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## Excluding false negative error in certification of quantum channels

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Certification of quantum channels is based on quantum hypothesis testing and involves also preparation of an input state and choosing the final measurement. This work primarily focuses on the scenario when the false negative error cannot occur, even if it leads to the growth of the probability of false positive error. We establish a condition when it is possible to exclude false negative error after a finite number of queries to the quantum channel in parallel, and we provide an upper bound on the number of queries. On top of that, we found a class of channels which allow for excluding false negative error after a finite number of queries in parallel, but cannot be distinguished unambiguously. Moreover, it will be proved that parallel certification scheme is always sufficient, however the number of steps may be decreased by the use of adaptive scheme. Finally, we consider examples of certification of various classes of quantum channels and measurements.

Being deceived is not a nice experience. People have been developing plenty of methods to protect themselves against being cheated and one of these methods concerns verification of objects, also quantum ones. The cornerstone for theoretical studies on discrimination of quantum objects was laid by Helstrom<sup>1</sup> a few decades ago.

In the era of Noisy Intermediate-Scale Quantum (NISQ) devices<sup>2,3</sup>, assuring the correctness of components is undeniably in the spotlight. A broad review of multipronged modern methods of certification as well as benchmarking of quantum states and processes can be found in the recent paper<sup>4</sup>. For a more introductory tutorial to the theory of system certification we refer the reader to<sup>5</sup>. Verification of quantum processes is often studied in the context of specific elements of quantum information processing tasks. Protocols for efficient certification of quantum processes, such as quantum gates and circuits, were recently studied in<sup>6–8</sup>.

Let us introduce the most general problem of verification studied in this work. Assume there are two known quantum channels and one of them is secretly chosen. Then, we are given the secretly chosen channel to verify which of the two channels it is. We are allowed to prepare any input state and apply the given channel on it. Finally, we can prepare any quantum measurement and measure the output state. Basing on the measurement's outcome we make a decision which of the two channels was secretly chosen. In this work we focus on the case when we are promised which of the channels is given. After performing some certification procedure we can either agree that the channel was the promised one or claim that we were cheated. We want to assure that we will always realize when we are cheated. It may happen though, that we appear to be too suspicious and claim that we were cheated when we were not.

There are three major theoretical approaches towards verification of quantum channels called minimum error discrimination, unambiguous discrimination and certification. All these three approaches can be generalized to the multiple-shot case, that is when the given channel can be used multiple times in various configurations. The most straightforward possibility is the parallel scheme and the most sophisticated is the adaptive scheme (where we are allowed to use any processing between the uses of the given channel).

The first approach is called minimum error discrimination (a.k.a. distinguishability or symmetric discrimination) and makes use of the distance between quantum channels expressed by the use of the diamond norm. In this scenario one wants to minimize the probability of making the erroneous decision using the bound on this probability given by the Holevo-Helstrom theorem<sup>1,9</sup>. Single-shot discrimination of unitary channels and von Neumann measurements were studied in<sup>10,11</sup> and<sup>12–14</sup> respectively. Parallel discrimination of quantum channels was studied eg. in<sup>15,16</sup>. It appeared that parallel discrimination scheme is optimal in the case of distinguishability of unitary channels<sup>17</sup> and von Neumann measurements<sup>18</sup>. In some cases however, the use of adaptive discrimination scheme can significantly improve the certification<sup>19,20</sup>. Advantages of the use of adaptive discrimination

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scheme in the asymptotic regime were studied in<sup>21</sup>. Fundamental and ultimate limits for quantum channel discrimination were derived in<sup>22,23</sup>. The works<sup>24,25</sup> address the problem of distinguishability of quantum channels in the context of resource theory.

In the second approach, that is unambiguous discrimination, there are three possible outcomes. Two of them designate quantum channels while the third option is the inconclusive result. In this approach, when the result indicated which channel was given, we know it for sure. There is a chance however, that we will obtain an inconclusive answer. Unambiguous discrimination of quantum channels was considered in<sup>26</sup>, while unambiguous discrimination of von Neumann measurements was explored in<sup>18</sup>. Studies on unambiguous discrimination of quantum channels took a great advantage of unambiguous discrimination of quantum states, which can be found e.g. in<sup>27–31</sup>.

The third approach, known as certification or asymmetric discrimination, is based on hypothesis testing. We are promised to be given one of the two channels and associate this channel with the null hypothesis,  $H_0$ . The other channel is associated with the alternative hypothesis,  $H_1$ . When making a decision whether to accept or to reject the null hypothesis, two types of errors may occur, that is we can come across false positive and false negative errors. In this work we consider the situation when we want to assure that false negative error will not occur, even if the probability of false positive error grows. A similar task of minimizing probability of false negative error having fixed bound on the probability of false positive was studied in the case of von Neumann measurements in<sup>32</sup>. Certification of quantum channels was studied in the asymptotic regime e.g. in<sup>21,24,33</sup>.

It should come as no surprise that in some cases perfect verification is not possible by any finite number of steps. Conditions for perfect minimum error discrimination of quantum operations were derived in<sup>34</sup>. Similar condition for unambiguous discrimination was proved in<sup>26</sup>. However, no such conditions have been stated for certification. In this work we derive a condition when we can exclude false negative error after a finite number of uses in parallel. This condition holds for arbitrary quantum channels and is expressed by the use of Kraus operators of these channels. We will provide an example of channels which can be certified in a finite number of queries in parallel, but cannot be distinguished unambiguously. Moreover, we will show that, in contrast to discrimination of quantum channels<sup>19,20</sup>, parallel certification scheme is always sufficient for certification, although the number of uses of the certified channel may not be optimal. On top of that, we will consider certification of quantum measurements and focus on the class of measurements with rank-one effects. The detailed derivation of the upper bound for the probability of false positive error will be presented for SIC POVMs.

This work is organized as follows. After introducing basic mathematical concepts in “Preliminaries” section, we present our main result, that is the condition when excluding false negative is possible in a finite number of uses in parallel, in Theorem 1 in “Parallel certification” section. Next, we apply this result to a specific subclass of quantum channels in “Certification of quantum measurements” section. Then, in “Adaptive certification and Stein setting” section we state, as Theorem 2, the condition when excluding false negative error is possible in the adaptive scheme. Finally, summary can be found in “Conclusions” section.

## Preliminaries

Let  $\mathcal{D}_d$  denote the set of quantum states of dimension  $d$ , that is the set of positive semidefinite operators having trace equal one. Throughout this paper quantum states will be denoted by lower-case Greek letters, usually  $\rho, \sigma, \tau$ . For any state  $\rho \in \mathcal{D}_d$  we can write its spectral decomposition as  $\rho = \sum_i p_i |\lambda_i\rangle\langle\lambda_i|$ . Having a set of quantum states  $\{\rho_1, \dots, \rho_m\}$  with spectral decompositions  $\rho_1 = \sum_{i_1} p_{i_1} |\lambda_{i_1}\rangle\langle\lambda_{i_1}|, \dots, \rho_m = \sum_{i_m} p_{i_m} |\lambda_{i_m}\rangle\langle\lambda_{i_m}|$  respectively, their support is defined as  $\text{supp}(\rho_1, \dots, \rho_m) := \text{span}\{|\lambda_{i_j}\rangle : p_{i_j} > 0\}$ . The set of unitary matrices of dimension  $d$  will be denoted  $\mathcal{U}_d$ .

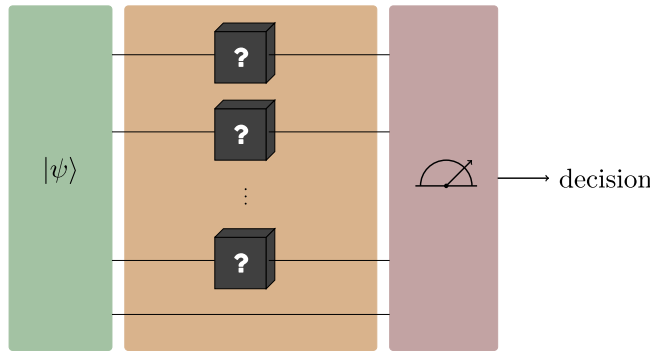
Quantum channels are linear maps which are completely positive and trace preserving. In this work we will often take advantage of the Kraus representations of channels. Let

$$\Phi_0(X) := \sum_{i=1}^k E_i X E_i^\dagger, \quad \Phi_1(X) := \sum_{j=1}^l F_j X F_j^\dagger \quad (1)$$

be the Kraus representations of the channels that will correspond to null and alternative hypotheses respectively. The sets of operators  $\{E_i\}_i$  and  $\{F_j\}_j$  are called Kraus operators of channels  $\Phi_0$  and  $\Phi_1$  respectively. We will use the notation  $\text{supp}(\Phi_0) := \text{span}\{E_i\}_i$ ,  $\text{supp}(\Phi_1) := \text{span}\{F_j\}_j$ , to denote the supports of quantum channels. Moreover, the notation  $\mathbb{1}$  will be used for the identity channel.

The most general quantum measurements, known also as POVMs (positive operator valued measure) are defined as a collection of positive semidefinite operators  $\mathcal{P} = \{M_1, \dots, M_m\}$  which fulfills the condition  $\sum_{i=1}^m M_i = \mathbb{1}_d$ , where  $\mathbb{1}_d$  denotes the identity matrix of dimension  $d$ . When a quantum state  $\rho$  is measured by the measurement  $\mathcal{P}$ , then the label  $i$  is obtained with probability  $\text{Tr}(E_i \rho)$  and the state  $\rho$  ceases to exist. A special class of quantum measurements are projective von Neumann measurements. These POVMs have rank-one effects of the form  $\{|u_1\rangle\langle u_1|, \dots, |u_d\rangle\langle u_d|\}$ , where vectors  $\{|u_i\rangle\}_{i=1}^d$  form an orthonormal basis and therefore they are columns of some unitary matrix  $U \in \mathcal{U}_d$ .

Now we proceed to describing the detailed scheme of certification. There are two quantum channels:  $\Phi_0$  and  $\Phi_1$ . We are promised that we are given  $\Phi_0$  but we are not sure and we want to verify it using hypothesis testing. We associate the channel  $\Phi_0$  with the null hypothesis  $H_0$  and we associate the other channel  $\Phi_1$  with the alternative hypothesis  $H_1$ . We consider the following scheme. We are allowed to prepare any (possibly entangled) input state and perform the given channel on it. Then, we prepare a binary measurement  $\{\Omega_0, \mathbb{1} - \Omega_0\}$  and measure the output state. If we obtain the label associated with the effect  $\Omega_0$ , then we decide that the certified channel was  $\Phi_0$  and we accept the null hypothesis. If we get the label associated with the effect  $\mathbb{1} - \Omega_0$ , then we decide that the certified channel was  $\Phi_1$  and therefore we reject the null hypothesis.



**Figure 1.** Parallel certification scheme.

The aim of certification is to make a decision whether to accept or to reject  $H_0$ . While making such a decision one can come upon two types of errors. The false positive error (also known as type I error) happens when we reject the null hypothesis when in fact it was true. The converse situation, that is accepting the null hypothesis when the alternative hypothesis was correct, is known as the false negative (or type II) error. In this work we will focus on the situation when the probability of the false negative error equals zero and we want to minimize the probability of false positive error.

Let us now take a closer look into the scheme of entanglement-assisted single-shot certification procedure. We begin with preparing an input state  $|\psi\rangle$  on the compound space. Then, we apply the certified channel extended by the identity channel on the input state, obtaining as the output the state either  $\rho_0^{|\psi\rangle} = (\Phi_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$ , if the given channel was  $\Phi_0$ , or  $\rho_1^{|\psi\rangle} = (\Phi_1 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$ , if the given channel was  $\Phi_1$ . Eventually, we perform the measurement  $\{\Omega_0, \mathbb{1} - \Omega_0\}$ , where the effect  $\Omega_0$  accepts hypothesis  $H_0$  and the effect  $\mathbb{1} - \Omega_0$  accepts the alternative hypothesis  $H_1$ .

Assuming that the input state  $|\psi\rangle$  and measurement effect  $\Omega_0$  have been fixed, the probability of making the false positive error is given by

$$p_1(|\psi\rangle, \Omega_0) := \text{Tr} \left( (\mathbb{1} - \Omega_0) \rho_0^{|\psi\rangle} \right) = 1 - \text{Tr} \left( \Omega_0 \rho_0^{|\psi\rangle} \right). \tag{2}$$

In a similar manner we have the probability of making the false negative error, that is

$$p_2(|\psi\rangle, \Omega_0) := \text{Tr} \left( \Omega_0 \rho_1^{|\psi\rangle} \right). \tag{3}$$

We will be interested in the situation when probability of the false negative error is equal to zero and we want to minimize the probability of false positive error. Therefore, we introduce the notation

$$p_1 := \min_{|\psi\rangle, \Omega_0} \{ p_1(|\psi\rangle, \Omega_0) : p_2(|\psi\rangle, \Omega_0) = 0 \} \tag{4}$$

for minimized probability of false positive error in the single-shot scenario.

For a given  $\epsilon > 0$ , we say that quantum channel  $\Phi_0$  can be  $\epsilon$ -certified against channel  $\Phi_1$  if there exist an input state  $|\psi\rangle$  and measurement effect  $\Omega_0$  such that  $p_2(|\psi\rangle, \Omega_0) = 0$  and  $p_1(|\psi\rangle, \Omega_0) \leq \epsilon$ . In other words, quantum channel  $\Phi_0$  can be  $\epsilon$ -certified against another channel  $\Phi_1$  if we can assure no false negative will occur and the probability of false positive error is smaller than  $\epsilon$ .

When performing the certification of quantum channels, we can use the channels many times in various configurations. Now we proceed to introducing notation needed for studying parallel and adaptive certification schemes.

**Parallel certification scheme.** Let  $N$  denote the number of uses of the quantum channel in parallel. A schematic representation of the scenario of parallel certification is depicted in Fig. 1. In this scheme we consider certifying tensor products of the channels. In other words, parallel certification of channels  $\Phi_0$  and  $\Phi_1$  can be seen as certifying channels  $\Phi_0^{\otimes N}$  and  $\Phi_1^{\otimes N}$  for some natural number  $N$ .

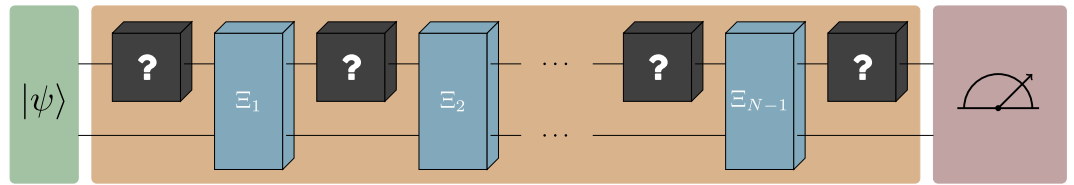
Let  $|\psi\rangle$  be the input state to the certification procedure. After applying the channel  $\Phi_0$   $N$  times in parallel, we obtain the output state

$$\sigma_0^{N,|\psi\rangle} = \left( \Phi_0^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|), \tag{5}$$

if the channel was  $\Phi_0$ , and similarly

$$\sigma_1^{N,|\psi\rangle} = \left( \Phi_1^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|), \tag{6}$$

if the channel was  $\Phi_1$ . In the same spirit let



**Figure 2.** Adaptive certification scheme. The processings  $\Xi_1, \dots, \Xi_{N-1}$  can be arbitrary quantum channels where we only assume that the first subsystem must fit the input of the black box. In particular, the processings can swap subsystems, and therefore one can obtain the parallel scheme as a special case of the adaptive discrimination scheme.

$$p_1^{\mathbb{P},N}(|\psi\rangle, \Omega_0) = \text{Tr} \left( (\mathbb{1} - \Omega_0) \sigma_0^{N,|\psi} \right), \quad p_2^{\mathbb{P},N}(|\psi\rangle, \Omega_0) = \text{Tr} \left( \Omega_0 \sigma_1^{N,|\psi} \right) \quad (7)$$

be the probabilities of occurring false positive and false negative errors respectively. When  $N = 1$ , then we arrive at single-shot certification. Therefore we will neglect the upper index and simply write  $p_1(|\psi\rangle, \Omega_0)$  and  $p_2(|\psi\rangle, \Omega_0)$ .

We introduce the notation

$$p_1^{\mathbb{P},N} := \min_{|\psi\rangle, \Omega_0} \left\{ p_1^{\mathbb{P},N}(|\psi\rangle, \Omega_0) : p_2^{\mathbb{P},N}(|\psi\rangle, \Omega_0) = 0 \right\} \quad (8)$$

for the minimized probability of false positive error in the parallel scheme.

We say that quantum channel  $\Phi_0$  can be certified against  $\Phi_1$  in the parallel scheme, if for every  $\epsilon > 0$  there exist a natural number  $N$ , an input state  $|\psi\rangle$  and measurement effect  $\Omega_0$  such that  $p_2^{\mathbb{P},N}(|\psi\rangle, \Omega_0) = 0$  and  $p_1^{\mathbb{P},N}(|\psi\rangle, \Omega_0) \leq \epsilon$ .

Let us now elaborate a bit on the number of steps needed for certification. Assume that we have fixed upper bound on the probability of false positive error,  $\epsilon > 0$ . We will be interested in calculating the minimal number of queries,  $N_\epsilon$ , for which  $p_2^{\mathbb{P},N}(|\psi\rangle, \Omega_0) = 0$  and  $p_1^{\mathbb{P},N}(|\psi\rangle, \Omega_0) \leq \epsilon$  for some input state  $|\psi\rangle$  and measurement effect  $\Omega_0$ . Such a number,  $N_\epsilon$ , will be called the *minimal number of steps needed for parallel certification*.

**Adaptive certification scheme.** Adaptive certification scheme allows for the use of processing between the uses of the certified channel, therefore this procedure is more complex than the parallel certification. However, when the processings only swap the subsystems, then the adaptive scheme may reduce to the parallel one.

Assume as previously that  $|\psi\rangle$  is the input state to the certification procedure in which the certified channel is used  $N$  times and any processing is allowed between the uses of this channel. The scheme of this procedure is presented in the Fig. 2. Having the input state  $|\psi\rangle$  on the compound register, we perform the certified channel (denoted by the black box with question mark) on one part of it. Having the output state we can perform some processing  $\Xi_1$  and therefore get prepared for the next use of the certified channel. Then again, we apply the certified channel on one register of the prepared state and again, we can perform processing  $\Xi_2$ . We repeat this procedure  $N - 1$  times. After the  $N$ -th use of the certified channel we obtain the state either  $\tau_0^{N,|\psi}$ , if the channel was  $\Phi_0$ , or  $\tau_1^{N,|\psi}$ , if the channel was  $\Phi_1$ . Then, we prepare a global measurement  $\{\Omega_0, \mathbb{1} - \Omega_0\}$  and apply it on the output state. Let

$$p_1^{\mathbb{A},N}(|\psi\rangle, \Omega_0) = \text{Tr} \left( (\mathbb{1} - \Omega_0) \tau_0^{N,|\psi} \right), \quad p_2^{\mathbb{A},N}(|\psi\rangle, \Omega_0) = \text{Tr} \left( \Omega_0 \tau_1^{N,|\psi} \right) \quad (9)$$

be the probabilities of the false positive and false negative errors in adaptive scheme, respectively, when the input state and the measurement effects were fixed. When  $N = 1$ , then we will neglect the upper index and simply write  $p_1(|\psi\rangle, \Omega_0)$  and  $p_2(|\psi\rangle, \Omega_0)$ .

We say that quantum channel  $\Phi_0$  can be certified against  $\Phi_1$  in the adaptive scheme, if for every  $\epsilon > 0$  there exist a natural number  $N$ , an input state  $|\psi\rangle$  and measurement effect  $\Omega_0$  such that  $p_2^{\mathbb{A},N}(|\psi\rangle, \Omega_0) = 0$  and  $p_1^{\mathbb{A},N}(|\psi\rangle, \Omega_0) \leq \epsilon$ .

For a fixed upper bound on the probability of false positive error,  $\epsilon$ , we introduce the *minimal number of steps needed for adaptive certification*,  $N_\epsilon$ , as the minimal number of steps after which  $p_2^{\mathbb{A},N}(|\psi\rangle, \Omega_0) = 0$  and  $p_1^{\mathbb{A},N}(|\psi\rangle, \Omega_0) \leq \epsilon$  for some input state  $|\psi\rangle$  and measurement effect  $\Omega_0$ .

### Parallel certification

Not all quantum channels can be discriminated perfectly after a finite number of queries. Conditions for perfect discrimination were states in the work<sup>34</sup>. Similar conditions for unambiguous discrimination were proved in<sup>26</sup>. In this section we will complement these results with the condition concerning parallel certification. More specifically, we will prove a simple necessary and sufficient condition when a quantum channel  $\Phi_0$  can be certified against some other channel  $\Phi_1$ . As the condition utilizes the notion of the support of a quantum channel, recall that it is defined as the span of their Kraus operators. The condition will be stated as Theorem 1, however its proof will be presented after introducing two technical lemmas.

In fact, the statement of Theorem 1 is a bit more general, that is it concerns the situation when the alternative hypothesis corresponds to a set of channels  $\{\Phi_1, \dots, \Phi_m\}$  having Kraus operators  $\{F_{j_1}^{(1)}\}_{j_1}, \dots, \{F_{j_m}^{(m)}\}_{j_m}$  respectively. We will use the notation  $\text{supp}(\Phi_1, \dots, \Phi_m) := \text{span}\{F_{j_1}^{(1)}, \dots, F_{j_m}^{(m)}\}_{j_1, \dots, j_m}$ .

**Theorem 1** Quantum channel  $\Phi_0$  can be certified against quantum channels  $\Phi_1, \dots, \Phi_m$  in the parallel scheme if and only if  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1, \dots, \Phi_m)$ .

Moreover, to ensure that the probability of false positive error is no greater than  $\epsilon$ , the number of steps needed for parallel certification is bounded by  $N_\epsilon \geq \left\lceil \frac{\log \epsilon}{\log p_1} \right\rceil$  where  $p_1$  is the upper bound on probability of false positive error in single-shot certification.

Before presenting the proof of this theorem we will introduce two lemmas. The proofs of lemmas are postponed to ‘‘Supplementary Appendix A’’. Lemma 1 states that if the inclusion does not hold for supports of the quantum channels, then the inclusion also does not hold for supports of output states assuming that the input state has full Schmidt rank. The proof of Lemma 1 is based on the proof in<sup>26</sup>[Theorem 1], which studies unambiguous discrimination among quantum operations.

**Lemma 1** Let  $\{|a_t\rangle\}_t$  and  $\{|b_t\rangle\}_t$  be two orthonormal bases and  $|\psi\rangle := \sum_t \lambda_t |a_t\rangle |b_t\rangle$  where  $\lambda_t > 0$  for every  $t$ . Let also  $\rho_0^{|\psi\rangle} = (\Phi_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$  and  $\rho_j^{|\psi\rangle} = (\Phi_j \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$  for  $j = 1, \dots, m$ . If  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1, \dots, \Phi_m)$ , then  $\text{supp}(\rho_0^{|\psi\rangle}) \not\subseteq \text{supp}(\rho_1^{|\psi\rangle}, \dots, \rho_m^{|\psi\rangle})$ .

Lemma 2 also concerns inclusions of supports. It states that if the inclusion of supports does not hold for some output states, then it does not hold also for supports of the channels.

**Lemma 2** With the notation as above, if there exists a natural number  $N$  and an input state  $|\psi\rangle$  such that  $\text{supp}(\sigma_0^{N,|\psi\rangle}) \not\subseteq \text{supp}(\sigma_1^{N,|\psi\rangle}, \dots, \sigma_m^{N,|\psi\rangle})$ , then  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1, \dots, \Phi_m)$ .

Finally, we are in position to present the proof of Theorem 1.

**Proof of Theorem 1** ( $\Leftarrow$ ) Let  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1, \dots, \Phi_m)$ . From Lemma 1 this implies  $\text{supp}(\rho_0^{|\psi\rangle}) \not\subseteq \text{supp}(\rho_1^{|\psi\rangle}, \dots, \rho_m^{|\psi\rangle})$  where the input state is  $|\psi\rangle = \sum_t \lambda_t |a_t\rangle |b_t\rangle$ . Hence we can always find a state  $|\phi_0\rangle$  for which

$$|\phi_0\rangle \not\in \text{supp}(\rho_0^{|\psi\rangle}) \quad \text{and} \quad |\phi_0\rangle \perp \text{supp}(\rho_1^{|\psi\rangle}, \dots, \rho_m^{|\psi\rangle}), \tag{10}$$

and therefore

$$\langle \phi_0 | \rho_0^{|\psi\rangle} | \phi_0 \rangle > 0 \quad \text{and} \quad \langle \phi_0 | \rho_i^{|\psi\rangle} | \phi_0 \rangle = 0 \tag{11}$$

for  $i = 1, \dots, m$ .

Now we consider the certification scheme by taking the measurement with effects  $\{\Omega_0, \mathbb{1} - \Omega_0\}$ . Without loss of generality we can assume that  $\Omega_0 := |\phi_0\rangle\langle\phi_0|$  is a rank-one operator. We calculate

$$\begin{aligned} \text{tr}(\Omega_0 \rho_0^{|\psi\rangle}) &= \langle \phi_0 | \rho_0^{|\psi\rangle} | \phi_0 \rangle > 0 \\ p_2(|\psi\rangle, \Omega_0) &= \sum_{i=1}^m \text{tr}(\Omega_0 \rho_i^{|\psi\rangle}) = \sum_{i=1}^m \langle \phi_0 | \rho_i^{|\psi\rangle} | \phi_0 \rangle = 0 \\ p_1(|\psi\rangle, \Omega_0) &= \text{tr}((\mathbb{1} - \Omega_0) \rho_0^{|\psi\rangle}) = 1 - \langle \phi_0 | \rho_0^{|\psi\rangle} | \phi_0 \rangle < 1. \end{aligned} \tag{12}$$

Hence after sufficiently many uses,  $N$ , of the certified channel in parallel (actually when  $N \geq \left\lceil \frac{\log \epsilon}{\log p_1} \right\rceil$ ) we obtain that  $\text{tr}(\Omega_1^{\otimes N} (\rho_0^{|\psi\rangle})^{\otimes N}) \leq \epsilon$  for any positive  $\epsilon$ . Therefore after  $N$  queries we will be able to exclude false negative error.

( $\Rightarrow$ ) Assume that  $\Phi_0$  can be certified against  $\Phi_1, \dots, \Phi_m$  in the parallel scenario. This means that there exist a natural number  $N$ , an input state  $|\psi\rangle$  and a positive operator (measurement effect)  $\Omega_0$  on the composite system such that

$$\begin{aligned} p_1^{\mathbb{P},N}(|\psi\rangle, \Omega_0) &= 1 - \text{tr}(\Omega_0 (\Phi_0^{\otimes N} \otimes \mathbb{1})(|\psi\rangle\langle\psi|)) \leq \epsilon < 1 \\ p_2^{\mathbb{P},N}(|\psi\rangle, \Omega_0) &= \sum_{i=1}^m \text{tr}(\Omega_0 (\Phi_i^{\otimes N} \otimes \mathbb{1})(|\psi\rangle\langle\psi|)) = 0. \end{aligned} \tag{13}$$

Therefore  $\text{tr} \left( \Omega_0 \left( \Phi_0^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|) \right) > 0$  and thus

$$\begin{aligned} \Omega_0 \perp \text{supp} \left( \left( \Phi_0^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|) \right) &= \text{span} \{ (E_{i_1} \otimes \dots \otimes E_{i_N} \otimes \mathbb{1}) |\psi\rangle \}_{i_1, \dots, i_N} \\ \Omega_0 \perp \text{supp} \left( \left( \Phi_1^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|), \dots, \left( \Phi_m^{\otimes N} \otimes \mathbb{1} \right) (|\psi\rangle\langle\psi|) \right) & \\ = \text{span} \{ (K_{l_1} \otimes \dots \otimes K_{l_N} \otimes \mathbb{1}) |\psi\rangle \}_{l_1, \dots, l_N}, & \end{aligned} \tag{14}$$

where  $\text{span} \{ F_{j_1}^{(1)}, \dots, F_{j_m}^{(m)} \}_{j_1, \dots, j_m} = \text{span} \{ K_l \}$ .

Hence

$$\text{span} \{ (E_{i_1} \otimes \dots \otimes E_{i_N} \otimes \mathbb{1}) |\psi\rangle \}_{i_1, \dots, i_N} \not\subseteq \text{span} \{ (K_{l_1} \otimes \dots \otimes K_{l_N} \otimes \mathbb{1}) |\psi\rangle \}_{l_1, \dots, l_N}. \tag{15}$$

The reminder of the proof follows directly from Lemma 2. □

It is worth mentioning that in the above proof the measurement effect  $\Omega_0$  is a rank-one projection operator. This is sufficient to prove that quantum channel  $\Phi_0$  can be certified against  $\Phi_1$  in the parallel scheme, but this is, in most of the cases, not optimal.

In the remaining of this section we will discuss two examples. The first example shows that if quantum channels can be certified in the parallel scheme, then it does not have to imply that they can be discriminated unambiguously. We will provide an explicit example of mixed-unitary channels which fulfill the condition from Theorem 1, and therefore can be certified in the parallel scheme, but cannot be discriminated unambiguously. In the second example we will consider the situation when the channel associated with the  $H_1$  hypothesis is the identity channel and derive an upper bound on the probability of false positive error.

**Channels which cannot be discriminated unambiguously but still can be certified.** In this subsection we will give an example of a class of channels which cannot be discriminated unambiguously, but they can be certified by a finite number of uses in the parallel scheme. The work<sup>26</sup> presents the condition when quantum channels can be unambiguously discriminated by a finite number of uses. More precisely, Theorem 2 therein states that if a set of quantum channels  $\mathcal{S} = \{ \Phi_i \}_i$  satisfies the condition  $\text{supp}(\Phi_i) \not\subseteq \text{supp}(\Phi_j)$  for every  $\Phi_i, \Phi_j \in \mathcal{S}$ , then they can be discriminated unambiguously in a finite number of uses.

Now we proceed to presenting our example. Let  $\Phi_0$  be a mixed unitary channel of the form

$$\Phi_0(\rho) = \sum_{i=1}^m p_i U_i \rho U_i^\dagger, \tag{16}$$

where  $p = (p_1, \dots, p_m)$  is a probability vector and  $\{U_1, \dots, U_m\}$  are unitary matrices. As the second channel we take a unitary channel of the form  $\Phi_1(\rho) = \tilde{U} \rho \tilde{U}^\dagger$ , where we make a crucial assumption that  $\tilde{U} \in \{U_1, \dots, U_m\}$ .

Therefore we have  $\text{supp}(\Phi_0) = \text{span} \{ \sqrt{p_i} U_i \}_i$ , while  $\text{supp}(\Phi_1) = \text{span} \{ \tilde{U} \}$ . In this example it can be easily seen that the condition for unambiguous discrimination is not fulfilled as  $\text{supp}(\Phi_1) \subseteq \text{supp}(\Phi_0)$ . Nevertheless, the condition from Theorem 1 is fulfilled as  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1)$ , and hence it is possible to exclude false negative error after a finite number of queries in parallel.

**Certification of arbitrary channel against the identity channel.** Assume that we want to certify channel  $\Phi_0$ , which Kraus operators are  $\{E_i\}_i$ , against the identity channel  $\Phi_1$  having Kraus operator  $\{ \mathbb{1} \}$ . We will show that as long as the channel  $\Phi_0$  is not the identity channel, it can always be certified against the identity channel in the parallel scheme.

**Proposition 1** *Every quantum channel (except the identity channel) can be certified against the identity channel in the parallel scheme.*

**Proof** Let  $|\psi\rangle$  be an input state. After applying the certified channels on it, we obtain the state either  $\rho_0^{|\psi\rangle} = (\Phi_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)$ , if the channel was  $\Phi_0$ , or  $\rho_1^{|\psi\rangle} = |\psi\rangle\langle\psi|$ , if the channels was  $\Phi_1$ . As the final measurement effect we can take  $\Omega_0 := \mathbb{1} - |\psi\rangle\langle\psi|$ , which is always orthogonal to  $\rho_1^{|\psi\rangle}$ , hence no false negative error will occur. Having the input state and final measurement fixed, we will calculate the probability of false positive error in the single-shot scheme

$$\begin{aligned} p_1(|\psi\rangle, \Omega_0) &= 1 - \text{Tr} \left( \Omega_0 \rho_0^{|\psi\rangle} \right) = 1 - \text{Tr} \left( (\mathbb{1} - |\psi\rangle\langle\psi|) \rho_0^{|\psi\rangle} \right) = \text{Tr} \left( |\psi\rangle\langle\psi| \rho_0^{|\psi\rangle} \right) \\ &= \langle \psi | ((\Phi_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)) | \psi \rangle < 1, \end{aligned} \tag{17}$$

where the last inequality follows from the fact that  $\Phi_0$  is not the identity channel. Therefore, after sufficiently many queries in the parallel scheme the probability of false positive error will be arbitrarily small. □

Note that the expression for the probability of false positive error in Eq. (17) is in fact the fidelity between the input state and the output of the channel  $\Phi_0$  extended by the identity channel. As we were not imposing



any specific assumptions on the input state, we can take the one which minimizes the expression in Eq. (17). Therefore, the probability of the false positive error in the single-shot certification yields

$$p_1 = \min_{|\psi\rangle} \langle \psi | ((\Phi_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|)) | \psi \rangle. \tag{18}$$

Eventually, to make sure that the probability of false positive error will not be greater than  $\epsilon$ , we will need  $N_\epsilon \geq \left\lceil \frac{\log \epsilon}{\log p_1} \right\rceil$  steps in the parallel scheme.

From the above considerations we can draw a simple conclusion concerning the situation when  $\Phi_0(X) = UXU^\dagger$  is a unitary channel. Then, as the unitary channel has only one Kraus operator, it holds that  $p_1 = \min_{|\psi\rangle} |\langle \psi | (U \otimes \mathbb{1}) | \psi \rangle|^2 = \min_{|\psi\rangle} |\langle \psi | U | \psi \rangle|^2 = v^2(U)$ , where  $v(U)$  is the distance from zero to the numerical range of the matrix  $U$ <sup>18,32</sup>. Thanks to this geometrical representation (see further<sup>18</sup>) one can deduce the connection between the probability of false positive error,  $p_1$ , and the probability of making an error in the unambiguous discrimination of unitary channels. More specifically, let  $p_{\text{error}}^u$  denote the probability of making an erroneous decision in unambiguous discrimination of unitary channels. Then, it holds that  $p_{\text{error}}^u = p_1^2$ . Therefore, in the case of certification of unitary channels the probability of making the false positive error is significantly smaller than the probability of erroneous unambiguous discrimination.

### Certification of quantum measurements

In this section we will take a closer look into the certification of quantum measurements. We will begin with general POVMs and later focus on the class of measurements with rank-one effects. Before stating the results, let us recall that every quantum measurement can be associated with quantum-classical channel defined as

$$\mathcal{P}(\rho) = \sum_i \text{tr}(M_i \rho) |i\rangle\langle i|, \tag{19}$$

where  $\{M_i\}_i$  are measurement's effects and  $\text{tr}(M_i \rho)$  is the probability of obtaining the  $i$ -th label.

The following proposition can be seen as a corollary from Theorem 1 as it gives a simple condition when we forbid false negative error. This condition is expressed in terms of inclusion of supports of the measurements' effects.

**Proposition 2** *Let  $\mathcal{P}_0$  and  $\mathcal{P}_1$  be POVMs with effects  $\{M_i\}_{i=1}^m$  and  $\{N_i\}_{i=1}^m$  respectively. Then  $\mathcal{P}_0$  can be certified against  $\mathcal{P}_1$  in the parallel scheme if and only if there exists a pair of effects  $M_i, N_i$  for which  $\text{supp}(M_i) \not\subseteq \text{supp}(N_i)$ .*

**Proof** Let

$$M_i = \sum_{k_i} \alpha_{k_i}^i |x_{k_i}^i\rangle\langle x_{k_i}^i| \tag{20}$$

be the spectral decomposition of  $M_i$  (where  $\alpha_{k_i}^i > 0$  for every  $k$ ). Then

$$\mathcal{P}_0(\rho) = \sum_i |i\rangle\langle i| \text{tr}(M_i \rho) = \sum_i \sum_{k_i} \alpha_{k_i}^i |i\rangle\langle x_{k_i}^i| \rho |x_{k_i}^i\rangle\langle i| \tag{21}$$

and hence the Kraus operators of  $\mathcal{P}_0$  are  $\left\{ \sqrt{\alpha_{k_i}^i} |i\rangle\langle x_{k_i}^i| \right\}_{k_i,i}$ . Analogously, the Kraus operators of  $\mathcal{P}_1$  are  $\left\{ \sqrt{\beta_{k_i}^i} |i\rangle\langle y_{k_i}^i| \right\}_{k_i,i}$ .

Therefore from Theorem 1 we have that  $\mathcal{P}_0$  can be certified against  $\mathcal{P}_1$  in the parallel scheme if and only if

$$\text{span} \left\{ \sqrt{\alpha_{k_i}^i} |i\rangle\langle x_{k_i}^i| \right\}_{k_i,i} \not\subseteq \text{span} \left\{ \sqrt{\beta_{k_i}^i} |i\rangle\langle y_{k_i}^i| \right\}_{k_i,i}, \tag{22}$$

that is when there exists a pair of effects  $M_i, N_i$  for which  $\text{supp}(M_i) \not\subseteq \text{supp}(N_i)$ . □

The above proposition holds for any pair of quantum measurements. In the case of POVMs with rank-one effects, the above condition can still be simplified to linear independence of vectors. This is stated as the following corollary.

**Corollary 1** *Let  $\mathcal{P}_0$  and  $\mathcal{P}_1$  be measurements with effects  $\{\alpha_i |x_i\rangle\langle x_i|\}_{i=1}^m$  and  $\{\beta_i |y_i\rangle\langle y_i|\}_{i=1}^m$  for  $\alpha_i, \beta_i \in (0, 1]$ , respectively. Then  $\mathcal{P}_0$  can be certified against  $\mathcal{P}_1$  in the parallel scheme if and only if there exists a pair of vectors  $|x_i\rangle, |y_i\rangle$  which are linearly independent.*

While studying the certification of measurements with rank-one effects, one cannot overlook their very important subclass, namely projective von Neumann measurements. These measurements have effects of the form  $\{|u_1\rangle\langle u_1|, \dots, |u_n\rangle\langle u_n|\}$ , where  $\{|u_i\rangle\}_i$  form an orthonormal basis. This class of measurements was studied in<sup>32</sup>, though in a slightly different context. The main result of that work was the expression for minimized probability of the false negative error, where the bound on the false positive error was assumed. In this work, however, we consider the situation when false negative error must be equal zero after sufficiently many uses. Nevertheless, from Corollary 1 we can draw a conclusion that any von Neumann measurement can be certified against some other von Neumann measurement if and only if the measurements are not the same.

**SIC POVMs.** Now we proceed to studying the certification of a special class of measurements with rank-one effects, that is symmetric informationally complete (SIC) POVMs<sup>35–38</sup>. We will directly calculate the bounds on the false positive error in the single-shot and parallel certification. We will be using the following notation. The SIC POVM  $\mathcal{P}_0$  with effects  $\{|x_i\rangle\langle x_i|\}_{i=1}^{d^2}$ , where  $|x_i\rangle\langle x_i| = \frac{1}{d}|\phi_i\rangle\langle\phi_i|$  and  $\|\phi_i\| = 1$ , will be associated with the  $H_0$  hypothesis. The SIC POVM  $\mathcal{P}_1$  corresponding to the alternative  $H_1$  hypothesis will have effects  $\{|y_i\rangle\langle y_i|\}_{i=1}^{d^2}$ , where  $|y_i\rangle\langle y_i| = \frac{1}{d}|\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|$  and  $\pi$  is a permutation of  $d^2$  elements. Moreover, the SIC condition assures that  $|\langle\phi_i|\phi_{\pi(i)}\rangle|^2 = \frac{1}{d+1}$  whenever  $i \neq \pi(i)$ .

**Remark 1** From Corollary 1 it follows that for a SIC POVMs  $\mathcal{P}_0$  can be certified against SIC POVM  $\mathcal{P}_1$  in the parallel scheme as long as  $\mathcal{P}_0 \neq \mathcal{P}_1$ .

Now we are working towards calculating the upper bound on the probability of the false positive error in single-shot certification of SIC POVMs. As the input state we take the maximally entangled state  $|\psi\rangle := \frac{1}{\sqrt{d}}|\mathbb{1}\rangle$ . If the measurement was  $\mathcal{P}_0$ , then the output state is

$$\rho_0^{|\psi\rangle} = (\mathcal{P}_0 \otimes \mathbb{1})(|\psi\rangle\langle\psi|) = \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \frac{1}{d}(|x_i\rangle\langle x_i|)^\top = \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \frac{1}{d^2}(|\phi_i\rangle\langle\phi_i|)^\top, \tag{23}$$

and similarly, if the measurement was  $\mathcal{P}_1$ , then the output state is

$$\rho_1^{|\psi\rangle} = \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \frac{1}{d^2}(|\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|)^\top. \tag{24}$$

As the output states have block-diagonal structure, we take the measurement effect to be in the block-diagonal form, that is

$$\Omega_0 := \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \Omega_i^\top, \tag{25}$$

where for every  $i$  we assume  $\Omega_i \perp |\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|$  to ensure that the probability of the false negative error is equal to zero. We calculate

$$\begin{aligned} \text{tr}(\Omega_0 \rho_0) &= \text{tr} \left( \left( \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \Omega_i^\top \right) \left( \sum_{j=1}^{d^2} |j\rangle\langle j| \otimes \frac{1}{d^2}(|\phi_j\rangle\langle\phi_j|)^\top \right) \right) \\ &= \text{tr} \left( \sum_{i=1}^{d^2} |i\rangle\langle i| \otimes \Omega_i^\top \frac{1}{d^2}(|\phi_i\rangle\langle\phi_i|)^\top \right) = \frac{1}{d^2} \sum_{i=1}^{d^2} \langle\phi_i|\Omega_i|\phi_i\rangle. \end{aligned} \tag{26}$$

Let  $k$  be the number of fixed points of the permutation  $\pi$ . Taking  $\Omega_i := \mathbb{1} - |\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|$  we obtain

$$\begin{aligned} \text{tr}(\Omega_0 \rho_0) &= \frac{1}{d^2} \sum_{i=1}^{d^2} \langle\phi_i|\Omega_i|\phi_i\rangle = \frac{1}{d^2} \sum_{i=1}^{d^2} \langle\phi_i|(\mathbb{1} - |\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|)|\phi_i\rangle \\ &= \frac{1}{d^2} \sum_{i=1}^{d^2} (1 - |\langle\phi_i|\phi_{\pi(i)}\rangle|^2) = \frac{1}{d^2} (d^2 - k) (1 - |\langle\phi_i|\phi_{\pi(i)}\rangle|^2) \\ &= \frac{1}{d^2} (d^2 - k) \left( 1 - \frac{1}{d+1} \right) = \frac{d^2 - k}{d^2 + d}. \end{aligned} \tag{27}$$

So far all the calculations were done for some fixed input state (maximally entangled state) and measurement effect  $\Omega_0$ , which give us actually only the upper bound on the probability of the false positive error. The current choice of  $\Omega_i = \mathbb{1} - |\phi_{\pi(i)}\rangle\langle\phi_{\pi(i)}|$  seems like a good candidate, but we do not know whether it is possible to find a better one. Using the notation for the probability of the false positive error introduced in Eq. (4) and (2) we can write our bound as

$$p_1 \leq p_1(|\psi\rangle, \Omega_0) = 1 - \text{tr}(\Omega_0 \rho_0) = \frac{d+k}{d^2+d}. \tag{28}$$

On top of that, if  $\pi$  does not have fixed points, that is when  $k = 0$ , we have  $p_1 \leq \frac{1}{d+1}$  and the number of steps needed for parallel certification is bounded by  $N_\epsilon \geq \left\lceil -\frac{\log \epsilon}{\log(d+1)} \right\rceil$ . In the case when the permutation  $\pi$  has one fixed point, that is when  $k = 1$ , it holds that  $p_1 \leq \frac{1}{d}$  and hence the number of steps needed for parallel certification can be bounded by  $N_\epsilon \geq \left\lceil -\frac{\log \epsilon}{\log d} \right\rceil$ .



**Parallel certification of SIC POVMs.** Let us consider a generalization of the results from previous subsection into the parallel scenario. We want to certify SIC POVMs  $\mathcal{P}_0$  and  $\mathcal{P}_1$  defined as in “SIC POVMs” section, however we assume that we are allowed to use the certified SIC POVM  $N$  times in parallel. In this setup we associate the  $H_0$  hypothesis with the measurement  $\mathcal{P}_0^{\otimes N}$ , and analogously we associate the  $H_1$  hypothesis with the measurement  $\mathcal{P}_1^{\otimes N}$ . It appears that the upper bound on false positive error is very similar to the upper bound for the single-shot case. Straightforward but lengthy and technical calculations give us

$$p_1^{\mathbb{P},N} \leq \left( \frac{d+k}{d^2+d} \right)^N. \quad (29)$$

The detailed derivation of this bound is relegated to “Supplementary Appendix B”.

### Adaptive certification and Stein setting

So far we were considering only the scheme in which the given channel is used a finite number of times in parallel. In this section we will focus on studying a more general scheme of certification, that is the adaptive certification. In the adaptive scenario, we use the given channel  $N$  times and between the uses we can perform some processing. It seems natural that the use of adaptive scheme instead of the simple parallel one should improve the certification. Surprisingly, in the case of von Neumann measurements the use of adaptive scheme gives no advantage over the parallel one<sup>18,32</sup>. In other cases it appears that the use of processing is indeed a necessary step towards perfect discrimination<sup>19,20</sup>.

Having the adaptive scheme as a generalization of the parallel one, let us take a step further and take a look into the asymptotic setting. In other words, let us discuss the situation when the number of uses of the certified channel tends to infinity. There are various settings known in the literature concerning asymptotic discrimination, like Stein and Hoeffding settings for asymmetric discrimination, as well as Chernoff and Han-Kobayashi settings for symmetric discrimination. In the context of this work we will discuss only the setting concerning asymmetric discrimination, however a concise introduction to all of these settings can be found e.g. in<sup>33</sup>. Arguably, the most well-known of these is the Hoeffding setting which assumes the bound on the false negative error to be decreasing exponentially, and its area of interest is characterizing the error exponent of probability of false positive error. Adaptive strategies for asymptotic discrimination in Hoeffding setting were recently explored in<sup>21</sup>. In the Stein setting, on the other hand, we assume a constraint on the probability of false positive error and study the error exponent of the false negative error. Let us define a non-asymptotic quantity

$$\zeta_n(\epsilon) := \sup_{\Omega_0, |\psi\rangle} \left\{ -\frac{1}{n} \log p_2^{\mathbb{A},n}(|\psi\rangle, \Omega_0) : p_1^{\mathbb{A},n}(|\psi\rangle, \Omega_0) \leq \epsilon \right\}, \quad (30)$$

which describes the behavior of probabilities of errors in adaptive discrimination scheme. The probability of false positive error after  $n$  queries is upper-bounded by some fixed  $\epsilon$ , and we are interested in studying how quickly the probability of false negative error decreases. Therefore we consider the logarithm of probability of false negative error divided by the number of queries. Finally, a supremum is taken over all possible adaptive strategies, that is we can choose the best input state, final measurement as well as the processings between uses of the certified channel.

Note that in the previous sections we were considering  $p_2^{\mathbb{A},N}$  instead of  $p_2^{\mathbb{A},n}$ , which is used in the Stein setting. The aim of this difference is to emphasize that in the Stein setting we study the situation in which the number of uses,  $n$ , tends to infinity. In contrary, in previous sections we were interested only in the case when the number of uses,  $N$ , was finite.

Having introduced the non-asymptotic quantity  $\zeta_n(\epsilon)$ , let us consider the case when the number of queries,  $n$ , tends to infinity. To do so, we define the upper limit of the Stein exponent as

$$\bar{\zeta}(\epsilon) := \limsup_{n \rightarrow \infty} \zeta_n(\epsilon). \quad (31)$$

Note that when  $\bar{\zeta}(\epsilon)$  is finite, then the probability of the false negative error for adaptive certification will not be equal to zero for any finite number of uses  $N$ . A very useful Remark 19 from<sup>33</sup> states that  $\bar{\zeta}(\epsilon)$  is finite if and only if

$$\text{supp}((\Phi_0 \otimes \mathbb{1})(|\psi_{\text{ent}}\rangle\langle\psi_{\text{ent}}|)) \subseteq \text{supp}((\Phi_1 \otimes \mathbb{1})(|\psi_{\text{ent}}\rangle\langle\psi_{\text{ent}}|)), \quad (32)$$

where  $|\psi_{\text{ent}}\rangle$  is the maximally entangled state.

Finally, we are in position to express the theorem stating the relation between adaptive and parallel certification.

**Theorem 2** *Quantum channel  $\Phi_0$  can be certified against quantum channel  $\Phi_1$  in the parallel scenario if and only if quantum channel  $\Phi_0$  can be certified against quantum channel  $\Phi_1$  in the adaptive scenario.*

Before presenting the proof of the Theorem we will state a useful lemma, which proof is postponed to “Supplementary Appendix A”.

**Lemma 3** *Let  $\bar{\zeta}(\epsilon)$  be as in Eq. (31). Then  $\bar{\zeta}(\epsilon)$  is finite if and only if  $\text{supp}(\Phi_0) \subseteq \text{supp}(\Phi_1)$ .*

**Proof of Theorem 2** When quantum channel  $\Phi_0$  can be certified against the channel  $\Phi_1$  in the parallel scenario, then naturally,  $\Phi_0$  can be certified against the channel  $\Phi_1$  in the adaptive scenario. Therefore it suffices to prove the reverse implication.

Assume that the channel  $\Phi_0$  can be certified against  $\Phi_1$  in the adaptive scenario. This means that  $\bar{\zeta}(\epsilon)$  is infinite. Hence from Lemma 3 it holds that  $\text{supp}(\Phi_0) \not\subseteq \text{supp}(\Phi_1)$ . Finally, from Theorem 1 we obtain that  $\Phi_0$  can be certified against  $\Phi_1$  in the parallel scheme.  $\square$

Theorem 2 states that if a quantum channel  $\Phi_0$  can be certified against  $\Phi_1$  in a finite number of queries, then the use of parallel scheme is always sufficient. Therefore it may appear that adaptive certification is of no value. Nevertheless, in some cases it still may be worth using adaptive certification to reduce the number of uses of the certified channel. For example in the case of SIC POVMs the use of adaptive scheme reduces the number of steps significantly<sup>20</sup>. A pair of qutrit SIC POVMs can be discriminated perfectly after two queries in adaptive scenario, therefore they can also be certified. Nevertheless, they cannot be discriminated perfectly after any finite number of queries in parallel. On the other hand, in the case of von Neumann measurements the number of steps is the same no matter which scheme is used<sup>18</sup>.

## Conclusions

As certification of quantum channels is in the NISQ era a task of significant importance, the main aim of this work was to give an insight into this problem from theoretical perspective. Certification was considered as an extension of quantum hypothesis testing, which includes also preparation of an input state and the final measurement. We primarily focused on multiple-shot schemes of certification, that is our areas of interest were mostly parallel and adaptive certification schemes. The parallel scheme consists in certifying tensor products of channels while adaptive scheme is the most general of all scenarios.

We derived a condition when after a finite number of queries in the parallel scenario one can assure that the false negative error will not occur. We pointed a class of channels which allow for excluding false negative error after a finite number of uses in parallel but cannot be discriminated unambiguously. On top of that, having a fixed upper bound on the probability of false positive error, we found a bound on the number of queries needed to make the probability of false positive error no greater than this fixed bound.

Moreover, we took into consideration the most general adaptive certification scheme and studied whether it can improve the certification. It turned out that the use of parallel certification scheme is always sufficient to assure that the false negative error will not occur after a finite number of queries. Nevertheless, the number of queries needed to have the probability of false positive error sufficiently small, may be decreased by using adaptive scheme.

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## References

- Helstrom, C. W. *Quantum Detection and Estimation Theory* Vol. 84 (Academic Press, New York, 1976).
- Preskill, J. Quantum computing in the NISQ era and beyond. *Quantum* **2**, 79 (2018).
- Bharti, K., Cervera-Lierta, A., Kyaw, T. H., Haug, T., Alperin-Lea, S., Anand, A., Degroote, M., Heimonen, H., Kottmann, J. S., Menke, T., et al. Noisy intermediate-scale quantum (NISQ) algorithms. arXiv preprint [arXiv:2101.08448](https://arxiv.org/abs/2101.08448) (2021).
- Eisert, J. et al. Quantum certification and benchmarking. *Nat. Rev. Phys.* **2**(7), 382–390 (2020).
- Kliesch, M. & Roth, I. Theory of quantum system certification. *PRX Quantum* **2**(1), 010201 (2021).
- Liu, Y.-C., Shang, J., Yu, X.-D. & Zhang, X. Efficient verification of quantum processes. *Phys. Rev. A* **101**(4), 042315 (2020).
- Zeng, P., Zhou, Y. & Liu, Z. Quantum gate verification and its application in property testing. *Phys. Rev. Res.* **2**(2), 023306 (2020).
- Zhu, H. & Zhang, H. Efficient verification of quantum gates with local operations. *Phys. Rev. A* **101**(4), 042316 (2020).
- Watrous, J. *The Theory of Quantum Information* (Cambridge University Press, Cambridge, 2018).
- Ziman, M. & Sedlák, M. Single-shot discrimination of quantum unitary processes. *J. Mod. Opt.* **57**(3), 253–259 (2010).
- Bae, J. Discrimination of two-qubit unitaries via local operations and classical communication. *Sci. Rep.* **5**(1), 1–8 (2015).
- Ji, Z., Feng, Y., Duan, R. & Ying, M. Identification and distance measures of measurement apparatus. *Phys. Rev. Lett.* **96**(20), 200401 (2006).
- Sedlák, M. & Ziman, M. Optimal single-shot strategies for discrimination of quantum measurements. *Phys. Rev. A* **90**(5), 052312 (2014).
- Puchała, Z., Paweł, Ł., Krawiec, A. & Kukulski, R. Strategies for optimal single-shot discrimination of quantum measurements. *Phys. Rev. A* **98**(4), 042103 (2018).
- Duan, R., Guo, C., Li, C.-K., & Li, Y. Parallel distinguishability of quantum operations. In: *2016 IEEE International Symposium on Information Theory (ISIT)*, pp. 2259–2263 (IEEE, 2016).
- Cao, T.-Q. et al. Minimal number of runs and the sequential scheme for local discrimination between special unitary operations. *Sci. Rep.* **6**(1), 1–6 (2016).
- Chiribella, G., D'Ariano, G. M. & Perinotti, P. Memory effects in quantum channel discrimination. *Phys. Rev. Lett.* **101**(18), 180501 (2008).
- Puchała, Z., Paweł, Ł., Krawiec, A., Kukulski, R. & Oszmaniec, M. Multiple-shot and unambiguous discrimination of von Neumann measurements. *Quantum* **5**, 425 (2021).
- Harrow, A. W., Hassidim, A., Leung, D. W. & Watrous, J. Adaptive versus nonadaptive strategies for quantum channel discrimination. *Phys. Rev. A* **81**(3), 032339 (2010).
- Krawiec, A., Paweł, Ł. & Puchała, Z. Discrimination of POVMs with rank-one effects. *Quantum Inf. Process.* **19**(12), 1–12 (2020).
- Salek, F., Hayashi, M., & Winter, A. When are adaptive strategies in asymptotic quantum channel discrimination useful?. arXiv preprint [arXiv:2011.06569](https://arxiv.org/abs/2011.06569) (2020).
- Pirandola, S., Laurenza, R., Lupo, C. & Pereira, J. L. Fundamental limits to quantum channel discrimination. *NPJ Quantum Inf.* **5**(1), 1–8 (2019).

23. Zhuang, Q. & Pirandola, S. Ultimate limits for multiple quantum channel discrimination. *Phys. Rev. Lett.* **125**(8), 080505 (2020).
24. Katariya, V., & Wilde, M. M. Evaluating the advantage of adaptive strategies for quantum channel distinguishability. arXiv preprint [arXiv:2001.05376](https://arxiv.org/abs/2001.05376) (2020).
25. Wang, X. & Wilde, M. M. Resource theory of asymmetric distinguishability for quantum channels. *Phys. Rev. Res.* **1**(3), 033169 (2019).
26. Wang, G. & Ying, M. Unambiguous discrimination among quantum operations. *Phys. Rev. A* **73**(4), 042301 (2006).
27. Feng, Y., Duan, R. & Ying, M. Unambiguous discrimination between mixed quantum states. *Phys. Rev. A* **70**(1), 012308 (2004).
28. Zhang, C., Feng, Y. & Ying, M. Unambiguous discrimination of mixed quantum states. *Phys. Lett. A* **353**(4), 300–306 (2006).
29. Herzog, U. & Bergou, J. A. Optimum unambiguous discrimination of two mixed quantum states. *Phys. Rev. A* **71**(5), 050301 (2005).
30. Bergou, J. A., Feldman, E. & Hillery, M. Optimal unambiguous discrimination of two subspaces as a case in mixed-state discrimination. *Phys. Rev. A* **73**(3), 032107 (2006).
31. Herzog, U. Discrimination of two mixed quantum states with maximum confidence and minimum probability of inconclusive results. *Phys. Rev. A* **79**(3), 032323 (2009).
32. Lewandowska, P., Krawiec, A., Kukulski, R., Pawela, Ł & Puchała, Z. On the optimal certification of von Neumann measurements. *Sci. Rep.* **11**(1), 1–16 (2021).
33. Wilde, M. M., Berta, M., Hirche, C. & Kaur, E. Amortized channel divergence for asymptotic quantum channel discrimination. *Lett. Math. Phys.* **110**(8), 2277–2336 (2020).
34. Duan, R., Feng, Y. & Ying, M. Perfect distinguishability of quantum operations. *Phys. Rev. Lett.* **103**(21), 210501 (2009).
35. Renes, J. M., Blume-Kohout, R., Scott, A. J. & Caves, C. M. Symmetric informationally complete quantum measurements. *J. Math. Phys.* **45**(6), 2171–2180 (2004).
36. Flamia, S. T. On SIC-POVMs in prime dimensions. *J. Phys. A Math. Gen.* **39**(43), 13483 (2006).
37. Zhu, H. SIC POVMs and Clifford groups in prime dimensions. *J. Phys. A Math. Theor.* **43**(30), 305305 (2010).
38. Appleby, D. Properties of the extended Clifford group with applications to SIC-POVMs and MUBs. arXiv preprint [arXiv:0909.5233](https://arxiv.org/abs/0909.5233) (2009).

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## Author contributions

All authors participated in proving the theorems constituting the results of this work. Ł.P. and Z.P. set the research objective and A.K. prepared the figures.

## Competing interests

The authors declare no competing interests.

## Additional information

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