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On solving the chlorine transport model via Laplace transform

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This paper analyzes the two-dimensional chlorine-transport model in pipes. The studied model is in the form of a second-order partial differential equation with a set of boundary conditions. Obtaining exact solution for the current model is a challenge due to the nature of the involved boundary conditions, especially, when applying the Laplace transform. However, such difficulties are solved via implementing the method of residues. The exact solution is obtained in terms of the Bessel functions. The expression for a dimensionless cup-mixing average concentration is also derived analytically. The proposed approach is validated via numerical examples for comparing the results with those in the literature. The present analysis/approach is effective/straightforward and can be further applied on other similar models under different boundary conditions.

The study of the water quality is a growing field of research due to its importance in industry and engineering sciences. As a result of its importance in the quality of drinking water, chlorine is used in most parts of the world as a guarantee for the distribution of safe drinking water^{1,2}. To ensure the quality of the water, part of the chlorine must remain to prevent the survival and growth of any microorganisms that may be present during the movement of water in the networks. Therefore, maintaining a certain level of chlorine ensures that no harmful by-products are formed within the distribution networks. In Ref.³, the percentage of chlorine concentration that ensures the avoidance of public health risks has been determined as 0.2 mg l⁻¹. Accordingly, the successful management of ensuring the quality of drinking water requires adherence to the aforementioned chlorine concentration and ensuring that this percentage does not rise above the specified limit. The basic model of chlorine transport was established by Biswas et al.⁴ in the form:

$$\frac{\partial u}{\partial x} = \frac{A_0}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u}{\partial r} \right) - A_1 u, \quad (1)$$

and

$$u(0, r) = 1, \quad 0 \leq r \leq 1, \quad (2)$$

$$\frac{\partial}{\partial r} u(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (3)$$

$$\frac{\partial}{\partial r} u(x, 1) + A_2 u(x, 1) = 0, \quad 0 \leq x \leq 1, \quad (4)$$

where $u(x, r)$ is the chlorine concentration, and A_0 , A_1 and A_2 are dimensionless parameters related to the chlorine decay. Further, details about the dimensionless parameters and derivation of the Eqs. (1)–(4) can be found in Ref.⁴. The purpose of the present work is to solve the system (1)–(4) through applying the Laplace transform (LT). The LT is a well-known and effective approach to solving various scientific models in physics and engineering^{5–10}. Ebaid and Al shari⁶ applied the LT on the ODE governing the heat transfer of nanofluids suspended with carbon-nanotubes. Ebaid et. al⁷ provided the analytic solution for a class of singular boundary value problems (SBVPs) via the LT. Khaled⁸ obtained the exact solution of the model describing the radiation

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effect on MHD Marangoni convection over a flat plate. Ebaid et. al⁹ solved a general class of SBVPs with applications in nanofluids via the LT. Bakodah and Ebaid¹⁰ addressed the Ambartsumian delay equation using the LT. A variety of other of LT applications in addition to other transforms can be found in Refs.¹¹⁻³⁰. In this paper we consider the application of the LT for the solution of the system (1)–(4). The paper is structured as follows. The LT approach is applied in “Application of LT” on Eqs. (1)–(4). Section “Analysis and exact solution of the chlorine decay model” is devoted to obtaining exact solution. Section “Discussion of results” analyses the results and discusses their physical meaning. In addition, the results are to be compared with those in Ref.⁴. Finally, Section “Conclusion” outlines the main conclusions.

Application of LT

Applying LT on Eq. (1) with respect to the variable x , we can write

$$L\left(\frac{\partial u(x, r)}{\partial x}\right) = L\left(\frac{A_0}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial u(x, r)}{\partial r}\right)\right) - L(A_1 u(x, r)), \tag{5}$$

which gives

$$sU(s, r) - u(0, r) = \frac{A_0}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial U(s, r)}{\partial r}\right) - A_1 U(s, r), \tag{6}$$

where $L(\cdot)$ and s denote the Laplace operator and variable, respectively. After introducing the BC given in (2), Eq. (6) becomes

$$(s + A_1)U(s, r) - 1 = \frac{A_0}{r} \frac{d}{dr} \left(\frac{1}{r} \frac{dU(s, r)}{dr}\right). \tag{7}$$

Re-arranging Eq. (7), yields

$$\frac{d^2 U(s, r)}{dr^2} + \frac{1}{r} \frac{dU(s, r)}{dr} - \left(\frac{s + A_1}{A_0}\right)U(s, r) = -\frac{1}{A_0}, \tag{8}$$

which is the well known Bessel differential equation with inhomogeneous part $(-\frac{1}{A_0})$. The solution of Eq. (8) is given by

$$U(s, r) = c_1 J_0\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) + c_2 Y_0\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) + \frac{1}{s + A_1}, \tag{9}$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are Bessel functions and c_1 and c_2 denote unknown constants, $i = \sqrt{-1}$. Since $u(x, r)$ and its LT, $U(s, r)$, must be bounded at $r = 0$, the value of c_2 must be zero since $Y_0\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) \rightarrow \infty$ as $r \rightarrow 0$. Thus, we can write

$$U(s, r) = c_1 J_0\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) + \frac{1}{s + A_1}, \tag{10}$$

and

$$\frac{dU(s, r)}{dr} = -c_1 i\sqrt{\frac{s + A_1}{A_0}} J_1\left(i\sqrt{\frac{s + A_1}{A_0}}r\right), \tag{11}$$

where $J'_0(\lambda r) = -\lambda J_1(\lambda r)$. From Eqs. (10) and (11), we obtain

$$\frac{dU(s, r)}{dr} + A_2 U(s, r) = c_1 \left[A_2 J_0\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) - i\sqrt{\frac{s + A_1}{A_0}} J_1\left(i\sqrt{\frac{s + A_1}{A_0}}r\right) \right] + \frac{A_2}{(s + A_1)}. \tag{12}$$

Applying LT on (4) yields

$$\left[\frac{dU(s, r)}{dr} + A_2 U(s, r) \right]_{r=1} = 0. \tag{13}$$

From Eqs. (12) and (13), we obtain

$$c_1 = -\frac{A_2}{(s + A_1) \left[A_2 J_0\left(i\sqrt{\frac{s + A_1}{A_0}}\right) - i\sqrt{\frac{s + A_1}{A_0}} J_1\left(i\sqrt{\frac{s + A_1}{A_0}}\right) \right]}. \tag{14}$$

Substituting (14) into (10) leads to

$$U(s, r) = -\frac{A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}} r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]} + \frac{1}{s+A_1}. \quad (15)$$

However, Eq. (15) can be written as

$$U(s, r) = -A_2 F(s, r) + \frac{1}{s+A_1}, \quad (16)$$

where

$$F(s, r) = \frac{J_0\left(i\sqrt{\frac{s+A_1}{A_0}} r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}. \quad (17)$$

Applying the inverse LT on Eq. (16), yields

$$L^{-1}(U(s, r)) = -A_2 L^{-1}(F(s, r)) + L^{-1}\left(\frac{1}{s+A_1}\right), \quad (18)$$

or

$$u(x, r) = -A_2 f(x, r) + e^{-A_1 x}, \quad (19)$$

where $f(x, r)$ is the inverse LT of $F(s, r)$ so that

$$f(x, r) = L^{-1}\left(\frac{J_0\left(i\sqrt{\frac{s+A_1}{A_0}} r\right)}{(s+A_1)\left[A_2 J_0\left(i\sqrt{\frac{s+A_1}{A_0}}\right) - i\sqrt{\frac{s+A_1}{A_0}} J_1\left(i\sqrt{\frac{s+A_1}{A_0}}\right)\right]}\right). \quad (20)$$

Analysis and exact solution of the chlorine decay model

Analysis. The below theorem introduces the method of residues when applied to calculating the inverse LT.

Theorem 1 (Method of residues³¹) *Let s_i are the poles of $F(s, r)$, then $f(x, r)$ (inverse LT of $F(s, r)$) is $f(x, r) = \sum_{i=1}^n \text{Res}(e^{s_i x} F(s_i, r))$ at all poles s_i .*

It will be shown later that the inverse LT of the function $F(s, r)$, defined in (17), using the residues, can be obtained in terms of Bessel functions with the help of their properties. In this regard, the Bessel functions $J_0(y)$, $J_1(y)$ and $J_2(y)$ are defined by the expressions:

$$J_0(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{y}{2}\right)^{2k}, \quad (21)$$

$$J_1(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{y}{2}\right)^{2k+1}, \quad (22)$$

$$J_2(y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+2)!} \left(\frac{y}{2}\right)^{2k+2}, \quad (23)$$

that satisfy the following properties:

$$\frac{d}{dy}(J_0(\lambda y)) = -\lambda J_1(\lambda y) \quad (24)$$

$$\frac{d}{dy}(J_1(\lambda y)) = \frac{\lambda}{2}(J_0(\lambda y) - J_2(\lambda y)), \quad (25)$$

$$yJ_2(y) + yJ_0(y) = 2J_1(y). \quad (26)$$

Exact solution of the chlorine decay model. The main challenge of this paper is to obtain the inverse LT of the expression in Eq. (20). The expression (20) is really complex due to the nature of the boundary conditions (2–4). This is because the denominator in expression (20) involves Bessel functions of first and second kind which leads to actual difficulties when deriving the inverse LT of the expression in Eq. (20). However, such difficulties are overcome through applying the method of residues as indicated below.

At first sight, the expression $(s + A_1) \left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]$ has simple zeros at $s = -A_1$ and $i \sqrt{\frac{s+A_1}{A_0}} = \lambda_1, \lambda_2, \dots, \lambda_n, \dots$, and thus we find simple poles at $s_1 = -A_1$ and $s_2 = -A_1 - A_0 \lambda_n^2, n = 1, 2, 3, \dots$. Therefore, the inverse LT of $F(s, r)$, i.e., $f(x, r)$ can be obtained from Theorem 1 by calculating the residues (Res) of $e^{sx} F(s, r)$ at $s_1 = -A_1$ and $s_2 = -A_1 - A_0 \lambda_n^2$, and then by taking their sum. At $s_1 = -A_1$, we have

$$\begin{aligned} (\text{Res})_{s_1} &= \lim_{s \rightarrow s_1} (s - s_1) e^{sx} F(s, r), \\ &= e^{-A_1 x} \lim_{s \rightarrow -A_1} \left(\frac{J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} r \right)}{\left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]} \right), \\ &= e^{-A_1 x} \left(\frac{J_0(0)}{[A_2 J_0(0) - 0]} \right), \\ &= \frac{e^{-A_1 x}}{A_2}, \quad \text{where } J_0(0) = 1. \end{aligned} \tag{27}$$

At $s_2 = -A_1 - A_0 \lambda_n^2$, we have

$$\begin{aligned} (\text{Res})_{s_2} &= \lim_{s \rightarrow s_2} \left(\frac{(s - s_2) e^{sx} J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} r \right)}{(s + A_1) \left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]} \right), \\ &= \lim_{s \rightarrow s_2} \left(\frac{(s + A_1 + A_0 \lambda_n^2)}{\left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]} \right) \times \\ &\quad \lim_{s \rightarrow s_2} \left(\frac{e^{sx} J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} r \right)}{s + A_1} \right), \\ &= \lim_{s \rightarrow s_2} (G(s, r)) \times \left(\frac{e^{-(A_1 + A_0 \lambda_n^2)x} J_0(-\lambda_n r)}{-A_0 \lambda_n^2} \right). \end{aligned} \tag{28}$$

The limit of $G(s, r)$ as $s \rightarrow s_2$ can be calculated using the L'Hospital's rule as follows

$$\begin{aligned} \lim_{s \rightarrow s_2} (G(s, r)) &= \frac{\lim_{s \rightarrow s_2} (s - s_2)}{\lim_{s \rightarrow s_2} \left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]} = \frac{0}{0}, \\ &= \frac{\lim_{s \rightarrow s_2} \frac{d}{ds} (s + A_1 + A_0 \lambda_n^2)}{\lim_{s \rightarrow s_2} \frac{d}{ds} \left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right]}, \\ &= \frac{1}{d}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} d &= \lim_{s \rightarrow s_2} \frac{d}{ds} \left[A_2 J_0 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) - i \sqrt{\frac{s+A_1}{A_0}} J_1 \left(i \sqrt{\frac{s+A_1}{A_0}} \right) \right], \\ &= \frac{1}{4i A_0 \lambda_n} [-2i(1 + A_2) J_1(-\lambda_n) + i \lambda_n (J_0(-\lambda_n) - J_2(-\lambda_n))]. \end{aligned} \tag{30}$$

Since the functions J_0 and J_2 are even and J_1 is odd, we obtain

$$\begin{aligned} d &= \frac{1}{4A_0 \lambda_n} [2(1 + A_2) J_1(\lambda_n) + \lambda_n (J_0(\lambda_n) - J_2(\lambda_n))], \\ &= \frac{1}{4A_0 \lambda_n} [2(1 + A_2) J_1(\lambda_n) + \lambda_n J_0(\lambda_n) - \lambda_n J_2(\lambda_n)], \\ &= \frac{1}{4A_0 \lambda_n} [2(1 + A_2) J_1(\lambda_n) + \lambda_n J_0(\lambda_n) - 2J_1(\lambda_n) + \lambda_n J_0(\lambda_n)], \\ &= \frac{1}{2A_0 \lambda_n} [A_2 J_1(\lambda_n) + \lambda_n J_0(\lambda_n)]. \end{aligned} \tag{31}$$

From (29) and (31), we get

$$\lim_{s \rightarrow s_2} (G(s, r)) = \frac{2A_0\lambda_n}{[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)]}. \quad (32)$$

Substituting (32) into (28), yields

$$\begin{aligned} (\text{Res})_{s_2} &= \left(\frac{e^{-(A_1+A_0\lambda_n^2)x}J_0(-\lambda_nr)}{-A_0\lambda_n^2} \right) \times \frac{2A_0\lambda_n}{[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)]}, \\ &= -\frac{2e^{-(A_1+A_0\lambda_n^2)x}J_0(\lambda_nr)}{\lambda_n[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)]}. \end{aligned} \quad (33)$$

Hence,

$$f(x, r) = \frac{e^{-A_1x}}{A_2} - 2 \sum_{n=1}^{\infty} \frac{e^{-(A_1+A_0\lambda_n^2)x}J_0(\lambda_nr)}{\lambda_n[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)]}. \quad (34)$$

Inserting (34) into (19) leads to

$$u(x, r) = 2 \sum_{n=1}^{\infty} \frac{A_2J_0(\lambda_nr)e^{-(A_1+A_0\lambda_n^2)x}}{\lambda_n[A_2J_1(\lambda_n) + \lambda_nJ_0(\lambda_n)]}, \quad (35)$$

where the symbols λ_n denote the roots of the equation:

$$A_2J_0(\lambda_n) - \lambda_nJ_1(\lambda_n) = 0. \quad (36)$$

Using (36), the solution (35) can be written as

$$u(x, r) = 2 \sum_{n=1}^{\infty} \frac{\lambda_nJ_1(\lambda_n)J_0(\lambda_nr)}{(A_2^2 + \lambda_n^2)J_0^2(\lambda_n)} e^{-(A_1+A_0\lambda_n^2)x}, \quad (37)$$

which agrees with solution in Ref.⁴ derived by the separation of variables technique.

Discussion of results

The dimensionless cup-mixing average concentration is defined by

$$u_{av} = 2 \int_0^1 u(x, r) r dr. \quad (38)$$

Substituting (37) into (38), yields

$$u_{av} = 4 \sum_{n=1}^{\infty} \frac{\lambda_nJ_1(\lambda_n)}{(A_2^2 + \lambda_n^2)J_0^2(\lambda_n)} e^{-(A_1+A_0\lambda_n^2)x} \int_0^1 rJ_0(\lambda_nr) dr, \quad (39)$$

or

$$u_{av} = 4 \sum_{n=1}^{\infty} \frac{J_1^2(\lambda_n)}{(A_2^2 + \lambda_n^2)J_0^2(\lambda_n)} e^{-(A_1+A_0\lambda_n^2)x}. \quad (40)$$

After including the relation (36) we obtain

$$u_{av} = 4 \sum_{n=1}^{\infty} \frac{A_2^2}{\lambda_n^2(A_2^2 + \lambda_n^2)} e^{-(A_1+A_0\lambda_n^2)x}. \quad (41)$$

If $A_2 \rightarrow \infty$, then

$$u_{av} = 4 \lim_{A_2 \rightarrow \infty} \left(\sum_{n=1}^{\infty} \frac{A_2^2}{\lambda_n^2(A_2^2 + \lambda_n^2)} e^{-(A_1+A_0\lambda_n^2)x} \right), \quad (42)$$

which gives

$$u_{av} = \sum_{n=1}^{\infty} \frac{4}{\lambda_n^2} e^{-(A_1+A_0\lambda_n^2)x}. \quad (43)$$

Moreover, if $A_2 \rightarrow 0$ then Eq. (19) implies

$$u(x, r) = e^{-A_1x}, \quad (44)$$

and the corresponding u_{av} is obtained from Eq. (38) as

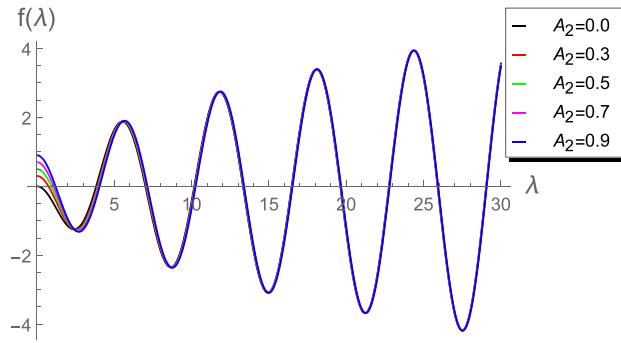


Figure 1. Plot of $f(\lambda)$ vs λ for $A_2 = \{0.0, 0.3, 0.5, 0.7, 0.9\}$.

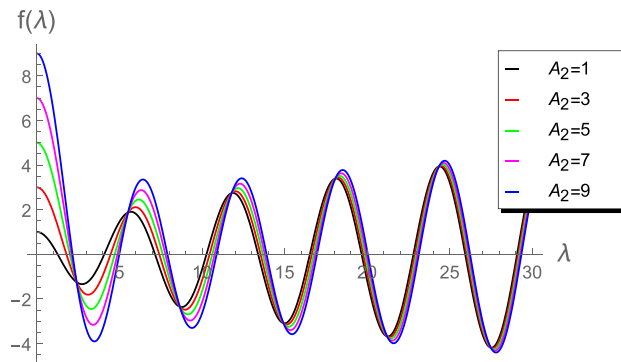


Figure 2. Plot of $f(\lambda)$ vs λ for $A_2 = \{1, 3, 5, 7, 9\}$.

$$u_{av} = 2 \int_0^1 e^{-A_1 x} r dr = e^{-A_1 x}. \tag{45}$$

Therefore, Eq. (41) gives the general expression for the u_{av} while Eqs. (43) and (45) are limiting cases. According to Biswas et al.⁴, three roots λ_1, λ_2 , and λ_3 of Eq. (36) were used. In addition, the following fitting functions were used to reproduce $(\lambda_1, \lambda_2, \lambda_3)$ in terms of A_2 at several ranges.

(i) For $0.01 \leq A_2 < 1$,

$$\lambda_1 = 1.29861(A_2)^{0.477433}, \lambda_2 = 4.00946(A_2)^{0.0119894}, \lambda_3 = 7.11555(A_2)^{0.00376107}, \tag{46}$$

(ii) For $1 \leq A_2 < 10$,

$$\lambda_1 = 1.30427(A_2)^{0.239289}, \lambda_2 = 4.05693(A_2)^{0.0927629}, \lambda_3 = 7.10846(A_2)^{0.0463785}, \tag{47}$$

(iii) For $10 \leq A_2 < 1000$,

$$\lambda_1 = 2.10218(A_2)^{0.021361}, \lambda_2 = 4.86441(A_2)^{0.0200514}, \lambda_3 = 7.71165(A_2)^{0.0182292}. \tag{48}$$

In order to have a numerical comparison between the current calculations of the first three roots λ_1, λ_2 , and λ_3 of Eq. (36) and the corresponding results in⁴ (using Eqs. (46)–(48)), we may write Eq. (36) as a function of λ in the form:

$$f(\lambda) = A_2 J_0(\lambda) - \lambda J_1(\lambda). \tag{49}$$

Figures 1, 2, 3, 4 and 5) highlight details about the roots of $f(\lambda)$ for different values of A_2 . We verify that we have an infinite number of roots. In addition, all the roots of $f(\lambda)$, excepting the first one, are nearly identical for small $A_2 \in (0.01, 1]$, as shown by Fig. 1. However, for the range $1 \leq A_2 < 10$, Fig. 2 reveals that the first seven roots are considerably different, while the others have approximately the same values. Figure 3 indicates that the first two roots are nearly identical, while the rest of roots are considerably different for the range $10 \leq A_2 < 45$. For higher values of A_2 , namely in the ranges $50 \leq A_2 < 90$ and $100 \leq A_2 < 900$, Figs. 4 and 5 reveal that the roots of $f(\lambda)$ are nearly identical. However, the results introduced in⁴ were mainly depend on obtaining the first three roots λ_1, λ_2 and λ_3 and, therefore, the proposed approach is more accurate.

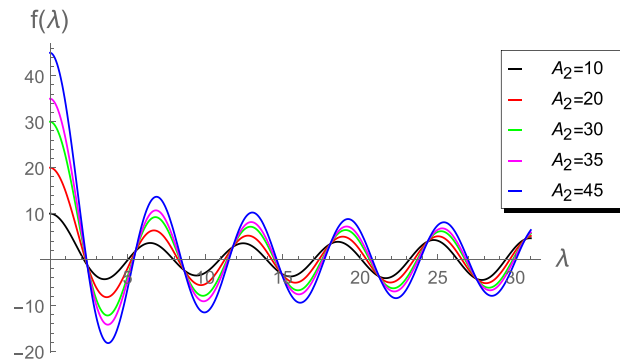


Figure 3. Plot of $f(\lambda)$ vs λ for $A_2 = \{10, 20, 30, 35, 45\}$.

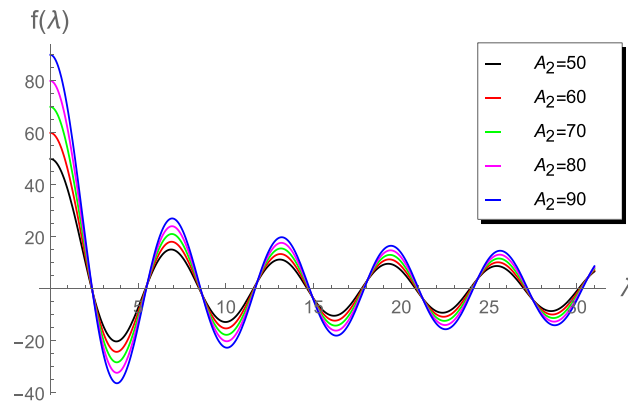


Figure 4. Plot of $f(\lambda)$ vs λ for $A_2 = \{50, 60, 70, 80, 90\}$.

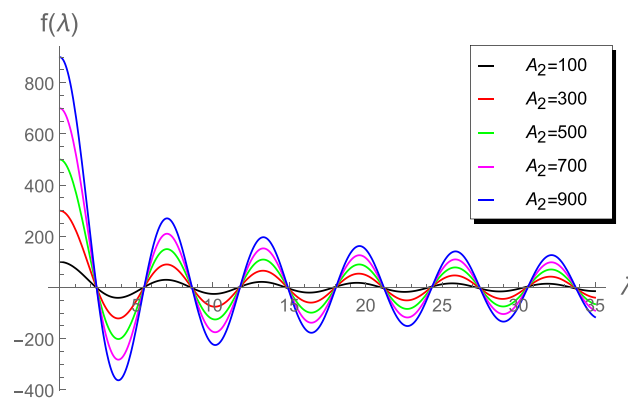


Figure 5. Plot of $f(\lambda)$ vs λ for $A_2 = \{100, 300, 500, 700, 900\}$.

Table 1 presents the three roots λ_1, λ_2 and λ_3 of Eq. (36) and the corresponding results from⁴ (using Eq. (46)) in the interval $A_2 \in (0.01, 1]$. The calculations of the present roots are accomplished through MATHEMATICA. The absolute errors listed in Table 1 show that the results presented in⁴ agree with the obtained ones only up to two/three digits at most. This means that the new approach leads to better results than those of⁴ after 3 decimal places. A similar conclusion is also obtained in Table 2 regarding λ_1, λ_2 and λ_3 for the range $1 \leq A_2 < 10$.

Table 3 shows that the absolute errors increase in the range $10 \leq A_2 < 1000$. Such differences in the values may lead to differences when calculating the chlorine concentration or the cup-mixing average concentration. The behavior of the cup-mixing average concentration u_{av} , at the outlet $x = 1$ of a pipe, versus A_1 are displayed in Figs. 6, 7, 8 and 9 for several values of A_0 and A_2 . These figures indicate that the u_{av} is always a decreasing function in the parameter A_1 . This means that the cup-mixing average concentration decays with increasing the parameter A_1 . In conclusion, the proposed approach gives a clear and precise solution of the mathematical model.

A_2	Present			Ref. ⁴			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.01	0.141245	3.83431	7.01701	0.144083	3.79409	6.99337	0.002838	0.040230	0.023645
0.10	0.441682	3.85771	7.02983	0.432559	3.90029	7.05419	0.009122	0.042576	0.024369
0.20	0.616975	3.88351	7.04403	0.602237	3.93283	7.07261	0.014738	0.049329	0.028579
0.50	0.940771	3.95937	7.08638	0.932732	3.97628	7.09702	0.008038	0.016907	0.010643

Table 1. Comparisons of present roots λ_1 , λ_2 , and λ_3 of Eq. (36) and the corresponding results in Ref.⁴ using Eqs. (46) ($0.01 \leq A_2 < 1$).

A_2	Present			Ref. ⁴			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
1	1.25578	4.07948	7.15580	1.30427	4.05693	7.10846	0.04849	0.02255	0.04734
2	1.59945	4.29096	7.28839	1.53957	4.32635	7.34069	0.05987	0.03539	0.05230
5	1.98981	4.71314	7.61771	1.91701	4.71016	7.65936	0.07281	0.00298	0.04166

Table 2. Comparisons of present roots λ_1 , λ_2 , and λ_3 of Eq. (36) and the corresponding results in Ref.⁴ using Eqs. (47) ($1 \leq A_2 < 10$).

A_2	Present			Ref. ⁴			Absolute error		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
10	2.17950	5.03321	7.95688	2.20816	5.09427	8.04223	0.02867	0.06105	0.08535
50	2.35724	5.41120	8.48399	2.28540	5.26135	8.28167	0.07185	0.14985	0.20231
100	2.38090	5.46521	8.56783	2.31949	5.33498	8.38698	0.06141	0.13023	0.18085

Table 3. Comparisons of present roots λ_1 , λ_2 , and λ_3 of Eq. (36) and the corresponding results in Ref.⁴ using Eqs. (48) ($10 \leq A_2 < 1000$).

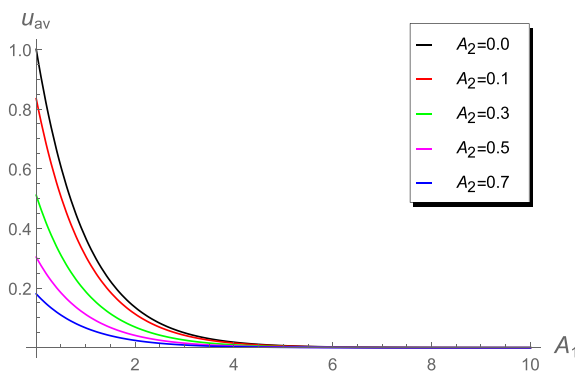


Figure 6. The u_{av} against A_1 at different values of A_2 , $A_0=1.4$.

Conclusion

The two-dimensional chlorine-model was theoretically solved. The exact solution was obtained by applying the LT. The method of residues was adopted to obtain the inverse LT of complex expressions and the solution was expressed in terms of Bessel functions of the first and the second kinds of order zero. The exact solutions agree with those published previously using the method of separation of variables. However, the obtained numerical results are superior to than those reported in⁴ due to the limitations on the calculation of the roots of Eq. (36). Previous studies conducted the numerical calculations based on various fitting curves to predict the values of such roots at prescribed values of A_2 . Such fitting scheme involves some numerical errors as verified in Tables (1-3), where the absolute errors may be relevant for many cases. The expression of the dimensionless cup-mixing average concentration was analytically derived. The results proved that the new approach gives reliable and accurate solutions of the problem.

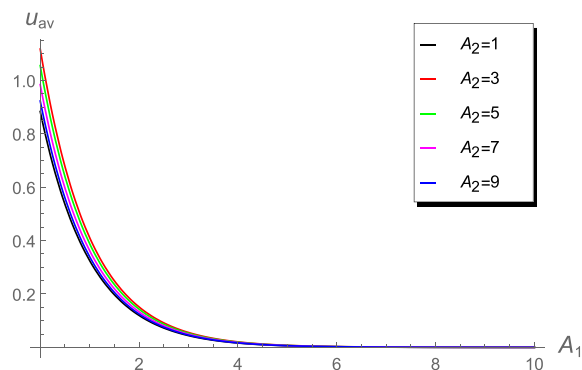


Figure 7. The u_{av} against A_1 at different values of A_2 , $A_0 = 1.4 \times 10^{-3}$.

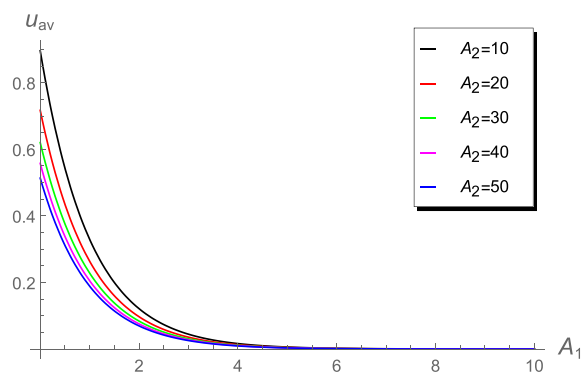


Figure 8. The u_{av} against A_1 at different values of A_2 , $A_0 = 1.4 \times 10^{-2}$.

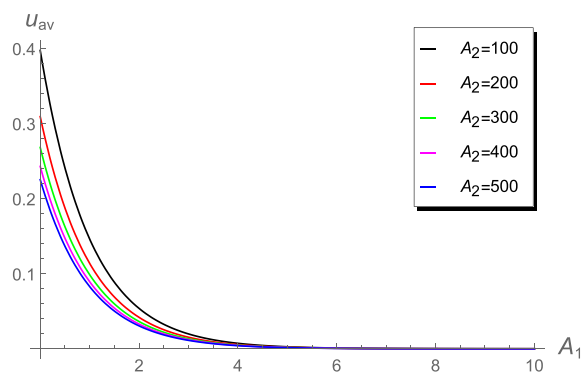


Figure 9. The u_{av} against A_1 at higher values of A_2 , $A_0 = 1.4 \times 10^{-2}$.

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A.F.A. and A.E. wrote the main manuscript text and E.A.A., Y.M.M., M.A., H.K.A., and P.A. prepared the figures and tables. All authors reviewed the manuscript.

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Competing interests

The authors declare no competing interests.

Additional information

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