta} - 1) \frac{\theta t_{i,j}^{\beta_l \theta} \ln(t_{i,j})}{w(t_{i,j})} - \lambda t_{i,j}^{\beta_l \theta} \ln(t_{i,j}) w^{\frac{1}{\theta}-1}(t_{i,j}) \right) \\
&- \sum_{j=1}^{w-a} \lambda(k-1) t_{n-k+1,j}^{\beta_l \theta} \ln(t_{n-k+1,j}) w^{\frac{1}{\theta}-1}(t_{n-k+1,j}) \\
&- \sum_{s=1}^{b} \lambda n r_s t_{n-k+1,s}^{\beta_l \theta} \ln(t_{n-k+1,s}) w^{\frac{1}{\theta}-1}(t_{n-k+1,s}),
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\frac{\partial^2 \ln \pi_{l+1}(\beta_l|\lambda, \beta_{3-l}, \theta, \boldsymbol{d})}{\partial \beta_l^2} \\
&= -\frac{N_l}{\beta_l^2} + \sum_{j=1}^{w-a} \sum_{i=1}^{n-k+1} \left( (\frac{1}{\theta} - 1) \frac{t_{i,j}^{\beta_l \theta} (\theta \ln(t_{i,j}))^2 w(t_{i,j}) - (t_{i,j}^{\beta_l \theta} \theta \ln(t_{i,j}))^2}{w^2(t_{i,j})} - \lambda w_2(t_{i,j}) \right)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{w-a}\lambda(k-1)w_2(t_{n-k+1,j})-\sum_{s=1}^{b}\lambda nr_s w_2(t_{n-k+1,s})\\
&=-\frac{N_l}{\beta_l^2}+\sum_{j=1}^{w-a}\sum_{i=1}^{n-k+1}\Big((\frac{1}{\theta}-1)\frac{t_{i,j}^{\beta_1\theta+\beta_2\theta}(\theta\ln(t_{i,j}))^2}{w^2(t_{i,j})}-\lambda w_2(t_{i,j})\Big)\\
&-\sum_{j=1}^{w-a}\lambda(k-1)w_2(t_{n-k+1,j})-\sum_{s=1}^{b}\lambda nr_s w_2(t_{n-k+1,s})<0,
\end{aligned}
\tag{15}
$$

where,

$$
\begin{aligned}
w_2(t)&=\theta t^{\beta_l\theta}(\ln t)^2(t^{\beta_1\theta}+t^{\beta_2\theta})^{\frac{1}{\theta}-1}+(1-\theta)(t^{\beta_l\theta})^2(\ln t)^2(t^{\beta_1\theta}+t^{\beta_2\theta})^{\frac{1}{\theta}-2}\\
&=t^{\beta_l\theta}(\ln t)^2(t^{\beta_1\theta}+t^{\beta_2\theta})^{\frac{1}{\theta}-2}(\theta(t^{\beta_1\theta}+t^{\beta_2\theta})+(1-\theta)t^{\beta_l\theta})>0.
\end{aligned}
$$

## Data availability

No new data was generated for the research described in the article.

## References


[1] V.V. Rykov, M.G. Sukharev, V.Y. Itkin, Investigations of the potential application of $k/n$ systems in oil and gas industry objects, J. Mar. Sci. Eng. (2020), https://doi.org/10.3390/jmse8110928.

[2] N. Zhang, M. Fouladirad, A. Barros, J. Zhang, Condition-based maintenance for a $k$-out-of-$n$ deteriorating system under periodic inspection with failure dependence, Eur. J. Oper. Res. 285 (1) (2020) 159–167.

[3] T.F. Xiahou, Y.X. Zheng, Y. Liu, H. Chen, Reliability modeling of modular $k$-out-of-$n$ systems with functional dependency: a case study of radar transmitter systems, Reliab. Eng. Syst. Saf. 233 (2023) 1–15.

[4] H. Pham, Optimal number of components for a parallel system with competing failure modes, Int. J. Syst. Sci. 23 (3) (1992) 449–455.

[5] S.A. Alghamdi, G.A. Abd-Elmougod, D. Kundu, M. Marin, Statistical inference of jointly type-II lifetime samples under Weibull competing risks models, Symmetry Basel (2022), https://doi.org/10.3390/sym14040701.

[6] A.S. Alghamdi, Statistical inferences of competing risks generalized half-logistic lifetime populations in presence of generalized type-I hybrid censoring scheme, Alex. Eng. J. 65 (2023) 699–708.

[7] A. Sklar, Fonctions de repartition a $n$ dimensions et leurs marges, Publ. Inst. Stat. Univ. Paris 8 (1959) 229–231.

[8] J. Liu, Z.Z. Yan, A circular-linear probabilistic model based on nonparametric Copula with applications to directional wind energy assessment, Entropy 26 (06) (2024), https://doi.org/10.3390/e26060487.

[9] D. Valle, D. Kaplan, Quantifying the impacts of dams on riverine hydrology under non-stationary conditions using incomplete data and Gaussian copula models, Sci. Total Environ. 677 (2019) 599–611.

[10] X.W. Zhang, H. Jiang, Application of copula function in financial risk analysis, Comput. Electr. Eng. 77 (2019) 376–388.

[11] M. Salehi, Z. Shishebor, M. Asadi, On the reliability modeling of weighted $k$-out-of-$n$ systems with randomly chosen components, Metrika 82 (05) (2019) 589–605.

[12] J. Xu, Z.L. Liang, Y.F. Li, K.B. Wang, Generalized condition-based maintenance optimization for multi-component systems considering stochastic dependency and imperfect maintenance, Reliab. Eng. Syst. Saf. (2021), https://doi.org/10.1016/j.ress.2021.107592.

[13] E. Mahmoudi, R.S. Meshkat, H. Torabi, Copula-based reliability for weighted-$k$-out-of-$n$ systems having randomly chosen components of $m$ different types, IEEE Trans. Reliab. 71 (02) (2021) 630–639.

[14] Z. Saberzadeh, M. Razmkhah, M. Amini, Bayesian reliability analysis of complex $k$-out-of-$n$ : $L$ systems under degradation performance, Reliab. Eng. Syst. Saf. (2023), https://doi.org/10.1016/j.ress.2022.109020.

[15] R. Admadi, Reliability and maintenance modeling for a load-sharing $k$-out-of-$n$ system subject to hidden failures, Comput. Ind. Eng. (2020), https://doi.org/10.1016/j.cie.2020.106894.

[16] S. Kanwal, M.H. Tahir, F. Jamal, M. Ameeq, J.T. Mendy, A weighted Weibull detection model for line transect sampling: application on wooden stake perpendicular distance data, Cogent Eng. (2024), https://doi.org/10.1080/23311916.2024.2303237.

[17] N. Hussain, M.H. Tahir, F. Jamal, M. Ameeq, S. Shadiq, J.T. Mendy, An acceptance sampling plan for the odd exponential-logarithmic Fréchet distribution: applications to quality control data, Cogent Eng. (2024), https://doi.org/10.1080/23311916.2024.2304497.

[18] K. Sher, M. Ameeq, M.M. Hassan, O. Albalawi, A. Afzal, Development of improved estimators of finite population mean in simple random sampling with dual auxiliaries and its application to real world problems, Heliyon 10 (10) (2023).

[19] S. Naz, M.H. Tahir, F. Jamal, M. Ameeq, S. Shafiq, J.T. Mendy, A group acceptance sampling plan based on flexible new Kumaraswamy exponential distribution: an application to quality control reliability, Cogent Eng. (2023), https://doi.org/10.1080/23311916.2023.2257945.

[20] M. Imran, H.S. Bakouch, M.H. Tahir, M. Ameeq, F. Jamal, J.T. Mendy, A new Bell-exponential model: properties and applications, Cogent Eng. (2023), https://doi.org/10.1080/23311916.2023.2281062.

[21] H. Panahi, N. Moradi, Estimation of the inverted exponentiated Rayleigh distribution based on adaptive type II progressive hybrid censored sample, J. Comput. Appl. Math. (2020), https://doi.org/10.1016/j.cam.2019.112345.

[22] C.M. Zhang, T.M. Cong, W.H. Gui, Order-restricted inference for generalized inverted exponential distribution under balanced joint progressive type-II censored data and its application on the breaking strength of jute fibers, Mathematics (2023), https://doi.org/10.3390/math11020329.

[23] M.S. Kotb, M.Z. Raqab, Statistical inference for modified Weibull distribution based on progressively type-II censored data, Math. Comput. Simul. 162 (2019) 233–248.

[24] M. Ameeq, M.H. Tahir, M.M. Hassan, F. Jamal, S. Shafiq, J.T. Mendy, A group acceptance sampling plan truncated life test for alpha power transformation inverted perks distribution based on quality control reliability, Cogent Eng. (2023), https://doi.org/10.1080/23311916.2023.2224137.

[25] S. Bera, N. Jana, Estimating reliability parameters for inverse Gaussian distributions under complete and progressively type-II censored samples, Qual. Technol. Quant. Manag. 20 (3) (2023) 334–359.

[26] S. Rostamian, N. Nematollahi, Estimation of stress-strength reliability in the inverse Gaussian distribution under progressively type II censored data, Math. Sci. 13 (2) (2019) 175–191.

[27] A.A. Farghal, S.K. Badr, H. Abu-Zindah, G.A. Ahd-Elmougod, Analysis of generalized inverted exponential competing risks model in presence of partially observed failure modes, Alex. Eng. J. 78 (2023) 74–87.

[28] R.B. Nelsen, An Introduction to Copulas, Springer, New York, 2006.
[29] H. Joe, Multivariate Models and Multivariate Dependence Concepts, Chapman & Hall, London, 1997.
[30] A. Hasankhani, S.M. Hakimi, Stochastic energy management of smart microgrid with intermittent renewable energy resources in electricity market, Energy (2021), https://doi.org/10.1016/j.energy.2020.119668.
[31] S. Latif, F. Mustafa, Bivariate joint distribution analysis of the flood characteristics under semiparametric copula distribution framework for the Kelantan River basin in Malaysia, J. Ocean Eng. Sci. 6 (2) (2021) 128–145.
[32] L.F. Song, B. Xu, X.J. Kong, D.G. Zou, X. Yu, R. Pang, Reliability analysis of $3D$ rockfill dam slope stability based on the copula function, Int. J. Geomech. (2021), https://doi.org/10.1061/(asce)gm.1943-5622.0001932.
[33] G.Q. Fang, R. Pan, Y.L. Hong, Copula-based reliability analysis of degrading systems with dependent failures, Reliab. Eng. Syst. Saf. (2020), https://doi.org/10.1016/j.ress.2019.106618.
[34] X.C. Bai, X.R. Li, N. Balakrishnan, M. He, Reliability analysis for dependent stress-strength reliability of multi-state system using improved generalized survival signature, Comput. Ind. Eng. (2022), https://doi.org/10.1016/j.cie.2022.107941.
[35] H. Joe, Asymptotic efficiency of the two-stage estimation method for copula-based models, J. Multivar. Anal. 94 (2) (2004) 401–419.
[36] W.R. Gilks, P. Wild, Adaptive rejection sampling for Gibbs sampling, J. R. Stat. Soc., Ser. C, Appl. Stat. 41 (2) (1992) 337–348.
[37] W. Nelson, Applied Life Data Analysis, John Wiley and Sons, Inc., 1982.