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Algebraic structure and basics of analysis of n-dimensional quaternionic space

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ABSTRACT

In this study, we focused on n-dimensional quaternionic space $\mathbb{H}^n$. To create the module structure, first part is devoted to define a metric depending on the product order relation of $\mathbb{R}^n$. The set of $\mathbb{H}^n$ has been rewritten with a different representation of n-vectors. Using this notation, formulations corresponding to the basic operations in $\mathbb{H}^n$ are obtained. By adhering these representations, module structure of $\mathbb{H}^n$ over the set of real ordered n-tuples is given. Afterwards, we gave limit, continuity and the derivative basics of quaternion valued functions of a real variable.

1. Introduction

Although most studies are on real space, many approaches differ when we consider the concept of dimension. For example, considering the limit, n-dimensional vector approximations offer a more rational solution. In this paper, we will examine real quaternions, a 4-dimensional number system, in n-dimensions.

In the literature, various ordering types are given on the n-dimensional real numbers space and also the product order is defined [1]. First of all, by considering the componentwise multiplication detailed in [2], we construct the ring structure of n-dimensional real numbers. Then, we define a metric depending on the order relation of the n-dimensional real numbers set [3].

Just as the elements of an n-dimensional real numbers space $\mathbb{R}^n$ are called real n-vectors, the elements of n-dimensional real quaternion space $\mathbb{H}^n$ can also be called quaternion n-vectors. The n-dimensional quaternionic space is a $4n$-dimensional vector space over the real numbers. We also deal with the properties of this vector space. Regarding this space, studies are carried out in areas such as symplectic geometry, Clifford algebra, etc [4, 5, 6, 7].

The module structure of n-dimensional quaternionic space has generally been studied over quaternions [4, 8, 9]. Different from them, by adapting the componentwise multiplication to the n-dimensional quaternionic space, the ring structure of $\mathbb{H}^n$ was established. Besides, vector space structure of n-dimensional quaternionic space over real space was formed. Thereafter, module structure of n-dimensional quaternionic space over the set of real ordered n-tuples was given using a different representation of quaternion n-vectors.

Along with these, in the last section, quaternion vector functions are defined, and according to the metric accepted, the limit, continuity and the derivative definitions for quaternion valued vector functions are analyzed.

2. Preliminaries

2.1. Real quaternion space

Real quaternions, first described by W. R. Hamilton in 1843, are 4-dimensional number system [10, 11]. Real quaternions have been studied in many areas such as algebra, geometry, physics, computer-aided design (CAD), put into practice and have provided technological improvements. This subsection is devoted to the basic definitions and results for real quaternions.

The set of all real quaternions $\mathbb{H}$ can be represented as

$$\mathbb{H} = \left\{ q \;\middle|\; q = a\vec{e}_0 + b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3,\ a,b,c,d \in \mathbb{R},\ \vec{e}_0,\vec{e}_1,\vec{e}_2,\vec{e}_3 \in \mathbb{R}^4 \right\}.$$

Each element of this set can be written as $q = S_q + V_q$, with the *real* (scalar) *part* $S_q = a\vec{e}_0$ and the *vector* (spatial, pure) *part* $V_q = b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3$.

Addition operation is defined as

$$q_1 \oplus q_2 = \left(a_1 + a_2\right)\vec{e}_0 + \left(b_1 + b_2\right)\vec{e}_1 + \left(c_1 + c_2\right)\vec{e}_2 + \left(d_1 + d_2\right)\vec{e}_3 \tag{1}$$

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and multiplication by a real scalar is defined as

$$\mu \odot q_1 = (\mu a_1)\vec{e}_0 + (\mu b_1)\vec{e}_1 + (\mu c_1)\vec{e}_2 + (\mu d_1)\vec{e}_3$$

for all $q_i = a_i\vec{e}_0 + b_i\vec{e}_1 + c_i\vec{e}_2 + d_i\vec{e}_3 \in \mathbb{H}$ $(i = 1, 2)$ and $\mu \in \mathbb{R}$.

Corollary 1. $\{\mathbb{H}, \oplus, \mathbb{R}, +, \cdot, \odot\}$ *is a vector space with* $\dim \mathbb{H} = 4$.

Multiplication of q_1 and q_2 is defined as

$$\begin{aligned} q_1 \times q_2 = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)\vec{e}_0 \\ & + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)\vec{e}_1 \\ & + (a_1c_2 + a_2c_1 - b_1d_2 + b_2d_1)\vec{e}_2 \\ & + (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)\vec{e}_3, \end{aligned} \tag{2}$$

for all $q_i = a_i\vec{e}_0 + b_i\vec{e}_1 + c_i\vec{e}_2 + d_i\vec{e}_3 \in \mathbb{H}$ $(i = 1, 2)$. Quaternion multiplication rewritten by accepting $q_1 = S_{q_1} + V_{q_1}$ and $q_2 = S_{q_2} + V_{q_2}$, such as

$$q_1 \times q_2 = S_{q_1}S_{q_2} - \left\langle V_{q_1}, V_{q_2} \right\rangle_{\mathbb{R}^3} + S_{q_1}V_{q_2} + S_{q_2}V_{q_1} + V_{q_1} \wedge_{\mathbb{R}^3} V_{q_2}$$

where, $\langle , \rangle_{\mathbb{R}^3}$ and $\wedge_{\mathbb{R}^3}$ are dot and vector products in $\mathbb{R}^3$, respectively.

Corollary 2. $(\mathbb{H}, \oplus, \times)$ *is a unitary ring. However, the quaternion multiplication is not commutative, so quaternions form a skew field.*

Conjugate of a quaternion $q = a\vec{e}_0 + b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3$ is defined as

$$\overline{q} = a\vec{e}_0 - b\vec{e}_1 - c\vec{e}_2 - d\vec{e}_3 = S_q - V_q. \tag{3}$$

From here, q is called *spatial quaternion* if $q \oplus \overline{q} = 0$ or $a = 0$. Also, all spatial quaternions set can be represented as

$$\mathbb{H}_P = \left\{ q \;\middle|\; q = b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3,\ b, c, d \in \mathbb{R},\ \vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^4 \right\}.$$

The real-valued inner product function $h : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is defined as

$$h(q_1, q_2) = \frac{1}{2}[q_1 \times \overline{q}_2 \oplus q_2 \times \overline{q}_1] = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2. \tag{4}$$

The quaternion-valued inner product function $\langle , \rangle_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is defined as

$$\begin{aligned} \langle q_1, q_2 \rangle_{\mathbb{H}} = q_1 \times \overline{q}_2 = {} & (a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)\vec{e}_0 \\ & + (-a_1b_2 + b_1a_2 - c_1d_2 + d_1c_2)\vec{e}_1 \\ & + (-a_1c_2 + c_1a_2 - d_1b_2 + b_1d_2)\vec{e}_2 \\ & + (-a_1d_2 + d_1a_2 - b_1c_2 + c_1b_2)\vec{e}_3. \end{aligned}$$

Norm of $q = a\vec{e}_0 + b\vec{e}_1 + c\vec{e}_2 + d\vec{e}_3$ is defined as

$$\begin{aligned} \|q\| & = \sqrt{h(q, q)} = \sqrt{\langle q, q \rangle_{\mathbb{H}}} = \sqrt{q \times \overline{q}} = \sqrt{\overline{q} \times q} \\ & = \sqrt{a^2 + b^2 + c^2 + d^2}. \end{aligned} \tag{5}$$

The quaternion q is called *unit real quaternion* if $\|q\| = 1$.

The inverse of a real quaternion is defined as

$$q^{-1} = \frac{\overline{q}}{\|q\|^2}.$$

Since inverse of a quaternion can also be defined as $q^{-1} = \|q\|^{-2}\,\overline{q}$ where $q \neq 0$, the space $\mathbb{H}$ is a division algebra.

In order to divide the quaternion $p \in \mathbb{H}$ to quaternion $q \in \mathbb{H}$, p must be multiplied with q^{-1}. However, since quaternion multiplication is not commutative, it is of two kinds: $p \times q^{-1}$ is called *right division of* p *with* q; $q^{-1} \times p$ is called *left division of* p *with* q.

2.2. n-Dimensional real space

Ordering is of great importance in many fields of mathematics. There are many different order types people have studied. One of them is *the product order* (also called the coordinatewise order or componentwise order) which is a partial ordering on a Cartesian product of ordered sets [1, 2].

The product order in $\mathbb{R}^n$ defined as

$$\vec{A} \leq \vec{B} \iff a_i \leq b_i$$

for all $\vec{A} = (a_i), \vec{B} = (b_i) \in \mathbb{R}^n$ $(1 \leq i \leq n)$.

The set of n-tuples of real numbers with componentwise addition and scalar multiplication is a real vector space. To describe the ring structure of space $\mathbb{R}^n$, a multiplication operation called *componentwise multiplication*, which is also used for lower dimensional real spaces, can be considered [2].

Component to component multiplication $\boxtimes_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined

$$\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B} = (a_1, a_2, \ldots, a_n) \boxtimes_{\mathbb{R}^n} (b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n) \tag{6}$$

for all $\vec{A} = (a_i), \vec{B} = (b_i) \in \mathbb{R}^n$.

Theorem 2.1. *Multiplication operation satisfies the properties as follows:*

(1) $\left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B}\right) \boxtimes_{\mathbb{R}^n} \vec{C} = \vec{A} \boxtimes_{\mathbb{R}^n} \left(\vec{B} \boxtimes_{\mathbb{R}^n} \vec{C}\right)$,
(2) $\vec{A} \boxtimes_{\mathbb{R}^n} \left(\vec{B} + \vec{C}\right) = \left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B}\right) + \left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{C}\right)$,
(3) $\left(\vec{A} + \vec{B}\right) \boxtimes_{\mathbb{R}^n} \vec{C} = \left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{C}\right) + \left(\vec{B} \boxtimes_{\mathbb{R}^n} \vec{C}\right)$,
(4) $\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B} = \vec{B} \boxtimes_{\mathbb{R}^n} \vec{A}$,
(5) $\vec{1} \boxtimes_{\mathbb{R}^n} \vec{A} = \vec{A}$

for all $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^n$ *and identity element is* $\vec{1} = (1, 1, \ldots, 1)$.

Proof. *(1)* For all $\vec{A} = (a_i), \vec{B} = (b_i), \vec{C} = (c_i) \in \mathbb{R}^n$,

$$\begin{aligned} \left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{B}\right) \boxtimes_{\mathbb{R}^n} \vec{C} & = [(a_1, a_2, \ldots, a_n) \boxtimes_{\mathbb{R}^n} (b_1, b_2, \ldots, b_n)] \boxtimes_{\mathbb{R}^n} (c_1, c_2, \ldots, c_n) \\ & = (a_1b_1, a_2b_2, \ldots, a_nb_n) \boxtimes_{\mathbb{R}^n} (c_1, c_2, \ldots, c_n) \\ & = ((a_1b_1)c_1, (a_2b_2)c_2, \ldots, (a_nb_n)c_n) \\ & = (a_1(b_1c_1), a_2(b_2c_2), \ldots, a_n(b_nc_n)) \\ & = (a_1, a_2, \ldots, a_n) \boxtimes_{\mathbb{R}^n} (b_1c_1, b_2c_2, \ldots, b_nc_n) \\ & = (a_1, a_2, \ldots, a_n) \boxtimes_{\mathbb{R}^n} [(b_1, b_2, \ldots, b_n) \boxtimes_{\mathbb{R}^n} (c_1, c_2, \ldots, c_n)] \\ & = \vec{A} \boxtimes_{\mathbb{R}^n} \left(\vec{B} \boxtimes_{\mathbb{R}^n} \vec{C}\right). \end{aligned}$$

Other proofs can be made similarly. □

Corollary 3. $\mathbb{R}^n$ *is a commutative and unitary ring with addition and componentwise multiplication.*

On the other hand, vectors in space $\mathbb{R}^n$ can be viewed as $1 \times n$ matrices and we can use Hadamard matrix product. With fixing a matrix size, the set of such matrices forms a ring under matrix addition and Hadamard multiplication [12]. According to the assumptions, it is encountered that this ring is not an integral domain since it contains zero divisors. To avoid this, we will give some definitions in Section 3.

3. Metric on $\mathbb{R}^n$

Definition 4. Inverse of an n-vector can be defined as

$$\frac{\vec{1}}{\vec{A}} = \vec{A}^{-1} = \left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right) = \left(a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\right)$$

for all $\vec{A} = (a_i) \in \mathbb{R}^n$ where $\forall a_i \neq 0$ $(1 \leq i \leq n)$.

Theorem 3.1. *Inverse operation satisfies the properties as follows:*

- $\vec{A}\boxtimes_{\mathbb{R}^n}\vec{A}^{-1}=\vec{A}^{-1}\boxtimes_{\mathbb{R}^n}\vec{A}=\vec{1}$,
- $(\vec{A}\boxtimes_{\mathbb{R}^n}\vec{B})^{-1}=\vec{A}^{-1}\boxtimes_{\mathbb{R}^n}\vec{B}^{-1}\quad(\forall a_i,b_i\neq 0)$.

Proof. Theorem can be easily proved by using (6). □

Definition 5. Division on $\mathbb{R}^n$ is defined as

$$\frac{\vec{A}}{\vec{B}}=\vec{A}\boxtimes_{\mathbb{R}^n}\vec{B}^{-1}=\vec{B}^{-1}\boxtimes_{\mathbb{R}^n}\vec{A}$$

for all $\vec{A},\vec{B}\in\mathbb{R}^n$ where $\vec{B}\neq\vec{0}$.

Definition 6. Consider the real exponential function $f_{a_i}:\mathbb{R}\to\mathbb{R}$, $f_{a_i}(x)=a_i^x$. A function $F_x:\mathbb{R}^n\to\mathbb{R}^n$ is defined by

$$F_x(\vec{A})=\vec{A}^x=(f_{a_1},f_{a_2},\dots,f_{a_n})=(a_1^x,a_2^x,\dots,a_n^x)$$

such that $\vec{A}>\vec{0}$ and $\forall a_i\neq 1$ for all $\vec{A}=(a_i)\in\mathbb{R}^n$ $(1\le i\le n)$. Clearly, this function is well-defined.

Theorem 3.2. *On $\mathbb{R}^n$ ($\mathbb{R}^n$-module), function $\|.\|:\mathbb{R}^n\to\mathbb{R}^n$,*

$$\left\|\vec{A}\right\|=\sqrt{\vec{A}\boxtimes_{\mathbb{R}^n}\vec{A}}=\left(\sqrt{a_1^2},\sqrt{a_2^2},\dots,\sqrt{a_n^2}\right)=(|a_1|,|a_2|,\dots,|a_n|)$$

for all $\vec{A}=(a_i)\in\mathbb{R}^n$ defines a norm. With this norm function,

$$d(\vec{A},\vec{B})=\left\|\vec{A}-\vec{B}\right\|$$

defines a metric, called the componentwise metric for $\vec{A},\vec{B}\in\mathbb{R}^n$.

Proof. We first check that the given formula defines a norm.

(N1) Every $|a_i|\ge 0$ for $\left\|\vec{A}\right\|=(|a_1|,|a_2|,\dots,|a_n|)$, then $\left\|\vec{A}\right\|\ge\vec{0}$.

(N2) Every $|a_i|=0$ for $\left\|\vec{A}\right\|=(|a_1|,|a_2|,\dots,|a_n|)=\vec{0}$, then $a_i=0$ and also $\vec{A}=\vec{0}$.

(N3) Suppose the scalar is $\vec{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$ we can write,

$$\begin{aligned}\left\|\vec{\lambda}\boxtimes_{\mathbb{R}^n}\vec{A}\right\|&=(|\lambda_1a_1|,|\lambda_2a_2|,\dots,|\lambda_na_n|)\\&=(|\lambda_1||a_1|,|\lambda_2||a_2|,\dots,|\lambda_n||a_n|)\\&=\left\|\vec{\lambda}\right\|\boxtimes_{\mathbb{R}^n}\left\|\vec{A}\right\|.\end{aligned}$$

Also, $\left\|\lambda\odot\vec{A}\right\|=|\lambda|\left\|\vec{A}\right\|$ obtained for $\lambda\in\mathbb{R}$.

(N4) With using the triangle inequality on $\mathbb{R}$,

$$\begin{aligned}\left\|\vec{A}+\vec{B}\right\|&=(|a_1+b_1|,|a_2+b_2|,\dots,|a_n+b_n|)\\&\le(|a_1|+|b_1|,|a_2|+|b_2|,\dots,|a_n|+|b_n|)=\left\|\vec{A}\right\|+\left\|\vec{B}\right\|\end{aligned}$$

for all $\vec{A}=(a_i)\in\mathbb{R}^n$ $(1\le i\le n)$. So, $d(\vec{A},\vec{B})$ is also a metric, as each norm defines a metric. □

4. *n*-Dimensional quaternionic space

The set of all real quaternion n-vectors $\mathbb{H}^n$ can be represented as

$$\mathbb{H}^n=\mathbb{H}\times\mathbb{H}\times\cdots\times\mathbb{H}=\left\{\vec{q}=(q_i)=(q_1,q_2,\dots q_n)\mid q_i\in\mathbb{H},\ 1\le i\le n\right\}.$$

Each component of this set are quaternions $q_i=S_{q_i}+V_{q_i}\in\mathbb{H}$, hence,

$$\vec{q}=\left(S_{q_1},S_{q_2},\dots,S_{q_n}\right)+\left(V_{q_1},V_{q_2},\dots,V_{q_n}\right)=S_{\vec{q}}+V_{\vec{q}}.$$

That is, each quaternion n-vector can be written as the sum of the *real part* $S_{\vec{q}}$ and the *vector part* $V_{\vec{q}}$. As a matter of course, the set of *spatial quaternion n-vectors* is defined as

$$\mathbb{H}_P^n=\mathbb{H}_P\times\mathbb{H}_P\times\cdots\times\mathbb{H}_P=\left\{\vec{q}=(q_i)=(q_1,q_2,\dots q_n)\mid q_i\in\mathbb{H}_P,\ 1\le i\le n\right\}.$$

Addition operation $\boxplus:\mathbb{H}^n\times\mathbb{H}^n\to\mathbb{H}^n$ is defined as

$$\vec{q}\boxplus\vec{p}=\left(q_1\oplus p_1,q_2\oplus p_2,\dots,q_n\oplus p_n\right)$$

and multiplication by a scalar operation $\odot:\mathbb{H}^n\times\mathbb{H}\to\mathbb{H}^n$ is defined as

$$\vec{q}\odot\lambda=(q_i)\odot\lambda=(q_i\times\lambda)=(q_1\times\lambda,q_2\times\lambda,\dots,q_n\times\lambda)\tag{7}$$

for all $\vec{q}=(q_i)=(q_1,q_2,\dots,q_n),\vec{p}=(p_i)=(p_1,p_2,\dots,p_n)\in\mathbb{H}^n$, $\lambda\in\mathbb{H}$.

Corollary 7. *$\{\mathbb{H}^n,\boxplus,\odot\}=\mathbb{H}^n$ is a right $\mathbb{H}$-module with $dim\mathbb{H}^n=n$.*

Inner product function $\langle,\rangle_{\mathbb{H}^n}:\mathbb{H}^n\times\mathbb{H}^n\to\mathbb{H}$ is defined as

$$\langle\vec{q},\vec{p}\rangle_{\mathbb{H}^n}=\sum_{i=1}^n\bar{q}_i\times p_i\tag{8}$$

for all $\vec{q}=(q_i)$, $\vec{p}=(p_i)\in\mathbb{H}^n$, $q_i,p_i\in\mathbb{H}$ $(1\le i\le n)$.

Norm function $\|.\|_{\mathbb{H}^n}:\mathbb{H}^n\to\mathbb{R}$ is defined as

$$\begin{aligned}\|\vec{q}\|_{\mathbb{H}^n}&=\sqrt{\langle\vec{q},\vec{q}\rangle_{\mathbb{H}^n}}=\sqrt{\sum_{i=1}^n\bar{q}_i\times q_i}=\sqrt{\sum_{i=1}^n\|q_i\|^2}\\&=\sqrt{\|q_1\|^2+\|q_2\|^2+\cdots+\|q_n\|^2}\end{aligned}$$

for all $\vec{q}=(q_i)\in\mathbb{H}^n$ [4, 5].

4.1. Ring structure of $\mathbb{H}^n$

Multiplication operation $\boxtimes:\mathbb{H}^n\times\mathbb{H}^n\to\mathbb{H}^n$ is defined as

$$\vec{q}\boxtimes\vec{p}=\left(q_1\times p_1,\ q_2\times p_2,\ \dots,\ q_n\times p_n\right)\tag{9}$$

for all $\vec{q}=(q_i)$, $\vec{p}=(p_i)\in\mathbb{H}^n$, $q_i,p_i\in\mathbb{H}$ $(1\le i\le n)$. This operation is called *quaternion n-vectors componentwise multiplication* on $\mathbb{H}^n$. It is clear that, here, $\oplus$ and $\times$ are the quaternion addition and multiplication, respectively, as we have noted by (1) and (2).

Corollary 8. *$(\mathbb{H}^n,\boxplus,\boxtimes)$ is a unitary ring.*

Conjugate of a quaternion n-vector is defined as

$$\bar{\vec{q}}=(\bar{q}_1,\bar{q}_2,\dots,\bar{q}_n)=\left(S_{q_1}-V_{q_1},S_{q_2}-V_{q_2},\dots,S_{q_n}-V_{q_n}\right)=S_{\vec{q}}-V_{\vec{q}}$$

for all $\vec{q}=(q_i)\in\mathbb{H}^n$.

Theorem 4.1. *Conjugate operation satisfies the following properties as follows:*

(1) $\overline{\vec{q}\boxplus\vec{p}}=\bar{\vec{q}}\boxplus\bar{\vec{p}}$,
(2) $\overline{\vec{q}\boxtimes\vec{p}}=\bar{\vec{p}}\boxtimes\bar{\vec{q}}$,
(3) $\bar{\bar{\vec{q}}}=\vec{q}$,
(4) $S_{\vec{q}}=\frac{\vec{q}\boxplus\bar{\vec{q}}}{2}$,
(5) $V_{\vec{q}}=\frac{\vec{q}-\bar{\vec{q}}}{2}$,
(6) $\bar{\vec{q}}=\vec{q}$, $\vec{q}=\left(S_{q_1},S_{q_2},\dots,S_{q_n}\right)=S_{\vec{q}}$,
(7) $\bar{\vec{q}}=-\vec{q}$, $\vec{q}=\left(V_{q_1},V_{q_2},\dots,V_{q_n}\right)=V_{\vec{q}}$

for all $\vec{q},\vec{p}\in\mathbb{H}^n$.

Proof. *(2)* With using quaternion n-vectors componentwise multiplication, we can write,

$$\overline{\vec{q}\boxtimes\vec{p}}=\left(\overline{q_1\times p_1},\overline{q_2\times p_2},\dots,\overline{q_n\times p_n}\right)$$

for all $\vec{q}=(q_i)$, $\vec{p}=(p_i)\in\mathbb{H}^n$. Considering the property of the conjugate of a quaternion given by (3) indicates

$$\overline{\vec{q}\boxtimes\vec{p}}=(\bar{p}_1\times\bar{q}_1,\bar{p}_2\times\bar{q}_2,\dots,\bar{p}_n\times\bar{q}_n)=\bar{\vec{p}}\boxtimes\bar{\vec{q}}.\quad\square$$

4.2. Vector space structure of n-dimensional quaternionic space over real space

We can write the multiplication operator $\boxdot : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n$ with real scalar on $\mathbb{H}^n$ as

$$\vec{q} \boxdot \lambda = (q_1 \odot \lambda, q_2 \odot \lambda, \ldots, q_n \odot \lambda) \tag{10}$$

for all $\vec{q} = (q_i) \in \mathbb{H}^n$ and $\lambda \in \mathbb{R}$. Clearly, this operation is compatible with (7), if scalars in $\mathbb{H}$ are restricted to scalars in $\mathbb{R}$. Hence, we can give the following corollary.

Corollary 9. *$\{\mathbb{H}^n, \boxplus, \boxdot\}$ is a vector space over the real space.*

Also, with the multiplication between basis, we write the elements in $\mathbb{H}^n$ using the basis $\vec{\varepsilon}_i = (0,0,\ldots,1,\ldots,0,0)$ $(1 \le i \le n)$ as

$$\begin{aligned}
\vec{q} &= (q_1, q_2, \ldots, q_n) \\
&= \vec{\varepsilon}_1 q_1 + \vec{\varepsilon}_2 q_2 + \cdots + \vec{\varepsilon}_n q_n \\
&= \vec{\varepsilon}_1 \left(a_1 + b_1\vec{e}_1 + c_1\vec{e}_2 + d_1\vec{e}_3\right) + \vec{\varepsilon}_2 \left(a_2 + b_2\vec{e}_1 + c_2\vec{e}_2 + d_2\vec{e}_3\right) + \ldots \\
&+ \vec{\varepsilon}_n \left(a_n + b_n\vec{e}_1 + c_n\vec{e}_2 + d_n\vec{e}_3\right) \\
&= a_1(\vec{\varepsilon}_1\vec{e}_0) + a_2(\vec{\varepsilon}_2\vec{e}_0) + \cdots + a_n(\vec{\varepsilon}_n\vec{e}_0) + b_1\left(\vec{\varepsilon}_1\vec{e}_1\right) + b_2\left(\vec{\varepsilon}_2\vec{e}_1\right) + \ldots \\
&+ b_n\left(\vec{\varepsilon}_n\vec{e}_1\right) \\
&+ c_1\left(\vec{\varepsilon}_1\vec{e}_2\right) + c_2\left(\vec{\varepsilon}_2\vec{e}_2\right) + \cdots + c_n\left(\vec{\varepsilon}_n\vec{e}_2\right) + d_1\left(\vec{\varepsilon}_1\vec{e}_3\right) + d_2\left(\vec{\varepsilon}_1\vec{e}_3\right) + \ldots \\
&+ d_n\left(\vec{\varepsilon}_1\vec{e}_3\right)
\end{aligned}$$

for all $q_i = a_i\vec{e}_0 + b_i\vec{e}_1 + c_i\vec{e}_2 + d_i\vec{e}_3 \in \mathbb{H}$ and $a_i, b_i, c_i, d_i \in \mathbb{R}$ $(1 \le i \le n)$. So,

$$\begin{aligned}
\mathbb{H}^n = span_{\mathbb{R}}\{&\vec{\varepsilon}_1\vec{e}_0, \vec{\varepsilon}_2\vec{e}_0, \ldots, \vec{\varepsilon}_n\vec{e}_0, \vec{\varepsilon}_1\vec{e}_1, \vec{\varepsilon}_2\vec{e}_1, \ldots, \vec{\varepsilon}_n\vec{e}_1, \vec{\varepsilon}_1\vec{e}_2, \vec{\varepsilon}_2\vec{e}_2, \ldots, \\
&\vec{\varepsilon}_n\vec{e}_2, \vec{\varepsilon}_1\vec{e}_3, \vec{\varepsilon}_2\vec{e}_3, \ldots, \vec{\varepsilon}_n\vec{e}_3\}
\end{aligned}$$

can be written and $dim\mathbb{H}^n = 4n$.

Definition 10. Inner product function $\langle , \rangle_{\mathbb{H}^n}^{\mathbb{R}} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ is defined as

$$\left\langle \vec{q}, \vec{p} \right\rangle_{\mathbb{H}^n}^{\mathbb{R}} = \sum_{i=1}^{n} h\left(q_i, p_i\right) = h\left(q_1, p_1\right) + h\left(q_2, p_2\right) + \cdots + h\left(q_n, p_n\right) \tag{11}$$

for all $\vec{q} = (q_i)$, $\vec{p} = (p_i) \in \mathbb{H}^n$.

Definition 11. Norm function $\|.\|_{\mathbb{H}^n}^{\mathbb{R}} : \mathbb{H}^n \to \mathbb{R}$ is defined as

$$\|\vec{q}\|_{\mathbb{H}^n}^{\mathbb{R}} = \sqrt{\left\langle \vec{q}, \vec{q} \right\rangle_{\mathbb{H}^n}^{\mathbb{R}}} = \sqrt{\sum_{i=1}^{n} h\left(q_i, p_i\right)} = \sqrt{\|q_1\|^2 + \|q_2\|^2 + \cdots + \|q_n\|^2}$$

for all $\vec{q} = (q_i) \in \mathbb{H}^n$.

4.3. A representation of quaternionic n-vector with real ordered n-tuples

A quaternion n-tuple in $\mathbb{H}^n$ is given as $\vec{q} = (q_1, q_2, \ldots, q_n)$. Now, rewrite an n-vector with the coefficients in $\mathbb{R}^n$. Suppose the quaternions,

$$\begin{aligned}
&q_1 = a_1\vec{e}_0 + b_1\vec{e}_1 + c_1\vec{e}_2 + d_1\vec{e}_3 \in \mathbb{H}, \\
&q_2 = a_2\vec{e}_0 + b_2\vec{e}_1 + c_2\vec{e}_2 + d_2\vec{e}_3 \in \mathbb{H}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&q_n = a_n\vec{e}_0 + b_n\vec{e}_1 + c_n\vec{e}_2 + d_n\vec{e}_3 \in \mathbb{H}
\end{aligned}$$

are given. We can write with taking the real n-vector coefficients into bases brackets

$$\begin{aligned}
\vec{q} &= \left(a_1, a_2, \ldots, a_n\right)\vec{e}_0 + \left(b_1, b_2, \ldots, b_n\right)\vec{e}_1 + \left(c_1, c_2, \ldots, c_n\right)\vec{e}_2 \\
&\quad + \left(d_1, d_2, \ldots, d_n\right)\vec{e}_3 \\
&= \vec{A}\vec{e}_0 + \vec{B}\vec{e}_1 + \vec{C}\vec{e}_2 + \vec{D}\vec{e}_3
\end{aligned} \tag{12}$$

where $\vec{A} = (a_1, a_2, \ldots, a_n)$, $\vec{B} = (b_1, b_2, \ldots, b_n)$, $\vec{C} = (c_1, c_2, \ldots, c_n)$, $\vec{D} = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$.

The set of quaternion n-vectors is also defined as

$$\mathbb{H}^n = \left\{ \vec{q} \;\middle|\; \vec{q} = \vec{A}\vec{e}_0 + \vec{B}\vec{e}_1 + \vec{C}\vec{e}_2 + \vec{D}\vec{e}_3,\ \vec{A}, \vec{B}, \vec{C}, \vec{D} \in \mathbb{R}^n \right\}.$$

Here, $S_{\vec{q}} = \vec{A}\vec{e}_0$ is the *scalar part* and $V_{\vec{q}} = \vec{B}\vec{e}_1 + \vec{C}\vec{e}_2 + \vec{D}\vec{e}_3$ is the *vector part* of quaternion n-vectors.

Now, we give a corollary for writing the formulas again of in (9) in terms of the representation (12).

Corollary 12. *Rearranging the multiplication of n-vectors is*

$$\begin{aligned}
\vec{q} \boxtimes \vec{p} &= \left(\vec{A}_1\vec{e}_0 + \vec{B}_1\vec{e}_1 + \vec{C}_1\vec{e}_2 + \vec{D}_1\vec{e}_3\right) \boxtimes \left(\vec{A}_2\vec{e}_0 + \vec{B}_2\vec{e}_1 + \vec{C}_2\vec{e}_2 + \vec{D}_2\vec{e}_3\right) \\
&= \left(\vec{A}_1 \boxtimes_{\mathbb{R}^n} \vec{A}_2 - \vec{B}_1 \boxtimes_{\mathbb{R}^n} \vec{B}_2 - \vec{C}_1 \boxtimes_{\mathbb{R}^n} \vec{C}_2 - \vec{D}_1 \boxtimes_{\mathbb{R}^n} \vec{D}_2\right)\vec{e}_0 \\
&+ \left(\vec{A}_1 \boxtimes_{\mathbb{R}^n} \vec{B}_2 + \vec{B}_1 \boxtimes_{\mathbb{R}^n} \vec{A}_2 + \vec{C}_1 \boxtimes_{\mathbb{R}^n} \vec{D}_2 - \vec{C}_2 \boxtimes_{\mathbb{R}^n} \vec{D}_1\right)\vec{e}_1 \\
&+ \left(\vec{A}_1 \boxtimes_{\mathbb{R}^n} \vec{C}_2 + \vec{C}_1 \boxtimes_{\mathbb{R}^n} \vec{A}_2 - \vec{B}_1 \boxtimes_{\mathbb{R}^n} \vec{D}_2 + \vec{B}_2 \boxtimes_{\mathbb{R}^n} \vec{D}_1\right)\vec{e}_2 \\
&+ \left(\vec{A}_1 \boxtimes_{\mathbb{R}^n} \vec{D}_2 + \vec{D}_1 \boxtimes_{\mathbb{R}^n} \vec{A}_2 + \vec{B}_1 \boxtimes_{\mathbb{R}^n} \vec{C}_2 - \vec{B}_2 \boxtimes_{\mathbb{R}^n} \vec{C}_1\right)\vec{e}_3
\end{aligned}$$

for all $\vec{q} = \vec{A}_1\vec{e}_0 + \vec{B}_1\vec{e}_1 + \vec{C}_1\vec{e}_2 + \vec{D}_1\vec{e}_3$, $\vec{p} = \vec{A}_2\vec{e}_0 + \vec{B}_2\vec{e}_1 + \vec{C}_2\vec{e}_2 + \vec{D}_2\vec{e}_3 \in \mathbb{H}^n$.

4.4. Module structure of n-dimensional quaternionic space over n-dimensional real space

This part, we first define a scalar multiplication operation that will allow us to construct the module structure. Then, we define the inner product and norm functions on this module.

Multiplication by real n-vector $\boxdot : \mathbb{H}^n \times \mathbb{R}^n \to \mathbb{H}^n$ is defined as

$$\begin{aligned}
\vec{q} \boxdot \vec{\lambda} &= \left(\vec{A}\vec{e}_0 + \vec{B}\vec{e}_1 + \vec{C}\vec{e}_2 + \vec{D}\vec{e}_3\right) \boxdot \vec{\lambda} \\
&= \left(\vec{A} \boxtimes_{\mathbb{R}^n} \vec{\lambda}\right)\vec{e}_0 + \left(\vec{B} \boxtimes_{\mathbb{R}^n} \vec{\lambda}\right)\vec{e}_1 + \left(\vec{C} \boxtimes_{\mathbb{R}^n} \vec{\lambda}\right)\vec{e}_2 + \left(\vec{D} \boxtimes_{\mathbb{R}^n} \vec{\lambda}\right)\vec{e}_3
\end{aligned}$$

for all $\vec{q} = \vec{A}\vec{e}_0 + \vec{B}\vec{e}_1 + \vec{C}\vec{e}_2 + \vec{D}\vec{e}_3 \in \mathbb{H}^n$ and $\vec{\lambda} \in \mathbb{R}^n$.

Multiplication operation is also defined as

$$\vec{q} \boxdot \vec{\lambda} = (q_1 \odot \lambda_1, q_2 \odot \lambda_2, \ldots, q_n \odot \lambda_n)$$

for all $\vec{q} = (q_i) \in \mathbb{H}^n$, $q_i \in \mathbb{H}$ and $\vec{\lambda} = (\lambda_i) \in \mathbb{R}^n$ $(1 \le i \le n)$.

Corollary 13. *$\{\mathbb{H}^n, \boxplus, \boxdot\}$ is a module over n-dimensional real space. Also, $\mathbb{H}^n = span_{\mathbb{R}^n}\left\{\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3\right\}$ is written. In that case,* $\dim \mathbb{H}^n = 4$.

Definition 14. Inner product $\langle\, , \rangle_{\mathbb{H}^n}^{\mathbb{R}^n} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}^n$ is defined as

$$\left\langle \vec{q}, \vec{p} \right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \frac{1}{2}\left[\left(\vec{q} \boxtimes \bar{\vec{p}}\right) \boxplus \left(\vec{p} \boxtimes \bar{\vec{q}}\right)\right] \tag{13}$$

for all $\vec{q}, \vec{p} \in \mathbb{H}^n$.

Corollary 15. *Inner product can also be written as*

$$\left\langle \vec{q}, \vec{p} \right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \left(h\left(q_1, p_1\right), h\left(q_2, p_2\right), \ldots, h\left(q_n, p_n\right)\right) \tag{14}$$

or

$$\left\langle \vec{q}, \vec{p} \right\rangle_{\mathbb{H}^n}^{\mathbb{R}^n} = \vec{A}_1 \boxtimes_{\mathbb{R}^n} \vec{A}_2 + \vec{B}_1 \boxtimes_{\mathbb{R}^n} \vec{B}_2 + \vec{C}_1 \boxtimes_{\mathbb{R}^n} \vec{C}_2 + \vec{D}_1 \boxtimes_{\mathbb{R}^n} \vec{D}_2 \tag{15}$$

for all $\vec{q} = \vec{A}_1\vec{e}_0 + \vec{B}_1\vec{e}_1 + \vec{C}_1\vec{e}_2 + \vec{D}_1\vec{e}_3, \vec{p} = \vec{A}_2\vec{e}_0 + \vec{B}_2\vec{e}_1 + \vec{C}_2\vec{e}_2 + \vec{D}_2\vec{e}_3 \in \mathbb{H}^n$.

Remark 1. Note that, equations (13), (14) and (15) are equal to each other.

Definition 16. Norm function $\|\cdot\|_{\mathbb{H}^n}^{\mathbb{R}^n} : \mathbb{H}^n \to \mathbb{R}^n$ is defined as

$$\|\vec{q}\|_{\mathbb{H}^n}^{\mathbb{R}^n} = \sqrt{\langle \vec{q}, \vec{q} \rangle_{\mathbb{H}^n}^{\mathbb{R}^n}} = \sqrt{\vec{q} \boxtimes \bar{\vec{q}}} = \left(\|q_1\|, \|q_2\|, \ldots, \|q_n\|\right) \tag{16}$$

for all $\vec{q}, \vec{p} \in \mathbb{H}^n$. Here, components are quaternion norm function real values given by (5).

Remark 2. Considering all the algebraic structures of the $\mathbb{H}^n$ space, it is noteworthy that the $\mathbb{H}^n$ space is $4n$-dimensional over the space of real numbers, n-dimensional over the space of quaternions, and 4-dimensional over the space of real n-vectors.

5. Limits, continuity, and the derivative in $\mathbb{H}^n$

This section, we will examine function $\vec{\gamma} : \mathbb{R} \to \mathbb{H}^n$ which are the quaternion n-vector valued vector functions. For this examination, which metric system we work with will play an important role in the domain of the defined function.

For the analysis concepts of such functions, we will use the $\mathbb{R}^n$-module structure of the $\mathbb{H}^n$ space. We discuss the limits, continuity and the derivatives of n-dimensional quaternionic space in detail with consider the componentwise metric on $\mathbb{H}^n$ ($\mathbb{R}^n$-module) as we have mentioned in Section 3.

Definition 17. Let $\mathbb{H}^n$ be the real quaternion n-vectors set. Let $I = [0, 1]$ be an interval in $\mathbb{R}$, $t \in I$ be real parameter and $\gamma_i : \mathbb{R} \to \mathbb{H}$ $(1 \leq i \leq n)$ be the quaternion valued functions of a real variable, then

$$\vec{\gamma} : \mathbb{R} \to \mathbb{H}^n,\ \vec{\gamma}(t) = \left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\right)$$

is called the quaternion valued vector function of a real variable.

Definition 18. If the vector $\vec{\gamma} = \vec{\gamma}(t)$ gives the same vector for all values of the parameter t, then the vector $\vec{\gamma}$ is called the constant n-vector.

Definition 19. Let $\vec{\gamma} : \mathbb{R} \to \mathbb{H}^n$ be a quaternion valued vector function of a real variable defined on some open interval that contains the number $a \in \mathbb{R}$, except possibly at a itself. Let $\vec{L}$ be an n-vector. The limit of $\vec{\gamma}$, as t value approaches a, is $\vec{L}$, denoted by

$$\lim_{t \to a} \vec{\gamma}(t) = \vec{L},$$

means that for every $\vec{\varepsilon} \succ \vec{0}$, there exists $\delta_i = \delta_i(\varepsilon_i) > 0$ $(1 \leq i \leq n)$, such that $0 < |t - a| < \delta_i$ and $t \in \mathbb{R}$ implies $\|\vec{\gamma}(t) - \vec{L}\|_{\mathbb{H}^n}^{\mathbb{R}^n} \prec \vec{\varepsilon}$.

Remark 3. The limit value obtained in the space $\mathbb{H}^n$ is a real n-vector.

Theorem 5.1. *Let $\gamma_i : \mathbb{R} \to \mathbb{H}$ $(1 \leq i \leq n)$ be a quaternion valued functions of a real variable. For all $L_i \in \mathbb{H}$ $(1 \leq i \leq n)$, according to the componentwise metric, limit of $\vec{\gamma}$ is $\lim_{t \to a} \vec{\gamma}(t) = \vec{L} = \left(L_1, L_2, \ldots, L_n\right) \in \mathbb{H}^n$ at point a if and only if $\lim_{t \to a} \gamma_i(t) = L_i$.*

Example 20. The function $\vec{\gamma} : \mathbb{R} \to \mathbb{H}^3$ defined by $\vec{\gamma}(t) = (2t + 2t^2 i, t^3 + tj, t + t^2 k)$ is given. Using the formal definition of limit, it can be observed that the limit of the function $\vec{\gamma}$ at point $t = 3$ is $\vec{L} = (6 + 18i, 27 + 3j, 3 + 9k)$.

Considering the componentwise metric on $\mathbb{H}^3$, for all $\vec{\varepsilon} \succ \vec{0}$ we will examine whether there exists $\delta_i > 0$ $(1 \leq i \leq 3)$ values for each component, such that $|t - 3| < \delta_i$ and $t \in \mathbb{R}$ implies $\|\vec{\gamma}(t) - \vec{L}\|_{\mathbb{H}^3}^{\mathbb{R}^3} \prec \vec{\varepsilon}$.

First, calculate $\|\vec{\gamma}(t) - \vec{L}\|_{\mathbb{H}^3}^{\mathbb{R}^3}$ by using (16),

$$\begin{aligned}\|\vec{\gamma}(t) - \vec{L}\|_{\mathbb{H}^3}^{\mathbb{R}^3} = &\big(\|(2t-6) + (2t^2 - 18)i\|, \|(t^3 - 27) + (t-3)j\|, \\ &\|(t-3) + (t^2 - 9)k\|\big).\end{aligned}$$

For the first component, by using (5)

$$\begin{aligned}\|(2t-6) + (2t^2 - 18)i\| &= \sqrt{(2t-6)^2 + (2t^2 - 18)^2} \\ &= 2|t-3|\sqrt{1 + (t+3)^2}\end{aligned}$$

we'll start by simplifying the inequality in an attempt to get to choose δ_1. We can assume $|t - 3| < \delta_1$ and when $\delta_1 = 1$, we have $\sqrt{1 + (t+3)^2} < \sqrt{50}$ for all $\varepsilon_1 > 0$, and

$$\|(2t-6) + (2t^2 - 18)i\| < |t-3| 2\sqrt{50} < \left(\frac{\varepsilon_1}{2\sqrt{50}}\right) 2\sqrt{50} = \varepsilon_1$$

guarantee that $\exists \delta_1 = \min\left\{1, \dfrac{\varepsilon_1}{2\sqrt{50}}\right\}$.

Make the same assumption for the second and the third components and we'll see the quaternion valued functions limit exists at point $t = 3$. So, by our definition of limit for quaternion valued vector function, we can write, for every $\varepsilon_i > 0$ there exists $\delta_i > 0$ $(1 \leq i \leq 3)$. For the function $\vec{\gamma}$, we obtained $\lim_{t \to 3} \vec{\gamma}(t) = (6 + 18i, 27 + 3j, 3 + 9k)$ according to the componentwise metric.

Definition 21. Let $\vec{\gamma}$ be a quaternion valued vector function of a real variable defined on $[c, d]$ real line. $\vec{\gamma}$ is continuous at a point $a \in [c, d]$ if the following three conditions are satisfied:

- $\vec{\gamma}(a)$ is defined;
- $\lim_{t \to a} \vec{\gamma}(t)$ exists; and
- $\lim_{t \to a} \vec{\gamma}(t) = \vec{\gamma}(a)$.

Also, $\vec{\gamma}$ function is continuous over an open interval $[c, d]$ if it is continuous at every point in the interval.

Definition 22. The derivative of the quaternion valued vector function of a real variable $\vec{\gamma} : \mathbb{R} \to \mathbb{H}^n$, $\vec{\gamma}(t) = \left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\right)$ at point $t_0 \in \mathbb{R}$ is defined as

$$\vec{\gamma}'\left(t_0\right) = \frac{d}{dt}\vec{\gamma}\left(t_0\right) = \lim_{h \to 0} \vec{\gamma}\left[\left(t_0 + h\right) - \vec{\gamma}\left(t_0\right)\right] \boxdot h^{-1}$$

for $h \in \mathbb{R}$. Here, multiplication is given by (10).

Theorem 5.2. *The derivative of a quaternion valued vector function of a real variable $\vec{\gamma} : \mathbb{R} \to \mathbb{H}^n$, $\vec{\gamma}(t) = \left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)\right)$ is obtained by differentiating the component functions separately and writing them as components of the derivative function with the help of the derivatives of the quaternion valued functions $\gamma_i : \mathbb{R} \to \mathbb{H}$ $(1 \leq i \leq n)$:*

$$\vec{\gamma}'(t) = \left(\gamma_1'(t), \gamma_2'(t), \ldots, \gamma_n'(t)\right).$$

Theorem 5.3. *$\vec{\gamma}$ and $\vec{\beta}$ be quaternion valued differentiable vector functions of a real variable, $\boxdot : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^n$ be the multiplication operation by a real scalar, λ be a scalar valued function. The following properties are satisfied:*

(1) $\dfrac{d}{dt}\left(\vec{\gamma}(t) \mp \vec{\beta}(t)\right) = \dfrac{d}{dt}\vec{\gamma}(t) \mp \dfrac{d}{dt}\vec{\beta}(t)$,

(2) $\dfrac{d}{dt}\left(\vec{\gamma}(t) \boxdot \lambda(t)\right) = \left(\dfrac{d}{dt}\vec{\gamma}(t)\right) \boxdot \lambda(t) \boxplus \vec{\gamma}(t) \boxdot \left(\dfrac{d}{dt}\lambda(t)\right)$,

(3) $\dfrac{d}{dt}\left[\vec{\gamma}(t) \boxtimes \vec{\beta}(t)\right] = \left(\dfrac{d}{dt}\vec{\gamma}(t) \boxtimes \vec{\beta}(t)\right) \boxplus \left(\vec{\gamma}(t) \boxtimes \dfrac{d}{dt}\vec{\beta}(t)\right)$,

(4) $\dfrac{d}{dt}\left\langle \vec{\gamma}(t), \vec{\beta}(t)\right\rangle = \left\langle \dfrac{d}{dt}\vec{\gamma}(t), \vec{\beta}(t)\right\rangle + \left\langle \vec{\gamma}(t), \dfrac{d}{dt}\vec{\beta}(t)\right\rangle$.

Remark 4. Inner product functions given by (8), (11) and (13) satisfy option *(4)* of Theorem 5.3.

6. Conclusions

In order to ensure the integrity, we would like to state that these definitions, theorems and results presented in this study are preliminary information in terms of our research in the field of differential geometry in n-dimensional quaternionic space.

It is aimed to add the theory of curves on n-dimensional quaternionic space to the literature, assuming that the results obtained and provided by the applications to be carried out in the next study confirm the results obtained in this study.

Declarations

Author contribution statement

D. Altun: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper. S. Yüce: Conceived and designed the experiments; Performed the experiments; Contributed reagents, materials, analysis tools or data.

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