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Research article

Derivation of Bessel function closed-form solutions in zero dimensional φ⁴-field theory

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ABSTRACT

The integral $\int_{-\infty}^{\infty} e^{-x^2-gx^4}dx$ is used as an introductory learning tool in the study of Quantum Field Theory and path integrals. Typically, it is analyzed via perturbation theory. Closed-form solutions have been quoted for which I could not find any derivation. Using a simple and elegant transformation, the close form solutions for the integral and its even positive integer moments can be obtained in terms of Bessel functions.

1. Introduction

A common integral in the preparatory studies of Quantum Field Theory (QFT), perturbation theory[1] and Feynman diagrams is the "toy model":

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2-gx^4} dx \quad g \geq 0. \tag{1}$$

The function $Z(g)$ is referred to as the partition function of i) *zero dimensional* φ^4*-field theory* 1,2 or alternatively ii) *the zero-dimensional anharmonic oscillator* [3]. The references [2,3] state (without derivation) the closed-form solution to the integral given by (1):

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] K_{1/4}\left(\frac{1}{8g}\right) \tag{2}$$

Note the *modified Bessel function of the second kind* K_ν , also called a MacDonald function [2], in equation (2). The function can be expressed as [4] for $|\arg z| < \pi/2$, i.e. $Re\, z > 0$:

$$K_\nu(z) = \int_0^\infty \cosh(\nu t) e^{-z \cosh t} dt. \tag{3}$$

The expression (2) is in agreement with the formula (Ch. 3.323 No. 3) in Ref. [5], which in turn refers to the formula (Ch. 4.5 No. 34) in Ref. [6]. No derivations of the formulae are stated in either [5] or [6]. In addition, we note the alternative formulation of the closed-form solution for $Z(g)$ in Ref. [2]:

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[1] Please refer to AppendixB.

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$$Z(g) = \left(\frac{1}{2g}\right)^{1/4} \exp\left[\frac{1}{8g}\right] D_{-1/2}\left(\frac{1}{2g}\right). \tag{4}$$

Equation (4) can be verified from (2) using the identity [7]:

$$D_{-1/2}(z) = \sqrt{\frac{z}{2\pi}} K_{1/4}\left(\frac{z^2}{4}\right)$$

Derivation.
We re-cast (1) as

$$Z(g) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2 - gx^4} dx, \tag{5}$$

and define a new variable transformation

$$x = \frac{1}{\sqrt{g}} \sinh\left(\frac{\xi}{4}\right) \tag{6}$$

$$dx = \frac{1}{4\sqrt{g}} \cosh\left(\frac{\xi}{4}\right) \tag{7}$$

When switching to the variable ξ, the limits in the integral in (5) are unchanged. Next, we consider the exponent in (5):

$$x^2 + gx^4 = x^2 \bullet \left[1 + gx^2\right] = g^{-1} \sinh^2\left(\frac{\xi}{4}\right) \bullet \left[1 + \sinh^2\left(\frac{\xi}{4}\right)\right] = g^{-1} \sinh^2\left(\frac{\xi}{4}\right) \bullet \cosh^2\left(\frac{\xi}{4}\right) = (4g)^{-1}$$
$$\bullet \sinh^2\left(\frac{\xi}{2}\right) = (4g)^{-1}\left[\cosh^2\left(\frac{\xi}{2}\right) - 1\right] = (4g)^{-1}\left[\frac{1}{2}(\cosh \xi + 1) - 1\right] = \frac{1}{8g}\cosh \xi - \frac{1}{8g}. \tag{8}$$

Using (6), (7), and (8), we re-write (5) as

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] \int_0^\infty \cosh\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g}\cosh \xi} d\xi. \tag{9}$$

The final step is to use the definition (3) to substitute for the integral in (9) to obtain the expression (2):

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] K_{1/4}\left(\frac{1}{8g}\right).$$

This is in agreement with the formula (Ch. 3.323 No. 3) in Ref. [5].

1.1Closed form expression for the moments

The integrand in the partition function (1) is an even function and should thus only have even moments given by the formula:

$$\langle x^{2n} \rangle = \frac{1}{Z(g)} \frac{2}{\sqrt{\pi}} \int_0^\infty x^{2n} e^{-x^2 - gx^4} dx. \tag{10}$$

We proceed as before using the variable transformations (6), (7) and (3) to re-write (10):

$$\langle x^{2n} \rangle = \mathcal{N} \int_0^\infty \sinh^{2n}\left(\frac{\xi}{4}\right) \cosh\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g}\cosh \xi} d\xi. \tag{11}$$

The normalization factor $\mathcal{N} = \mathcal{N}(n, g)$ can be expressed as:

$$\mathcal{N}(n, g) = \frac{1}{Z(g)} \frac{e^{1/8g}}{\sqrt{4\pi g^{2n+1}}} = \left[g^n \bullet K_{1/4}\left(\frac{1}{8g}\right)\right]^{-1}$$

To proceed we make use of

$$\sinh^{2n} x = \left[\cosh^2 x - 1\right]^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \cosh^{2k} x,$$

to transform (11) to

$$\langle x^{2n} \rangle = \mathcal{N} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_0^\infty \cosh^{2k+1}\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g}\cosh \xi} d\xi. \tag{12}$$

We first use the trigonometric identity for odd powers of cosine [8] and then apply Osborn's rule [9] to convert the identity to the hyperbolic analogue:

$$\cosh^{2k+1} x = \frac{1}{4^k} \sum_{m=0}^{k} \binom{2k+1}{m} \cosh([2k+1-2m] \bullet x).$$

We can now finish up (12):

$$\langle x^{2n} \rangle = \mathscr{N} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}}{4^k} \sum_{m=0}^{k} \binom{2k+1}{m} \int_0^\infty \cosh\left(\frac{[2(k-m)+1] \bullet \xi}{4}\right) e^{-\frac{1}{8g}\cosh \xi} d\xi,$$

and simplify the integral above using (3)

$$\langle x^{2n} \rangle = \mathscr{N} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}}{4^k} \sum_{m=0}^{k} \binom{2k+1}{m} K_{\frac{2(k-m)+1}{4}}\left(\frac{1}{8g}\right). \tag{13}$$

In general, we see that (13) is of the form

$$\langle x^{2n} \rangle = \left[(4g)^n \bullet K_{1/4}\left(\frac{1}{8g}\right)\right]^{-1} \sum_{k=0}^{n} c_m^{(n)} K_{\frac{2m+1}{4}}\left(\frac{1}{8g}\right), \tag{14}$$

where with a little work we see that the coefficients $c_m^{(n)}$ in (14) can be expressed as:

$$c_m^{(n)} = \sum_{k=m}^{n} (-1)^{n-k} 4^{n-k} \binom{n}{k} \binom{2k+1}{k-m} \tag{15}$$

The coefficients in (15) can be simplified when starting from the top in descending order - for example the first few entries are

$$c_n^{(n)} = 1 c_{n-1}^{(n)} = 1 - 2n \; c_{n-2}^{(n)} = n(2n-3) \; c_{n-3}^{(n)} = \frac{n(1-2n)(2n-5)}{3}$$

The even moments $2n$ for $n = 1, 2, 3$ are then given by:

$$\langle x^2 \rangle = \frac{1}{4g}\left[\frac{K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} - 1\right] \langle x^4 \rangle = \frac{1}{16g^2}\left[\frac{K_{5/4}\left(\frac{1}{8g}\right) - 3K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} + 2\right] \langle x^6 \rangle = \frac{1}{64g^3}\left[\frac{K_{7/4}\left(\frac{1}{8g}\right) - 5K_{5/4}\left(\frac{1}{8g}\right) + 9K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} - 5\right]$$

2. Discussion

As I was unable to find a derivation of the closed form expressions for φ^4-field theory in zero dimensions, I set about deriving the expression on my own. Along the way the trick I used to derived the expression also enables one to write a closed form expression for the even positive integer moments. I hope these results can lead to further insights on resummation methods used in the perturbative approach explored in the works [1–3,10] and references therein. I also discovered an erratum in one of the quoted formulas [6] for which the correction is mentioned in appendix. I hope this short letter will be a useful reference for practitioners and students of field theory and statistical physics.

Author contribution statement

Ranjiva M. Munasinghe: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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Appendices.

A Erratum in Erdélyi et al.

The formula in Ref. [6] (Ch. 4.5 No. 34) states that

$$\int_0^{\infty}(2t)^{-3/4}e^{-2a^{1/2}t^{1/2}}e^{-pt}dt=\left[\frac{a}{2p}\right]^{1/2}\exp\left[\frac{a}{2p}\right]K_{1/4}\left(\frac{a}{2p}\right) \tag{16}$$

We also note the conditions for (16) are stated as $|\arg a|<\ \pi$ $| and $Re\,p>0$ [6].
We start by using the substitution $t=x^4$ to transform the LHS of (16) to:

$$2^{5/4}\int_0^{\infty}e^{-2a^{1/2}x^2-px^4}dx$$

We now use a modified version of the transformation in (6)

$$x=\frac{4a^{1/2}}{p}\sinh\left(\frac{\xi}{4}\right),$$

which leads to the correct version of (16):

$$\int_0^{\infty}(2t)^{-3/4}e^{-2a^{1/2}t^{1/2}}e^{-pt}dt=\left[\frac{a}{2p^2}\right]^{1/4}\exp\left[\frac{a}{2p}\right]K_{1/4}\left(\frac{a}{2p}\right)$$

B Perturbative Treatment

Perturbative expansions for $Z(g)$ in (1) can be derived by expanding the exponential in the integral (1) and interchanging the order of the resulting summation and integration. In the *weak coupling* limit $g\to 0$ one obtains the divergent asymptotic expansion [1–3]:

$$Z(g)\ \sim\sum_{n=0}^{N}(-1)^n\frac{\Gamma(2n+1/2)}{n!\sqrt{\pi}}g^n \tag{17}$$

In the *strong coupling* limit $g\to\infty$ we obtain the convergent expansion [3]:

$$Z(g)\ \sim g^{-1/4}\sum_{n=0}^{N}(-1)^n\frac{\Gamma(n/2+1/4)}{2n!\sqrt{\pi}}g^{-n/2} \tag{18}$$

Both expansions, (17) and (18), can also be obtained from (1) using the appropriate expansion of $K_\nu(z)$ [2].

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