Research Article

Autism & Developmental Language Impairments
Volume 6: 1–14

DOI: 10.1177/23969415211045324
journals.sagepub.com/home/dli
SAGE

Teaching addition strategies to students with learning difficulties

Irene Polo-Blanco and Eva M González López
Universidad de Cantabria, Santander, Spain

Abstract

Background & aims: In recent years, there has been an increased interest in analyzing the mathematical performance of students with learning difficulties in order to provide them with teaching methods adapted to their needs. In particular, the importance of studying the type of informal strategy that students use when solving problems has been highlighted. Observing how these strategies emerge and develop in children with learning difficulties is crucial, as it allows us to understand how they develop a subsequent understanding of arithmetic operations. In this paper we study the effect of explicit instruction in addition strategies, focusing on the minimum addend strategy, and analyze the difficulties that arise during this process.
Methods: An adapted multiple-probe design across students with a microgenetic approach was employed to assess the effectiveness of the teaching instruction and the acquisition of the minimum addend strategy while solving addition word problems. The participants were three primary-school children (two boys and one girl) with learning difficulties, one of them diagnosed with autism spectrum disorder. The instruction on the minimum addend strategy was sequenced into levels of abstraction based on the addends represented with and without manipulatives.
Results: The results show that the three participants were able to acquire the minimum addend strategy and transfer it to two-step problems. They all showed difficulties during the instructional process, with quantity comparison difficulties predominating. The instruction provided to address these and other difficulties is detailed for each participant.
Conclusions: The teaching of the minimum addend strategy has proven effective, and all three students acquired it throughout the instruction. The results concerning the student diagnosed with autism spectrum disorder are especially interesting given the lack of studies that focus on the strategies employed by students with this disorder to solve arithmetic problems. In this sense, the use of the microgenetic approach was especially useful to observe the type of spontaneous strategies used by this participant, and how they varied in response to the instruction.
Implications: Each study participant faced different difficulties and needed different periods of time to assimilate the new strategy. Conclusions are drawn for educators to help children with learning difficulties advance to more sophisticated strategies, so they can acquire these and subsequent mathematical concepts.

Keywords

Mathematics, learning difficulties, microgenetic, problem solving, strategies.

Research suggests that at least 12% of school-age students exhibit difficulties over the course of learning mathematics (Geary, 2013). It is important to work from an early age to overcome difficulties in learning mathematical concepts, as many of these difficulties can persist also in the face of new knowledge (Montague, 2007). Studies that have evaluated the strategies manifested by students with learning difficulties (LD) show that these are often less varied and effective than those of their typically-performing peers (Geary, 1990). This means that they could benefit from specific instruction to acquire these strategies (Siegler, 1988; Timmermans & Van Lieshout, 2003). Despite this, there is a lack of research on the effect of explicitly teaching advanced strategies to students with LD. For this reason, we propose a study with primary-school students with LD in order to help them develop effective addition strategies.

Corresponding author:
Irene Polo-Blanco, Departamento Matemáticas Estadística y Computación, Facultad de Ciencias, Universidad de Cantabria, Avda de los Castros s/n, 39005, Santander, Spain.
Email: irene.polo@unican.es

Specifically, we first identify the participant's solution strategies when solving addition word problems independently (spontaneous strategies), and then provide instruction aimed at acquiring the minimum addend strategy.

Literature review

Various researchers have shown that, from a very early age, young children exhibit informal mathematical knowledge that will develop over time (Geary, 1994). For example, they are able to acquire the concepts of cardinality and/or ordinality (Wynn, 1990) and anticipate correct solutions to addition and subtraction. This informal mathematics is an intermediate step between intuitive and formal mathematics (Baroody & Tiilikainen, 2003). Some researchers have studied the relationship between some numerical abilities and informal mathematical knowledge. For example, Libertus et al. (2013) showed that children's approximation abilities correlated with and predicted informal, but not formal, mathematics abilities.

Accordingly, researchers have delved into the skills of children involving non-symbolic, approximate arithmetic, and how they relate to subsequent mathematical learning (Gilmore et al., 2007, 2010). For example, Gilmore et al. (2007) carried out a study involving children who had mastered verbal counting but had not yet been taught symbolic arithmetic. They showed that children were able to solve problems of symbolic approximate arithmetic without resorting to guessing or comparison strategies. Their findings suggest that children retrieve their non-symbolic number knowledge when they solve new approximate symbolic arithmetic problems.

Various studies in the literature have analyzed the type of informal strategy that typically-performing students manifest when solving basic arithmetic operations. Observing how these strategies emerge and develop in children is crucial, as it allows us to understand how they develop a subsequent understanding of arithmetic operations. Following this line of research, Cookson and Moser (1980) conducted a longitudinal study of strategies with children from Grade 1 to Grade 3 when solving addition and subtraction problems. At the beginning of the study, the students primarily exhibited the counting all strategy, and to a lesser extent the first addend or max strategy (counting up from the first addend). Over time, the use of these strategies decreased as the use of the minimum addend strategy increased (counting up from the larger addend), until they eventually acquired number facts (Cookson & Moser, 1980).

Teaching addition facts to children with learning difficulties

In the case of students with LD, the study of these informal strategies is especially relevant because it provides information that will allow for adequate interventions (Geary, 1990). Research in this area shows that students with LD often use fewer and less effective strategies than their typically-performing peers when solving different mathematical reasoning tasks, which may make it difficult to formally learn arithmetic operations and other mathematical concepts later (Geary, 1990; Geary et al., 2004; Siegler, 2007).

The most effective informal addition strategy is the minimum addend (hereinafter, MA) strategy: the student identifies the larger addend and starts counting up from that cardinal value the number of units of the smaller addend. This strategy is an essential predictor of success in mathematics (Siegler, 1988). Although most students learn this strategy over time, other students with learning difficulties may not master it, so it is necessary to provide them with appropriate instructions so they can learn how to execute this and other effective strategies (Montague, 2007; Powell et al., 2009).

In order to contribute to this effort, some studies have attempted to provide instructions for teaching MA to children with LD. Such is the case of the study by Tournaki (2003), who compared the drill-and-practice approach with that of teaching the MA strategy to students with LD and general education students. The study shows how only the students with LD who followed the MA strategy method improved significantly, whereas general education students showed improvement with both methods.

Microgenetic approach

One approach that has proved especially useful in explaining how strategies are developed is the microgenetic approach. This approach is frequently used to investigate how learning takes place in children in very different areas. It has been applied successfully in areas such as arithmetic, scientific reasoning, spoken and written language, motor activity, memory and reading (Blöte et al., 2004; Fletcher et al., 1998; Siegler, 2006). Some of the advantages of this approach are: the change can be observed while it is happening; several aspects of the change can be studied (the sequence of behaviors, how quickly they occur, the degree of generalization, individual differences and their causes); it makes it possible to detect variability in the behavior of different individuals under similar circumstances (or tasks); and finally, it offers flexibility because it can be used for highly diverse concepts and from different theoretical positions (Siegler, 2006).

In the particular case of arithmetic strategies, this approach can be used to identify the trial in which the child uses a certain strategy for the first time, as well as what leads him to discovery, and how it is generalized to other contexts (Blöte et al., 2004; Fletcher et al., 1998; Siegler & Stern, 1998). This is particularly useful for observing how strategies are developed in response to instruction, and, therefore, can be used to understand how the instruction exerts its effects (Siegler, 2006).

The main criteria that define the microgenetic approach are: (1) the complete period during which the change in behavior takes place must be observed; (2) the density of observations of the behavior in question must be high; and (3) the data from these observations are examined through a "trial by trial" analysis of the behavior, both qualitative and quantitative, to study the aspects that cause the change (Siegler, 2006). Specifically, the microgenetic approach suggests that cognitive change can be analyzed from five dimensions, which, in the case of developing arithmetic strategies, can be specified as follows (Siegler & Stern, 1998; Zhang et al., 2011): source of change (refers to the causes that make children adopt new strategies); path of change (refers to the sequence of different strategies that students manifest as they progress); rate of change (refers to the amount of time or experience that elapses between the first use of a strategy and its consistent use); breadth of change (refers to how a new strategy is generalized to other problems); and variability of change (considers the differences between children in the other dimensions, and the changing set of strategies used by each child).

Several studies have used the microgenetic approach to analyze how arithmetic operations strategies evolve in typically-developing children (Siegler, 1988), although few researchers have used this method to study how children with LD develop these strategies (Huffman et al., 2004). Of note in the area of developing strategies in response to instruction in children with LD are the works of Zhang et al. (2011, 2014), who propose methods for teaching advanced multiplication strategies to children with LD. The microgenetic approach allowed these authors to observe how the multiplication strategies evolved during the instruction. In particular, the results showed that the unitary counting was frequent during the baseline sessions, but it decreased during the teaching experiment to be replaced by the double counting strategy. In order to describe participants' performance and strategic development, five dimensions (source, path, rate, breadth and variability) of change were analyzed according to the framework of microgenetic studies (Siegler, 2006). The authors note the need to conduct more studies with this approach in children with LD to establish a functional relation between the intervention and the changes in the students' performance (Zhang et al., 2014).

Research questions

In view of what was gleaned from the literature, in this paper we propose studying the strategies exhibited by three students with LD in mathematics when solving addition problems, and how instruction focused on teaching the MA strategy affects the acquisition of new strategies. In particular, the following research questions are posed:

1. What strategies do students with LD exhibit when solving arithmetic addition problems in the baseline sessions?
2. How does the instruction favor these students' acquisition of new strategies (in particular the MA strategy) and what difficulties arise during this process?
3. Do they generalize the new strategies to problems with two operations?
4. To what extent do they retain the acquired knowledge over time?

Method

Design and data collection

An adapted multiple-probe design across students with a microgenetic approach was employed to assess the effectiveness of the teaching instruction and the acquisition of new strategies while solving addition word problems by three primary school students with LD. In order to provide a detailed analysis of people's learning process during the intervention, microgenetic studies generally employ single-participant designs (Zhang et al., 2011) and often focus on a small number of individuals (Siegler, 2006).

During baseline, several probes were performed consisting of change and combine addition problems. After a stable baseline was observed, the first student was provided the instruction. Once the student modified the type of strategy used, the instruction began with the next student, and so on. The phases of the experiment included: baseline, instruction, generalization to two-step problems and maintenance.

As in other studies of strategies that follow a microgenetic approach, evaluations were performed during the intervention in order to observe the acquisition of new strategies (Blöte et al., 2004; Siegler, 2006; Zhang et al., 2011). This method requires trial by trial observation and coding, so all the sessions were videotaped. All phases of the study were carried out individually, in weekly sessions, with each student in a classroom in the school without distractions. Nine instructional sessions were held with each student, each lasting approximately 30 min. At the end of each instruction session, the authors watched the videos and planned the next instruction to give each student.

Participants and setting

Three children with LD and a teacher participated in the study. The teacher who conducted the experiment held a degree in teaching, specializing in special education, and had previous experience with children with LD. The three children, who have been given the pseudonyms Peter, Jane and Robert, had been identified by their tutors as struggling with math and were receiving learning support. They were in different grades of primary education in the same mainstream school. Table 1 shows the demographic information on the three participants.

Peter was a 7-year-old boy who was repeating the first grade of primary school. At two years of age, the child was detected with a bilateral hearing loss with a hearing threshold of around 70 dB, and started using hearing implants. He showed a development delay in language, particularly in the acquisition of vocabulary (at 5: 9 years he showed age of 4:11 in vocabulary according to the Vavel test, Spanish Vocabulary Assessment Test, 6–9 years) and a global development delay. Jane was a 7-year-old girl enrolled in the second grade of primary school. In addition to pedagogical and language therapy, she received two hours of physical therapy per week. Robert was a 10-year-old boy diagnosed with autism spectrum disorder with severe symptoms. He was in fourth grade with an individual significant curriculum adaptation with contents and objectives corresponding to the first year of primary school. He showed a good attitude towards schoolwork, but had serious difficulties in directing or maintaining attention.

Table 1. Student demographics.

Variable	Student		
	Peter	Jane	Robert
Gender	Male	Female	Male
Ethnicity	Caucasian	Caucasian	Caucasian
Age (years:months)	7:9	7:8	10:1
Disability category	GDD	GDD	ASD
Schooling	Mainstream	Mainstream	Mainstream
Grade	1st	2nd	4th
Socioeconomic status*	Low	Med	Low
Learning support	8 h/week	10 h/week	8 h/week
IQ score (RPM)	71 (low)	45 (very low)	58 (very low)
Literacy skills	Prolec-r	Prolec-r	Prolec-r
Oral comprehension	Very low	Very low	Very low
Reading comprehension	Very low	Low	Low
Mathematics Achievement	TEMA-3	TEMA-3	TEMA-3
Total score	34	21	18
Equivalent age (y:m)	6:0	5:1	4:9
Number skills	70%	43%	48%
Number-Comparison	67%	50%	0%
Calculation skills	24%	18%	12%
Concepts	47%	24%	29%
Number facts	0%	0%	0%

Note. GDD: Global Developmental Delay, ASD: Autism Spectrum Disorder, RPM: Raven's Progressive Matrices test (Raven, 2015), Prolec-r: Reading Processes Assessment Battery, revised version (Cuetos et al., 2014), TEMA-3: Test of Early Mathematics Ability (Ginsburg & Baroody, 2007). *Based on parents' profession and level of education, as per Hollingshead's occupational scale (Hollingshead, 1975).

Measures and tasks

The problems designed for the study were change and combine addition problems with the unknown in the final position, as these are the easiest to solve (Carpenter & Moser, 1984). An example of a problem considered in the instruction is the following combine addition problem: "*I bought 4 strawberry and 9 mint candies. How many candies did I buy in total?*". The number sets considered in the problems contained addends with no result higher than 15, and they did not contain repeated addends (4 + 4 = 8) or sums equal to 10 (6 + 4 = 10) in order to avoid easy numerical combinations (Carpenter & Moser, 1984). In order to distinguish between the first addend and minimum addend strategies, and in keeping with the guidelines of similar studies (Carpenter & Moser, 1984), the smaller addend always appeared first in the problem statement.

For the purpose of assessing generalization to more complex problems, two-step problems were introduced after the 4th session. These problems where of the type change-change and combine-combine with the unknown in the final position, and required two sums for their resolution (e.g., "*On a farm there are 2 cows, 3 horses, and 7 sheep. How many animals are there in total on the farm?*").

Procedures

Baseline. The three students completed three pre-tests during the baseline that contained change and combine addition problems. The students were asked to solve the problems independently. They were given access to manipulatives and were encouraged to use them during the problem-solving process as needed. No help or feedback was provided during the baseline phase.

Experimental procedures. After completing the baseline sessions, the participants were given the instruction sequentially. An adapted Strategic Training Program from Zhang et al. (2014) was followed, which consisted of: (1) progress monitoring and appropriately selected task assignment, (2) encouraging students to use their own solution strategies before explicit instruction on MA, (3) providing feedback to the student and (4) explicit instruction with a focus on the MA strategy. This approach consisted on providing the students addition problems of two different types (change and combine). They were also given access to manipulatives and encouraged to use them as needed. Each problem was first solved by them independently and the solution strategy they used was recorded. After that, the instructor provided feedback to the students both when the resolution was correct and when it was not, which has often proved beneficial to the students' learning (Zhang et al., 2014). Finally, the explicit instruction with a focus on the MA strategy was carried out.

To demonstrate the MA strategy, a sequence adapted from Tournaki (2003, Appendix A) was followed: (i) read the problem, (ii) choose largest amount, (iii) add the smallest to that, and (iv) what is the final result? In addition, in keeping with the sequencing of strategies identified by authors such as Baroody and Tiilikainen (2003) in terms of the representation of addends, the MA strategy was demonstrated using a three-stage process: level 1 (two addends represented with manipulatives), level 2 (one addend represented with manipulatives) and level 3 (no addends represented with manipulatives). In order to provide visual support for working with the MA strategy at each of these levels, instructional sequences were designed using simple messages and adapted enhanced language material (pictographic symbols). The sequence of instruction for the three levels is illustrated in Figure 1).

In order to evaluate the generalization of the instruction to more complex problems, from the 4th session on, two-step problems were introduced provided that the following criteria were met: manifest a successful MA strategy at least 60% of the time, and achieve at least 50% right answers in the previous two sessions. In these problems, the instruction adhered to guidelines similar to those used for one-step problems. First, the largest of the three numbers was identified. Then, depending on the level of abstraction being used to execute the MA strategy (see Figure 1), the student was guided to use the manipulatives to represent some of the numbers in the problem before eventually adding the three addends.

Four weeks after finishing the instruction, each student completed a post-test similar to those in the baseline phase, consisting of change and combine one-step problems (maintenance).

Below is an instructional scenario with Robert, in which he first exhibits level-1 knowledge of the MA strategy (two addends represented with manipulatives)

Coding. The students' performance while solving each problem was recorded in terms of the type of solution strategy and the success of its implementation. Specifically, the following strategies were considered: incorrect strategies (guess, given number, incorrect operation), counting all, first addend, minimum addend. For each session, the frequency of each solution strategy spontaneously exhibited by the participants was calculated, as was the success rate of each. Behavioral aspects of the participants during each session were also recorded, which was used to adapt the instruction in subsequent sessions.

Reliability. Interobserver reliability data were collected during the baseline, instruction and maintenance phases. An experienced mathematics education teacher, who did not know the hypotheses of the study, recoded 30% of the students' strategies and performance. Interobserver agreement was calculated by dividing the number of agreements by the number of agreements plus disagreements and multiplying by 100. It ranged from 91% to 100% during baseline, 84% to 91% during instruction, and 83% during maintenance. The mean interobserver reliability agreement for strategy categorization for each student across all phases was 86% for Peter, 89% for Jane and 86% for Robert. The mean interobserver reliability agreement for solution accuracy was 100% for Peter, 96% for Jane and 95% for Robert.

Procedural reliability measured the instructor's performance regarding the planned behaviors, which were: the instructor: (1) provides the agreed number of problems, with the agreed amounts; (2) provides the agreed manipulatives for the session; (3) lets the students solve problems independently; (4) [only for instructional sessions] after an unsuccessful attempt by the student, demonstrates how to solve it using the MA strategy at the corresponding level of abstraction (level 1, 2 or 3); (5) [only for instructional sessions] congratulates the student and/or encourages him/her once the problem is solved. Procedural agreement was calculated by dividing the number of observed teacher behaviors by the number of planned behaviors and multiplying by 100 for 30% of the sessions across all phases. Procedural reliability was 100% for Peter and Jane and 95% for Robert.

Results

Figure 2 shows the participants' performance in terms of the strategies used and accuracy during the baseline, intervention and maintenance phases. In addition, Table 2 shows the strategies used by participants on trials immediately before and after each child's first use of the MA Strategy.

To delve into how this strategy was acquired by each participant, its manifestation and success at each level was analyzed (see Figures 3–5).

Baseline

The participants used few strategies during baseline, with different success rates between them. None of the three students ever exhibited the MA strategy during baseline. Peter employed incorrect strategies in the first baseline session, and the first addend strategy in the resolution of all problems of the following two sessions. He had a high success rate (78%) throughout baseline. The mistakes were due to the use of incorrect strategies such as guessing and, on isolated occasions, to counting errors when using the first addend strategy. Jane exhibited the counting all strategy when solving 88% of the problems, which resulted in the right answer 63% of the time. The main errors noted were the incorrect execution of the counting all strategy and the use of the improper strategy of guessing. Robert did not solve any of the problems correctly during baseline, and

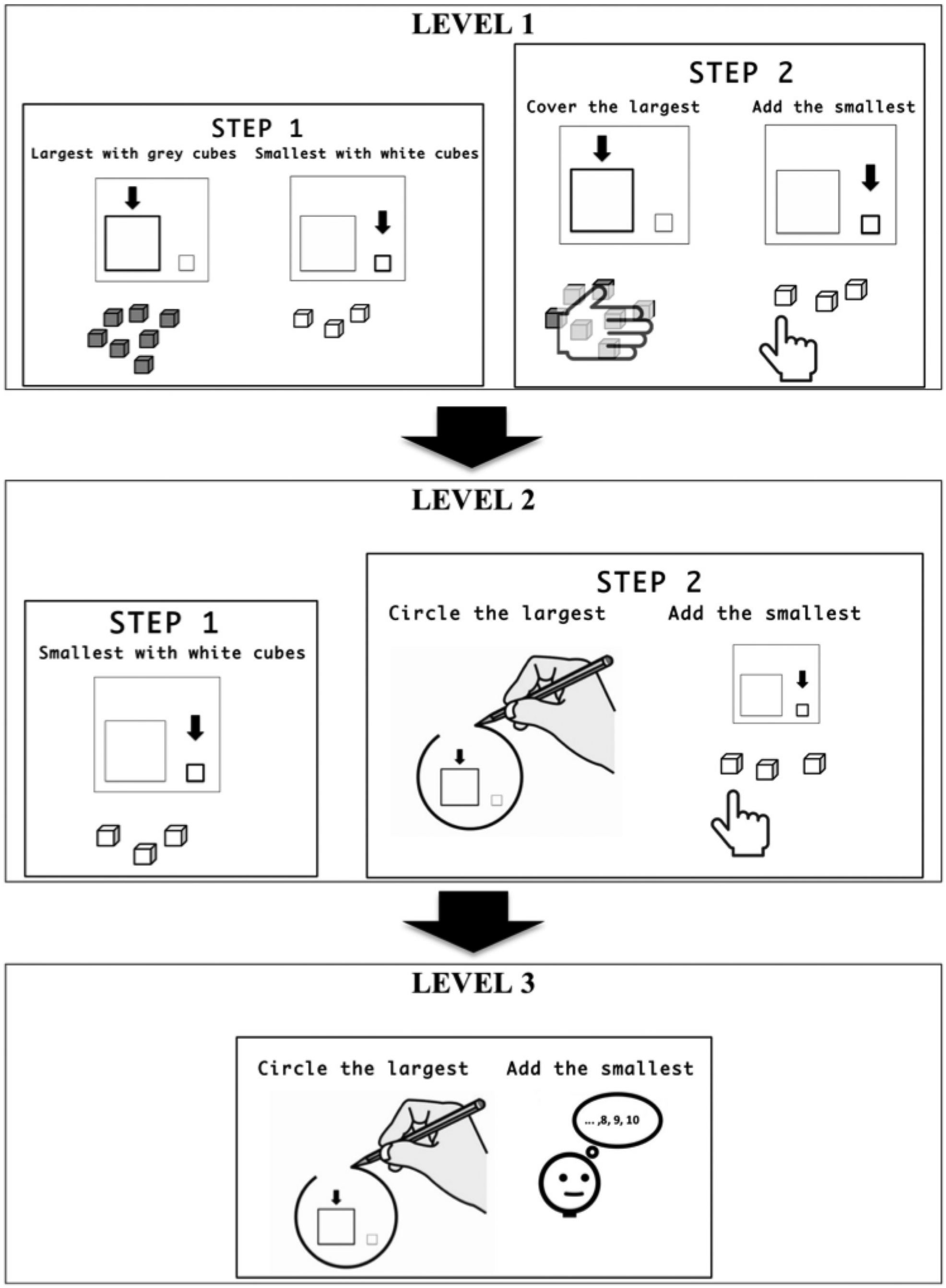

Figure 1. Instructional steps in each of the three levels of the MA strategy: level 1 (two addends represented with manipulatives); level 2 (one addend represented with manipulatives); and level 3 (no addends represented with manipulatives). Pictographic symbols adapted from ARASAAC (http://arasaac.org), created by Sergio Palao and distributed under Creative Commons license (BY-NC-SA).

mostly exhibited incorrect strategies (guessing, or given number).

Instruction

A functional relation between the teaching experiment and the development of new strategies was observed: first the students showed a stable baseline in terms of the variety of strategies, and used new strategies only after the intervention began.

Peter. The student had a success rate of 85% to 100% during all the instructional sessions. Peter was instructed on the use of the MA strategy in the first instructional session. He immediately showed an understanding of the strategy and used it to solve almost every problem in this

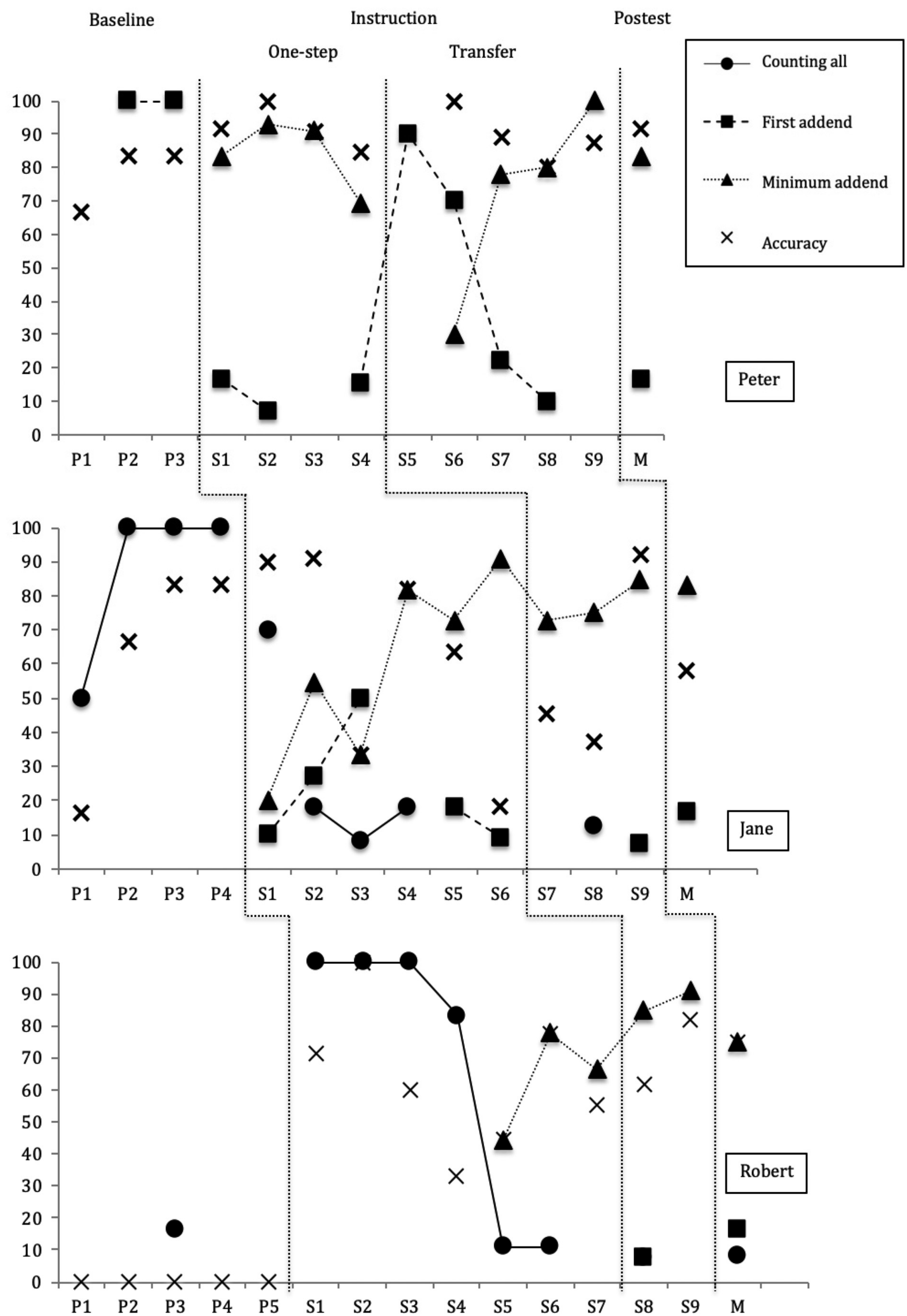

Figure 2. Percentage of strategy types and accuracy during the baseline, intervention and maintenance sessions of the three participants.

session (see Figure 2). This strategy was initially employed at level 1 (representing both addends with manipulatives, see Figure 3), although he quickly stopped using the manipulatives. He exhibited the MA strategy at level 3 (no addend with manipulatives) during the third session (see Figure 3). The student was highly motivated during the first four sessions and responsive to the instructions that were given on how to solve the problems.

In order to evaluate the generalization of the MA strategy, and after satisfying the pre-defined criteria (having exhibited this strategy at least 60% of the time, with a 50% success rate in the last two sessions), two-step problems were introduced in the fifth session. Initially, Peter did not transfer the MA strategy to these problems, but instead went back to the first addend strategy (see Figure 2), with a high success rate (85%). In the sixth session, Peter transitioned from the first addend strategy to the MA strategy, and eventually stopped using the first one altogether. A high success rate (in excess of 85%) was maintained throughout the experience. It should be noted that although he finally used the MA strategy in all the two-step problems, he again resorted to manipulatives to represent some of the addends in the one- and two-step problems, thus regressing to levels 1 and 2 in the execution of the strategy. The following graph shows the frequency of the MA strategy, differentiated by the levels used to represent the addends during the instructional and maintenance sessions.

The incorrect answers during the instructional sessions were due in most cases to counting errors when executing the MA strategy. The types of errors and the number of trials per instructional and maintenance sessions are listed in Table 3.

As Table 3 shows, on some occasions Peter also made comparison errors when executing the MA strategy. He made these errors when adding the same addend twice, which we interpret as a difficulty when identifying the larger and smaller amount. The errors related to forgetting data occurred when two-step problems were introduced and Peter forgot to add one of the three addends.

Table 2. Example of instruction on the MA strategy with robert.

Tutor: We will read the problem together. Remember: What did we have to do first?
Robert: (looking at the instruction sequence) Count out the larger number using gray cubes.
Tutor: Very good, and which one is larger?
Robert: 6! [takes 6 gray cubes]
Tutor: Great. What now?
Robert: Count out the smaller one using white cubes.
Tutor: Very good, and which one is smaller?
Robert: 3 [takes 3 white cubes]
Tutor: Very good. And what do we do next?
Robert: [Looking at the sequence] Now we cover the gray ones, right? [Covers the gray ones with his hand and counts the white ones:] 1, 2, 3. But I counted wrong, didn't I?
Tutor: You forgot the gray ones! How many were under your hand?
Robert: Oh, that's right. There were 6.
Tutor: So now add the white ones.
Robert: [Touching his hand] 6, [touching a white cube] 7, [another] 8, and [another] 9.
Tutor: Very good! See how well you did?

Jane. She had a success rate above 50% during most of the instructional sessions. During the first instructional session, Jane was instructed on the MA strategy; however, she resorted most of the time to the counting all strategy, representing both addends with manipulatives, and only spontaneously exhibited the MA strategy at level 1 in the last two problems of the first session. During this session, it was very important to insist on the proper representation of the addends with the cubes, since she occasionally made coordination errors when counting. During the second session, she relied more on the MA strategy, which she successfully used in half of the problems. As a result, in the third session, the teacher demonstrated the MA strategy at level 2 (only one of the addends represented with manipulatives). Figure 4 shows the frequency of the MA strategy used by Jane during the instructional and maintenance sessions, differentiating between the three levels in the representation of the addends.

During the third session, she made numerous errors when executing the MA at level 2 that we attribute to difficulties

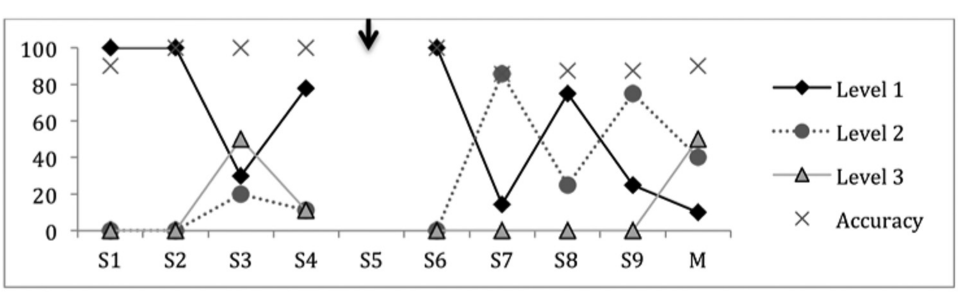

Figure 3. Percentage of each level in the use of the MA strategy by Peter during the instruction and maintenance sessions. The arrow indicates the session where two-step problems are introduced.

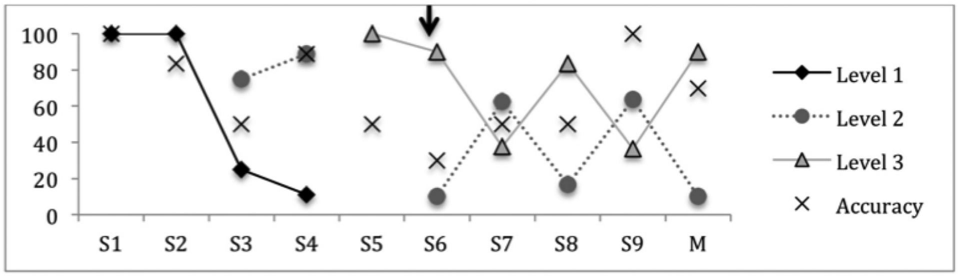

Figure 4. Percentage of each level in the use of the MA strategy by Jane during the instruction and maintenance sessions. The arrow indicates the session where two-step problems are introduced.

Table 3. Errors by Peter during instruction and maintenance sessions.

	Error type				
Session	Counting error	Forgetting data	Comparison error	Total number of errors	Total number of trials
S1			1	1	12
S2				0	14
S3	1			1	11
S4	2			2	13
S5		1		1	10
S6				0	10
S7			1	1	9
S8	1	1		2	10
S9	1			1	8
M			1	1	12

comparing the addends. Specifically, she circled the larger addend and represented the smaller addend but starting counting from the latter, obtaining as a result double this addend. The table below shows the number and types of errors per session. As we can see, the number of comparison errors decreased, although she continued to make numerous counting mistakes throughout the instruction and maintenance sessions.

Since the student maintained a success rate of over 60% during sessions 4 and 5, two-step problems were introduced in session 6. From the beginning, the student was able to transfer the MA strategy to the two-step problems, although the percentage of right answers decreased considerably (see Figure 2). This was due primarily to mental counting mistakes that had not manifested themselves with the manipulatives. Although the student was encouraged to resort again to the manipulatives when executing the MA strategy in two-step problems - as explained in the procedures-, the girl insisted on not using them, responding on more than one occasion: “I want to do it in my head, I don’t like it with the manipulatives.” In addition to the counting errors mentioned, the student forgot to add one of the addends in some of the two-step problems (see Table 4).

Robert. From the beginning Robert showed a success rate above 50%, which he maintained during most instructional sessions. Due to the number of counting errors (both partitioning and coordination) exhibited during the baseline by this student when representing the addends, the instruction focused on working with the counting all strategy with manipulatives to help him overcome these errors and allow him to progress toward instruction in the MA strategy. The student was receptive and spontaneously exhibited the counting all strategy from the first session, using it to achieve a high success rate in the first two sessions (see Figure 2). In order to prepare him for the instruction in the MA strategy, he was instructed to represent the larger addend with the gray cubes, and the smaller one with the white cubes (see Step 1, Level 1 in Figure 1). Sometimes he was unable to distinguishing the smaller addend from the larger one, so a preliminary step was added in which he was asked to represent both quantities by matching the set of cubes, thus coordinating the elements of the smaller addend with those of the larger one.

From the third session, the instruction in the MA strategy began at level 1, although the subject did not exhibit it spontaneously until the fifth session (see Figures 2 and 5).

The student spontaneously exhibited the MA strategy for the first time, at level 1, when solving the third problem in the fifth session (see Table 1 for the description of this instruction scenario). From this moment on, the student exhibited a significant improvement in terms of interest and concentration, which he maintained until the end of the study.

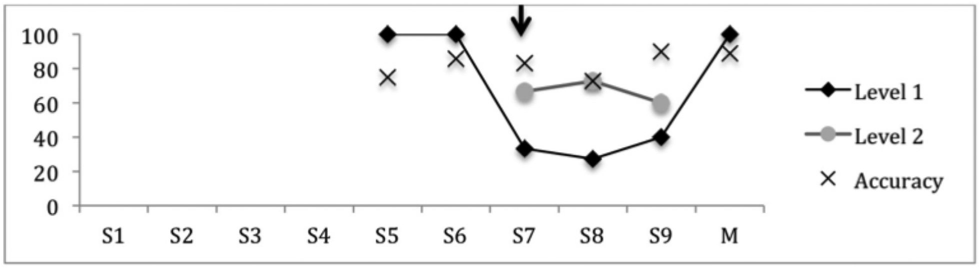

Figure 5. Percentage of each level in the use of the MA strategy by Robert during the instruction and maintenance sessions. The arrow indicates the session where two-step problems are introduced.

Table 4. Errors by Jane during instruction and maintenance sessions.

	Error type				
Session	Counting error	Forgetting data	Comparison error	Total number of errors	Total number of trials
S1	1			1	10
S2	1			1	11
S3	1		7	8	12
S4	1		1	2	11
S5	4			4	11
S6	9			9	11
S7	3	2	1	6	11
S8	3	2		5	8
S9	1			1	13
M	4		1	5	12

After this point, he almost completely abandoned the counting all strategy and used the MA strategy, which he adapted without difficulty to the two-step problems after session 7, with a high success rate. The student wanted to resort to the manipulatives to represent both addends (level 1) or one of them (level 2) in every session, as shown in Figure 5.

During the instructional sessions, Robert made several mistakes. The most frequent one (as in Jane's case) was, after representing both addends with the appropriate number of cubes, adding the larger addend to itself, thus doubling it (*comparison error*). He also made numerous counting errors in almost every session, including maintenance, and he forgot an addend in two of the two-step problems. The type and number of errors per session are given in Table 5.

Maintenance

During the post-test (four weeks after finishing the instruction), Peter exhibited the MA strategy in 83% of the problems, and resorted to the first addend strategy on all other occasions. He maintained a high percentage of correct answers (92%). Jane's approach was similar to Peter's in the use of strategies during the post-test (83% MA strategy, and first addend in the rest). Although the percentage of correct answers by Peter with respect to baseline increased, it did not exceed 60%. Robert exhibited the MA strategy in 75% of the post-test problems and resorted to the counting all and first addend strategies in the remaining problems. This resulted in a large increase in the number of right answers, from 0% in baseline to 75% in the post-test.

It is interesting to note that the three subjects used the MA strategy again in the trials immediately after their first use. Table 6 shows the strategy used in the four trials immediately before and after the initial use of the MA by the three subjects. Note, for example, how Peter used this strategy again in three of the four trials after his first use, and Jane only in the one immediate after it, before again returning to the counting all or first addend strategies. In Robert's case, he used it again in the three trials immediately after his initial use.

Discussion and conclusions

This paper presents the effect of explicit instruction in advanced strategies, focusing on the MA strategy, in three children with LD (one of them with ASD). The instruction in the strategy was sequenced into levels of abstraction based on the addends represented with and without manipulatives, which helped the students better acquire the knowledge. We also observed the three students develop strategies in terms of: changes in the set of existing

Table 5. Errors by Robert during instruction and maintenance sessions.

	Error type					
Session	Counting error	Guessing	Forgetting data	Comparison error	Total number of errors	Total number of trials
S1	2	2			4	7
S2					0	4
S3	1			1	2	5
S4	2			2	4	6
S5	1			4	5	9
S6	1			1	2	9
S7	1			3	4	9
S8	1		1	3	5	13
S9	1		1		2	11
M	3				3	12

Table 6. Strategies used in trials immediately before and after each child's first use of the MA strategy (trial 0).

Trial	−4	−3	−2	−1	0	1	2	3	4
Peter	CA	FA	FA	FA	MA	FA	MA	MA	MA
Jane	CA	CA	CA	CA	MA	MA	CA	FA	FA
Robert	IS	IS	IS	IS	MA	MA	MA	MA	CA

Note. CA: Counting all, FA: First addend, MA: Minimum addend, IS: Incorrect Strategy.

strategies, construction of new strategies and disuse of old ones (Blöte et al., 2004; Siegler, 2006).

In relation to the sequence of strategies that students showed as they gained proficiency, various strategies were observed. It is worth distinguishing here between the study subjects. In the case of Peter and Jane, the two students used a sequence of strategies that was similar to those of typically-developing students (Cookson & Moser, 1980): both exhibited the counting all strategy at the beginning, combined with the first addend strategy, before the MA strategy finally prevailed. It should also be noted that both subjects exhibited the MA strategy in the first instructional session where it was explicitly demonstrated. In the case of Robert (diagnosed with ASD), a smaller variety of strategies was observed: he went directly from using the counting all strategy exclusively, to gradually introducing the MA strategy, exhibiting practically no other effective strategies, like the first addend strategy. Moreover, the first time he exhibited the MA strategy was not during the first session when it was taught (session 3), but two sessions later. In this dimension we also consider the sequence of levels in the manifestation of the MA strategy. In this sense, the three subjects showed a progression from a concrete to an abstract level in terms of the representation of the two addends that coincided with previous studies on strategies (Baroody & Tiilikainen, 2003), although not all attained or manifested all three levels of abstraction. The use of incorrect strategies, such as guessing or given number, was significantly reduced in all three subjects during the course of the instruction.

In relation to how the strategy was generalized to other problems, we again observe different behaviors in the three subjects. Peter consistently acquired the MA strategy from the first session, but almost completely stopped using it with the introduction of two-step problems during two sessions, and did not generalize its use until the seventh session. As for Jane, it was not until the fourth session that she consistently used the MA strategy and generalized its use to solve two-step problems, although this did considerably reduce her success rate. Finally, although Robert took longer to exhibit the MA strategy spontaneously, it replaced the counting all strategy almost from the beginning, and he used the MA quite successfully for the most complicated problems until the end of the study.

In general, the students resorted to increasingly abstract counting strategies over the course of the instruction. They barely resorted to guessing strategies to solve the problems, consistent with the work of Gilmore et al., 2007, which showed that they are able to solve symbolic addition problems without resorting to this strategy. When executing the MA strategy, all three participants exhibited comparison errors. Comparison is regarded as one of the main skills in index numerical magnitude processing (Holloway and Ansari, 2009). Various studies have shown a link between numerical comparison and mathematical competency in both typically and atypically developing children (Gilmore et al., 2010; Rousselle & Noël, 2007). In our work, all the participants in the study exhibited difficulties comparing quantities, although it was more evident in the two students with the lowest mathematical skills. This is consistent with studies that have demonstrated that the basic processing of numerical magnitude is affected in children who present with mathematical difficulties (Holloway & Ansari, 2009; Rousselle & Noël, 2007).

The use of manipulatives to represent addends helped overcome the difficulties mentioned, by allowing the transition from symbolic to non-symbolic representations of addends.

This confirms other studies that have shown that children with mathematics disabilities had difficulty comparing Arabic digits (i.e., symbolic number magnitude), but not comparing collections (i.e., non-symbolic number magnitude) (Rousselle & Noël, 2007). More generally, the benefits of using concrete manipulatives during instruction coincide with previous research on teaching arithmetic operations to students with learning difficulties (Baroody & Tiilikainen, 2003), and specifically with ASD (Bruno et al., 2021; Hart & Cleary, 2015; Polo-Blanco et al., 2019; 2021). The use of sequences based on augmentative language to teach strategies to students with this disorder also proved beneficial, as other studies show (Hart & Cleary, 2015; Mirenda, 2003).

Our results also coincide with prior research (Zhang et al., 2011) in the sense that the three subjects exhibited a wider variety of strategies during the instruction than during the baseline. As mentioned earlier, they also showed differences in how they manifested the MA strategy (some in the first session when explicitly demonstrated by the tutor, and others later) and the variety in their use of other strategies (less variety in the case of the subject with ASD).

The results concerning the student with ASD are especially interesting given the lack of studies that focus on the strategies employed by students with this disorder to solve arithmetic problems (Hart & Cleary, 2015; Polo-Blanco et al., 2019; in press). In this sense, the use of the microgenetic approach was especially useful to observe the type of spontaneous strategies used by the subject with ASD, and how they varied in response to the instruction. Although our study provides limited data on only one subject, the results show notable behaviors, such as the later but more consistent acquisition of this MA strategy, and a lower variability in the use of strategies, aspects that can be further studied in future research with students with this disorder.

In relation to the implications to classroom practices, teachers must weigh the considerable evidence showing that many students with learning difficulties, including autism, have problems learning the basic arithmetic operations used in everyday life, which makes it important for educators to know methods to teach them (Geary, 1990). Accordingly, it is very important that educators of children with LD, and in particular of those diagnosed with ASD, observe the strategies employed by children before the arithmetic operations are formalized, and provide tasks that help them acquire more advanced strategies (Zhang et al., 2011). Also, given the frequency of two forms of error (counting and comparison) observed, and given the fact that a high rate of error was observed at maintenance in one of the students, it could also be useful in further research to attempt intervention specifically designed to address these skills outside the context of arithmetic.

In this study, the students received instruction on the MA strategy, sequenced into levels of abstraction. As we have seen, adaptations were made in terms of the instruction given to each student that are interesting to consider and that are simple to implement. In the case of the student with ASD, the counting all strategy had to be drilled beforehand to overcome counting errors, and the matching strategy to compare magnitudes. It was also essential to carefully observe each student's preference for the level of abstraction of the MA strategy. While the student with ASD seemed comfortable with the more concrete levels of representation of the addends (levels 1 and 2), the other two students soon showed a desire to forego (at least partially) the manipulatives, so they were encouraged and helped to execute the strategy at higher levels of abstraction.

One aspect that emerges from this work is that each study participant needed different periods of time to assimilate the new strategies. Numerous studies involving students with learning difficulties have shown that they require more time to acquire new mathematical concepts, meaning that teachers must plan carefully and make the necessary temporary adaptations to each student's needs.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

This work was supported by projects EDU2017-84276-R and PID2019-105677RB-I00 / AEI/10.13039/501100011033.

ORCID iD

Irene Polo-Blanco https://orcid.org/0000-0001-6425-6337

References

Baroody, A. J., & Tiilikainen, S. (2003). Two perspective on addition development. In Baroody, A. J., y Dowker, A. (Eds.), *The development of arithmetic concepts and skills. Constructing adaptative expertise* (pp. 75–125). Erlbaum.

Blöte, A., Van Otterloo, S., Stevenson, C., & Veenman, M. (2004). Discovery and maintenance of the many-to-one counting strategy in 4-year-olds: A microgenetic study. *British Journal of Developmental Psychology*, *22*(1), 83–102. https://doi.org/10.1348/026151004772901122

Bruno, A., Polo-Blanco, I., González M. J., & González-Sánchez, J. (2021). Strategies for solving addition problems using modified schema-based instruction in students with intellectual disabilities. *Mathematics [Special Issue: Research on Powerful Ideas for Enriching School Mathematical Learning]*, *9*(1814), 1–17. https://doi.org/10.3390/math9151814

Carpenter, T., & Moser, J. (1984). Acquisition of addition and subtraction concepts in grades one through three. *Journal for*

Research in Mathematics Education, *15*(3), 179–202. https://doi.org/10.2307/748348

Cookson, C., & Moser, J. M. (1980). *Coordinated study individual interview procedures (working paper No. 290)*. Wisconsin Research and Development Center for Individualized Schooling.

Cuetos, F., Rodríguez, B., Ruano, E., & Arribas, D. (2014). *Prolec-r. Batería de evaluación de los procesos lectores, revisada (5.ª edición). [assessment battery of the reading processes, revised (5th edition)]*. TEA Ediciones.

Fletcher, K. L., Huffman, L. F., Bray, N. W., & Grupe, L. A. (1998). The use of the microgenetic method with children with disabilities: Discovering competence. *Early Education & Development*, *9*(4), 358–373. https://doi.org/10.1207/s15566935eed0904_3

Geary, D. C. (1990). A componential analysis of an early learning deficit in mathematics. *Journal of Experimental Child Psychology*, *49*(3), 363–383. https://doi.org/10.1016/0022-0965(90)90065-G

Geary, D. C. (1994). *Childreńs mathematical development: Research and practical applications*. American Psychological Association. https://doi.org/10.1037/10163-000.

Geary, D. C. (2013). Learning disabilities in mathematics: Recent advances. In Swanson, H. L., Harris, K., & Graham, S. (Eds.), *Handbook of learning disabilities* (2nd ed., pp. 239–255). Guilford Press.

Geary, D., Hoard, M., Craven, J., & Desto, M. (2004). Strategy choices in simple and complex addition: Contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology*, *88*(2), 121–151. https://doi.org/10.1016/s0022-0965(04)00033-5

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and mathematics achievement in the first year of formal schooling. *Cognition*, *115*(3), 394–406. https://doi.org/10.1016/j.cognition.2010.02.002

Gilmore, C., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature*, *447*(7144), 589–591. https://doi.org/10.1038/nature05850

Ginsburg, H., & Baroody, A. (2007). *Test of early mathematics ability (3rd ed.). [spanish language version, adapted by M.C. Núñez y I. Lozano]*. TEA Ediciones.

Hart, J. E., & Cleary, S. (2015). Review of evidence-based mathematics interventions for students with autism spectrum disorders. *Education and Training in Autism and Developmental Disabilities*, *50*(2), 172–185. Retrieved from http://www.jstor.-org/ stable/24827533.

Hollingshead, A. B. (1975). *Four Factor Index of Social Status*, Unpublished Working Paper, Yale University, New Haven, CT.

Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's Mathematics achievement. *Journal of Experimental Child Psychology*, *103*(1), 17–29. https://doi.org/10.1016/j.jecp.2008.04.001

Huffman, L., Fletcher, K., Bray, N. W., & Grupe, L. A. (2004). Similarities and differences in addition strategies of children with and without mental retardation. *Education and Training in Developmental Disabilities*, *39*(4), 317–325. Retrieved from http://www.jstor.org/stable/23880211.

Libertus, M. E., Feigenson, L., & Halberda, J. (2013). Numerical approximation abilities correlate with and predict informal butnot formal school mathematics abilities. *Journal of Experimental Child Psychology*, *116*(4), 829–838. https://doi.org/10.1016/j.jecp.2013.08.003

Mirenda, P. (2003). Toward functional augmentative and alternative communication for students with autism: Manual signs, graphic symbols, and voice output communication aids. *Language, Speech, and Hearing Services in Schools*, *34*(3), 203–216. https://doi.org/10.1044/0161-1461(2003/017)

Montague, M. (2007). Self-regulation and mathematics instruction. *Learning Disabilities Research and Instruction*, *22*(1), 75–83. https://doi.org/10.1111/j.1540-5826.2007.00232.x

Polo-Blanco, I., González, M. J., & Bruno, A. (2019). An exploratory study on strategies and errors of a student with autism spectrum disorder when solving partitive division problems. *Brazilian Journal of Special Education*, *25*(2), 247–264. https://doi.org/10.1590/s1413-65382519000200005

Polo-Blanco, I., González, M. J., & Bruno, A. (2021). Influencia del contexto en problemas de multiplicación y división: estudio de caso de un alumno con autismo. [Influence of the context on multiplication and division problems: A case study of a student with autism]. *Siglo Cero*, *52*(1), 59–78. https://doi.org/10.14201/scero20215215978

Polo-Blanco, I., Van Vaerenbergh, S., Bruno, A., & González, M. J. (in press). Conceptual model-based approach to teaching multiplication and division word-problem solving to a student with autism spectrum disorder. *Education and Training in Autism and Developmental Disabilities*.

Powell, S. R., Fuchs, L. S., Fuchs, D., Cirino, P. T., & Fletcher, J. M. (2009). Effects of fact retrieval tutoring on third-grade students with math difficulties with and without reading difficulties. *Learning Disabilities Research & Practice*, *24*(1), 1–11. https://doi.org/10.1111/j.1540-5826.2008.01272.x

Raven, J. (2015). *Test de matrices progresivas: Escala general; series A, B, C, D y E. [progressive matrices test: General scale; series A, B, C, D y E.]*. TEA Ediciones.

Rousselle, L., & Noël, M. P. (2007). Basic numerical skills in children with mathematics learning disabilities: A comparison of symbolic vs. non-symbolic number magnitude processing. *Cognition*, *102*(3), 361–395. https://doi.org/10.1016/j.cognition.2006.01.005

Siegler, R. S. (1988). Individual differences in strategy choices: Good students, not-so-good students, and perfectionists. *Child Development*, *59*(4), 833–851. https://doi.org/10.2307/1130252

Siegler, R. S. (2006). Microgenetic analyses of learning. En W. Damon & R. M. Lerner (Series Eds.), D. Kuhn & R. S. Siegler (Vol. Eds.), *Handbook of child psychology: Vol. 2. Cognition, perception, and language* (6th ed., pp. 464–510). John Wiley. 10.1002/9780470147658.chpsy0211.

Siegler, R. S. (2007). Cognitive variability. *Developmental Science*, *10*(1), 104–109. https://doi.org/10.1111/j.1467-7687.2007.00571.x

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of*

Experimental Psychology: General, *127*(4), 377–397. https://doi.org/10.1037/0096-3445.127.4.377

Timmermans, R. E., & Van Lieshout, E. C. D. M. (2003). Influence of instruction in mathematics for low performing students on strategy use. *European Journal of Special Needs Education*, *18*(1), 5–16. https://doi.org/10.1080/0885625032000042285

Tournaki, N. (2003). The differential effects of teaching addition through strategy instruction versus drill and practice to students with and without learning disabilities. *Journal of Learning Disabilities*, *36*(5), 449–458. https://doi.org/10.1177/00222194030360050601

Wynn, K. (1990). Childreńs understanding of counting. *Cognition*, *36*(2), 155–192. https://doi.org/10.1016/0010-0277(90)90003-3

Zhang, D., Xin, Y. P., Harris, K., & Ding, Y. (2014). Improving multiplication strategic development in children with math difficulties. *Learning Disability Quarterly*, *37*(1), 15–30. https://doi.org/10.1177/0731948713500146

Zhang, D., Xin, Y. P., & Si, L. (2011). Transition from intuitive to advanced strategies in multiplicative reasoning for students with math difficulties. *The Journal of Special Education*, *47*(1), 50–64. https://doi.org/10.1177/0022466911399098