Article

Steady-State Thermodynamics of a Cascaded Collision Model

Lu Li, Zhong-Xiao Man *[ID] and Yun-Jie Xia

Shandong Provincial Key Laboratory of Laser Polarization and Information Technology, Department of Physics, Qufu Normal University, Qufu 273165, China; liluqfsfdx@163.com (L.L.); yjxia@qfnu.edu.cn (Y.-J.X.)
* Correspondence: zxman@qfnu.edu.cn

Abstract: We study the steady-state thermodynamics of a cascaded collision model where two subsystems S_1 and S_2 collide successively with an environment R in the cascaded fashion. We first formulate general expressions of thermodynamics quantities and identify the nonlocal forms of work and heat that result from cascaded interactions of the system with the common environment. Focusing on a concrete system of two qubits, we then show that, to be able to unidirectionally influence the thermodynamics of S_2, the former interaction of $S_1 - R$ should not be energy conserving. We finally demonstrate that the steady-state coherence generated in the cascaded model is a kind of useful resource in extracting work, quantified by ergotropy, from the system. Our results provide a comprehensive understanding on the thermodynamics of the cascaded model and a possible way to achieve the unidirectional control on the thermodynamics process in the steady-state regime.

Keywords: collision model; quantum thermodynamics; cascaded model

1. Introduction

Recent years have seen increasing interests in the study of quantum thermodynamics (QT) [1,2] which exploits an open quantum system [3] as a working substance to implement thermodynamic tasks. The main purposes of QT are to examine fundamental laws of classical thermodynamics in the quantum level and to reveal the influences of various quantum resources and/or quantum effects on thermodynamics processes, among others. The applications of quantum resources such as quantum coherence and correlation in QT have attracted much attention since the pioneering work of Scully et al. [4]. In Ref. [4], by conveying the atomic beam with coherence through the cavity field and interacting with the field mode for a short duration, Scully et al. have shown that the field can arrive at a larger temperature than the situation where the atoms are prepared in regular thermal states. As a result, the efficiency of the photonic Carnot engine driven by a coherent bath can outperform its classical counterpart [4]. Subsequently, the effects of quantum coherence and correlation have been applied to improve the performance of thermal machines [5–14], enhance the extraction of work [15–17], and increase the thermalization temperature of a quantum system [18–20].

Apart from specific quantum resources and quantum effects, researchers also try to explore quantum thermodynamics by engineering the couplings of a quantum system with external environments. The simultaneous couplings of several systems with a common environment lead to the noise-induced interference, which proves to be useful in enhancing the thermodynamics tasks [21–24]. Manzano et al. show that it is possible to improve the machine performance by virtue of common noise sources [22]. Albeit surrounded by a common environment, unidirectional (one-way) influence of two systems, say, S_1 and S_2, can be achieved by the so-called cascaded model [25–29]. In this model, S_1 and S_2 successively interact with the common environment R in such a way that the interaction of $S_1 - R$ takes place first, which is then followed by $S_2 - R$ resulting in the one-way influence of S_1 to the dynamics of S_2. The cascaded model is an efficient tool to depict the ordered interactions of individual subsystems with a common environment. For instance, in the

cavity QED, the unidirectional exchange of information (energy) between a linear array of cavities and the successively injected atoms can be described by this framework [26]. The cascaded fashion of interaction is also applied in QT to study the dynamics of heat current [25,28,29]. In Ref. [25], Lorenzo et al. explore the non-Markovian dynamics of heat current and the effects of quantum correlations in dissipative cascaded systems. It turns out that, in clear contrast to the configuration with each subsystem being independently coupled to the reservoir, the heat flow under the cascaded interaction exhibits a non-exponential time behavior [25]. In these studies of the cascaded model [25–29], the system–environment interactions are assumed to be energy conserving so that all the energy changes are accounted for by heat without the contribution of work. In this paper, we relax the assumption of energy conservation of the system–environment interaction to make a comprehensive exploration on the thermodynamics of cascaded model involving both heat and work.

The quantum master equation (QME) is the most popular tool in the study of the dynamics of an open quantum system even with several approximations. Since QME cannot capture information about the global state of the system and environment, applications of QME in dealing with QT in some cases might lead to the occurrence of thermodynamic inconsistencies [30–41]. In Ref. [41], Levy and Kosloff have considered a model consisting of two subsystems embedded in two independent thermal reservoirs with different temperatures. It is found that the heat flows automatically from cold to hot reservoirs, i.e., the second law of thermodynamics is violated, if the local QME is used to describe the system's dynamics [41]. A possible way to overcome this weakness is the collision model (CM) [42] where the environment is modeled as a collection of identically prepared ancillas and at each time step the system interacts/collides with a fresh ancilla. The CM has been used as an alternative tool in the study of dynamics of an open quantum system [43–50] for a long time. In particular, the CM is more efficient in the simulation of non-Markovian dynamics through several strategies, such as the introduction of either initial correlations between ancillas or ancilla–ancilla collisions between two successive system–ancilla collisions [51–61]. An advantage of the CM over the QME is that it can track the information of environment. In the field of QT [62–74], CM resolves some thermodynamics inconsistencies and makes the most fundamental definitions of thermodynamic quantities, such as heat and work, possible. It has been recognized that, in the CM, a certain amount of work should be supplied to maintain the successive collisions of the system and environment [62]. By taking the extra work cost of maintaining the successive collisions into account, the local QME is shown to comply again with thermodynamics [63]. By means of CM, the effects of non-Markovianity on the laws of thermodynamics and on the performances of thermal machines have also been studied [28,75–79].

In this work, we address the thermodynamics of a cascaded model where two subsystems S_1 and S_2 collide successively with an environment R. Thanks to the framework of CM, we can construct general formulations of thermodynamics quantities from the most fundamental definitions. Nonlocal forms of work and heat are identified as a result of cascaded interaction of the subsystems with the common environment. We then demonstrate the features of steady-state thermodynamics concentrating on a two-qubit system. It turns out that, only when the interaction of $S_1 - R$ does not satisfy strict energy conservation, can it have a one-way effect on the thermodynamics regarding S_2. We also show that the cascaded interaction leads to steady-state coherence of the system, which can be recognized as a kind of resource in extracting useful work from the system.

2. The Model and Master Equation

We consider that the system S consists of two uncoupled subsystems S_1 and S_2 with the free Hamiltonian $\hat{H}_{S_i}$ ($i = 1, 2$). In terms of CM, the environment R is modeled as a series of identical ancillas described by the Hamiltonian $\hat{H}_R$. Note that we use $\hat{H}_R$ to represent both the environment and ancillas therein. The system–environment interactions adopt the cascaded manner in such a way that at each time step S_1 collides with an ancilla

for a short duration τ, which is followed by another collision of S_2 with that ancilla for the same time τ, as depicted in Figure 1. The environment is assumed to be sufficiently large so that the system never collides twice with the same ancilla. Although there is no direct coupling between these two subsystems, the cascaded model results in unidirectional influences of S_1 to S_2 but no backward actions.
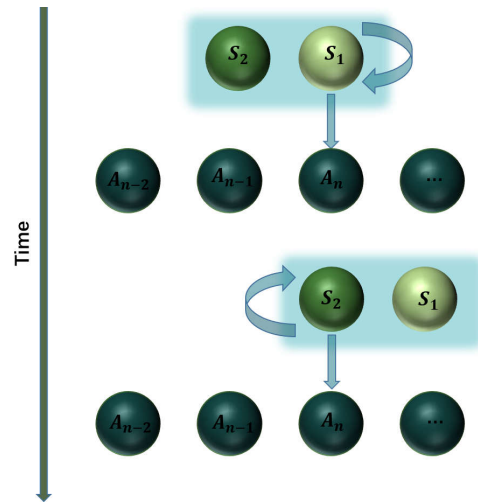

Figure 1. Schematic diagram of the cascaded model. The system consists of two subsystems S_1 and S_2, while the environment is modeled as a collection of ancillas with the nth one being labeled as A_n. In the nth round of the collision, S_1 collides with A_n for a duration τ, which is then followed by another collision of S_2 with A_n for the same duration.

The total Hamiltonian of the system plus environment can be expressed as

$$\hat{H}_{tot}(t) = \hat{H}_S + \hat{H}_R + \sum_{i=1}^{2} \lambda_i(t) \hat{H}_{int}^{(i)} \tag{1}$$

where $\hat{H}_S = \sum_{i=1}^{2} \hat{H}_{S_i}$, $\hat{H}_{int}^{(i)} \equiv \hat{V}_{int}^{(i)}/\sqrt{\tau}$ is the interaction Hamiltonian of S_i with R and we have scaled it by the interaction time τ for the convenience of taking continuous time limit although not necessary. The step function $\lambda_i(t)$ in Equation (1) denotes the time-dependence of the system–environment collisions and has the value 1 when $t \in [(n - 2 + i)\tau, (n - 1 + i)\tau]$ with $n \geq 1$ the number of collisions and zero otherwise. After a round of collision, the state ρ_S of the system at time t will be transformed to ρ_S' at time $t + 2\tau$ as

$$\rho_S' = \text{tr}_R \left\{ \hat{U}_2(\tau)\hat{U}_1(\tau)\rho_S \otimes \rho_R^{th} \hat{U}_1^{\dagger}(\tau)\hat{U}_2^{\dagger}(\tau) \right\}, \tag{2}$$

in which $\hat{U}_i(\tau) = e^{-i\tau(\hat{H}_{S_i} + \hat{H}_R + \hat{H}_{int}^{(i)})}$ is the unitary time evolution operator and $\rho_R^{th} = e^{-\beta_R \hat{H}_R}/Z_R$ is the initial state of the environment, which has been assumed to be prepared in the thermal state at inverse temperature β_R with $Z_R = \text{tr}\left\{ e^{-\beta_R \hat{H}_R} \right\}$ as the corresponding partition function. We set $\hbar = k_B = 1$ here and throughout the paper. By expanding $\hat{U}_i(\tau)$ to the second order of τ, we derive the master equation governing the system's dynamics as

$$
\begin{aligned}
\dot{\rho}_S &= \lim_{\tau \to 0} (\rho_S' - \rho_S)/\tau \\
&= -i[\hat{H}_S, \rho_S] + \sum_{i=1}^{2} \mathcal{L}_i(\rho_S) + \mathcal{D}_{12}(\rho_S),
\end{aligned} \tag{3}
$$

where

$$\mathcal{L}_i(\rho_S) = -\frac{1}{2}\mathrm{tr}_R\left\{\left[\hat{V}_{int}^{(i)}, \left[\hat{V}_{int}^{(i)}, \rho_S \otimes \rho_R^{th}\right]\right]\right\} \tag{4}$$

stands for the local dissipation of S_i being consistent with the situation when only S_i exists without the involvement of other subsystems, and

$$\mathcal{D}_{12}(\rho_S) = -\mathrm{tr}_R\left\{\left[\hat{V}_{int}^{(2)}, \left[\hat{V}_{int}^{(1)}, \rho_S \otimes \rho_R^{th}\right]\right]\right\} \tag{5}$$

characterizes the collective actions of the environment on the two subsystems owning to the cascaded interactions.

3. Thermodynamic Quantities of the Cascaded Model

In quantum thermodynamics, the work is generally defined as the change of energy induced by the change of the time-dependent Hamiltonian of the total system. In the CM, the successive couplings and decouplings of the system with the environment lead to the time dependence of the interaction Hamiltonian, as shown in Equation (1); therefore, the energetic cost of sustaining such successive collisions is supplied in the form of work. In a round of collisions started from t to $t + 2\tau$, the work performed on the system can be formulated as

$$\begin{aligned}
\Delta W &= \mathrm{tr}_{SR}\left\{\hat{H}_{tot}(t+2\tau)\rho'_{SR}\right\} - \mathrm{tr}_{SR}\left\{\hat{H}_{tot}(t)\rho_{SR}\right\} \\
&= \int_t^{t+2\tau} ds \frac{\partial}{\partial s}\mathrm{tr}_{SR}\left\{\hat{H}_{tot}(s)\rho_{SR}\right\} \\
&= \int_t^{t+2\tau} ds \,\mathrm{tr}_{SR}\left\{\frac{\partial \hat{H}_{tot}(s)}{\partial s}\rho_{SR}\right\} + \int_t^{t+2\tau} ds \,\mathrm{tr}_{SR}\left\{\hat{H}_{tot}(s)\frac{\partial \rho_{SR}}{\partial s}\right\},
\end{aligned} \tag{6}$$

in which $\rho_{SR} = \rho_S \otimes \rho_R^{th}$ and $\rho'_{SR} = \hat{U}_2(\tau)\hat{U}_1(\tau)\rho_{SR}\hat{U}_1^\dagger(\tau)\hat{U}_2^\dagger(\tau)$ are the total state of the system and environment at time t and $t + 2\tau$, respectively. Since $\mathrm{tr}_{SR}\left\{\hat{H}_{tot}(s)\frac{\partial \rho_{SR}}{\partial s}\right\} = 0$, the formulation (6) is further reduced to

$$\begin{aligned}
\Delta W &= \int_t^{t+2\tau}\left\langle\frac{\partial\hat{H}_{tot}(s)}{\partial s}\right\rangle_{\rho_{SR}} ds \\
&= \int_t^{t+\tau}\frac{\partial\lambda_1(s)}{\partial s}\left\langle\hat{H}_{int}^{(1)}\right\rangle_{\rho_{SR}} ds + \int_{t+\tau}^{t+2\tau}\frac{\partial\lambda_2(s)}{\partial s}\left\langle\hat{H}_{int}^{(2)}\right\rangle_{\rho_{SR}} ds \\
&\equiv \sum_{i=1}^2 \Delta W_i + \Delta W_{12},
\end{aligned} \tag{7}$$

where $\langle\cdot\rangle_\varrho \equiv \mathrm{tr}\{\cdot\varrho\}$ and the two components ΔW_i and ΔW_{12} constituting the total work ΔW can be formulated as

$$\Delta W_i = \frac{\tau^2}{2}\mathrm{tr}_{SR}\left\{\left[\hat{H}_{int}^{(i)}, \left[\hat{H}_{S_i} + \hat{H}_R, \hat{H}_{int}^{(i)}\right]\right]\rho_S \otimes \rho_R^{th}\right\} \tag{8}$$

and

$$\Delta W_{12} = \tau^2\mathrm{tr}_{SR}\left\{\left[\hat{H}_{int}^{(1)}, \left[\hat{H}_{S_2} + \hat{H}_R, \hat{H}_{int}^{(2)}\right]\right]\rho_S \otimes \rho_R^{th}\right\}. \tag{9}$$

We thus identify two types of work, i.e., the local one ΔW_i and nonlocal one ΔW_{12}, which sustain the local collisions of S_i with R and the cascaded collisions of S_1 and S_2 with R, respectively. Although ΔW_i can be formally derived by the collision model as if only S_i exists in the absence of the other one, ΔW_{12} embodies the unique one-way influence of the cascaded model.

By means of Equations (7)–(9) and after taking the continuous time limit, we obtain the current of work as

$$\dot{W} = \lim_{\tau \to 0} \frac{\Delta W}{\tau} = \sum_{i=1}^{2} \dot{W}_i + \dot{W}_{12} \tag{10}$$

with

$$\dot{W}_i = \frac{1}{2} \mathrm{tr}_{SR} \left\{ \left[\hat{V}_{int}^{(i)}, \left[\hat{H}_{S_i} + \hat{H}_R, \hat{V}_{int}^{(i)} \right] \right] \rho_S \otimes \rho_R^{th} \right\} \tag{11}$$

and

$$\dot{W}_{12} = \mathrm{tr}_{SR} \left\{ \left[\hat{V}_{int}^{(1)}, \left[\hat{H}_{S_2} + \hat{H}_R, \hat{V}_{int}^{(2)} \right] \right] \rho_S \otimes \rho_R^{th} \right\}. \tag{12}$$

The heat in a collision can be unambiguously defined as the change in the energy of environment being of the form

$$\begin{aligned} \Delta Q &= \mathrm{tr}_{SR} \left\{ \hat{H}_R \left(\rho_{SR}' - \rho_{SR} \right) \right\} \\ &= \sum_{i=1}^{2} \Delta Q_i + \Delta Q_{12} \end{aligned} \tag{13}$$

with

$$\Delta Q_i = \frac{\tau^2}{2} \mathrm{tr}_{SR} \left\{ \left[\hat{H}_{int}^{(i)}, \left[\hat{H}_R, \hat{H}_{int}^{(i)} \right] \right] \rho_S \otimes \rho_R^{th} \right\} \tag{14}$$

and

$$\Delta Q_{12} = \tau^2 \mathrm{tr}_{SR} \left\{ \left[\hat{H}_{int}^{(1)}, \left[\hat{H}_R, \hat{H}_{int}^{(2)} \right] \right] \rho_S \otimes \rho_R^{th} \right\}. \tag{15}$$

Obviously, the total heat ΔQ can be divided into local heat ΔQ_i and nonlocal heat ΔQ_{12}, which are related to the local collision of S_i with R and the nonlocal collision of S_1 and S_2 with R. Though ΔQ_{12} is generated by the collective collisions, it completely contributes to the heat of S_2 due to the unidirectional effect of the cascaded model [25].

By taking continuous time limit, the heat current can be derived as

$$\dot{Q} = \lim_{\tau \to 0} \frac{\Delta Q}{\tau} = \sum_{i=1}^{2} \dot{Q}_i + \dot{Q}_{12} \tag{16}$$

where

$$\dot{Q}_i = \frac{1}{2} \mathrm{tr}_{SR} \left\{ \left[\hat{V}_{int}^{(i)}, \left[\hat{H}_R, \hat{V}_{int}^{(i)} \right] \right] \rho_S \otimes \rho_R^{th} \right\} \tag{17}$$

and

$$\dot{Q}_{12} = \mathrm{tr}_{SR} \left\{ \left[\hat{V}_{int}^{(1)}, \left[\hat{H}_R, \hat{V}_{int}^{(2)} \right] \right] \rho_S \otimes \rho_R^{th} \right\}. \tag{18}$$

With the expression of internal energy of the system

$$\begin{aligned} \Delta U &= \mathrm{tr}_{SR} \left\{ \hat{H}_S \left(\rho_{SR}' - \rho_{SR} \right) \right\} \\ &= \frac{\tau^2}{2} \sum_{i} \mathrm{tr}_{SR} \left\{ \left[\hat{H}_{int}^{(i)}, \left[\hat{H}_{S_i}, \hat{H}_{int}^{(i)} \right] \right] \rho_S \otimes \rho_R^{th} \right\} \\ &\quad + \tau^2 \mathrm{tr}_{SR} \left\{ \left[\hat{H}_{int}^{(1)}, \left[\hat{H}_{S_2}, \hat{H}_{int}^{(2)} \right] \right] \rho_S \otimes \rho_R^{th} \right\}, \end{aligned} \tag{19}$$

we can confirm that the derived thermodynamics quantities fulfill the first law of thermodynamics, i.e., $\Delta U = \Delta W - \Delta Q$. Note that, by definition, the negative ΔQ means the heat flowing from the environment to the system.

4. Demonstration by Two-Level System

In order to demonstrate our results, we consider a fundamental configuration that both the system and ancillas are two-level systems (qubits) described by the free Hamiltonians $\hat{H}_{S_i} = \frac{\omega_i}{2} \hat{\sigma}_{S_i}^z$ and $\hat{H}_R = \frac{\omega_R}{2} \hat{\sigma}_R^z$, respectively, with ω_i (ω_R) the frequency of subsystem S_i (ancilla R). The interaction between S_i and R is given as

$$
\begin{aligned}
\hat{H}_{int}^{(i)} &= \frac{1}{\sqrt{\tau}} \hat{V}_{int}^{(i)} \\
&= \frac{1}{\sqrt{\tau}} (J_i^x \hat{\sigma}_{S_i}^x \otimes \hat{\sigma}_R^x + J_i^y \hat{\sigma}_{S_i}^y \otimes \hat{\sigma}_R^y),
\end{aligned}
\tag{20}
$$

where $\hat{\sigma}_A^z$, $\hat{\sigma}_A^x$ and $\hat{\sigma}_A^y$ are the usual Pauli operators acting on A.

The system's dynamics is governed by the master Equation (3) with the dissipation terms being of the forms

$$
\begin{aligned}
\mathcal{L}_i(\rho_S) &= (J_i^x)^2 \left(\hat{\sigma}_{S_i}^x \rho_S \hat{\sigma}_{S_i}^x - \frac{1}{2} \left[\rho_S, \hat{\sigma}_{S_i}^x \hat{\sigma}_{S_i}^x \right]_+ \right) \\
&\quad + \left(J_i^y \right)^2 \left(\hat{\sigma}_{S_i}^y \rho_S \hat{\sigma}_{S_i}^y - \frac{1}{2} \left[\rho_S, \hat{\sigma}_{S_i}^y \hat{\sigma}_{S_i}^y \right]_+ \right) \\
&\quad - i J_i^x J_i^y \langle \hat{\sigma}_R^z \rangle_{\rho_R} \left(\hat{\sigma}_{S_i}^x \rho_S \hat{\sigma}_{S_i}^y - \frac{1}{2} \left[\rho_S, \hat{\sigma}_{S_i}^y \hat{\sigma}_{S_i}^x \right]_+ \right) \\
&\quad + i J_i^y J_i^x \langle \hat{\sigma}_R^z \rangle_{\rho_R} \left(\hat{\sigma}_{S_i}^y \rho_S \hat{\sigma}_{S_i}^x - \frac{1}{2} \left[\rho_S, \hat{\sigma}_{S_i}^x \hat{\sigma}_{S_i}^y \right]_+ \right)
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\mathcal{D}_{12}(\rho_S) &= i \langle \hat{\sigma}_R^z \rangle_{\rho_R} \left(J_1^x J_2^y \left[\hat{\sigma}_{S_2}^y, \rho_S \hat{\sigma}_{S_1}^x \right] + J_1^y J_2^x \left[\hat{\sigma}_{S_1}^y, \rho_S \hat{\sigma}_{S_2}^x \right] \right) \\
&\quad - i \langle \hat{\sigma}_R^z \rangle_{\rho_R} \left(J_1^y J_2^x \left[\hat{\sigma}_{S_2}^x, \rho_S \hat{\sigma}_{S_1}^y \right] + J_1^x J_2^y \left[\hat{\sigma}_{S_1}^x, \rho_S \hat{\sigma}_{S_2}^y \right] \right) \\
&\quad + J_1^x J_2^x \left[\hat{\sigma}_{S_2}^x, \left[\rho_S, \hat{\sigma}_{S_1}^x \right] \right] + J_1^y J_2^y \left[\hat{\sigma}_{S_2}^y, \left[\rho_S, \hat{\sigma}_{S_1}^y \right] \right].
\end{aligned}
\tag{22}
$$

The currents of local and nonlocal heat are derived as

$$
\dot{Q}_i = \omega_R \left(2 J_i^x J_i^y \left\langle \sigma_{S_i}^z \right\rangle_{\rho_S} - \left((J_i^x)^2 + (J_i^y)^2 \right) \langle \sigma_R^z \rangle_{\rho_R} \right),
\tag{23}
$$

and

$$
\dot{Q}_{12} = -2 \omega_R \langle \sigma_R^z \rangle_{\rho_R} \left(J_1^x J_2^x \left\langle \sigma_{S_1}^x \sigma_{S_2}^x \right\rangle_{\rho_S} + J_1^y J_2^y \left\langle \sigma_{S_1}^y \sigma_{S_2}^y \right\rangle_{\rho_S} \right),
\tag{24}
$$

while the currents of local and nonlocal work are expressed as

$$
\begin{aligned}
\dot{W}_i &= \omega_i \left(2 J_i^x J_i^y \langle \sigma_R^z \rangle_{\rho_R} - \left((J_i^x)^2 + (J_i^y)^2 \right) \left\langle \sigma_{S_i}^z \right\rangle_{\rho_S} \right) \\
&\quad + \omega_R \left(2 J_i^x J_i^y \left\langle \sigma_{S_i}^z \right\rangle_{\rho_S} - \left((J_i^x)^2 + (J_i^y)^2 \right) \langle \sigma_R^z \rangle_{\rho_R} \right)
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
\dot{W}_{12} &= 2 J_1^x \left(\omega_2 J_2^y - \omega_R J_2^x \right) \left\langle \sigma_{S_1}^x \sigma_{S_2}^x \right\rangle_{\rho_S} \langle \sigma_R^z \rangle_{\rho_R} \\
&\quad + 2 J_1^y \left(\omega_2 J_2^x - \omega_R J_2^y \right) \left\langle \sigma_{S_1}^y \sigma_{S_2}^y \right\rangle_{\rho_S} \langle \sigma_R^z \rangle_{\rho_R}.
\end{aligned}
\tag{26}
$$

From the expressions of local currents of heat and work, i.e., $\dot{Q}_i$ in Equation (23) and $\dot{W}_i$ in (25), we can see that the prior interaction of $S_1 - R$ will exert a one-way influence on the local currents of S_2 if any, through affecting the reduced state of S_2 and eventually the term $\left\langle \sigma_{S_2}^z \right\rangle_{\rho_S}$. The condition for the disappearance of $\dot{Q}_i$ is $J_i^x = J_i^y$ and meanwhile $\omega_i \beta_i = \omega_R \beta_R$ (i.e., $\left\langle \sigma_{S_i}^z \right\rangle_{\rho_S} = \langle \sigma_R^z \rangle_{\rho_R}$) with β_i the steady-state inverse temperature of S_i. As for $\dot{W}_i$, it will vanish when the interaction of $S_i - R$ satisfies strict energy conservation, namely, $\omega_i = \omega_R$ and $J_i^x = J_i^y$. From Equations (24) and (26), we can see that the nonlocal currents of heat and

work are closely related to the establishment of correlations between S_1 and S_2 in terms of $\left\langle \sigma_{S_1}^{x(y)} \sigma_{S_2}^{x(y)} \right\rangle_{\rho_S}$. Moreover, the formulation of (26) indicates that the nonlocal work current $\dot{W}_{12}$ is bound to vanish if $\omega_2 = \omega_R$ and $J_2^x = J_2^y$, i.e., $\dot{W}_2 = 0$; however, this does not imply that $\dot{W}_{12}$ is only determined by the interaction of $S_2 - R$ since even when $\dot{W}_2 \neq 0$, a finite nonzero nonlocal work current $\dot{W}_{12}$ requires the existence of correlations of S_1 and S_2.

In the following, we shall demonstrate the behaviors of steady-state currents of heat and work in detail by considering whether the interactions of $S_1 - R$ and $S_2 - R$ satisfy energy conservation or not. A simple situation is that both interactions of $S_1 - R$ and $S_2 - R$ are energy preserving in the sense that $J_i^x = J_i^y$ and $\omega_i = \omega_R$ for $i = 1, 2$. The system will then relax towards the equilibrium stationary state (ESS) [25]

$$\rho_S(\infty) = \frac{e^{-\beta_R \hat{H}_{S_1}}}{Z_1} \otimes \frac{e^{-\beta_R \hat{H}_{S_2}}}{Z_2}, \tag{27}$$

so that both the currents of heat and work vanish in the stationary regime. Albeit the steady-state feature in this case is trivial, the memory effects on the dynamics of heat current exhibit rich phenomenons as discussed in Ref. [25].

If the interaction of $S_1 - R$ is energy preserving with $J_1^x = J_1^y$ and $\omega_1 = \omega_R$, we find that apart from the vanishing currents of work and heat regarding S_1, i.e., $\dot{W}_1 = \dot{Q}_1 = 0$, the nonlocal currents also become zero, i.e., $\dot{W}_{12} = \dot{Q}_{12} = 0$ as no correlations can be constructed in this case. Moreover, the prior interaction of $S_1 - R$ does not exert any influences on S_2 in other words, the steady-state currents of S_2 are not influenced by S_1 manifesting behaviors as if the interaction of $S_1 - R$ does not exist; therefore, only the energy conservation of $S_1 - R$ does not hold, as it has an impact on S_2, which is to be discussed in the following.

4.1. The Interaction of $S_2 - R$ Is Energy-Preserving

We first consider the situation that the interaction of $S_1 - R$ is not energy-conserving, while the interaction of $S_2 - R$ still satisfies strict energy conservation with $J_2^x = J_2^y$ and $\omega_2 = \omega_R$. In this case, the steady-state correlation between S_1 and S_2 can be established, which makes S_2 fail to reach ESS although the interaction of $S_2 - R$ is energy-conserving. Eventually, both S_1 and S_2 reach NESS with effective inverse temperatures β_{eff1} and β_{eff2} that could deviate from β_R of the environment to different extents. The effective temperature of S_i can be defined as $T_{effi} = 1/\beta_{effi} = \omega_i / \ln(p_i^g / p_i^e)$ with $p_i^g (p_i^e)$ being the stationary population of the ground (excited) state of the subsystem S_i. As shown in Equations (25) and (26), the strict energy conservation of the interaction of $S_2 - R$ means $\dot{W}_2 = \dot{W}_{12} = 0$, namely, no work is provided through the interaction of $S_2 - R$; therefore, the NESS of S_2, alternatively speaking, the total NESS of S_1 and S_2 is sustained by the work invested via S_1. We also note that in this case $\dot{Q}_2 \neq 0$; nevertheless, the nonlocal heat current $\dot{Q}_{12} = -\dot{Q}_2$ so that the total heat current associated with S_2 stays zero, being consistent with the vanishing $\dot{W}_2$ and $\dot{W}_{12}$.

In Figure 2a,b, we illustrate deviations of effective inverse temperatures of S_1 and S_2 from that of the environment in terms of the ratios β_{eff1}/β_R and β_{eff2}/β_R. A comparison between β_{eff1}/β_R and β_{eff2}/β_R in Figure 2a,b shows that S_1 can reach ESS with $\beta_{eff1} = \beta_R$ only at the point of $J_1^x = J_1^y$ and the negative temperatures are achieved when $J_1^y/J_1^x < 0$, whereas S_2 can arrive at ESS with $\beta_{eff2} = \beta_R$ at the two points of $J_1^x = \pm J_1^y$. The nonconservation of energy for $S_1 - R$ induces steady-state correlations between S_1 and S_2, which is quantified by the l_1-norm of coherence defined as $C = \sum_{l \neq m} |\rho_{lm}|$ with ρ_{lm} the matrix elements of density operator ρ [80]. Apart from the effective temperature, the existence of correlation is also a signature of the system reaching NESS. The coherence of S_1 and S_2 as a function of J_1^y/J_1^x is demonstrated in Figure 2c. By comparing Figure 2a, 2b, and 2c, we observe that the farther the individual subsystem S_i ($i = 1, 2$) deviates from equilibrium, i.e., the smaller the ratios $|\beta_{effi}/\beta| < 1$, the larger the coherence. We also note

that the lower the temperature of environment, the larger the coherence. The NESS of the total system, which is characterized now by both the effective temperatures of individual subsystems and the coherence, should be sustained by the work current supplied by an external agent. In Figure 2d, our displays of the work current show that the work cost is consistent with the extent of the system deviating from the ESS, namely, the smaller the ratio $|\beta_{effi}/\beta_R|$ and the larger the coherence, the more the work current.
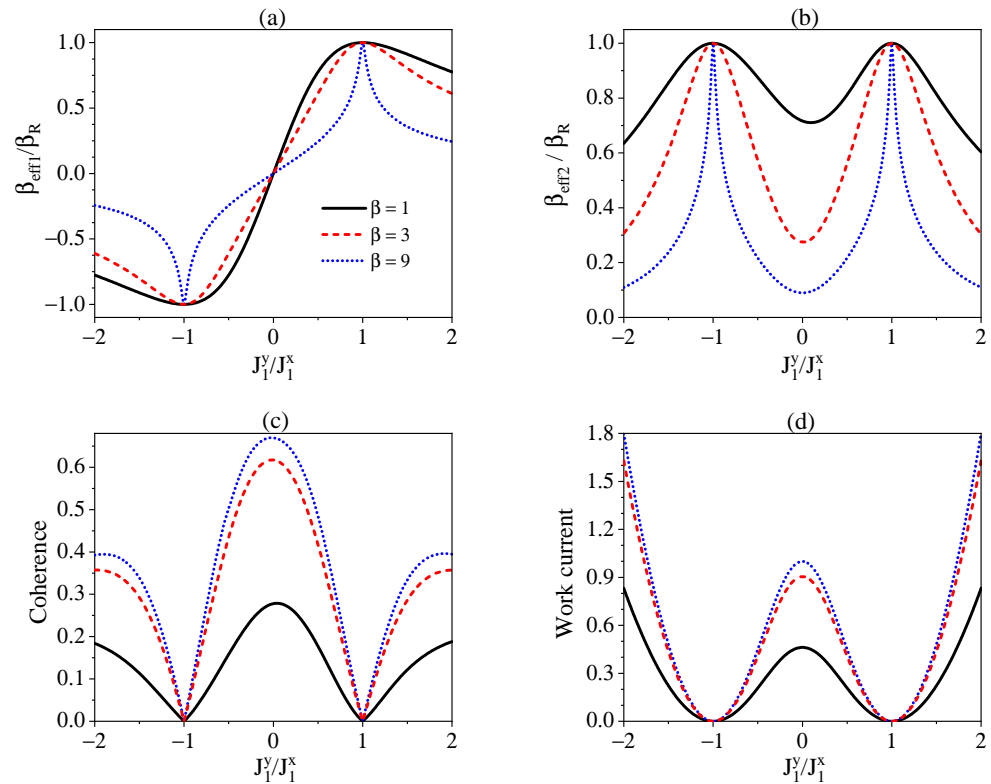

Figure 2. The ratios of effective inverse temperatures β_{eff1} and β_{eff2} of S_1 (**a**) and S_2 (**b**) to that of environment β_R, the coherence of the system (**c**), and the work current (**d**) as a function of J_1^y/J_1^x for different β_R. We set $J_2^y = J_2^x = \omega$ and $\omega_1 = \omega_2 = \omega_R = \omega$.

4.2. Both the Interactions of $S_1 - R$ and $S_2 - R$ Are Not Energy-Preserving

We finally consider the scenario where both the interactions of $S_1 - R$ and $S_2 - R$ are not energy-preserving, for which the nonlocal steady-state currents of work and heat might appear as indicated in Equations (24) and (26). Here, we are interested in the one-way influences of the interaction of $S_1 - R$ on the steady-state currents of S_2. For this purpose, we demonstrate in Figure 3a the local and nonlocal currents regarding both S_1 and S_2 against the interaction of $S_1 - R$ in terms of J_1^y/J_1^x. We can clearly observe the variations of $\dot{W}_2$ and $\dot{W}_{12}$ with respect to J_1^y/J_1^x implying influences of the interaction of $S_1 - R$ on the work currents of S_2. Moreover, the nonlocal current $\dot{W}_{12}$ can be transformed between positive and negative values, which means that the interaction of $S_1 - R$ is able to control the direction of $\dot{W}_{12}$. The currents of heat exhibit similar behaviors to that of the work, which is not shown here. We also note that the symmetric interactions of $S_1 - R$ and $S_2 - R$ with $J_1^x = J_2^x = J^x$ and $J_1^y = J_2^y = J^y$ can lead to $\dot{W}_1 = \dot{W}_2$ and $\dot{Q}_1 = \dot{Q}_2$, as demonstrate in Figure 3b. This implies that the local currents of work and heat for the symmetric interactions are the same as that would be obtained in the independent interactions for each subsystem with the environment. Since in this case the nonlocal current of work $\dot{W}_{12}$ always take opposite directions to the local ones, the total cost of work in the presence of cascaded interactions, i.e., $\dot{W}_{tot}^{cas} = \dot{W}_1 + \dot{W}_2 + \dot{W}_{12}$, are always less than that with only independent interactions, i.e., $\dot{W}_{tot}^{ind} = \dot{W}_1 + \dot{W}_2$, in the sense that $\dot{W}_{tot}^{cas} < \dot{W}_{tot}^{ind}$.
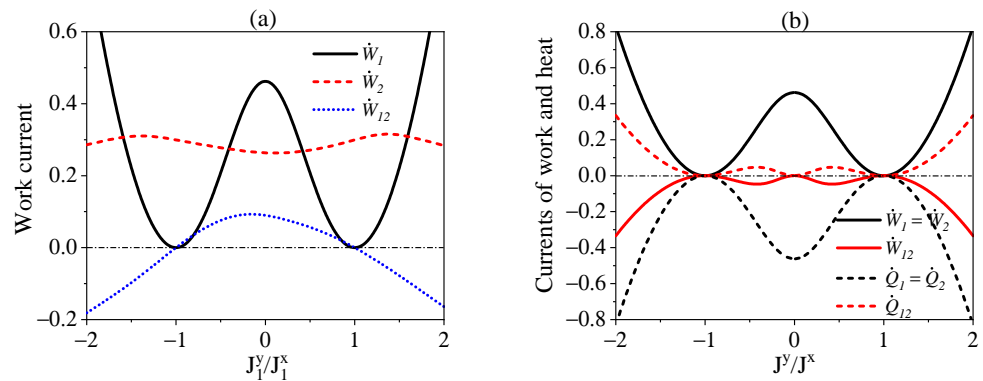
Figure 3. (**a**) The work currents against J_1^y / J_1^x for $J_2^x = 0.6\omega$, $J_2^y = 1.2\omega$. (**b**) The currents of heat and work against J^y / J^x for the symmetric couplings of $S_1 - R$ and $S_2 - R$ with $J_2^x = J_1^x = J^x$, $J_2^y = J_1^y = J^y$. We set $\omega_1 = \omega_2 = \omega_R = \omega$ and $\beta_R = \omega$.

5. The Extractable Work in Cascaded Model

From above discussions, we recognize that steady-state coherence of the system can be available due to cascaded interactions of subsystems with the common environment, exhibiting striking contrast to the situation of independent interactions where no any coherence can be generated; therefore, it is interesting to study the role of coherence in extracting useful work from the system. The maximum work that can be extracted from a quantum system via cyclic and unitary operations is quantified by the so-called ergotropy [81], which is given as the difference between the energy of the initial state and that of the final state with the minimum average energy through all possible unitary operations. For a quantum system described by Hamiltonian $\hat{H} = \sum_{k=1}^{d} \varepsilon_k |\varepsilon_k\rangle \langle \varepsilon_k|$ and density operator $\rho = \sum_{l=1}^{d} r_l |r_l\rangle \langle r_l|$ such that $\varepsilon_k \leq \varepsilon_{k+1}$ and $r_l \geq r_{l+1}$, the ergotropy can be defined as

$$\mathcal{E}(\rho) = \mathrm{Tr}\{\hat{H}\rho\} - \mathrm{Tr}\left\{\hat{H}\hat{U}_{min}\rho\hat{U}_{min}^{\dagger}\right\} = \mathrm{Tr}\left\{\hat{H}(\rho - P_\rho)\right\} \tag{28}$$

where $P_\rho = \sum_l r_l |\varepsilon_l\rangle \langle \varepsilon_l|$ is called the passive state. By plugging the explicit form of P_ρ in (28), we obtain the well-known expression of ergotropy as

$$\mathcal{E}(\rho) = \sum_{k,l} r_l \varepsilon_k \left(|\langle r_l | \varepsilon_k \rangle|^2 - \delta_{kl} \right). \tag{29}$$

Here, we make a comparison for the ergotropy in two configurations, namely, the cascaded interactions and the independent interactions of S_1 and S_2 with R. For this purpose, we consider the symmetric interactions of S_1 and S_2 with R with $J_1^x = J_2^x = J^x$ and $J_1^y = J_2^y = J^y$. Moreover, for the independent interactions, we define the total ergotropy as $\mathcal{E}^{ind}(\rho) = \mathcal{E}(\rho_{S_1}) + \mathcal{E}(\rho_{S_2})$ with ρ_{S_i} ($i = 1, 2$) the state of S_i. The behavior of ergotropy is illustrated in Figure 4a for different temperatures of the environment. We observe that at the region of $J^y / J^x < 0$ a finite nonzero ergotropy appears for both cascaded and independent interactions of $S_1 - R$ and $S_2 - R$ with magnitude of the former case is always larger than that of the latter case. By contrast, for the region of $J^y / J^x > 0$, the ergotropy retains a nonzero value only for the cascaded interactions. The results can be understood by recalling that the ergotropy under independent interactions is contributed totally by the population inversions of the subsystems with negative effective temperatures, which can occur only in the region of $J^y / J^x < 0$ (cf. Figure 2a). By contrast, the ergotropy under cascaded interactions is contributed by both the population inversions of individual subsystems and the coherence of the total system. As a result, the ergotropy of cascaded interactions in the region of $J^y / J^x < 0$ is always larger than that of independent interactions and arises also in the region of $J^y / J^x > 0$ where the correlations can be established. To have a visualized picture, we plot the coherence of system under the cascaded interaction in Figure 4b. A comparison between Figure 4a and Figure 4b actually confirms our explanations according

to the following observations. At the point of $J^y/J^x = -1$, the coherence becomes zero so that the ergotropy in cascaded and independent interactions coincide with each other. In the region of $J^y/J^x > 0$, the ergotropy is completely contributed by the coherence and the larger the coherence with the lower temperature of the environment, the larger the ergotropy. At the point of $J^y/J^x = 1$, the zero coherence leads to vanishing ergotropy. Moreover, the larger the coherence-based ergotropy, the greater the difference for the ergotropy in these two configurations, as shown in the region of $J^y/J^x < 0$.
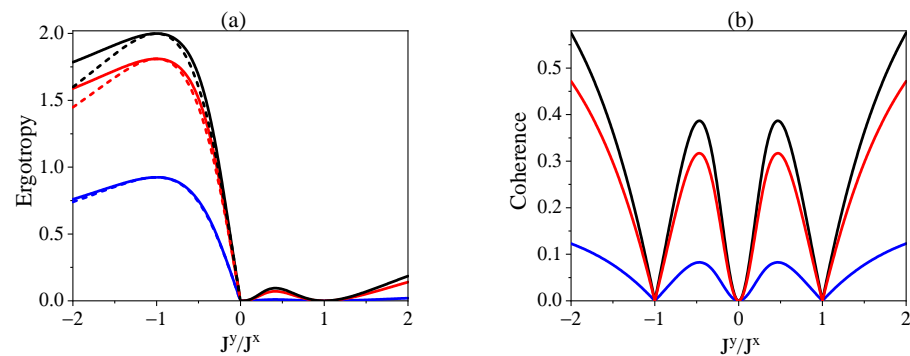

Figure 4. (**a**) The ergotropy for cascaded interactions (solid lines) and independent interactions (dashed lines) of S_1 and S_2 with R against J^y/J^x. (**b**) The corresponding coherence of the system under cascaded interactions. We set symmetric couplings of $S_1 - R$ and $S_2 - R$ with $J_1^x = J_2^x = J^x$ and $J_1^y = J_2^y = J^y$. The other parameters are set as $\beta_R = 9\omega$ (black lines), $\beta_R = 3\omega$ (red lines), $\beta_R = \omega$ (blue lines), and $\omega_1 = \omega_2 = \omega_R = \omega$.

6. Conclusions

In conclusion, by virtue of collision model, we have studied the thermodynamics of a bipartite system with subsystems S_1 and S_2 interacting with an environment R in the cascaded fashion, namely, the environment R collides/interacts firstly with S_1 and subsequently with S_2. Thanks to the framework of the collision model, we have constructed the general forms of work and heat of the system in both discrete steps and continuous time limit from their most fundamental definitions. The constructed formulations allow us to identify the local and nonlocal components of the work and heat and discuss their features due to the cascaded interactions. Focusing on the two-qubit system and concrete form of system–environment interactions, we have demonstrated our results and revealed the necessary condition for the unidirectional influence of the prior interaction of $S_1 - R$ to the thermodynamics of S_2 in the steady-state regime. It turns out that the one-way influence occurs only when the interaction of $S_1 - R$ does not fulfill energy conservation. With the influence of prior interaction, subsystem S_2 cannot arrive at ESS even when the interaction of $S_2 - R$ satisfies strict energy conservation. In case the interaction of $S_2 - R$ does not satisfy energy conservation at the same time, the interaction of $S_1 - R$ will have an impact on both the local work regarding S_2 and nonlocal work. We also show that the steady-state coherence generated by the cascaded interaction is a useful resource in extracting work in terms of ergotropy from the system. Our results thus reveal the unique thermodynamics features in the cascaded model and particularly provide a possible way to achieve the one-way control on the thermodynamics process in the steady-state regime.

Author Contributions: Conceptualization, Z.-X.M. and Y.-J.X.; methodology, L.L.; software, L.L.; writing—original draft preparation, L.L.; writing—review and editing, Z.-X.M.; supervision, Z.-X.M. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gemma, G.; Michel, M.; Mahler, G. *Quantum Thermodynamics*; Springer: Berlin/Heidelberg, Germany, 2004.
2. Kosloff, R. Quantum Thermodynamics: A Dynamical Viewpoint. *Entropy* **2013**, *15*, 2100. [CrossRef]
3. Breuer, H.; Petruccione, F. *The Theory of Open Quantum Systems*; Oxford University Press: Oxford, NY, USA, 2002.
4. Scully, M.O.; Zubairy, M.S.; Agarwal, G.S.; Walther, H. Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence. *Science* **2003**, *299*, 862. [CrossRef]
5. Türkpençe, D.; Müstecaplıoğlu, O.E. Quantum fuel with multilevel atomic coherence for ultrahigh specific work in a photonic Carnot engine. *Phys. Rev. E* **2016**, *93*, 012145. [CrossRef] [PubMed]
6. Shi, Y.H.; Shi, H.L.; Wang, X.H.; Hu, M.L.; Liu, S.Y.; Yang, W.L.; Fan, H. Quantum coherence in a quantum heat engine. *J. Phys. A* **2020**, *53*, 085301. [CrossRef]
7. Mehta, V.; Johal, R.S. Quantum Otto engine with exchange coupling in the presence of level degeneracy. *Phys. Rev. E* **2017**, *96*, 032110. [CrossRef]
8. Dillenschneider, R.; Lutz, E. Energetics of quantum correlations. *EPL (Europhys. Lett.)* **2009**, *88*, 50003. [CrossRef]
9. Brandner, K.; Bauer, M.; Schmid, M.T.; Seifert, U. Coherence-enhanced efficiency of feedback-driven quantum engines. *New J. Phys.* **2015**, *17*, 065006. [CrossRef]
10. Brunner, N.; Huber, M.; Linden, N.; Popescu, S.; Silva, R.; Skrzypczyk, P. Entanglement enhances cooling in microscopic quantum refrigerators. *Phys. Rev. E* **2014**, *89*, 032115. [CrossRef]
11. Niedenzu, W.; Gelbwaser-Klimovsky, D.; Kofman, A.G.; Kurizki, G. On the operation of machines powered by quantum non-thermal baths. *New J. Phys.* **2016**, *18*, 083012. [CrossRef]
12. Uzdin, R. Coherence-Induced Reversibility and Collective Operation of Quantum Heat Machines via Coherence Recycling. *Phys. Rev. Appl.* **2016**, *6*, 024004. [CrossRef]
13. Doyeux, P.; Leggio, B.; Messina, R.; Antezza, M. Quantum thermal machine acting on a many-body quantum system: Role of correlations in thermodynamic tasks. *Phys. Rev. E* **2016**, *93*, 022134. [CrossRef] [PubMed]
14. Latune, C.; Sinayskiy, I.; Petruccione, F. Quantum coherence, many-body correlations, and non-thermal effects for autonomous thermal machines. *Sci. Rep.* **2019**, *9*, 3191. [CrossRef]
15. Åberg, J. Catalytic Coherence. *Phys. Rev. Lett.* **2014**, *113*, 150402. [CrossRef] [PubMed]
16. Korzekwa, K.; Lostaglio, M.; Oppenheim, J.; Jennings, D. The extraction of work from quantum coherence. *New J. Phys.* **2016**, *18*, 023045. [CrossRef]
17. Li, H.; Zou, J.; Yu, W.L.; Xu, B.M.; Li, J.G.; Shao, B. Quantum coherence rather than quantum correlations reflect the effects of a reservoir on a system's work capability. *Phys. Rev. E* **2014**, *89*, 052132. [CrossRef]
18. Liao, J.Q.; Dong, H.; Sun, C.P. Single-particle machine for quantum thermalization. *Phys. Rev. A* **2010**, *81*, 052121. [CrossRef]
19. Manatuly, A.; Niedenzu, W.; Román-Ancheyta, R.; Çakmak, B.; Müstecaplıoğlu, O.E.; Kurizki, G. Collectively enhanced thermalization via multiqubit collisions. *Phys. Rev. E* **2019**, *99*, 042145. [CrossRef] [PubMed]
20. Dağ, C.B.; Niedenzu, W.; Ozaydin, F.; Müstecaplıoğlu, O.E.; Kurizki, G. Temperature Control in Dissipative Cavities by Entangled Dimers. *J. Phys. Chem. C* **2019**, *123*, 4035. [CrossRef]
21. Karimi, B.; Pekola, J.P. Correlated versus uncorrelated noise acting on a quantum refrigerator. *Phys. Rev. B* **2017**, *96*, 115408. [CrossRef]
22. Manzano, G.; Giorgi, G.L.; Fazio, R.; Zambrini, R. Boosting the performance of small autonomous refrigerators via common environmental effects. *New J. Phys.* **2019**, *21*, 123026. [CrossRef]
23. Wang, C.; Xu, D.; Liu, H.; Gao, X. Thermal rectification and heat amplification in a nonequilibrium V-type three-level system. *Phys. Rev. E* **2019**, *99*, 042102. [CrossRef]
24. Man, Z.X.; Tavakoli, A.; Brask, J.B.; Xia, Y.J. Improving autonomous thermal entanglement generation using a common reservoir. *Phys. Scr.* **2019**, *94*, 075101. [CrossRef]
25. Lorenzo, S.; Farace, A.; Ciccarello, F.; Palma, G.M.; Giovannetti, V. Heat flux and quantum correlations in dissipative cascaded systems. *Phys. Rev. A* **2015**, *91*, 022121. [CrossRef]
26. Giovannetti, V.; Palma, G.M. Master Equations for Correlated Quantum Channels. *Phys. Rev. Lett.* **2012**, *108*, 040401. [CrossRef]
27. Giovannetti, V.; Palma, G.M. Master equation for cascade quantum channels: A collisional approach. *J. Phys. B* **2012**, *45*, 154003. [CrossRef]
28. Lorenzo, S.; McCloskey, R.; Ciccarello, F.; Paternostro, M.; Palma, G.M. Landauer's Principle in Multipartite Open Quantum System Dynamics. *Phys. Rev. Lett.* **2015**, *115*, 120403. [CrossRef]
29. Man, Z.X.; Xia, Y.J.; An, N.B. Heat fluxes in a two-qubit cascaded system due to coherences of a non-thermal bath. *J. Phys. B* **2020**, *53*, 205505. [CrossRef]

30. Stockburger, J.T.; Motz, T. Thermodynamic deficiencies of some simple lindblad operators. *Fortschr. Phys.* **2016**, *65*, 1600067. [CrossRef]

31. Kołodyński, J.; Brask, J.B.; Perarnau-Llobet, M.; Bylicka, B. Adding dynamical generators in quantum master equations. *Phys. Rev. A* **2018**, *97*, 062124. [CrossRef]

32. Correa, L.A.; Palao, J.P.; Alonso, D. Internal dissipation and heat leaks in quantum thermodynamic cycles. *Phys. Rev. E* **2015**, *92*, 032136. [CrossRef]

33. Hofer, P.P.; Perarnau-Llobet, M.; Miranda, L.D.M.; Haack, G.; Silva, R.; Brask, J.B.; Brunner, N. Markovian master equations for quantum thermal machines: local versus global approach. *New J. Phys.* **2017**, *19*, 123037. [CrossRef]

34. Mitchison, M.T.; Plenio, M.B. Non-additive dissipation in open quantum networks out of equilibrium. *New J. Phys.* **2018**, *20*, 033005. [CrossRef]

35. Naseem, M.T.; Xuereb, A.; Müstecaplıoğlu, O.E. Thermodynamic consistency of the optomechanical master equation. *Phys. Rev. A* **2018**, *98*, 052123. [CrossRef]

36. Purkayastha, A.; Dhar, A.; Kulkarni, M. Out-of-equilibrium open quantum systems: A comparison of approximate quantum master equation approaches with exact results. *Phys. Rev. A* **2016**, *93*, 062114. [CrossRef]

37. Wichterich, H.; Henrich, M.J.; Breuer, H.P.; Gemmer, J.; Michel, M. Modeling heat transport through completely positive maps. *Phys. Rev. E* **2007**, *76*, 031115. [CrossRef]

38. Cattaneo, M.; Giorgi, G.L.; Maniscalco, S.; Zambrini, R. Local versus global master equation with common and separate baths: Superiority of the global approach in partial secular approximation. *New J. Phys.* **2019**, *21*, 113045. [CrossRef]

39. Scali, S.; Anders, J.; Correa, L.A. Local master equations bypass the secular approximation. *arXiv* **2009**, arXiv:2009.11324.

40. Hewgill, A.; De Chiara, G.; Imparato, A. Quantum thermodynamically consistent local master equations. *Phys. Rev. Res.* **2021**, *3*, 013165. [CrossRef]

41. Levy, A.; Kosloff, R. The local approach to quantum transport may violate the second law of thermodynamics. *EPL (Europhys. Lett.)* **2014**, *107*, 20004. [CrossRef]

42. Rau, J. Relaxation Phenomena in Spin and Harmonic Oscillator Systems. *Phys. Rev.* **1963**, *129*, 1880. [CrossRef]

43. Cattaneo, M.; De Chiara, G.; Maniscalco, S.; Zambrini, R.; Giorgi, G.L. Collision Models Can Efficiently Simulate Any Multipartite Markovian Quantum Dynamics. *Phys. Rev. Lett.* **2021**, *126*, 130403. [CrossRef]

44. Scarani, V.; Ziman, M.; Stelmachovic, P.; Gisin, N.; Bužek, V. Thermalizing Quantum Machines: Dissipation and Entanglement. *Phys. Rev. Lett.* **2002**, *88*, 097905. [CrossRef] [PubMed]

45. Ziman, M.; Stelmachovic, P.; Bužek, V.; Hillery, M.; Scarani, V.; Gisin, N. Diluting quantum information: An analysis of information transfer in system-reservoir interactions. *Phys. Rev. A* **2002**, *65*, 042105. [CrossRef]

46. Ziman, M.; Bužek, V. All (qubit) decoherences: Complete characterization and physical implementation. *Phys. Rev. A* **2005**, *72*, 022110. [CrossRef]

47. Benenti, G.; Palma, G.M. Reversible and irreversible dynamics of a qubit interacting with a small environment. *Phys. Rev. A* **2007**, *75*, 052110. [CrossRef]

48. Gennaro, G.; Benenti, G.; Palma, G.M. Relaxation due to random collisions with a many-qudit environment. *Phys. Rev. A* **2009**, *79*, 022105. [CrossRef]

49. Campbell, S.; Vacchini, B. Collision models in open system dynamics: A versatile tool for deeper insights? *EPL* **2021**, *133*, 60001. [CrossRef]

50. Ciccarello, F.; Lorenzo, S.; Giovannetti, V.; Palma, G.M. Quantum collision models: Open system dynamics from repeated interactions. *Phys. Rep.* **2022**, *954*, 1–70. [CrossRef]

51. Ciccarello, F.; Palma, G.M.; Giovannetti, V. Collision-model-based approach to non-Markovian quantum dynamics. *Phys. Rev. A* **2013**, *87*, 040103. [CrossRef]

52. Kretschmer, S.; Luoma, K.; Strunz, W.T. Collision model for non-Markovian quantum dynamics. *Phys. Rev. A* **2016**, *94*, 012106. [CrossRef]

53. Lorenzo, S.; Ciccarello, F.; Palma, G.M. Class of exact memory-kernel master equations. *Phys. Rev. A* **2016**, *93*, 052111. [CrossRef]

54. Bernardes, N.K.; Carvalho, A.R.R.; Monken, C.H.; Santos, M.F. Coarse graining a non-Markovian collisional model. *Phys. Rev. A* **2017**, *95*, 032117. [CrossRef]

55. Çakmak, B.; Pezzutto, M.; Paternostro, M.; Müstecaplıoğlu, O.E. Non-Markovianity, coherence, and system-environment correlations in a long-range collision model. *Phys. Rev. A* **2017**, *96*, 022109. [CrossRef]

56. Lorenzo, S.; Ciccarello, F.; Palma, G.M. Composite quantum collision models. *Phys. Rev. A* **2017**, *96*, 032107. [CrossRef]

57. Filippov, S.N.; Piilo, J.; Maniscalco, S.; Ziman, M. Divisibility of quantum dynamical maps and collision models. *Phys. Rev. A* **2017**, *96*, 032111. [CrossRef]

58. Mccloskey, R.; Paternostro, M. Non-Markovianity and system-environment correlations in a microscopic collision model. *Phys. Rev. A* **2014**, *89*, 052120. [CrossRef]

59. Bernardes, N.K.; Carvalho, A.R.R.; Monken, C.H.; Santos, M.F. Environmental correlations and Markovian to non-Markovian transitions in collisional models. *Phys. Rev. A* **2014**, *90*, 032111. [CrossRef]

60. Jin, J.; Yu, C.S. Non-Markovianity in the collision model with environmental block. *New J. Phys.* **2018**, *20*, 053026. [CrossRef]

61. Man, Z.X.; Xia, Y.J.; Lo Franco, R. Temperature effects on quantum non-Markovianity via collision models. *Phys. Rev. A* **2018**, *97*, 062104. [CrossRef]

62. Barra, F. The thermodynamic cost of driving quantum systems by their boundaries. *Sci. Rep.* **2015**, *5*, 14873. [CrossRef]
63. Chiara, G.D.; Landi, G.; Hewgill, A.; Reid, B.; Ferraro, A.; Roncaglia, A.J.; Antezza, M. Reconciliation of quantum local master equations with thermodynamics. *New J. Phys.* **2018**, *20*, 113024. [CrossRef]
64. Karevski, D.; Platini, T. Quantum Nonequilibrium Steady States Induced by Repeated Interactions. *Phys. Rev. Lett.* **2009**, *102*, 207207. [CrossRef]
65. Barra, F.; Lledó, C. Stochastic thermodynamics of quantum maps with and without equilibrium. *Phys. Rev. E* **2017**, *96*, 052114. [CrossRef]
66. Cusumano, S.; Cavina, V.; Keck, M.; De Pasquale, A.; Giovannetti, V. Entropy production and asymptotic factorization via thermalization: A collisional model approach. *Phys. Rev. A* **2018**, *98*, 032119. [CrossRef]
67. Seah, S.; Nimmrichter, S.; Scarani, V. Nonequilibrium dynamics with finite-time repeated interactions. *Phys. Rev. E* **2019**, *99*, 042103. [CrossRef]
68. Arısoy, O.; Campbell, S.; Müstecaplıoğlu, O.E. Thermalization of Finite Many-Body Systems by a Collision Model. *Entropy* **2019**, *21*, 1182. [CrossRef]
69. Landi, G.T.; Novais, E.; de Oliveira, M.J.; Karevski, D. Flux rectification in the quantum *XXZ* chain. *Phys. Rev. E* **2014**, *90*, 042142. [CrossRef]
70. Seah, S.; Nimmrichter, S.; Grimmer, D.; Santos, J.P.; Scarani, V.; Landi, G.T. Collisional Quantum Thermometry. *Phys. Rev. Lett.* **2019**, *123*, 180602. [CrossRef]
71. Barra, F. Dissipative Charging of a Quantum Battery. *Phys. Rev. Lett.* **2019**, *122*, 210601. [CrossRef]
72. Dağ, C.B.; Niedenzu, W.; Müstecaplıoğlu, O.E.; Kurizki, G. Multiatom Quantum Coherences in Micromasers as Fuel for Thermal and Nonthermal Machines. *Entropy* **2016**, *18*, 244. [CrossRef]
73. Hewgill, A.; González, J.O.; Palao, J.P.; Alonso, D.; Ferraro, A.; De Chiara, G. Three-qubit refrigerator with two-body interactions. *Phys. Rev. E* **2020**, *101*, 012109. [CrossRef] [PubMed]
74. De Chiara, G.; Antezza, M. Quantum machines powered by correlated baths. *Phys. Rev. Res.* **2020**, *2*, 033315. [CrossRef]
75. Pezzutto, M.; Paternostro, M.; Omar, Y. Implications of non-Markovian quantum dynamics for the Landauer bound. *New J. Phys.* **2016**, *18*, 123018. [CrossRef]
76. Man, Z.X.; Xia, Y.J.; Lo Franco, R. Validity of the Landauer principle and quantum memory effects via collisional models. *Phys. Rev. A* **2019**, *99*, 042106. [CrossRef]
77. Zhang, Q.; Man, Z.X.; Xia, Y.J. Non-Markovianity and the Landauer principle in composite thermal environments. *Phys. Rev. A* **2021**, *103*, 032201. [CrossRef]
78. Pezzutto, M.; Paternostro, M.; Omar, Y. An out-of-equilibrium non-Markovian quantum heat engine. *Quantum Sci. Technol.* **2019**, *4*, 025002. [CrossRef]
79. Abah, O.; Paternostro, M. Implications of non-Markovian dynamics on information-driven engine. *J. Phys. Commun.* **2020**, *4*, 085016. [CrossRef]
80. Baumgratz, T.; Cramer, M.; Plenio, M.B. Quantifying Coherence. *Phys. Rev. Lett.* **2014**, *113*, 140401. [CrossRef]
81. Allahverdyan, A.E.; Balian, R.; Nieuwenhuizen, T.M. Maximal work extraction from finite quantum systems. *EPL (Europhys. Lett.)* **2004**, *67*, 565. [CrossRef]