Springer **Plus**

**Open Access**

CrossMark

# Existence and exponential stability of positive almost periodic solution for Nicholson's blowflies models on time scales

Yongkun Li[*] and Bing Li

*Correspondence:
yklie@ynu.edu.cn
Department of Mathematics,
Yunnan University,
Kunming 650091, Yunnan,
People's Republic of China

## Abstract

In this paper, we first give a new definition of almost periodic time scales, two new definitions of almost periodic functions on time scales and investigate some basic properties of them. Then, as an application, by using a fixed point theorem in Banach space and the time scale calculus theory, we obtain some sufficient conditions for the existence and exponential stability of positive almost periodic solutions for a class of Nicholson's blowflies models on time scales. Finally, we present an illustrative example to show the effectiveness of obtained results. Our results show that under a simple condition the continuous-time Nicholson's blowflies model and its discrete-time analogue have the same dynamical behaviors.

**Keywords:** Almost periodic solution, Exponential stability, Nicholsons blowflies model, Almost periodic time scales

**Mathematics Subject Classification:** 34N05, 34K14, 34K20, 92D25

## Background

To describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained in Nicholson ([1954](#)), Gurney et al. ([1980](#)) proposed the following delay differential Equation model:

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)}, \tag{1}$$

where $p$ is the maximum per capita daily egg production rate, $1/a$ is the size at which the blowfly population reproduces at its maximum rate, $\delta$ is the per capita daily adult death rate, and $\tau$ is the generation time. Since Eq. ([1](#)) explains Nicholson's data of blowfly more accurately, the model and its modifications have been now refereed to as Nicholson's Blowflies model. The theory of the Nicholson's blowflies equation has made a remarkable progress in the past 40 years with main results scattered in many research papers. Many important results on the qualitative properties of the model such as the existence of positive solutions, positive periodic solutions, positive almost periodic solutions and positive pseudo almost periodic solutions, the persistence, the permanence, the oscillation and the stability for the classical Nicholson's model and its generalizations have been established in the literature (Chen [2003](#); Li and Du [2008](#); Liu [2010](#), [2014a](#);

Saker and Agarwal 2002; Zhou 2013; Yi and Zou 2008; Liu and Gong 2011; Hien 2014; Chérif 2015; Duan and Huang 2015; Yao 2015a; Shao 2012). For example, to describe the models of marine protected areas and B-cell chronic lymphocytic leukemia dynamics that are examples of Nicholson-type delay differential systems, Berezansky et al. (2011) and Wang et al. (2011) studied the following Nicholson-type delay system:

$$
\begin{cases}
N_1'(t) = -\alpha_1(t)N_1(t) + \beta_1(t)N_2(t) + \sum_{j=1}^{m} c_{1j}(t)N_1(t - \tau_{1j}(t))e^{-\gamma_{ij}(t)N_1(t-\tau_{1j}(t))}, \\
N_2'(t) = -\alpha_2(t)N_2(t) + \beta_2(t)N_1(t) + \sum_{j=1}^{m} c_{2j}(t)N_2(t - \tau_{1j}(t))e^{-\gamma_{ij}(t)N_2(t-\tau_{1j}(t))},
\end{cases}
$$

where $\alpha_i, \beta_i, c_{ij}, \gamma_{ij}, \tau_{ij} \in C(\mathbb{R}, (0, +\infty))$, $i = 1, 2, j = 1, 2, \ldots, m$; in Faria (2011), the authors discussed some aspects of the global dynamics for a Nicholson's blowflies model with patch structure given by

$$
x_i'(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{m} \beta_{ij}x_i(t - \tau_{ij})e^{-x_i(t-\tau_{ij})}, \quad i = 1, 2, \ldots, n.
$$

In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. Hence, the effects of almost periodic environment on evolutionary theory have been the object of intensive analysis by numerous authors and some of these results for the Nicholson's blowflies model can be found in Alzabut (2010), Chen and Liu (2011), Long (2012), Wang (2013), Liu and Meng (2012), Xu (2014), Liu (2014b), Ding and Alzabut (2015).

Besides, although most models are described by differential equations, the discrete-time models governed by difference equations are more appropriate than the continuous ones when the size of the population is rarely small, or the population has non-overlapping generations. Hence, it is also important to study the dynamics of the discrete-time Nicholson's blowflies model. Recently, authors of Yao (2014), Alzabut (2013) studied the existence and exponential convergence of almost periodic solutions for the discrete Nicholson's blowflies model, respectively. In fact, it is troublesome to study the dynamics for continuous systems and their corresponding discrete ones respectively, therefore, it is significant to study that on time scales, which was initiated by Stefan Hilger (see Hilger 1990) in order to unify continuous and discrete cases. However, to the best of our knowledge, very few results are available on the existence and stability of positive almost periodic solutions for the Nicholson's blowflies model on time scales except (Li and Yang 2012). But Li and Yang (2012) only considered the asymptotical stability of the model and the exponential stability is stronger than asymptotical stability among different stabilities.

On the other hand, in order to study the almost periodic dynamic equations on time scales, a concept of almost periodic time scales was proposed in Li and Wang (2011a). Based on this concept, almost periodic functions Li and Wang (2011a), pseudo almost periodic functions (Li and Wang 2012), almost automorphic functions (Lizama and Mesquita 2013a), weighted pseudo almost automorphic functions (Wang and Li 2013), weighted piecewise pseudo almost automorphic functions (Wang and Agarwal 2014a)

and almost periodic set-valued functions (Hong and Peng 2016) on on time scales were defined successively. Also, some works have been done under the concept of almost periodic time scales (see Lizama and Mesquita 2013b; Lizama et al. 2014; Li and Yang 2014; Liang et al. 2014; Gao et al. 2014; Yao 2015b; Mophou et al. 2014; Zhou et al. 2015). Although the concept of almost periodic time scales in Li and Wang (2011a) can unify the continuous and discrete situations effectively, it is very restrictive. This excludes many interesting time scales. Therefore, it is a challenging and important problem in theories and applications to find new concepts of almost periodic time scales (Li and Wang 2011b; Wang and Agarwal 2014b; Li and Li 2015; Li et al. 2015a, b).

Motivated by the above discussion, our main purpose of this paper is firstly to propose a new definition of almost periodic time scales, two new definitions of almost periodic functions on time scales and study some basic properties of them. Then, as an application, we study the existence and global exponential stability of positive almost periodic solutions for the following Nicholson's blowflies model with patch structure and multiple time-varying delays on time scales:

$$
\begin{aligned}
x_i^\Delta(t) = -c_i(t)x_i(t) + &\sum_{k=1,k\neq i}^{n} b_{ik}(t)x_k(t) \\
&+ \sum_{j=1}^{n} \beta_{ij}(t)x_i(t - \tau_{ij}(t))e^{-\alpha_{ij}(t)x_i(t-\tau_{ij}(t))}, \quad i = 1, 2, \ldots, n,
\end{aligned}
\tag{2}
$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale, $x_i(t)$ denotes the density of the species in patch $i$, $b_{ik}(k \neq i)$ is the migration coefficient from patch $k$ to patch $i$ and the natural growth in each patch is of Nicholson-type.

For convenience, for a positive almost periodic function $f : \mathbb{T} \to \mathbb{R}$, we denote $f^+ = \sup_{t\in\mathbb{T}} f(t), f^- = \inf_{t\in\mathbb{T}} f(t)$. Due to the biological meaning of (2), we just consider the following initial condition:

$$
\varphi_i(s) > 0, \ s \in [t_0 - \theta, t_0]_\mathbb{T}, \ t_0 \in \mathbb{T}, \quad i = 1, 2, \ldots, n,
\tag{3}
$$

where $\theta = \max_{(i,j)} \sup_{t\in\mathbb{T}}\{\tau_{ij}(t)\}, [t_0 - \theta, t_0]_\mathbb{T} = [t_0 - \theta, t_0] \cap \mathbb{T}$.

This paper is organized as follows: In "Preliminaries", we introduce some notations and definitions which are needed in later sections. In "Almost periodic time scales and almost periodic functions on time scales" section, we give a new definition of almost periodic time scales and two new definitions of almost periodic functions on time scales, and we state and prove some basic properties of them. In "Positive almost periodic solutions for the Nicholson's blowflies model" section, we establish some sufficient conditions for the existence and exponential stability of positive almost periodic solutions of (2). In "An example" section, we give an example to illustrate the feasibility of our results obtained in previous sections. We draw a conclusion in "Conclusion" section.

## Preliminaries

In this section, we shall first recall some definitions and state some results which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If $f$ is continuous at each right-dense point and each left-dense point, then $f$ is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If $y$ is continuous, then $y$ is right-dense continuous, and if $y$ is delta differentiable at $t$, then $y$ is continuous at $t$.

Let $y$ be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s = Y(t) - Y(a)$.

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and *rd*-continuous functions $r : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{r \in \mathcal{R} : 1 + \mu(t)r(t) > 0, \forall t \in \mathbb{T}\}$.

**Lemma 1**  (Bohner and Peterson 2001) *Suppose that $p \in \mathcal{R}^+$, then*

*(i)    $e_p(t, s) > 0$, for all $t, s \in \mathbb{T}$;*
*(ii)   if $p(t) \leq q(t)$ for all $t \geq s, t, s \in \mathbb{T}$, then $e_p(t, s) \leq e_q(t, s)$ for all $t \geq s$.*

**Definition 1**  (Fink 1974) A subset $S$ of $\mathbb{R}$ is called relatively dense if there exists a positive number $L$ such that $[a, a + L] \cap S \neq \phi$ for all $a \in \mathbb{R}$. The number $L$ is called the inclusion length.

**Definition 2**  (Li and Wang 2011a) A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

The following definition is a slightly modified version of Definition 3.10 in Li and Wang (2011a).

**Definition 3**  Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the $\varepsilon$-translation set of $f$

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t + \tau, x) - f(t, x)| < \varepsilon, \quad \forall(t, x) \in \mathbb{T} \times S\}$$

is relatively dense for all $\varepsilon > 0$ and for each compact subset $S$ of $D$; that is, for any given $\varepsilon > 0$ and each compact subset $S$ of $D$, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t + \tau, x) - f(t, x)| < \varepsilon, \quad \forall (t, x) \in \mathbb{T} \times S.$$

$\tau$ is called the $\varepsilon$-translation number of $f$.

## Almost periodic time scales and almost periodic functions on time scales

In this section, we will give a new definition of almost periodic time scales and two new definitions of almost periodic functions on time scales, and we will investigate some basic properties of them. Our new definition of almost periodic time scales is as follows:

**Definition 4** A time scale $\mathbb{T}$ is called an almost periodic time scale if the set

$$\Pi_0 := \{\tau \in \mathbb{R} : \mathbb{T}_{\pm\tau} \neq \emptyset\} \neq \{0\},$$

where $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} - \tau\} = \mathbb{T} \cap \{t - \tau : t \in \mathbb{T}\}$, and there exists a set $\Pi_1$ satisfies

(i)    $0 \in \Pi_1 \subseteq \Pi_0$,
(ii)    $\Pi(\Pi_1) \setminus \{0\} \neq \emptyset$,
(iii)    $\widetilde{\mathbb{T}} := \mathbb{T}(\Pi) = \bigcap_{\tau \in \Pi} \mathbb{T}_\tau \neq \emptyset$,
       where $\Pi := \Pi(\Pi_1) = \{\tau \in \Pi_1 \subseteq \Pi_0 : \sigma \pm \tau \in \Pi_1, \forall \sigma \in \Pi_1\}$.

Clearly, if $t \in \mathbb{T}_\tau$, then $t + \tau \in \mathbb{T}$. If $t \in \widetilde{\mathbb{T}}$, then $t + \tau \in \mathbb{T}$ for $\tau \in \Pi$.

*Remark 1* Obviously, if $\mathbb{T}$ is an almost periodic time scale under Definition 4, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$. If $\mathbb{T}$ is an almost periodic time scale under Definition 2, then $\mathbb{T}$ is also an almost periodic time scale under Definition 4 and in this case, $\widetilde{\mathbb{T}} = \mathbb{T}$.

*Example 1* Let $\mathbb{T} = \mathbb{Z} \cup \{\frac{1}{4}\}$. Take $\Pi_1 = \{\tau \in \mathbb{T} : \mathbb{T}_\tau \neq \emptyset, \mathbb{T}_\tau \neq \{0\}\} \subseteq \Pi_0$, then for every $\tau \in \mathbb{Z}$, we have $\mathbb{T}_\tau = \mathbb{Z}$ and $\mathbb{T}_{\frac{1}{4}} = \{0\}$. Hence $\Pi = \mathbb{Z}$ and $\widetilde{\mathbb{T}} = \bigcap_{\tau \in \Pi} \mathbb{T}_\tau = \mathbb{Z} \neq \emptyset$. So, $\mathbb{T}$ is an almost periodic time scale under Definition 4 but it is not an almost periodic time scale under Definition 2.

**Lemma 2** *If $\mathbb{T}$ is an almost periodic time scales under Definition 4, then $\widetilde{\mathbb{T}}$ is an almost periodic time scale under Definition 2.*

*Proof* By contradiction, suppose that there exists a $t_0 \in \widetilde{\mathbb{T}}$ such that for every $\tau \in \Pi \setminus \{0\}$, $t_0 + \tau \notin \widetilde{\mathbb{T}}$ or $t_0 - \tau \notin \widetilde{\mathbb{T}}$.

   **Case (i)** If $t_0 + \tau \notin \widetilde{\mathbb{T}}$, then there exists a $\tau_{t_0} \in \Pi$ such that $t_0 + \tau \notin \mathbb{T}_{\tau_{t_0}}$. On one hand, since $t_0 + \tau \in \mathbb{T}$, $t_0 + \tau + \tau_{t_0} \notin \mathbb{T}$. On the other hand, since $t_0 \in \widetilde{\mathbb{T}}$ and $\tau + \tau_{t_0} \in \Pi$, $t_0 + \tau + \tau_{t_0} \in \mathbb{T}$. This is a contradiction.
   **Case (ii)** If $t_0 - \tau \notin \widetilde{\mathbb{T}}$, then there exists a $\tilde{\tau}_{t_0} \in \Pi$ such that $t_0 - \tau \notin \mathbb{T}_{\tilde{\tau}_{t_0}}$. On one hand, since $t_0 - \tau \in \mathbb{T}$, $t_0 - \tau + \tilde{\tau}_{t_0} \notin \mathbb{T}$. On the other hand, since $t_0 \in \widetilde{\mathbb{T}}$ and $-\tau + \tilde{\tau}_{t_0} \in \Pi$, $t_0 - \tau + \tilde{\tau}_{t_0} \in \mathbb{T}$. This is a contradiction.

Therefore, for every $t \in \widetilde{\mathbb{T}}$, there exists a $\tau \in \Pi \backslash \{0\}$ such that $t \pm \tau \in \widetilde{\mathbb{T}}$. Hence, $\mathbb{T}$ is an almost periodic time scale under Definition 2. The proof is complete. □

Throughout this section, $\mathbb{E}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$, $D$ denotes an open set in $\mathbb{E}^n$ or $D = \mathbb{E}^n$, $S$ denotes an arbitrary compact subset of $D$.

From Li and Wang ([2011a](#)), under Definitions 2 and 3, we know that if we denote by $BUC(\mathbb{T} \times D, \mathbb{R}^n)$ the collection of all bounded uniformly continuous functions from $\mathbb{T} \times S$ to $\mathbb{R}^n$, then

$$AP(\mathbb{T} \times D, \mathbb{R}^n) \subset BUC(\mathbb{T} \times D, \mathbb{R}^n), \tag{4}$$

where $AP(\mathbb{T} \times D, \mathbb{R}^n)$ are the collection of all almost periodic functions in $t \in \mathbb{T}$ uniformly for $x \in D$. It is well known that if we let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, (4) is valid. So, for simplicity, we give the following definition:

**Definition 5** Let $\mathbb{T}$ be an almost periodic time scale under sense of Definition 4. A function $f \in BUC(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the $\varepsilon$-translation set of $f$

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t + \tau, x) - f(t, x)| < \varepsilon, \quad \forall (t, x) \in \widetilde{\mathbb{T}} \times S\}$$

is relatively dense for all $\varepsilon > 0$ and for each compact subset $S$ of $D$; that is, for any given $\varepsilon > 0$ and each compact subset $S$ of $D$, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t + \tau, x) - f(t, x)| < \varepsilon, \quad \forall (t, x) \in \widetilde{\mathbb{T}} \times S.$$

This $\tau$ is called the $\varepsilon$-translation number of $f$.

*Remark 2* If $\mathbb{T} = \mathbb{R}$, then $\widetilde{\mathbb{T}} = \mathbb{R}$, in this case, if we take $\Pi = \mathbb{R}$, then Definition 5 is actually equivalent to the definition of the uniformly almost periodic functions in Ref. Fink ([1974](#)). If $\mathbb{T} = \mathbb{Z}$, then $\widetilde{\mathbb{T}} = \mathbb{Z}$, in this case, if we take $\Pi = \mathbb{Z}$, then Definition 5 is actually equivalent to the definition of the uniformly almost periodic sequences in Fink and Seifert ([1969](#)), David and Cristina ([2004](#)).

*Example 2* Let $\mathbb{T} = \mathbb{Z} \cup \{\frac{1}{4}\}$, according to Example 3.1, $\mathbb{T}$ is an almost periodic time scale under Definition 4. Take $f(t, x) = 2x^2 + \sin 2t + \cos \sqrt{3} t$ for $(t, x) \in \mathbb{T} \times \mathbb{R}$. Then $f$ is an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in \mathbb{R}$ under Definition 5.

For convenience, we denote by $AP(\mathbb{T} \times D, \mathbb{E}^n)$ the set of all functions that are almost periodic in $t$ uniformly for $x \in D$ and denote by $AP(\mathbb{T})$ the set of all functions that are almost periodic in $t \in \mathbb{T}$, and introduce some notations: Let $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be two sequences. Then $\beta \subset \alpha$ means that $\beta$ is a subsequence of $\alpha$; $\alpha + \beta = \{\alpha_n + \beta_n\}$; $-\alpha = \{-\alpha_n\}$; and $\alpha$ and $\beta$ are common subsequences of $\alpha'$ and $\beta'$, respectively, means that $\alpha_n = \alpha'_{n(k)}$ and $\beta_n = \beta'_{n(k)}$ for some given function $n(k)$. We introduce the translation operator $T$, $T_\alpha f(t, x) = g(t, x)$ means that $g(t, x) = \lim_{n \to +\infty} f(t + \alpha_n, x)$ and is written only when the limit exists. The mode of convergence, e.g. pointwise, uniform, etc., will be specified at each use of the symbol.

Similar to the proofs of Theorem 3.14, Theorem 3.21 and Theorem 3.22 in Li and Wang ([2011a](#)), respectively, one can prove the following three theorems.

**Theorem 1**   *Let $f \in UBC(\mathbb{T} \times D, \mathbb{E}^n)$, if for any sequence $\alpha' \subset \Pi$, there exists $\alpha \subset \alpha'$ such that $T_\alpha f$ exists uniformly on $\widetilde{\mathbb{T}} \times S$, then $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$.*

**Theorem 2**   *If $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, then for any $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon, S)$, for any $a \in \mathbb{R}$, there exist a constant $\eta > 0$ and $\alpha \in \mathbb{R}$ such that $\big([\alpha, \alpha + \eta] \cap \Pi\big) \subset [a, a + L]$ and $\big([\alpha, \alpha + \eta] \cap \Pi\big) \subset E(\varepsilon, f, S)$.*

**Theorem 3**   *If $f, g \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, then for any $\varepsilon > 0$, $E(f, \varepsilon, S) \cap E(g, \varepsilon, S)$ is nonempty relatively dense.*

According to Definition 5, one can easily prove

**Theorem 4**   *If $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, then for any $\alpha \in \mathbb{R}, b \in \Pi$, functions $\alpha f, f(t + b, \cdot) \in AP(\mathbb{T} \times D, \mathbb{E}^n)$.*

Similar to the proofs of Theorem 3.24, Theorem 3.27, Theorem 3.28 and Theorem 3.29 in Li and Wang ([2011a](#)), respectively, one can prove the following four theorems.

**Theorem 5**   *If $f, g \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, then $f + g, fg \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, if $\inf_{t \in \mathbb{T}} |g(t, x)| > 0$, then $f/g \in AP(\mathbb{T} \times D, \mathbb{E}^n)$.*

**Theorem 6**   *If $f_n \in AP(\mathbb{T} \times D, \mathbb{E}^n)(n = 1, 2, \ldots)$ and the sequence $\{f_n\}$ uniformly converges to $f$ on $\mathbb{T} \times S$, then $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$.*

**Theorem 7**   *If $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, denote $F(t, x) = \int_0^t f(s, x) \Delta s$, then $F \in AP(\mathbb{T} \times D, \mathbb{E}^n)$ if and only if $F$ is bounded on $\mathbb{T} \times S$.*

**Theorem 8**   *If $f \in AP(\mathbb{T} \times D, \mathbb{E}^n)$, $F(\cdot)$ is uniformly continuous on the value field of $f$, then $F \circ f$ is almost periodic in $t$ uniformly for $x \in D$.*

By Definition 5, one can easily prove

**Theorem 9**   *Let $f : \mathbb{R} \to \mathbb{R}$ satisfies Lipschitz condition and $\varphi(t) \in AP(\mathbb{T})$, then $f(\varphi(t)) \in AP(\mathbb{T})$.*

**Definition 6**   (Li and Wang [2011b](#)) Let $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$, the linear system

$$x^\Delta(t) = A(t)x(t), \quad t \in \mathbb{T} \tag{5}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant $k, \alpha$, projection $P$, and the fundamental solution matrix $X(t)$ of ([5](#)), satisfying

$$|X(t)PX^{-1}(\sigma(s))| \le k e_{\ominus\alpha}(t, \sigma(s)), \ s, t \in \mathbb{T}, \ t \ge \sigma(s),$$
$$|X(t)(I - P)X^{-1}(\sigma(s))| \le k e_{\ominus\alpha}(\sigma(s), t), \ s, t \in \mathbb{T}, \ t \le \sigma(s),$$

where $|\cdot|$ is a matrix norm on $\mathbb{T}$, that is, if $A = (a_{ij})_{n\times n}$, then we can take $|A| = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}$.

Similar to the proof of Lemma 2.15 in Li and Wang (2011b), one can easily show that

**Lemma 3** *Let $a_{ii}(t)$ be an uniformly bounded rd -continuous function on $\mathbb{T}$, where $a_{ii}(t) > 0, -a_{ii}(t) \in \mathcal{R}^+$ for every $t \in \mathbb{T}$ and*

$$\min_{1\le i\le n}\{\inf_{t\in\mathbb{T}} a_{ii}(t)\} > 0,$$

*then the linear system*

$$x^\Delta(t) = diag(-a_{11}(t), -a_{22}(t), \ldots, -a_{nn}(t))x(t)$$

*admits an exponential dichotomy on $\mathbb{T}$.*

According to Lemma 2, $\widetilde{\mathbb{T}}$ is an almost periodic time scales under Definition 2, we denote the forward and the backward jump operators of $\widetilde{\mathbb{T}}$ by $\widetilde{\sigma}$ and $\widetilde{\rho}$, respectively.

**Lemma 4** *If $t$ is a right-dense point of $\widetilde{\mathbb{T}}$, then $t$ is also a right-dense point of $\mathbb{T}$.*

*Proof* Let $t$ be a right-dense point of $\widetilde{\mathbb{T}}$, then

$$t = \widetilde{\sigma}(t) = \inf\{s \in \widetilde{\mathbb{T}} : s > t\} \ge \inf\{s \in \mathbb{T} : s > t\} = \sigma(t).$$

Since $\sigma(t) \ge t, t = \sigma(t)$. The proof is complete. $\qquad\square$

Similar to the proof of Lemma 4, one can prove the following lemma.

**Lemma 5** *If $t$ is a left-dense point of $\widetilde{\mathbb{T}}$, then $t$ is also a left-dense point of $\mathbb{T}$.*

For each $f \in C(\mathbb{T}, \mathbb{R})$, we define $\widetilde{f} : \widetilde{\mathbb{T}} \to \mathbb{R}$ by $\widetilde{f}(t) = f(t)$ for $t \in \widetilde{\mathbb{T}}$. From Lemmas 4 and 5, we can get that $\widetilde{f} \in C(\widetilde{\mathbb{T}}, \mathbb{R})$. Therefore, $F$ defined by

$$F(t) := \int_{t_0}^{t} \widetilde{f}(\tau)\widetilde{\Delta}\tau, \ t_0, t \in \widetilde{\mathbb{T}}$$

is an antiderivative of $f$ on $\widetilde{\mathbb{T}}$, where $\widetilde{\Delta}$ denotes the $\Delta$-derivative on $\widetilde{\mathbb{T}}$.

Set $\widetilde{\Pi} = \{\tau \in \Pi : t \pm \tau \in \widetilde{\mathbb{T}}\}$. We give our second definition of almost periodic functions on time scales as follows.

**Definition 7** Let $\mathbb{T}$ be an almost periodic time scale under sense of Definition 4. A function $f \in BUC(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the $\varepsilon$-translation set of $f$

$$E\{\varepsilon, f, S\} = \{\tau \in \widetilde{\Pi} : |f(t + \tau, x) - f(t, x)| < \varepsilon, \quad \forall(t, x) \in \widetilde{\mathbb{T}} \times S\}$$

is relatively dense for all $\varepsilon > 0$ and for each compact subset $S$ of $D$; that is, for any given $\varepsilon > 0$ and each compact subset $S$ of $D$, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau,x) - f(t,x)| < \varepsilon, \quad \forall (t,x) \in \widetilde{\mathbb{T}} \times S.$$

This $\tau$ is called the $\varepsilon$-translation number of $f$.

*Remark 3*  It is clear that if a function is an almost periodic function under Definition 5, then it is also an almost periodic function under Definition 7.

*Remark 4*  Since $\widetilde{\mathbb{T}}$ is an almost periodic time scales under Definition 2, under Definition 5, all the results obtained in Li and Wang (2011a) remain valid when we restrict our discussion to $\widetilde{\mathbb{T}}$.

In the following, we restrict our discuss under Definition 7.

Consider the following almost periodic system:

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T}, \tag{6}$$

where $A(t)$ is a $n \times n$ almost periodic matrix function, $f(t)$ is a $n$-dimensional almost periodic vector function.

Similar to Lemma 2.13 in Li and Wang (2011b), one can easily get

**Lemma 6**  *If linear system* (5) *admits an exponential dichotomy, then system* (6) *has a bounded solution $x(t)$ as follows:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \ t \in \mathbb{T},$$

*where $X(t)$ is the fundamental solution matrix of* (5).

By Theorem 4.19 in Li and Wang (2011a), we have

**Lemma 7**  *Let $A(t)$ be an almost periodic matrix function and $f(t)$ be an almost periodic vector function. If* (5) *admits an exponential dichotomy, then* (6) *has a unique almost periodic solution:*

$$x(t) = \int_{-\infty}^{t} \widetilde{X}(t)P\widetilde{X}^{-1}(\widetilde{\sigma}(s))\widetilde{f}(s)\widetilde{\Delta}s - \int_{t}^{+\infty} \widetilde{X}(t)(I-P)\widetilde{X}^{-1}(\widetilde{\sigma}(s))\widetilde{f}(s)\widetilde{\Delta}s, \ t \in \widetilde{\mathbb{T}},$$

*where $\widetilde{X}(t)$ is the restriction of the fundamental solution matrix of* (5) *on $\widetilde{\mathbb{T}}$.*

From Definition 5 and Lemmas 6 and 7, one can easily get the following lemma.

**Lemma 8**  *If linear system* (5) *admits an exponential dichotomy, then system* (6) *has an almost periodic solution $x(t)$ can be expressed as:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \ t \in \mathbb{T},$$

*where $X(t)$ is the fundamental solution matrix of* (5).

## Positive almost periodic solutions for the Nicholson's blowflies model

In this section, we will state and prove the sufficient conditions for the existence and exponential stability of positive almost periodic solutions of (2). Throughout this section, we restrict our discussion under Definition 7.

Set $\mathbb{B} = \{\varphi \in C(\mathbb{T}, \mathbb{R}^n) : \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \text{ is an almost periodic function on } \mathbb{T}\}$ with the norm $||\varphi||_\mathbb{B} = \sup\limits_{t \in \mathbb{T}} ||\varphi(t)||$, where $||\varphi(t)|| = \max\limits_{1 \le i \le n} |\varphi_i(t)|$, then $\mathbb{B}$ is a Banach space. Denote $\mathbb{C} = C([t_0 - \theta, t_0]_\mathbb{T}, \mathbb{R}^n)$ and $C\{A_1, A_2\} = \{\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathbb{C} : A_1 \le \varphi_i(s) \le A_2, s \in [t_0 - \theta, t_0]_\mathbb{T}, i = 1, 2, \ldots, n\}$, where $0 < A_1 < A_2$ are constants.

In the proofs of our results of this section, we need the following facts: The function $xe^{-x}$ decreases on $[1, +\infty)$.

**Lemma 9** *Assume that the following conditions hold.*

$(H_1)$      $c_i, b_{ik}, \beta_{ij}, \alpha_{ij}, \tau_{ij} \in AP(\mathbb{T}, \mathbb{R}^+)$ and $c_i^- > 0, b_{ik}^- > 0, \beta_{ij}^- > 0, \alpha_{ij}^- > 0,$
         $t - \tau_{ij}(t) \in \mathbb{T}, i, k, j = 1, 2, \ldots, n.$

$(H_2)$      $\sum_{k=1, k \ne i}^n \frac{b_{ik}^+}{c_i^-} < 1, \quad i = 1, 2, \ldots, n.$

$(H_3)$      *There exist positive constants* $A_1, A_2$ *satisfy*

$$A_2 > \max_{1 \le i \le n} \left\{ \left[ 1 - \sum_{k=1, k \ne i}^n \frac{b_{ik}^+}{c_i^-} \right]^{-1} \sum_{j=1}^n \frac{\beta_{ij}^+}{c_i^- \alpha_{ij}^- e} \right\}$$

*and*

$$\min_{1 \le i \le n} \left\{ \left[ 1 - \sum_{k=1}^n \frac{b_{ik}^-}{c_i^+} \right]^{-1} \sum_{j=1}^n A_2 \frac{\beta_{ij}^-}{c_i^+} e^{-\alpha_{ij}^+ A_2} \right\} > A_1 \ge \frac{1}{\min\limits_{1 \le i, j \le n} \{\alpha_{ij}^-\}}.$$

*Then the solution* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *of* (2) *with the initial value* $\varphi \in C\{A_1, A_2\}$ *satisfies*

$$A_1 < x_i(t) < A_2, \ t \in [t_0, +\infty)_\mathbb{T}, \quad i = 1, 2, \ldots, n.$$

*Proof*   Let $x(t) = x(t; t_0, \varphi)$, where $\varphi \in C\{A_1, A_2\}$. At first, we prove that

$$0 < x_i(t) < A_2, \ t \in [t_0, \eta(\varphi))_\mathbb{T}, \quad i = 1, 2, \ldots, n, \tag{7}$$

where $[t_0, \eta(\varphi))_\mathbb{T}$ is the maximal right-interval of existence of $x(t; t_0, \varphi)$. By way of contradiction, assume that (7) does not hold. Then, there exists $i_0 \in \{1, 2, \ldots, n\}$ and the first time $t_1 \in [t_0, \eta(\varphi))_\mathbb{T}$ such that

$$x_{i_0}(t_1) \ge A_2, \ x_{i_0}(t) < A_2, \ t \in [t_0 - \theta, t_1)_\mathbb{T},$$
$$x_k(t) \le A_2, \text{ for } k \ne i_0, \ t \in [t_0 - \theta, t_1]_\mathbb{T}, \quad k = 1, 2, \ldots, n.$$

Therefore, there must be a positive constant $a \geq 1$ such that

$$
x_{i_0}(t_1) = aA_2, \ x_{i_0}(t) < aA_2, \ t \in [t_0 - \theta, t_1)_{\mathbb{T}},
$$
$$
x_k(t) \leq aA_2, \ \text{for } k \neq i_0, \ t \in [t_0 - \theta, t_1]_{\mathbb{T}}, \quad k = 1, 2, \ldots, n.
$$

In view of the fact that $\sup\limits_{u \geq 0} ue^{-u} = \frac{1}{e}$ and $a \geq 1$, we can obtain

$$
0 \leq x_{i_0}^{\Delta}(t_1) = -c_{i_0}(t_1)x_{i_0}(t_1) + \sum_{k=1,k \neq i_0}^{n} b_{i_0 k}(t_1)x_k(t_1)
$$
$$
+ \sum_{j=1}^{n} \frac{\beta_{i_0 j}(t_1)}{\alpha_{i_0 j}(t_1)} \alpha_{i_0 j}(t_1) x_{i_0}(t_1 - \tau_{i_0 j}(t_1)) e^{-\alpha_{i_0 j}(t_0)x_{i_0}(t_0 - \tau_{i_0 j}(t_0))}
$$
$$
\leq -c_{i_0}^{-} aA_2 + \sum_{k=1,k \neq i_0}^{n} b_{i_0 k}^{+} aA_2 + \sum_{j=1}^{n} \frac{\beta_{i_0 j}^{+}}{\alpha_{i_0 j}^{-}} \cdot \frac{1}{e}
$$
$$
\leq ac_{i_0}^{-} \left( -A_2 + \sum_{k=1,k \neq i_0}^{n} \frac{A_2 b_{i_0 k}^{+}}{c_{i_0}^{-}} + \sum_{j=1}^{n} \frac{\beta_{i_0 j}^{+}}{c_{i_0}^{-} \alpha_{i_0 j}^{-} e} \right) < 0,
$$

which is a contradiction and hence (7) holds. Next, we show that

$$
x_i(t) > A_1, \ t \in [t_0, \eta(\varphi))_{\mathbb{T}}, \quad i = 1, 2, \ldots, n. \tag{8}
$$

By way of contradiction, assume that (8) does not hold. Then, there exists $i_1 \in \{1, 2, \ldots, n\}$ and the first time $t_2 \in [t_0, \eta(\varphi))_{\mathbb{T}}$ such that

$$
x_{i_1}(t_2) \leq A_1, \ x_{i_1}(t) > A_1, \ t \in [t_0 - \theta, t_2)_{\mathbb{T}},
$$
$$
x_k(t) \geq A_1, \ \text{for } k \neq i_1, \quad t \in [t_0 - \theta, t_2]_{\mathbb{T}}, \quad k = 1, 2, \ldots, n.
$$

Therefore, there must be a positive constant $c \leq 1$ such that

$$
x_{i_1}(t_2) = cA_1, \ x_{i_1}(t) > cA_1, \quad t \in [t_0 - \theta, t_2)_{\mathbb{T}},
$$
$$
x_k(t) \geq cA_1, \ \text{for } k \neq i_1, \ t \in [t_0 - \theta, t_2]_{\mathbb{T}}, \quad k = 1, 2, \ldots, n.
$$

Noticing that $c \leq 1$, it follows that

$$
0 \geq x_{i_1}^{\Delta}(t_2) = -c_{i_1}(t_2)x_{i_1}(t_2) + \sum_{k=1,k \neq i_1}^{n} b_{i_1 k}(t_2)x_k(t_2)
$$
$$
+ \sum_{j=1}^{n} \beta_{i_1 j}(t_2)x_{i_1}(t_2 - \tau_{i_1 j}(t_2)) e^{-\alpha_{i_1 j}(t_2)x_{i_1}(t_2 - \tau_{i_1 j}(t_2))}
$$
$$
\geq -c_{i_1}^{+} cA_1 + \sum_{k=1,k \neq i_1}^{n} b_{i_1 k}^{-} cA_1 + \sum_{j=1}^{n} A_2 \frac{\alpha_{i_1 j}^{+} \beta_{i_1 j}^{-}}{\alpha_{i_1 j}^{+}} e^{-\alpha_{i_1 j}^{+} A_2}
$$
$$
= cc_{i_1}^{+} \left( -A_1 + \sum_{k=1,k \neq i_1}^{n} A_1 \frac{b_{i_1 k}^{-}}{c_{i_1}^{+}} + \sum_{j=1}^{n} A_2 \frac{\beta_{i_1 j}^{-}}{c_{i_1}^{+}} e^{-\alpha_{i_1 j}^{+} A_2} \right) > 0,
$$

which is a contradiction and hence (8) holds. Similar to the proof of Theorem 2.3.1 in Hale and Verduyn Lunel (1993), we easily obtain $\eta(\varphi) = +\infty$. This completes the proof.

□

*Remark 5* If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$, so, $-c_i \in \mathcal{R}^+$. If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$, so, $-c_i \in \mathcal{R}^+$ if and only if $c_i < 1$.

**Theorem 10** *Assume that $(H_1)$ and $(H_3)$ hold. Suppose further that*

($H_4$) $-c_i \in \mathcal{R}^+$, where $\mathcal{R}^+$ denotes the set of positive regressive functions, $i = 1, 2, \ldots, n$.

($H_5$) $\sum_{k=1,k\neq i}^n b_{ik}^+ + \sum_{j=1}^n \frac{\beta_{ij}^+}{e^2} < c_i^-$, $i = 1, 2, \ldots, n$.

*Then system* (2) *has a positive almost periodic solution in the region* $\mathbb{B}^* = \{\varphi| \ \varphi \in \mathbb{B}, A_1 \leq \varphi_i(t) \leq A_2, t \in \mathbb{T}, \quad i = 1, 2, \ldots, n\}$.

*Proof* For any given $\varphi \in \mathbb{B}$, we consider the following almost periodic dynamic system:

$$
\begin{aligned}
x_i^\Delta(t) = & -c_i(t)x_i(t) + \sum_{k=1,k\neq i}^n b_{ik}(t)\varphi_k(t) \\
& + \sum_{j=1}^n \beta_{ij}(t)\varphi_i(t - \tau_{ij}(t))e^{-\alpha_{ij}(t)\varphi_i(t-\tau_{ij}(t))}, \quad i = 1, 2, \ldots, n.
\end{aligned}
\tag{9}
$$

Since $\min_{1 \leq i \leq n}\{c_i^-\} > 0, t \in \mathbb{T}$, it follows from Lemma 3 that the linear system

$$
x_i^\Delta(t) = -c_i(t)x_i(t), \quad i = 1, 2, \ldots, n
$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, by Lemma 8, we obtain that system (9) has an almost periodic solution $x_\varphi = (x_{\varphi_1}, x_{\varphi_2}, \ldots, x_{\varphi_n})$, where

$$
\begin{aligned}
x_{\varphi_i}(t) = & \int_{-\infty}^t e_{-c_i}(t, \sigma(s)) \left[ \sum_{k=1,k\neq i}^n b_{ik}(s)\varphi_k(s) \right. \\
& + \left. \sum_{j=1}^n \beta_{ij}(s)\varphi_i(s - \tau_{ij}(s))e^{-\alpha_{ij}(s)\varphi_i(s-\tau_{ij}(s))} \right] \Delta s, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

Define a mapping $T : \mathbb{B}^* \to \mathbb{B}^*$ by

$$
T\varphi(t) = x_\varphi(t), \ \forall \varphi \in \mathbb{B}^*.
$$

Obviously, $\mathbb{B}^* = \{\varphi| \ \varphi \in \mathbb{B}, A_1 \leq \varphi_i(t) \leq A_2, t \in \mathbb{T}, \quad i = 1, 2, \ldots, n\}$ is a closed subset of $\mathbb{B}$. For any $\varphi \in \mathbb{B}^*$, by use of $(H_2)$, we have

$$
\begin{aligned}
x_{\varphi_i}(t) & \leq \int_{-\infty}^t e_{-c_i^-}(t, \sigma(s)) \left[ \sum_{k=1,k\neq i}^n b_{ik}^+ A_2 + \sum_{j=1}^n \frac{\beta_{ij}^+}{\alpha_{ij}^-} \times \frac{1}{e} \right] \Delta s \\
& \leq \frac{1}{c_i^-} \left[ \sum_{k=1,k\neq i}^n b_{ik}^+ A_2 + \sum_{j=1}^n \frac{\beta_{ij}^+}{\alpha_{ij}^-} \times \frac{1}{e} \right] \\
& \leq A_2, \quad i = 1, 2, \ldots, n
\end{aligned}
$$

and we also have

$$
\begin{aligned}
x_{\varphi_i}(t) &\geq \int_{-\infty}^{t} e_{-c_i^+}(t, \sigma(s)) \left[ \sum_{k=1,k\neq i}^{n} A_1 b_{ik}^- + \sum_{j=1}^{n} \beta_{ij}^- \varphi_i(s - \tau_{ij}(s)) e^{-\alpha_{ij}^+ \varphi_i(s-\tau_{ij}(s))} \right] \Delta s \\
&\geq \frac{1}{c_i^+} \left[ \sum_{k=1,k\neq i}^{n} A_1 b_{ik}^- + \sum_{j=1}^{n} A_2 \beta_{ij}^- e^{-\alpha_{ij}^+ A_2} \right] \\
&\geq A_1, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

Therefore, the mapping $T$ is a self-mapping from $\mathbb{B}^*$ to $\mathbb{B}^*$.

Next, we prove that the mapping $T$ is a contraction mapping on $\mathbb{B}^*$. Since $\sup_{u \geq 1} |\frac{1-u}{e^u}| = \frac{1}{e^2}$, we find that

$$
\begin{aligned}
|xe^{-x} - ye^{-y}| &= \left| \frac{1 - (x + \xi(y-x))}{e^{x+\xi(y-x)}} \right| |x - y| \\
&\leq \frac{1}{e^2} |x - y|, \; x, y \geq 1, \, 0 < \xi < 1.
\end{aligned}
$$

For any $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T, \psi = (\psi_1, \psi_2, \ldots, \psi_n)^T \in \mathbb{B}^*$, we obtain that

$$
\begin{aligned}
&|(T\varphi)_i(t) - (T\psi)_i(t)| \\
&\leq \left| \int_{-\infty}^{t} e_{-c_i}(t, \sigma(s)) \sum_{k=1,k\neq i}^{n} b_{ik}(s) \big( \varphi_k(s) - \psi_k(s) \big) \Delta s \right| \\
&\quad + \left| \int_{-\infty}^{t} e_{-c_i}(t, \sigma(s)) \sum_{j=1}^{n} \beta_{ij}(s) \bigg( \varphi_i(s - \tau_{ij}(s)) e^{-\alpha_{ij}(s)\varphi_i(s-\tau_{ij}(s))} \right. \\
&\quad \left. - \psi_i(s - \tau_{ij}(s)) e^{-\alpha_{ij}(s)\psi_i(s-\tau_{ij}(s))} \bigg) \Delta s \right| \\
&\leq \frac{1}{c_i^-} \sum_{k=1,k\neq i}^{n} b_{ik}^+ \|\varphi - \psi\|_{\mathbb{B}} + \left| \int_{-\infty}^{t} e_{-c_i}(t, \sigma(s)) \sum_{j=1}^{n} \frac{\beta_{ij}(s)}{\alpha_{ij}(s)} \big( \alpha_{ij}(s)\varphi_i(s - \tau_{ij}(s)) \right. \\
&\quad \left. \times e^{-\alpha_{ij}(s)\varphi_i(s-\tau_{ij}(s))} - \alpha_{ij}(s)\psi_i(s - \tau_{ij}(s)) e^{-\alpha_{ij}(s)\psi_i(s-\tau_{ij}(s))} \big) \Delta s \right| \\
&\leq \left( \frac{1}{c_i^-} \sum_{k=1,k\neq i}^{n} b_{ik}^+ + \sum_{j=1}^{n} \frac{\beta_{ij}^+}{c_i^- e^2} \right) \|\varphi - \psi\|_{\mathbb{B}}, \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

It follows that

$$
\|T\phi - T\psi\|_{\mathbb{B}} < \max_{1 \leq i \leq n} \left\{ \frac{1}{c_i^-} \sum_{k=1,k\neq i}^{n} b_{ik}^+ + \sum_{j=1}^{n} \frac{\beta_{ij}^+}{c_i^- e^2} \right\} \|\varphi - \psi\|_{\mathbb{B}},
$$

which implies that $T$ is a contraction. By the fixed point theorem in Banach space, $T$ has a unique fixed point $\varphi^* \in \mathbb{B}^*$ such that $T\varphi^* = \varphi^*$. In view of (9), we see that $\varphi^*$ is a solution of (2). Therefore, (2) has a positive almost periodic solution in the region $\mathbb{B}^*$. This completes the proof. □

**Definition 8** Let $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ be an almost periodic solution of (2) with initial value $\varphi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \ldots, \varphi_n^*(s))^T \in C\{A_1, A_2\}$. If

there exist positive constants $\lambda$ with $\ominus \lambda \in \mathcal{R}^+$ and $M > 1$ such that such that for an arbitrary solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of (2) with initial value $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \in C\{A_1, A_2\}$ satisfies

$$||x(t) - x^*(t)|| \leq M||\varphi - \varphi^*||_\infty e_{\ominus \lambda}(t, t_0), \ t_0 \in [-\theta, \infty)_{\mathbb{T}}, \ t \geq t_0,$$

where $||\varphi - \varphi^*||_\infty = \max_{1 \leq i \leq n} \left\{ \sup_{t \in [t_0 - \theta, t_0]} |\varphi_i(t) - \varphi_i^*(t)| \right\}$ for $\varphi, \psi \in C\{A_1, A_2\}$. Then the solution $x^*(t)$ is said to be exponentially stable.

**Theorem 11** Assume that $(H_1)$, $(H_3)$–$(H_5)$ hold. Then the positive almost periodic solution $x^*(t)$ in the region $\mathbb{B}^*$ of (2) is unique and exponentially stable.

*Proof* By Theorem 10, (2) has a positive almost periodic solution $x_i^*(t)$ in the region $\mathbb{B}^*$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be any arbitrary solution of (2) with initial value $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \in C\{A_1, A_2\}$. Then it follows from (2) that for $t \geq t_0, i = 1, 2, \ldots, n$,

$$(x_i(t) - x_i^*(t))^\Delta = -c_i(t)(x_i(t) - x_i^*(t)) + \sum_{k=1, k\neq i}^{n} b_{ik}(t)(x_k(t) - x_k^*(t))$$

$$+ \sum_{j=1}^{n} \beta_{ij}(t)\left[x_i(t - \tau_{ij}(t))e^{-\alpha_{ij}(t)x_i(t-\tau_{ij}(t))} - x_i^*(t - \tau_{ij}(t))e^{-\alpha_{ij}(t)x_i^*(t-\tau_{ij}(t))}\right].$$

$$(10)$$

The initial condition of (10) is

$$\psi_i(s) = \varphi_i(s) - x_i^*(s), \ s \in [t_0 - \theta, t_0]_{\mathbb{T}}, \quad i = 1, 2, \ldots, n.$$

For convenience, we denote $u_i(t) = x_i(t) - x_i^*(t)$, $i = 1, 2, \ldots, n$. Then, by (10), we have

$$u_i(t) = u_i(t_0)e_{-c_i}(t, t_0) + \int_{t_0}^{t} e_{-c_i}(t, \sigma(s)) \sum_{k=1, k\neq i}^{n} b_{ik}(s)u_k(s)\Delta s$$

$$+ \int_{t_0}^{t} e_{-c_i}(t, \sigma(s)) \sum_{j=1}^{n} \beta_{ij}(s)\left[x_i(s - \tau_{ij}(s))e^{-\alpha_{ij}(s)x_i(s-\tau_{ij}(s))}\right. \quad (11)$$

$$\left. - x_i^*(s - \tau_{ij}(s))e^{-\alpha_{ij}(s)x_i^*(s-\tau_{ij}(s))}\right]\Delta s, \ t \geq t_0, \quad i = 1, 2, \ldots, n.$$

For $\omega \in \mathbb{R}$, let $\Gamma_i(\omega)$ be defined by

$$\Gamma_i(\omega) = c_i^- - \omega - \exp\{\omega \sup_{s \in \mathbb{T}} \mu(s)\}\left( \sum_{k=1, k\neq i}^{n} b_{ik}^+ + \frac{1}{e^2}\sum_{j=1}^{n} \beta_{ij}^+ \exp\{\omega \tau_{ij}^+\}\right), \quad i = 1, 2, \ldots, n.$$

In view of $(H_2)$, we have that

$$\Gamma_i(0) = c_i^- - \left( \sum_{k=1, k\neq i}^{n} b_{ik}^+ + \frac{1}{e^2}\sum_{j=1}^{n} \beta_{ij}^+ \right) > 0, \quad i = 1, 2, \ldots, n.$$

Since $\Gamma_i(\omega)$ is continuous on $[0, +\infty)$ and $\Gamma_i(\omega) \to -\infty$ as $\omega \to +\infty$, so there exists $\omega_i > 0$ such that $\Gamma_i(\omega_i) = 0$ and $\Gamma_i(\omega) > 0$ for $\omega \in (0, \omega_i)$, $i = 1, 2, \ldots, n$. By choosing a positive constant $a = \min\{\omega_1, \omega_2, \ldots, \omega_n\}$, we have $\Gamma_i(a) \geq 0$, $i = 1, 2, \ldots, n$. Hence, we can choose a positive constant $0 < \alpha < \min\{a, \min\limits_{1 \leq i \leq n}\{c_i^-\}\}$ such that

$$\Gamma_i(\alpha) > 0, \quad i = 1, 2, \ldots, n,$$

which implies that

$$\frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_i^- - \alpha} \left( \sum_{k=1, k \neq i}^{n} b_{ik}^+ + \frac{1}{e^2} \sum_{j=1}^{n} \beta_{ij}^+ \exp\{\alpha \tau_{ij}^+\} \right) < 1, \quad i = 1, 2, \ldots, n.$$

Take

$$M = \max_{1 \leq i \leq n} \left\{ \frac{c_i^-}{\sum_{k=1, k \neq i}^{n} b_{ik}^+ + \frac{1}{e^2} \sum_{j=1}^{n} \beta_{ij}^+} \right\}.$$

It follows from $(H_5)$ that $M > 1$. Besides, we can obtain that

$$\frac{1}{M} < \frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_i^- - \alpha} \left( \sum_{k=1, k \neq i}^{n} b_{ik}^+ + \frac{1}{e^2} \sum_{j=1}^{n} \beta_{ij}^+ \exp\{\alpha \tau_{ij}^+\} \right).$$

In addition, noticing that $e_{\ominus \alpha}(t, t_0) \geq 1$ for $t \in [t_0 - \theta, t_0]_{\mathbb{T}}$. Hence, it is obvious that

$$\|u(t)\| \leq M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \quad \forall t \in [t_0 - \theta, t_0]_{\mathbb{T}}.$$

We claim that

$$\|u(t)\| \leq M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{12}$$

To prove this claim, we show that for any $p > 1$, the following inequality holds

$$\|u(t)\| < p M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, \tag{13}$$

which implies that, for $i = 1, 2, \ldots, n$, we have

$$|u_i(t)| < p M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{14}$$

By way of contradiction, assume that (14) is not true. Then there exists $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ and $i_0 \in \{1, 2, \ldots, n\}$ such that

$$|u_{i_0}(t_1)| \geq p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_{\infty}, \ |u_{i_0}(t)| < p M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \ t \in (t_0, t_1)_{\mathbb{T}},$$
$$|u_k(t)| \leq p M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \ \text{for } k \neq i_0, \ t \in (t_0, t_1]_{\mathbb{T}}, \quad k = 1, 2, \ldots, n.$$

Therefore, there must be a constant $q \geq 1$ such that

$$|u_{i_0}(t_1)| = q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_{\infty}, \ |u_{i_0}(t)| < q p M e_{\ominus \alpha}(t, t_0) \|\psi\|_{\infty}, \ t \in (t_0, t_1)_{\mathbb{T}},$$
$$|u_k(t)| < q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_{\infty}, \ \text{for } k \neq i_0, \ t \in (t_0, t_1]_{\mathbb{T}}, \quad k = 1, 2, \ldots, n.$$

According to (11), we have

$$
\begin{aligned}
|u_{i_0}(t_1)| = &\Bigg| u_{i_0}(t_0)e_{-c_{i_0}}(t_1, t_0) + \int_{t_0}^{t_1} e_{-c_{i_0}}(t_1, \sigma(s)) \sum_{k=1, k \neq i_0}^{n} b_{i_0 k}(s) u_k(s) \Delta s \\
&+ \int_{t_0}^{t_1} e_{-c_{i_0}}(t_1, \sigma(s)) \sum_{j=1}^{n} \beta_{i_0 j}(s) \big[ x_{i_0}(s - \tau_{i_0 j}(s)) e^{-\alpha_{i_0 j}(s) x_{i_0}(s - \tau_{i_0 j}(s))} \\
&- x_{i_0}^*(s - \tau_{i_0 j}(s)) e^{-\alpha_{i_0 j}(s) x_{i_0}^*(s - \tau_{i_0 j}(s))} \big] \Delta s \Bigg| \\
\leq & e_{-c_{i_0}}(t_1, t_0) \|\psi\|_\infty + q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \\
& \times \int_{t_0}^{t_1} e_{-c_{i_0}}(t_1, \sigma(s)) e_\alpha(t_1, \sigma(s)) \Bigg( \sum_{k=1, k \neq i_0}^{n} b_{i_0 k}^+ e_\alpha(\sigma(s), s) \\
& + \sum_{j=1}^{m} \frac{\beta_{i_0 j}^+}{e^2} e_\alpha(\sigma(s), s - \tau_{i_0 j}(s)) \Bigg) \Delta s \\
\leq & e_{-c_{i_0}}(t_1, t_0) \|\psi\|_\infty + q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \\
& \times \int_{t_0}^{t_1} e_{-c_{i_0} \oplus \alpha}(t_1, \sigma(s)) \Bigg( \sum_{k=1, k \neq i}^{n} b_{i_0 k}^+ \exp\{\alpha \sup_{s \in \mathbb{T}} \mu(s)\} \\
& + \sum_{j=1}^{m} \frac{\beta_{i_0 j}^+}{e^2} \exp\{\alpha(\tau_{i_0 j}^+ + \sup_{s \in \mathbb{T}} \mu(s))\} \Bigg) \Delta s \\
= & e_{-c_{i_0}}(t_1, t_0) \|\psi\|_\infty + q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \exp\{\alpha \sup_{s \in \mathbb{T}} \mu(s)\} \Bigg( \sum_{k=1, k \neq i_0}^{n} b_{i_0 k}^+ \\
& + \sum_{j=1}^{m} \frac{\beta_{i_0 j}^+}{e^2} \exp\{\alpha \tau_{i_0 j}^+\} \Bigg) \int_{t_0}^{t_1} e_{-c_{i_0} \oplus \alpha}(t_1, \sigma(s)) \Delta s \\
= & q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \Bigg\{ \frac{1}{q p M} e_{-c_{i_0} \oplus \alpha}(t_1, t_0) + \exp\{\alpha \sup_{s \in \mathbb{T}} \mu(s)\} \Bigg( \sum_{k=1, k \neq i_0}^{n} b_{i_0 k}^+ \\
& + \sum_{j=1}^{m} \frac{\beta_{i_0 j}^+}{e^2} \exp\{\alpha \tau_{i_0 j}^+\} \Bigg) \int_{t_0}^{t_1} e_{-c_{i_0} \oplus \alpha}(t_1, \sigma(s)) \Delta s \Bigg\} \\
< & q p M e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \Bigg\{ \frac{1}{q p M} e_{-(c_{i_0}^- - \alpha)}(t_1, t_0) + \exp\{\alpha \sup_{s \in \mathbb{T}} \mu(s)\} \Bigg( \sum_{k=1, k \neq i_0}^{n} b_{i_0 k}^+ \\
& + \sum_{j=1}^{m} \frac{\beta_{i_0 j}^+}{e^2} \exp\{\alpha \tau_{i_0 j}^+\} \Bigg) \frac{1}{-(c_{i_0}^- - \alpha)} \int_{t_0}^{t_1} \big( -(c_{i_0}^- - \alpha)\big) e_{-(c_{i_0}^- - \alpha)}(t_1, \sigma(s)) \Delta s \Bigg\}
\end{aligned}
$$

$$
\leq qpM e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \Bigg\{ \bigg[ \frac{1}{qpM} - \frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_{i_0}^- - \alpha} \Big( \sum_{k=1, k \neq i_0}^n b_{i_0 k}^+
$$

$$
+ \frac{1}{e^2} \sum_{j=1}^n \beta_{ij}^+ \exp\{\alpha \tau_{i_0 j}^+\} \Big) \bigg] e_{-(c_{i_0}^- - \alpha)}(t_1, t_0) + \frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_{i_0}^- - \alpha} \Big( \sum_{k=1, k \neq i_0}^n b_{i_0 k}^+
$$

$$
+ \frac{1}{e^2} \sum_{j=1}^n \beta_{i_0 j}^+ \exp\{\alpha \tau_{i_0 j}^+\} \Big) \Bigg\}
$$

$$
< qpM e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty \Bigg\{ \bigg[ \frac{1}{M} - \frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_{i_0}^- - \alpha} \Big( \sum_{k=1, k \neq i_0}^n b_{i_0 k}^+
$$

$$
+ \frac{1}{e^2} \sum_{j=1}^n \beta_{i_0 j}^+ \exp\{\alpha \tau_{i_0 j}^+\} \Big) \bigg] e_{-(c_{i_0}^- - \alpha)}(t_1, t_0) + \frac{\exp\{\alpha \sup\limits_{s \in \mathbb{T}} \mu(s)\}}{c_{i_0}^- - \alpha} \Big( \sum_{k=1, k \neq i_0}^n b_{i_0 k}^+
$$

$$
+ \frac{1}{e^2} \sum_{j=1}^n \beta_{i_0 j}^+ \exp\{\alpha \tau_{i_0 j}^+\} \Big) \Bigg\}
$$

$$
< qpM e_{\ominus \alpha}(t_1, t_0) \|\psi\|_\infty,
$$

which is a contradiction. Therefore, (14) and (13) hold. Let $p \to 1$, then (12) holds. Hence, we have that

$$
\|u(t)\| \leq M \|\psi\|_\infty e_{\ominus \alpha}(t, t_0), \quad t \in [t_0, +\infty)_\mathbb{T},
$$

which implies that the positive almost periodic solution $x^*(t)$ of (2) is exponentially stable. The exponential stability of $x^*(t)$ implies that the uniqueness of the positive almost periodic solution. The proof is complete. □

*Remark 6* It is easy to see that under definitions of almost periodic time scales and almost periodic functions in Li and Wang (2011a), the conclusions of Theorems 10 and 11 are true.
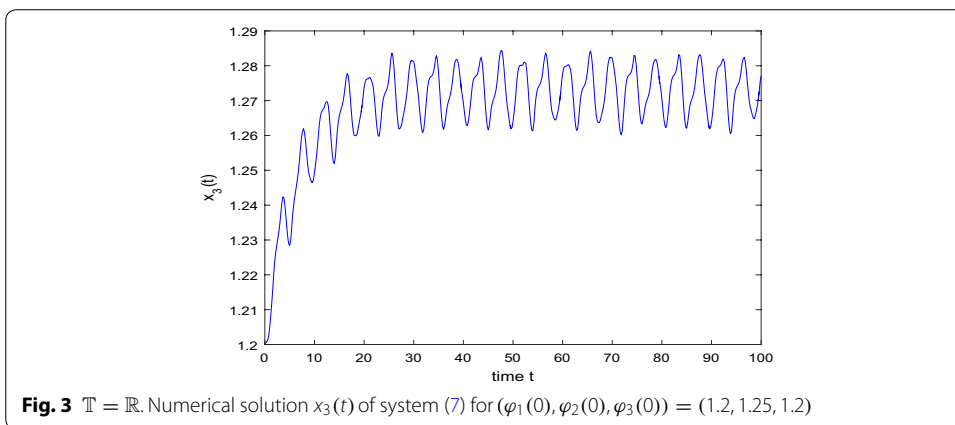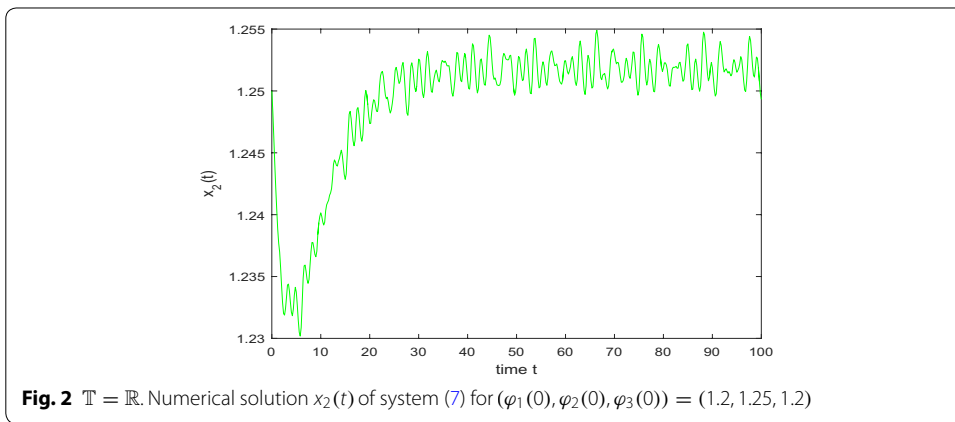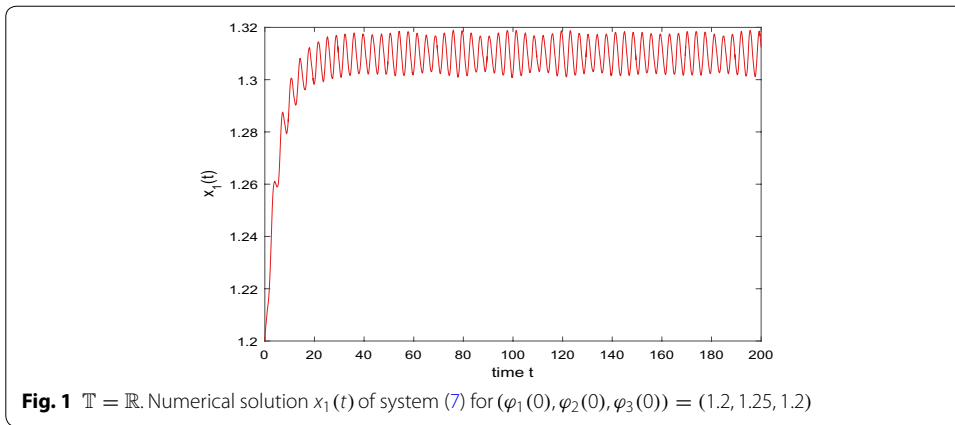
*Remark 7* From Remark 5, Theorems 10 and 11, we can easily see that if $c_i(t) < 1, i = 1, 2, \ldots, n$, then the continuous-time Nicholson's blowflies models and the discrete-time analogue have the same dynamical behaviors. This fact provides a theoretical basis for the numerical simulation of continuous-time Nicholson's blowflies models.
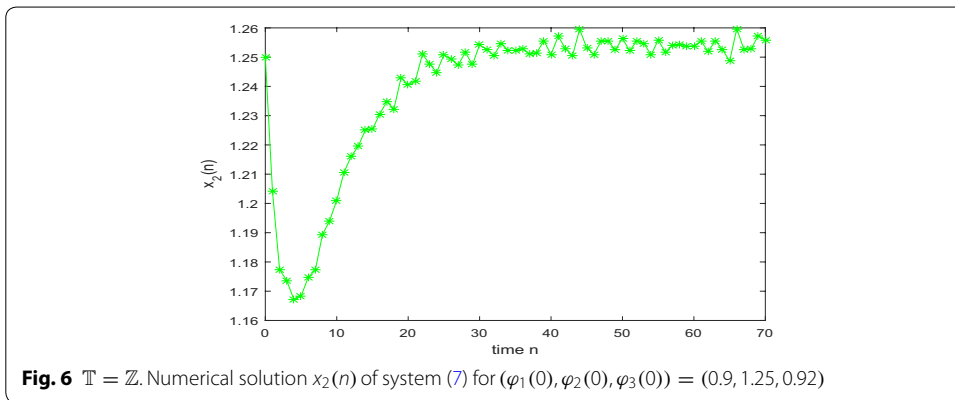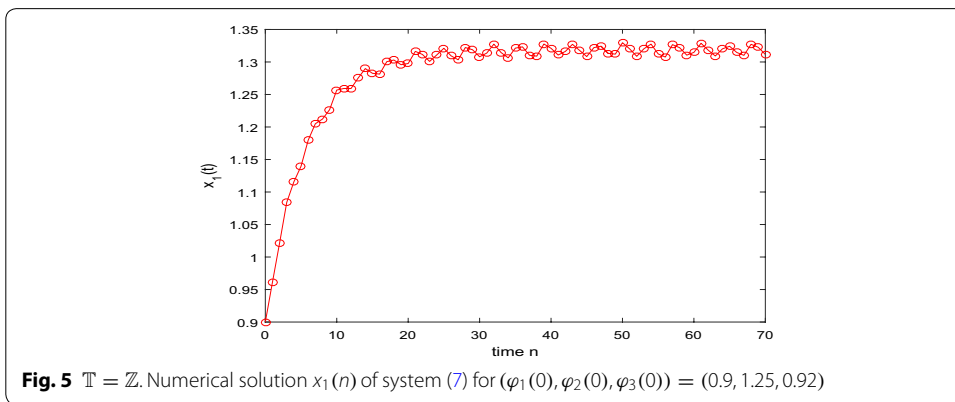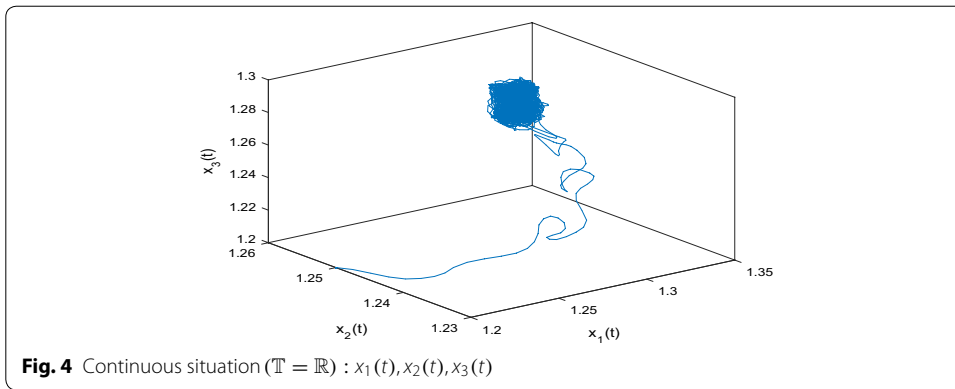
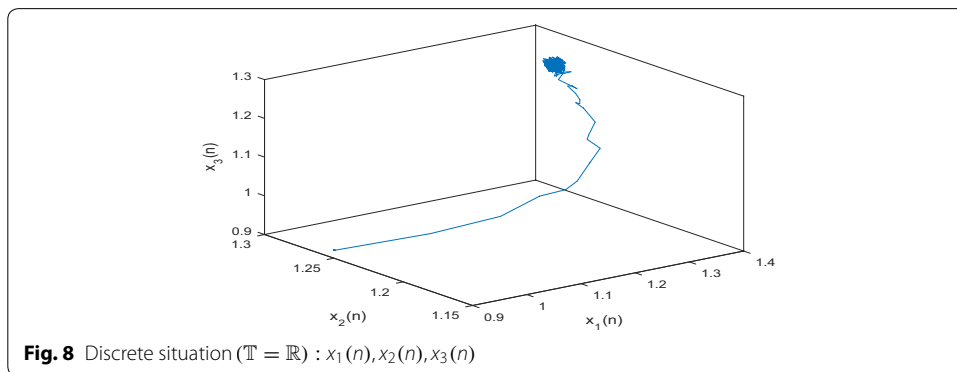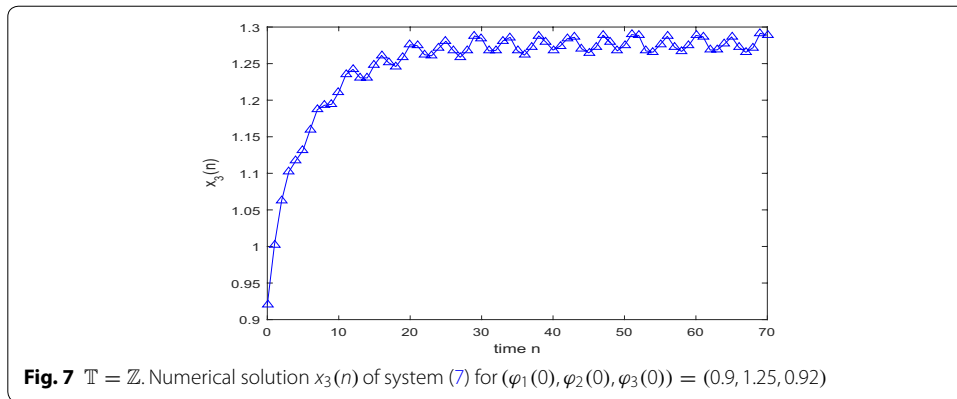*Remark 8* Our results and methods of this paper are different from those in Li and Yang (2012).

*Remark 9* When $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, our results of this section are also new. If we take $\mathbb{T} = \mathbb{R}, A_1 = 1, A_2 = e$, then Lemma 9, Theorems 10 and 11 improve Lemma 2.4, Theorems 2 and 3 in Wang et al. (2011), respectively.

## An example

In this section, we present an example to illustrate the feasibility of our results obtained in previous sections.

**Fig. 1** $\mathbb{T} = \mathbb{R}$. Numerical solution $x_1(t)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (1.2, 1.25, 1.2)$



**Fig. 2** $\mathbb{T} = \mathbb{R}$. Numerical solution $x_2(t)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (1.2, 1.25, 1.2)$



**Fig. 3** $\mathbb{T} = \mathbb{R}$. Numerical solution $x_3(t)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (1.2, 1.25, 1.2)$

**Fig. 4** Continuous situation $(\mathbb{T} = \mathbb{R}) : x_1(t), x_2(t), x_3(t)$



**Fig. 5** $\mathbb{T} = \mathbb{Z}$. Numerical solution $x_1(n)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (0.9, 1.25, 0.92)$



**Fig. 6** $\mathbb{T} = \mathbb{Z}$. Numerical solution $x_2(n)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (0.9, 1.25, 0.92)$

**Fig. 7** $\mathbb{T} = \mathbb{Z}$. Numerical solution $x_3(n)$ of system (7) for $(\varphi_1(0), \varphi_2(0), \varphi_3(0)) = (0.9, 1.25, 0.92)$



**Fig. 8** Discrete situation $(\mathbb{T} = \mathbb{R})$ : $x_1(n), x_2(n), x_3(n)$

*Example 3*  In system (2), let $n = 3$ and take coefficients as follows:

$$c_1(t) = 0.21 + 0.01\sin\sqrt{3}t, \ c_2(t) = 0.24 + 0.008|\sin 2t|, \ c_3(t) = 0.41 + 0.01\cos\sqrt{2}t,$$

$$b_{12}(t) = 0.04 - 0.001|\cos\pi t|, \ b_{13}(t) = 0.07 - 0.002|\cos\sqrt{3}t|, \ b_{21}(t) = 0.06 - 0.002|\cos\sqrt{3}t|,$$

$$b_{23}(t) = 0.06 - 0.001|\sin\sqrt{2}t|, \ b_{31}(t) = 0.17 - 0.01|\sin\sqrt{3}t|, \ b_{32}(t) = 0.14 - 0.01|\cos\sqrt{2}t|,$$

$$\beta_{11}(t) = 0.09 - 0.01|\sin\pi t|, \ \beta_{12}(t) = 0.16 - 0.01|\cos\sqrt{3}t|, \ \beta_{13}(t) = 0.16 - 0.01|\sin t|,$$

$$\beta_{21}(t) = 0.15 - 0.001|\cos\pi t|, \ \beta_{22}(t) = 0.19 - 0.009|\cos\sqrt{3}t|, \ \beta_{23}(t) = 0.09 + 0.01|\cos t|,$$

$$\beta_{31}(t) = 0.16 - 0.002|\cos t|, \ \beta_{32}(t) = 0.13 - 0.001|\cos\sqrt{2}t|, \ \beta_{33}(t) = 0.11 - 0.008|\sin t|,$$

$$\alpha_{11}(t) = \alpha_{12}(t) = \alpha_{13}(t) = 0.999 + 0.001|\sin\sqrt{3}t|,$$

$$\alpha_{21}(t) = 0.998 + 0.002\sin\sqrt{2}t, \ \alpha_{22}(t) = 0.998 + 0.002\cos\sqrt{2}t, \ \alpha_{23}(t) = 0.998 + 0.002\sin\pi t,$$

$$\alpha_{31}(t) = 0.998 + 0.002|\sin t|, \ \alpha_{32}(t) = 0.998 + 0.002|\sin\sqrt{3}t|, \ \alpha_{33}(t) = 0.998 + 0.002\left|\sin\left(\frac{4}{3}t\right)\right|,$$

$$\tau_{11}(t) = e^{0.2|\sin\pi t|}, \ \tau_{12}(t) = e^{0.4|\cos(\pi t + \frac{\pi}{2})|}, \ \tau_{13}(t) = e^{0.5|\sin\pi t|},$$

$$\tau_{21}(t) = e^{0.2|\cos(\pi t + \frac{\pi}{2})|}, \ \tau_{22}(t) = e^{0.3|\sin 3\pi t|}, \ \tau_{23}(t) = e^{0.4|\cos 2\pi t|},$$

$$\tau_{31}(t) = e^{0.5|\sin(\pi t + \frac{3\pi}{2})|}, \ \tau_{32}(t) = e^{0.3|\cos(\pi t + \frac{\pi}{2})|}, \ \tau_{33}(t) = e^{0.5|\cos 3\pi t|}.$$

By calculating, we have

$$c_1^- = 0.2, \ c_1^+ = 0.22, \ c_2^- = 0.24, \ c_2^+ = 0.248, \ c_3^- = 0.4, \ c_3^+ = 0.43,$$
$$b_{12}^- = 0.039, \ b_{12}^+ = 0.04, \ b_{13}^- = 0.068, \ b_{13}^+ = 0.07, \ b_{21}^- = 0.058, \ b_{21}^+ = 0.06,$$
$$b_{23}^- = 0.059, \ b_{23}^+ = 0.06, \ b_{31}^- = 0.16, \ b_{31}^+ = 0.17, \ b_{32}^- = 0.13, \ b_{32}^+ = 0.14,$$
$$\beta_{11}^- = 0.08, \ \beta_{11}^+ = 0.09, \ \beta_{12}^- = 0.15, \ \beta_{12}^+ = 0.16, \ \beta_{13}^- = 0.15, \ \beta_{13}^+ = 0.16,$$
$$\beta_{21}^- = 0.149, \ \beta_{21}^+ = 0.15, \ \beta_{22}^- = 0.181, \ \beta_{22}^+ = 0.19, \ \beta_{23}^- = 0.09, \ \beta_{23}^+ = 0.1,$$
$$\beta_{31}^- = 0.158, \ \beta_{31}^+ = 0.16, \ \beta_{32}^- = 0.129, \ \beta_{32}^+ = 0.13, \ \beta_{33}^- = 0.103, \ \beta_{33}^+ = 0.11,$$
$$\alpha_{11}^- = \alpha_{12}^- = \alpha_{13}^- = 0.999, \ \alpha_{11}^+ = \alpha_{12}^+ = \alpha_{13}^+ = 1, \ \alpha_{21}^- = \alpha_{22}^- = \alpha_{23}^- = 0.996,$$
$$\alpha_{21}^+ = \alpha_{22}^+ = \alpha_{23}^+ = 1, \ \alpha_{31}^- = \alpha_{32}^- = \alpha_{33}^- = 0.998, \ \alpha_{31}^+ = \alpha_{32}^+ = \alpha_{33}^+ = 1.$$

Hence

$$\sum_{k=1, k\neq 1}^{3} \frac{b_{1k}^+}{c_1^-} = \frac{b_{12}^+}{c_1^-} + \frac{b_{13}^+}{c_1^-} = \frac{0.04 + 0.07}{0.2} = \frac{11}{20} < 1,$$

$$\sum_{k=1, k\neq 2}^{3} \frac{b_{2k}^+}{c_2^-} = \frac{b_{21}^+}{c_2^-} + \frac{b_{23}^+}{c_2^-} = \frac{0.06 + 0.06}{0.24} = \frac{1}{2} < 1,$$

$$\sum_{k=1, k\neq 3}^{3} \frac{b_{3k}^+}{c_3^-} = \frac{b_{31}^+}{c_3^-} + \frac{b_{32}^+}{c_3^-} = \frac{0.17 + 0.14}{0.4} = \frac{31}{40} < 1,$$

$$b_{12}^+ + b_{13}^+ + \frac{\beta_{11}^+}{e^2} + \frac{\beta_{12}^+}{e^2} + \frac{\beta_{13}^+}{e^2} = 0.04 + 0.07 + \frac{0.09}{e^2} + \frac{0.16}{e^2} + \frac{0.16}{e^2} \approx 0.165 < c_1^- = 0.2,$$

$$b_{21}^+ + b_{23}^+ + \frac{\beta_{21}^+}{e^2} + \frac{\beta_{22}^+}{e^2} + \frac{\beta_{23}^+}{e^2} = 0.06 + 0.06 + \frac{0.15}{e^2} + \frac{0.19}{e^2} + \frac{0.1}{e^2} \approx 0.1795 < c_2^- = 0.24,$$

$$b_{31}^+ + b_{32}^+ + \frac{\beta_{31}^+}{e^2} + \frac{\beta_{32}^+}{e^2} + \frac{\beta_{33}^+}{e^2} = 0.17 + 0.14 + \frac{0.16}{e^2} + \frac{0.13}{e^2} + \frac{0.11}{e^2} \approx 0.364 < c_3^- = 0.4.$$

We can check that for any $A_2 = 1.68$, we have

$$A_2 > \max_{1 \leq i \leq 3} \left\{ \left[ 1 - \sum_{k=1, k\neq i}^{3} \frac{b_{ik}^+}{c_i^-} \right]^{-1} \sum_{j=1}^{3} \frac{\beta_{ij}^+}{c_i^- \alpha_{ij}^- e} \right\} = \max_{1 \leq i \leq 3} \{1.678, 1.354, 1.638\} = 1.678.$$

and

$$\min_{1 \leq i \leq 3} \left\{ \left[ 1 - \sum_{k=1, k\neq i}^{3} \frac{b_{ik}^-}{c_i^+} \right]^{-1} \sum_{j=1}^{3} A_2 \frac{\beta_{ij}^-}{c_i^+} e^{-\alpha_{ij}^+ A_2} \right\}$$

$$= \min_{1 \leq i \leq 3} \{1.053, 1.025, 1.104\}$$

$$= 1.025 > A_1 > \frac{1}{\min\limits_{1 \leq i,j \leq 3} \alpha_{ij}^-} = \frac{1}{0.996} \approx 1.004.$$

For $0 \leq \mu(t) \leq 1$, we have $1 - \mu(t)c_i(t) > 0$, therefore, whether $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, we have $-c_i \in \mathcal{R}^+$ and condition $(H_4)$ is satisfied.

If $-c_i \in \mathcal{R}^+$, that is, $1 - c_i(t)\mu(t) > 0, i = 1, 2, 3$, then it is easy to verify that all conditions of Theorem 11 are satisfied. Therefore, the system in Example 4.1 has a unique positive almost periodic solution in the region $\mathbb{B}^* = \{\varphi| \ \varphi \in \mathbb{B}, 1.004 < A_1 \leq \varphi_i(t) \leq 1.68, t \in \mathbb{T}, \quad i = 1, 2, \ldots, n\}$, which is exponentially stable.

Especially, if we take $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, then $1 - c_i(t)\mu(t) > 0, i = 1, 2, 3$. Hence, in this case, the continuous-time Nicholson's blowflies model (2) and its discrete-time analogue have the same dynamical behaviors (see Figs. 1,2, 3, 4, 5, 6, 7, 8).

*Remark 10*   Non of the results obtained in Chérif (2015), Duan and Huang (2015), Yao (2015a), Wang et al. (2011), Alzabut (2010), Chen and Liu (2011), Long (2012), Wang (2013), Liu and Meng (2012), Xu (2014), Ding and Alzabut (2015), Yao (2014), Alzabut (2013) can be used to obtain the results of Example 3.

## Conclusion

In this paper, we proposed a new concept of almost periodic time scales, two new definitions of almost periodic functions on time scales and investigated some basic properties of them, which can unify the continuous and the discrete cases effectively. As an application, we obtain some sufficient conditions for the existence and exponential stability of positive almost periodic solutions for a Nicholson's blowflies model on time scales. Our methods and results of this paper may be used to study almost periodicity of general dynamic equations on time scales. Besides, based on our this new concept of almost periodic time scales, one can further study the problems of pseudo almost periodic functions, pseudo almost automorphic functions and pseudo almost periodic set-valued functions on times as well as the problems of pseudo almost periodic, pseudo almost automorphic and pseudo almost periodic set-valued dynamic systems on times and so on.

## References

Alzabut JO (2010) Almost periodic solutions for an impulsive delay Nicholson's blowflies model. J Comput Appl Math 234:233–239

Alzabut JO (2013) Existence and exponential convergence of almost periodic aolutions for a discrete Nicholson's blow-flies model with nonlinear harvesting term. Math Sci Lett 2(3):201–207

Berezansky L, Idels L, Troib L (2011) Global dynamics of Nicholson-type delay systems with applications. Nonlinear Anal Real World Appl 12(1):436–445

Bohner M, Peterson A (2001) Dynamic equations on time scales, an introduction with applications. Birkhäuser, Boston

Chen Y (2003) Periodic solutions of delayed periodic Nicholson's blowflies models. Can Appl Math Q 11:23–28

Chen W, Liu BW (2011) Positive almost periodic solution for a class of Nicholson's blowflies model with multiple time-varying delays. J Comput Appl Math 235:2090–2097

Chérif F (2015) Pseudo almost periodic solution of Nicholson's blowflies model with mixed delays. Appl Math Model 39:5152–5163

David C, Cristina M (2004) Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations. Nonlinear Anal 56(4):465–484

Ding HS, Alzabut J (2015) Existence of positive almost periodic solutions for a Nicholson's blowflies model. Electron J Differ Equ 2015(180):1–6

Duan L, Huang L (2015) Pseudo almost periodic dynamics of delay Nicholson's blowflies model with a linear harvesting term. Math Methods Appl Sci 38(6):1178–1189

Faria T (2011) Global asymptotic behaviour for a Nicholson model with patch structure and multiple delays. Nonlinear Anal 74:7033–7046

Fink AM (1974) Almost periodic differential equations. Springer, Berlin

Fink AM, Seifert G (1969) Liapunov functions and almost periodic solutions for almost periodic systems. J Differ Equ 5:307–313

Gao J, Wang QR, Zhang LW (2014) Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales. Appl Math Comput 237:639–649

Gurney WSC, Blythe SP, Nisbet RM (1980) Nicholson's blowflies revisited. Nature 287:17–21

Hale JK, Verduyn Lunel SM (1993) Introduction to functional differential equations. Springer, New York

Hien LV (2014) Global asymptotic behaviour of positive solutions to a non-autonomous Nicholson's blowflies model with delays. J Biol Dyn 8(1):135–144

Hilger S (1990) Analysis on measure chains—a unified approach to continuous and discrete calculus. Results Math 18:18–56

Hong SH, Peng YZ (2016) Almost periodicity of set-valued functions and set dynamic equations on time scales. Inf Sci 330:157–174

Li J, Du C (2008) Existence of positive periodic solutions for a generalized Nicholson's blowflies model. J Comput Appl Math 221:226–233

Li YK, Li B (2015) Almost periodic time scales and almost periodic functions on time scales. J Appl Math 2015(730672):8

Li YK, Wang C (2011a) Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales. Abstr Appl Anal 2011(341520):22

Li YK, Wang C (2011b) Almost periodic functions on time scales and applications. Discrete Dyn Nat Soc. Article ID 727068

Li Y, Wang C (2012) Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales. Adv Differ Equ 2012:77

Li YK, Yang L (2012) Existence and stability of almost periodic solutions for Nicholson's blowflies models with patch structure and linear harvesting terms on time scales. Asian Eur J Math 5(3):1250038

Li Y, Yang L (2014) Almost automorphic solution for neutral type high-order Hopfield neural networks with delays in leakage terms on time scales. Appl Math Comput 242:679–693

Li YK, Zhao LL, Yang L (2015a) $C^1$-Almost periodic solutions of BAM neural networks with time-varying delays on time scales. Sci World J. Article ID 727329

Li YK, Li B, Meng XF (2015b) Almost automorphic funtions on time scales and almost automorphic solutions to shunting inhibitory cellular neural networks on time scales. J Nonlinear Sci Appl 8:1190–1211

Liang T, Yang Y, Liu Y, Li L (2014) Existence and global exponential stability of almost periodic solutions to Cohen–Grossberg neural networks with distributed delays on time scales. Neurocomputing 123:207–215

Liu BW (2010) Global stability of a class of Nicholson's blowflies model with patch structure and multiple time-varying delays. Nonlinear Anal Real World Appl 11:2557–2562

Liu BW (2014a) Global exponential stability of positive periodic solutions for a delayed Nicholson's blowflies model. J Math Anal Appl 412:212–221

Liu BW (2014b) Positive periodic solutions for a nonlinear density-dependent mortality Nicholson's blowflies model. Kodai Math J 37:157–173

Liu B, Gong S (2011) Permanence for Nicholson-type delay systems with nonlinear density-dependent mortality terms. Nonlinear Anal Real World Appl 12:1931–1937

Liu X, Meng J (2012) The positive almost periodic solution for Nicholson-type delay systems with linear harvesting terms. Appl Math Model 36:3289–3298

Lizama C, Mesquita JG (2013a) Almost automorphic solutions of dynamic equations on time scales. J Funct Anal 265:2267–2311

Lizama C, Mesquita JG (2013b) Asymptotically almost automorphic solutions of dynamic equations on time scales. J Math Anal Appl 407:339–349

Lizama C, Mesquita JG, Ponce R (2014) A connection between almost periodic functions defined on timescales and $\mathbb{R}$. Appl Anal 93:2547–2558

Long F (2012) Positive almost periodic solution for a class of Nicholson's blowflies model with a linear harvesting term. Nonlinear Anal Real World Appl 13:686–693

Mophou G, N'Guérékata GM, Milce A (2014) Almost automorphic functions of order and applications to dynamic equations on time scales. Discrete Dyn Nat Soc 2014(410210):13

Nicholson AJ (1954) An outline of the dynamics of animal populations. Aust J Zool 2:9–65

Saker S, Agarwal S (2002) Oscillation and global attractivity in a periodic Nicholson's blowflies model. Math Comput Model 35:719–731

Shao JY (2012) Global exponential stability of non-autonomous Nicholson-type delay systems. Nonlinear Anal Real World Appl 13:790–793

Wang LJ (2013) Almost periodic solution for Nicholson's blowflies model with patch structure and linear harvesting terms. Appl Math Model 37:2153–2165

Wang C, Agarwal RP (2014a) Weighted piecewise pseudo almost automorphic functions with applications to abstract impulsive dynamic equations on time scales. Adv Differ Equ 2014:153

Wang C, Agarwal RP (2014b) A further study of almost periodic time scales with some notes and applications. Abstr Appl Anal. Article ID 267384

Wang C, Li Y (2013) Weighted pseudo almost automorphic functions with applications to abstract dynamic equations on time scales. Ann Pol Math 108:225–240

Wang WT, Wang LJ, Chen W (2011) Existence and exponential stability of positive almost periodic solution for Nicholson-type delay systems. Nonlinear Anal Real World Appl 12:1938–1949

Xu YL (2014) Existence and global exponential stability of positive almost periodic solutions for a delayed Nicholson's blowflies model. J Korean Math Soc 51:473–493

Yao ZJ (2014) Existence and exponential convergence of almost periodic positive solution for Nicholson's blowflies discrete model with linear harvesting term. Math Methods Appl Sci 37:2354–2362

Yao ZJ (2015a) Existence and exponential stability of the unique positive almost periodic solution for impulsive Nicholson's blowflies model with linear harvesting term. Appl Math Model 39:7124–7133

Yao Z (2015b) Existence and global exponential stability of an almost periodic solution for a host-macroparasite equation on time scales. Adv Differ Equ 2015:41

Yi TS, Zou X (2008) Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: a non-monotone case. J Differ Equ 245(11):3376–3388

Zhou Q (2013) The positive periodic solution for Nicholson-type delay system with linear harvesting terms. Appl Math Model 37:5581–5590

Zhou H, Zhou Z, Jiang W (2015) Almost periodic solutions for neutral type BAM neural networks with distributed leakage delays on time scales. Neurocomputing 157:223–230