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Research article

Mathematical analysis and control optimization of soluble and insoluble water pollutants dispersion

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ABSTRACT

One of the major and abundant environmental elements is water, which life existence on earth is depending on. However, degradation of water quality arising from natural occurrences due to anthropogenic that resulted in eutrophication needs to be guided. Indiscriminate harmful waste discharge and dispersion into the water bodies leads to water pollution. As such, constant management and monitoring of water resources for an enhanced quality and quantity is essential. Therefore, this study employed a coupled derivative mathematical model to investigate soluble and insoluble water pollutants. The insoluble pollutants are converted to soluble pollutants through an applied control or treatment. To optimize the water quality, parametric sensitivities in relation to reproduction number are examined. From the computational analysis carried out, it was revealed that treatment still remains the only remedy for safe water.

1. Introduction

A concrete abstract description of systems with the help of mathematical language and concepts defined by mathematical modeling. Mathematical formulations or models are applicable in social sciences, agricultural administration, medicine, engineering, and other areas of specialization [1,2]. Deterministic or stochastic, nonlinear or linear models have been developed to forecast the epidemic outbreaks and control, Bonyah et al. [3]. Mathematical models remain an effective way for infectious diseases to be dynamically studied in a population and host. According to Yavuz and Ozdemir [4], the contagious diseases mathematical model needs a combination of vaccination, containment, quarantine and medication. A compartmental, epidemiological, coupled derivative model has been developed for diseases such as malaria, Ebola, COVID-19, HBV, HIV, and so on, which has attracted attention due to an effective control prediction [4–9]. Epidemiological models have been developed for water pollution, dispersion and treatment due to its need for life sustenance. Reaction-diffusion models are widely used for soluble pollutants that undergo chemical transformations. Crank [10] and Sposito [11] described how such models can capture the complex interplay of chemical reactions and diffusion processes, particularly for nutrients like nitrates and phosphates. However, these models typically exclude insoluble pollutants, which do not react in the same manner.

Water is an element of the environment and a universal chemical solvent that can dissolve several substances. It is a primary component of cells, essential for biological processes and a meaningful part of living organisms. Also, it is functional to life survival and

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vital in different geological processes, weather patterns, and earth ecosystems [12,13]. The water sources such as streams, lagoons, lakes, rivers and others are used for industrial, domestic, and irrigation purposes. Meanwhile, human activities often pollute and affect the quality and quantity of good water assessment, Shah et al. [14]. Thus, pollution describes susceptible substances that affect ecological systems, harm living resources, damage structures, and threaten human health, as stated by Sharma and Bhattacharya [15]. Water pollutants are caused by natural processes and human activities, which contaminate the water bodies and lead to water-borne diseases and environmental degradation. Water-borne diseases like respiratory infections, rashes, gastroenteritis, hepatitis, pink eyes, ear-arch and many more can be contracted from polluted water [16,17]. Among the water pollutants are insecticides, pesticides, untreated sewage, domestic waste, industrial waste, oil spillage and many more, which are equally harmful to aquatic animals, Yeleliere et al. [18]. The water pollutants can be categorized into soluble and insoluble pollutants, as presented by Saravanan et al. [19]. Water pollutants such as soaps, chlorides, nitrates, and so on are grouped as soluble pollutants, while pollutants such as polythene bags, oils, plastics, and many more are grouped as insoluble pollutants [14,19].

Many pollutants affect marine organisms, damage the human nervous system, and consequently signal serious threats to human life and ecosystems. Thus, contaminated water breeds mosquitoes, a malaria-causing agent and other agents like viruses, parasites and bacteria, Pandey et al. [20]. However, awareness of water pollution has increased recently, but water resource exploitation and sustainability need to be improved to prevent water pollution and water-borne diseases, Boelee et al. [21]. Several scholars have studied water pollution dynamics among is Agusto and Bamigbola [22], who applied a numerical Crank-Nicholson technique to a derivative water pollution and treatment model. The results show that pollutant concentration reduced greatly in each case as treatment was applied. Shah et al. [23] presented a mathematical formulation for the forest resource pollutants dispersion with control maximization strategies. It was revealed that forest resource pollutant density is reduced directly by the harvest from the wood-based factories, but the forest resources are affected directly by the non-wood-based factories. Parsaie and Haghiabi [24] conducted a computational artificial neural network to study river pollution transmission. A longitudinal pollution coefficient of transmission was predicted through the adopted technique. Pimpunchat et al. [25] studied river aeration pollution and its control via a mathematical model. The outcomes depicted that river pollutants insertion rate could be economically dead as the needed oxygen for fish survival is below standard. More earlier reports on water pollution and remedies can be obtained in [26–28]. Water pollution is a critical environmental issue that affects ecosystems, human health, and economies globally. Pollutants in water bodies can originate from industrial activities, agricultural runoff, urban waste, and natural processes. The harmful effects of water pollution range from habitat destruction, loss of biodiversity, and disruption of food chains to serious public health risks like waterborne diseases and the accumulation of toxic substances in humans through contaminated drinking water and food sources.

The various reports on water pollution and the essentials of clear water for microorganisms, aquatic animals and human beings' survival motivated the research. Thus, this study aims to examine soluble and insoluble water pollutant dispersion and control optimization through a bio-mathematical model analysis. There is a clear research gap in developing integrated models and control strategies that simultaneously address soluble and insoluble pollutants in complex, dynamic environments. Moreover, the interaction between these pollutants, the long-term behavior of insoluble substances, and the integration of real-time data into control models remain under-explored. Addressing these gaps through hybrid modeling approaches and advanced optimization techniques will contribute significantly to improving water quality management. The significance of water in agricultural activities, domestic and engineering processes, and others cannot be overstressed. As such, the study tends to analyze the pollution stability equilibrium point at the pollutants-free equilibrium and endemic and optimize the pollutants control and treatment strategies in reducing pollution of water bodies. Thus, the study developed an integrated mathematical model that captures the dispersion dynamics of both soluble and insoluble pollutants, accounting for their interaction mechanisms, such as adsorption, desorption, and sedimentation.

2. The descriptions model formulation

Consider a compartmental mathematical model for the soluble and insoluble propagation of water pollutions with suitable treatment. A discrete cubic volume compartmental derivative model for the polluted water sources (P_w), susceptible-soluble water pollutants (S_w), infected-insoluble water pollutants (I_w) and treated-insoluble water sources (T_w) is formulated. In reality, many water bodies are contaminated with both types of pollutants, and they often interact in complex ways. As schematically demonstrated in Fig. 1, the dispersion of the water pollutants model is developed according to [14,29], considering the assumptions that the water body is treated as a homogeneous medium, meaning that its properties are uniform throughout the domain of interest, the pollutants are assumed to be sufficiently diluted such that their interactions with the water body do not affect the flow properties or create significant nonlinear effects, the insoluble pollutants, biodegradation is often neglected in the modeling framework, assuming that the primary processes affecting these pollutants are physical, and the physical properties of the fluid, such as viscosity and density, are assumed to remain constant during the modeling period, irrespective of changes in pollutant concentration or flow conditions.

Appropriate dynamical parameters are considered in the development of the nonlinear coupled derivative model. The mechanisms for the mitigation of insoluble pollutants spread across the water regime are optimized for an effective control. Thus, the derivative dimensional model is given as:

$$\frac{dP_w}{dt} = \alpha - \beta P_w S_w - \gamma P_w I_w + \phi\beta I_w - \varphi P_w, \qquad P_{w0} \geq 0, \tag{1}$$

$$\frac{dS_w}{dt} = \beta P_w S_w + \lambda I_w - (\epsilon + \varphi) S_w, \qquad S_{w0} \geq 0, \tag{2}$$

$$\frac{dI_w}{dt} = \gamma P_w I_w - \phi\beta I_w - (\lambda + \delta + \varphi) I_w, \qquad I_{w0} \geq 0, \tag{3}$$

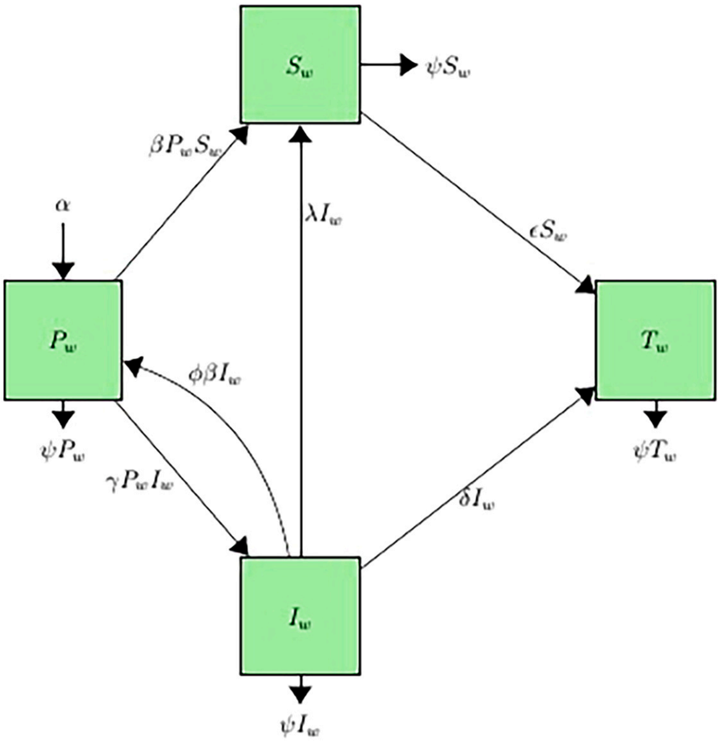

Fig. 1. Model transmission configuration.

$$\frac{dT_w}{dt} = \epsilon S_w + \delta I_w - \varphi T_w, \qquad T_{w0} \geq 0, \tag{4}$$

where $P_w(0) = P_{w0}, S_w(0) = S_{w0}, I_w(0) = I_{w0}, T_w(0) = T_{w0}$ are the initial constraints.

The embedded term α defines water pollutants rate, β and γ are soluble and insoluble water pollutants transmission rate separately. The parameter φ denotes compartmental water pollutants removable rate, λ describes insoluble water pollutants treatment rate, ϕ connotes water pollution rate of insoluble water pollutants, and ϵ and δ are the compartmental cubic volume treatment for soluble and insoluble water pollutants respectively.

2.1. Analysis of the mathematical model

The solution positivity and feasibility of the basic system of equations (1) to (4) are described in this section.

2.1.1. Solution positivity

For meaningful epidemiological translation, the water pollutants dispersion system of equations (1) to (4) are investigated for solution positivity. Thus, it is necessary to establish the non-negativity of all its stated variables for all $t \geq 0$.

Theorem 2.1. *The solution to the non-negative initial conditions of the coupled equations (1) to (4) of* (P_w, S_w, I_w, T_w) *are bounded uniformly on* $[0, \infty)$.

Proof. With non-negative initial conditions, the solutions to (P_w, S_w, I_w, T_w) are assumed to exist and unique. Taking equation (1) from the derivative modeled equations and simplify to become:

$$\frac{dP_w}{dt} = \alpha + \phi\beta I_w - (\beta S_w + \gamma I_w + \varphi)P_w,$$

removing the term $\alpha + \phi\beta I_w$ since it is non-negative to have

$$\frac{dP_w}{dt} \geq -(\beta S_w + \gamma I_w + \varphi)P_w. \tag{5}$$

Integrating both sides of equation (5) and apply separation of variables method, thus gives

$$\int \frac{dP_w}{P_w} \geq -\int (\beta S_w + \gamma I_w + \varphi)dt,$$
$$ln|P_w| \geq -(\beta S_w + \gamma I_w + \varphi)t + K,$$

$$P_w(t) \geq e^{-(\beta S_w+\gamma I_w+\varphi)t+K},$$

apply the initial conditions at $t=0$; hence, resulted to

$$P_w(t) \geq P_{w0}e^{-(\beta S_w+\gamma I_w+\varphi)t} \geq 0, \text{ since } (\beta S_w + \gamma I_w + \varphi) > 0.$$

Therefore, $P_w \geq 0$ for $t \geq 0$. Also, taking equation (2) from the derivative model,

$$\frac{dS_w}{dt} = \beta P_w S_w + \lambda I_w - (\epsilon + \varphi)S_w,$$

due to non-negativity, the term $\beta P_w S_w + \lambda I_w$ is removed, then the equation reduced to:

$$\frac{dS_w}{dt} \geq -(\epsilon + \varphi)S_w. \tag{6}$$

Solving equation (6) by using variable separation method to have

$$\int \frac{dS_w}{S_w} \geq -\int (\epsilon + \varphi)dt,$$

$$ln|S_w| \geq -(\epsilon + \varphi)t + K,$$

$$S_w(t) \geq e^{-(\epsilon+\varphi)t+K},$$

introducing the $S_w(t)$ initial condition at $t=0$,

$$S_w(t) \geq S_{w0}e^{-(\epsilon+\varphi)t} \geq 0 \text{ since } (\epsilon + \varphi) > 0.$$

Similarly, investigate the solution positivity of equation (3) given as

$$\frac{dI_w}{dt} = \gamma P_w I_w - (\phi\gamma + \epsilon + \varphi + \lambda)I_w,$$

remove the term $\gamma P_w I_w$ since it is non-negative, thus reduced to

$$\frac{dI_w}{dt} \geq -(\phi\gamma + \epsilon + \varphi + \lambda)I_w. \tag{7}$$

Integrating both sides of equation (7) via variables separating method to have

$$\int \frac{dI_w}{I_w} \geq -\int (\phi\gamma + \epsilon + \varphi + \lambda)dt,$$

$$ln|I_w| \geq -(\phi\gamma + \epsilon + \varphi + \lambda)t + K,$$

$$I_w(t) \geq e^{(\phi\gamma+\epsilon+\varphi+\lambda)t+K},$$

using the initial condition when $t=0$, this resulted to

$$I_w(t) \geq I_{w0}e^{-(\phi\gamma+\epsilon+\varphi+\lambda)t} \geq 0, \text{since } (\phi\gamma + \epsilon + \varphi + \lambda) > 0.$$

Finally, taking equation (4) from the modeled equations, such that

$$\frac{dT_w}{dt} = \epsilon S_w + \delta I_w - \varphi T_w,$$

ignoring the non-negative term $\epsilon S_w + \delta I_w$, thus the equation gives

$$\frac{dT_w}{dt} \geq -\varphi T_w \tag{8}$$

By applying separation of variables via integrating both sides of equation (8) to have

$$\int \frac{dT_w}{dt} \geq -\int \varphi dt,$$

$$ln|T_w| \geq -\varphi t + K,$$

$$T_w(t) \geq e^{-\varphi t+K},$$

employ the initial condition at $t=0$, thus gives

$$T_w(t) \geq T_{w0}e^{\varphi t} \geq 0, \text{since } \varphi > 0.$$

Therefore, at $t > 0$, P_w, S_w, I_w, T_w are all non-negative i.e. all give positive values. Hence, the proof.

2.1.2. Solution feasibility

The solution feasibility set for the modeled derivative equations (1) to (4) can be obtained through its positively invariant set give as

$$\Lambda = \left\{ (P_w, S_w, I_w, T_w) \in R_+^4 \, : \, A(t) \le \frac{\alpha}{\varphi} \right\}.$$

Lemma 2.1. *With respect to the dynamical system of the water pollutants transmission equations (1) to (4), the region* Λ *is positively invariant.*

Proof. To demonstrate the positively invariant region, $A(t) \le \frac{\alpha}{\varphi}$ as $t \to \infty$ shall be established. The total population from the model is $A = P_w + S_w + I_w + T_w$, thus differentiate A and the right hand side with respect to t to have an equation satisfying equations (1) to (4):

$$\begin{aligned}
A' &= P_w' + S_w' + I_w' + T_w',\\
&= \alpha - \beta P_w S_w - \gamma P_w I_w + \phi\beta I_w - \varphi P_w + \beta P_w S_w\\
&+ \lambda I_w - (\epsilon + \varphi) S_w + \gamma P_w I_w - \phi\beta I_w - (\delta + \varphi + \delta) I_w\\
&+ \epsilon S_w + \delta I_w - \varphi T_w,\\
&= \alpha - \varphi P_w - \varphi S_w + \varphi I_w - \varphi T_w,\\
&= \alpha - \varphi A, \text{ since } A = P_w + S_w + I_w + T_w,
\end{aligned}$$

therefore, this gives

$$\begin{aligned}
&\frac{dA}{dt} = \alpha - \varphi A,\\
&\frac{dA}{dt} + \varphi A = \alpha.
\end{aligned} \tag{9}$$

Apply the idea of integrating factor to equation (9), solving it gives

$$A(t) = \frac{\alpha}{\varphi} + Ce^{-\varphi t}, \tag{10}$$

where C is a constant, introducing the initial condition $A(0) = A_0$ on equation (10), this gives

$$A(0) = \frac{\alpha}{\varphi} + Ce^{-\varphi(0)} = A_0, \text{ where } C = A_0 - \frac{\alpha}{\varphi}$$

Hence, equation (10) becomes

$$A(t) = \frac{\alpha}{\varphi}\left(1 - e^{-\varphi t}\right) + A_0 e^{-\varphi t}, \tag{11}$$

taking the limit as $t \to \infty$, thus equation (11) gives

$$lim_{t\to\infty} A(t) \le \frac{\alpha}{\varphi}. \tag{12}$$

Hence, equations (1) to (4) feasible region is obtained as equation (12), which can be described as $\Lambda = \left\{(P_w, S_w, I_w, T_w) \in R_+^4 \, : \, A = \frac{\alpha}{\varphi}\right\}$ is positively invariant. Thus, the water pollutants transmission model is epidemiologically and mathematically posted.

2.2. Pollutants free equilibrium [PFE]

The concept of pollutants-free equilibrium is important in water pollution models as it represents the ideal target for water quality restoration efforts. While this equilibrium may be difficult to achieve in practice, mathematical models provide insights into the conditions necessary for the reduction of both soluble and insoluble pollutants, and the strategies required to move towards a cleaner, more sustainable state in aquatic ecosystems. The PFE can be determined from the derivative equations (1) to (4) by setting

$$\frac{dP_w}{dt} = \frac{dS_w}{dt} = \frac{dI_w}{dt} = \frac{dT_w}{dt} = 0.$$

Taking from equation (3),

$$\begin{aligned}
&I_w' = \gamma P_w I_w - \phi\beta I_w - (\delta + \varphi - \lambda) I_w = 0,\\
&(\gamma P_w - \phi\beta - \delta - \varphi - \lambda) I_w = 0,\\
&I_w = 0.
\end{aligned}$$

From equation (2), the equilibrium point is obtained as follows

$$S'_w = \beta P_w S_w + \lambda I_w - (\epsilon + \varphi)S_w = 0,$$
$$\lambda I_w - (\epsilon + \varphi - \beta P_w)S_w = 0, \text{ since } I_w = 0,$$
$$S_w = 0.$$

Likewise, from equation (4), it can be gotten as follows

$$T'_w = \epsilon S_w + \delta I_w - \varphi T_w = 0,$$
$$\varphi T_w = 0, \text{ since } I_w = S_w = 0,$$
$$T_w = 0.$$

Similarly, taking from equation (1), the equilibrium point is determined

$$P'_w = \alpha - \beta P_w S_w - \gamma P_w I_w + \phi\beta I_w - \varphi p_w = 0,$$
$$\alpha - \varphi P_w = 0, \text{ since } I_w = S_w = 0,$$
$$P_w = \frac{\alpha}{\varphi}.$$
$$(P_w^0, S_w^0, I_w^0, T_w^0) = \left(\frac{\alpha}{\varphi}, 0, 0, 0\right). \tag{13}$$

Therefore, the PFE is obtained as equation (13).

2.3. The pollutants basic reproduction number (R_0)

This is the epidemiology models threshold quantity, which describes the secondary cases expected number that a pollution or infection produced with a single primary case in water body or host population such that everyone is susceptible as stated by Adeniyi et al. [30] and Oke et al. [31]. Thus, when $R_0 > 0$ implies high water pollution or infection dispersion rate, and when $R_0 < 0$ indicates that the pollution or infection will disappear or die overtime.

2.3.1. Evaluation of R_0 via next generation matrix

The R_0 can be determined using next generation matrix with distinct finitely several groups of pollutants are recognized. The next generation matrix method properly position terms in infected or polluted group to the vectors G and H. The compartmental new infection appearance rate is G and compartmental out and in individual transfer rate that need to be negated. As illustrated by Pandey et al. [32], differentiating G and H with respect to variables subset obtained from the free equilibrium pollutant gives Jacobian matrices. Thus, resulted in the matrices G and H separately.

Let $X' = (P_w, S_w, I_w)$, the system of equations can be express in matrix form as:

$$\frac{dX}{dt} = u_r(X) - v_r(X), \; r = S_w, I_w,$$

where

$$X = \begin{pmatrix} S_w \\ I_w \end{pmatrix}, \qquad u_r(X) = \begin{pmatrix} \beta P_w S_w \\ \beta P_w I_w \end{pmatrix}, \qquad v_r(X) = \begin{pmatrix} -\lambda I_w + (\varphi + \epsilon)S_w \\ \phi\beta I_w + (\lambda + \varphi + \delta)I_w \end{pmatrix}.$$

The pollutants propagation matrix G and transition matrix H denoted as:

$$G = \left[\frac{\partial u_r(X)}{\partial X_n}\right], H = \left[\frac{\partial v_r(X)}{\partial X_n}\right], \text{ which resulted to,}$$
$$G = \begin{pmatrix} \beta P_w & 0 \\ 0 & \gamma P_w \end{pmatrix}, \qquad H = \begin{pmatrix} \varphi + \epsilon & -\lambda \\ 0 & \beta\phi + \lambda + \varphi + \delta \end{pmatrix},$$

Hence, the PFE Jacobian matrix for G and H is given as:

$$G = \begin{pmatrix} \frac{\alpha\beta}{\varphi} & 0 \\ 0 & \frac{\alpha\gamma}{\varphi} \end{pmatrix}, \text{ and } \qquad H = \begin{pmatrix} \varphi + \epsilon & -\lambda \\ 0 & \beta\phi + \lambda + \varphi + \delta \end{pmatrix},$$

Therefore, the next generation matrix is presented as GH^{-1}, such that H gives

$$H^{-1} = \begin{pmatrix} \frac{1}{\varphi+\epsilon} & \frac{\lambda}{(\varphi+\epsilon)(\beta\phi+\lambda+\varphi+\delta)} \\ 0 & \frac{1}{\beta\phi+\lambda+\varphi+\delta} \end{pmatrix},$$

Thus,

$$GH^{-1} = \begin{pmatrix} \frac{\alpha\beta\beta}{\varphi(\varphi+\epsilon}) & \frac{\alpha\beta\lambda}{\varphi(\varphi+\epsilon)(\beta\phi+\lambda+\varphi+\delta)} \\ 0 & \frac{\alpha\beta}{\varphi(\beta\phi+\lambda+\varphi+\delta)} \end{pmatrix}.$$

Determine the eigenvalues of the characteristic equation to obtain R_0,

$$det(GH^{-1} - \chi I) = 0,$$

$$det\begin{pmatrix} \frac{\alpha\beta\beta}{\varphi(\varphi+\epsilon)} - \chi & \frac{\alpha\beta\lambda}{\varphi(\varphi+\epsilon)(\beta\phi+\lambda+\varphi+\delta)} \\ 0 & \frac{\alpha\beta}{\varphi(\beta\phi+\lambda+\varphi+\delta)} - \chi \end{pmatrix} = 0. \tag{14}$$

Hence, equation (14) gives the eigenvalues of the PFE and gotten as

$$\left(\frac{\alpha\beta}{\varphi(\varphi+\epsilon)}, \frac{\alpha\gamma}{\varphi(\beta\phi+\lambda+\varphi+\delta)}\right).$$

The two distinct eigenvalues are:

$$R_1 = \frac{\alpha\beta}{\varphi(\varphi+\epsilon)} \text{and} R_2 = \frac{\alpha\gamma}{\varphi(\beta\phi+\lambda+\varphi+\delta)}.$$

As such, the basic reproduction ratio or number is described as

$$R_0 = max(R_1, R_2).$$

3. Existence of equilibrium point stability

Stability analysis is the best tool for classifying equilibrium point of a dynamical system, be it the local and the global stability.

3.1. Local stability of the PFE

The PFE is asymptotically stable locally if $R_0 < 1$ and unstable if otherwise. To show that the pollutants basic reproduction number R_0 is asymptotically stable or not at the PFE P_0. The Jacobian matrix is used to determine the stability of each equilibrium points.

$$\tau = \begin{bmatrix} -\beta S_w - \gamma I_w - \varphi & -\beta P_w & -\gamma P_w + \beta\phi & 0 \\ \beta S_w & \beta P_w - \varphi - \epsilon & \lambda & 0 \\ \gamma I_w & 0 & \gamma P_w - \beta\phi - \lambda - \varphi - \delta & 0 \\ 0 & \epsilon & \delta & -\varphi \end{bmatrix}. \tag{15}$$

Evaluating the Jacobian matrix of equation (15) at the PFE P_0

$$\tau_{P_0} = \begin{bmatrix} \varphi & \frac{-\beta\alpha}{\varphi} & \frac{-\gamma\alpha}{\varphi} + \beta\phi & 0 \\ 0 & \frac{\beta\alpha}{\varphi} - \varphi - \epsilon & \lambda & 0 \\ 0 & 0 & \frac{\gamma\alpha}{\varphi} - \beta\phi - \lambda - \varphi - \delta & 0 \\ 0 & \epsilon & \delta & -\varphi \end{bmatrix}. \tag{16}$$

The eigenvalues of equation (16) give $\chi_1 = \chi_2 = -\varphi < 0$. Reducing the matrix and finding the determinant to obtain the other eigen values.

$$\tau_{P_0} = \begin{bmatrix} \frac{\beta\alpha}{\varphi} - \varphi - \epsilon & \lambda \\ 0 & \frac{\gamma\alpha}{\varphi} - \beta\phi - \lambda - \varphi - \delta \end{bmatrix}. \tag{17}$$

Hence, from equation (17), the remnant eigenvalues are:

$$\chi_3 = \frac{\beta\alpha}{\varphi} - \varphi - \epsilon$$
$$\chi_4 = \frac{\gamma\alpha}{\varphi} - \beta\phi - \lambda - \varphi - \delta.$$

For PFE to be locally asymptotically stable, the eigenvalues χ_3 & χ_4 must be negative. i.e., $\frac{\beta\alpha}{\varphi} - \varphi - \epsilon < 0 \to \frac{\beta\alpha}{\varphi} < \varphi + \epsilon$ This can be rewritten as: $\frac{\beta\alpha}{\varphi(\varphi+\epsilon)} < 1$ The PFE is locally asymptotically unstable if $\frac{\beta\alpha}{\varphi} - \varphi - \epsilon > 0 \to \frac{\beta\alpha}{\varphi} > \varphi + \epsilon$ This can be rewritten as: $\frac{\beta\alpha}{\varphi(\varphi+\epsilon)} > 1$ Therefore,

$$R_1 = \frac{\beta\alpha}{\varphi(\varphi+\epsilon)} \tag{18}$$

Also, for the χ_4. It is stable if $\frac{\gamma\alpha}{\varphi} - \beta\phi - \lambda - \varphi - \delta < 0 \to \frac{\gamma\alpha}{\varphi} < \beta\phi + \lambda + \varphi + \delta$. This can also be written as: $\frac{\gamma\alpha}{\varphi(\beta\phi+\lambda+\varphi+\delta)} < 1$.

It is asymptotically unstable if, $\frac{\gamma\alpha}{\varphi} - \beta\phi - \lambda - \varphi - \delta > 0 \to \frac{\gamma\alpha}{\varphi} > \beta\phi + \lambda + \varphi + \delta$. This can also be written as: $\frac{\gamma\alpha}{\varphi(\beta\phi+\lambda+\varphi+\delta)} > 1$.

Therefore,

$$R_2 = \frac{\gamma\alpha}{\varphi(\beta\phi+\lambda+\varphi+\delta)} \tag{19}$$

Thus, the combination of equations (18) and (19) gives R_0, i.e. $R_1 + R_2 = R_0$

Hence, the PFE is locally asymptotically stable if $R_0 < 1$ or unstable if $R_0 > 1$.

It is vital to demonstrate that the PFE is globally-asymptotically stable in order to guarantee that the removal of pollutants is independent of the starting size of the infected water sources.

3.2. Global stability of the PFE

Castilla-Chavez et al. [33] demonstrated the global asymptotic stability of the PFE.

$$\Gamma' = Y(X, I) \tag{20}$$

$$\Xi' = Z(X, I), Z(X, 0) = 0 \tag{21}$$

If the two requirements in equations (20) and (21) are met, then PFE becomes globally asymptotic stable.

Also, $X \in R^2$ represents the number of uncontaminated water sources and $I \in R^2$ represents the number of contaminated water sources. $P_0 = (P_w^0, S_w^0, I_w^0, T_w^0) = (X^*, 0, 0, 0)$ denotes PFE.

To ensure the global stability of the PFE, two conditions must be met:

$$(\Upsilon 1) \qquad \text{for} \quad X' = Y(X, 0), \quad X^* \text{globally stable} \tag{22}$$

$$(\Upsilon 2) \qquad Z(X, I) = DI - \hat{Z}(X, I), \hat{Z}(X, I) \geq 0 \ \ \forall (X, I) \in B, \tag{23}$$

where $D = P_1 Z(X, 0)$ is an M-matrix with non-negative off diagonal entries, and B are the systems of equation. The following theorem applies if the system of equations (1)-(4) meets the conditions in (22) and (23). Observe that

$$\hat{Z}(X, I) = DI - I'$$

where $I = (S_w, I_w)$ and $I' = (S_w', I_w')$ given by equations (2) and (3) respectively.

Theorem. *The PFE* $P_0 = \left(\frac{\alpha}{\varphi}, 0, 0, 0\right)$ *is globally asymptotically stable if* $R_0 < 1$ *and the condition in (23) is met.*

Proof. The systems of equations in equations (14) will be rewritten in the form of (20) and (21), and then $X \in R^2 = (P_w, T_w)$ with $X_1 = P_w$ and $X_2 = T_w$, $I \in R^2 = (S_w, I_w)$,

$$X' = Y(X, 0) = \begin{pmatrix} \alpha - \varphi P_w \\ 0 \end{pmatrix}$$
$$X_1' = \alpha - \varphi P_w$$

is a linear differential equation and its solution is given as:

$$P_w(t) = \frac{\alpha}{\varphi} - \left(\frac{\alpha}{\varphi} - P_{w_0}(t)\right) e^{-ut}$$

Clearly, $P_w(t) \to \frac{\alpha}{\varphi}$ *as* $t \to \infty$, since $e^{-\infty} = 0$, $\forall P_{w_0}$. Hence X_1^* is globally stable asymptotically.

So that

$$Z(X,I)=\begin{pmatrix}\beta P_w S_w+\lambda I_w-(\epsilon+\varphi)S_w\\ \gamma P_w I_w-(\delta+\lambda+\varphi)I_w\end{pmatrix}$$

and

$$D=\begin{pmatrix}\beta-(\epsilon+\varphi) & \lambda\\ 0 & \gamma-(\beta\phi+\delta+\lambda+\varphi)\end{pmatrix} \tag{24}$$

Clearly, (24) is a non negative diagonal M-matrix. Therefore:

$$\hat{Z}(X,I)=\begin{pmatrix}\widehat{Z_1}(X,I)\\ \widehat{Z_2}(X,I)\end{pmatrix}=\begin{pmatrix}\beta S_w(1-P_w)\\ \gamma I_w(1-P_w)\end{pmatrix} \tag{25}$$

$(\Upsilon 2)$ is satisfied whenever $P_w<1$ is satisfied, but $(\Upsilon 2)$ is unsatisfied when $P_w<0$.

Hence from equation (25), the PFE P_0 may be stable globally or not.

3.3. Existence of the pollutant endemic equilibrium points

The endemic equilibrium points of the pollutant exist and can be denoted as: P_w^*, S_w^*, I_w^*, T_w^* which is usually calculated by setting the equations (1)-(4) to zero and stating the states variables as a function of the transmission coefficients, according to [27,30,31]. The endemic equilibrium point here is divided into P_1^*, a situation where the insoluble pollutants in the water do not exist and P_2^*, where all types of pollutants exist in the water.

The endemic equilibrium are given below:

$$P_1^*=\left(\frac{\varphi+\epsilon}{\beta},-\frac{\varphi^2+\epsilon\varphi-\alpha\beta}{\beta(\varphi+\epsilon)},0,-\frac{\epsilon(\varphi^2+\epsilon\varphi-\alpha\beta)}{\beta\varphi(\varphi+\epsilon)}\right)$$

$$P_2^*=(P_w^*,S_w^*,I_w^*,T_w^*)$$

where

$$P_w^*=\frac{\lambda+\varphi+\delta+\gamma\phi}{\gamma},$$

$$S_w^*=\frac{\lambda(\lambda\varphi-\alpha\gamma+\varphi\delta+\varphi^2+\gamma\varphi\phi}{a},$$

$$I_w^*=-\frac{(\beta\lambda+\beta\varphi-\gamma\varphi+\beta\delta-\gamma\epsilon+\beta\gamma\phi)}{\gamma\lambda}S_w^*,$$

$$T_w^*=-\frac{(\beta\delta^2+\beta\lambda\delta-\gamma\lambda\epsilon+\beta\varphi\delta-\gamma\varphi\delta-\gamma\epsilon\delta+\beta\gamma\phi\delta)}{\varphi\gamma\lambda}S_w^*,$$

where

$$a=(\beta\varphi^2+\gamma\varphi^2+\beta\delta^2+\beta\lambda\varphi-\gamma\lambda\varphi+\beta\lambda\delta-\gamma\lambda\epsilon+2\beta\varphi\delta-\gamma\varphi\epsilon-\gamma\varphi\delta-\gamma\epsilon\delta+\beta\gamma\varphi\phi+\beta\gamma\phi\delta)$$

3.3.1. Local stability of the endemic equilibrium

Theorem. *The endemic equilibrium point is said to be locally stable asymptotically if $R_0>1$ and unstable if $R_0<1$.*

To show that R_0 is stable or unstable at the endemic equilibrium P^*. The local stability of the equilibrium points can be determined by evaluating the Jacobian matrix of the system in (1)-(4). Evaluating (15) at endemic equilibrium point P^*, to give

$$\tau_{P*}=\begin{bmatrix}-\frac{\lambda b_1(\beta+\lambda b_2)}{a}-\varphi & -\frac{\beta(\gamma\phi+\lambda+\varphi+\delta)}{\gamma} & -(\lambda+\varphi+\delta) & 0\\ \frac{\beta\lambda b_1}{a} & \frac{\beta(\gamma\phi+\lambda+\varphi+\delta)}{\gamma}-\varphi-\epsilon & \lambda & 0\\ \frac{\delta^2 b_1 b_2}{a} & 0 & 0 & 0\\ 0 & \epsilon & \delta & -\varphi\end{bmatrix}. \tag{26}$$

The eigenvalues are gotten from the characteristics equation $|\tau_{P^*}-\chi I|=0$ of (26). Clearly the first eigenvalue is: $\chi_1=\varphi<0$. The subsequent eigenvalues are gotten from the remaining 3×3 matrix given below:

$$\tau_{P^*} = \begin{bmatrix} -\frac{\lambda b_1(\beta+\lambda b_2)}{a} - \varphi & -\frac{\beta(\gamma\phi+\lambda+\varphi+\delta)}{\gamma} & -(\lambda+\varphi+\delta) \\ \frac{\beta\lambda b_1}{a} & \frac{\beta(\gamma\phi+\lambda+\varphi+\delta)}{\gamma} - \varphi - \epsilon & \lambda \\ \frac{\delta^2 b_1 b_2}{a} & 0 & 0 \end{bmatrix}. \tag{27}$$

Equation (27) has the characteristics polynomial given as:

$$a_0\chi^3 + a_1\chi^2 + a_2\chi + a_3 = 0,$$

where

$$a = (\beta\varphi^2 + \gamma\varphi^2 + \beta\delta^2 + \beta\lambda\varphi - \gamma\lambda\varphi + \beta\lambda\delta - \gamma\lambda\epsilon + 2\beta\varphi\delta - \gamma\varphi\epsilon - \gamma\varphi\delta - \gamma\epsilon\delta + \beta\gamma\varphi\phi + \beta\gamma\phi\delta),$$

$$b_1 = (-\alpha\gamma + \gamma\varphi\phi + \lambda\varphi + \varphi^2 + \varepsilon\delta),$$

$$\gamma\varphi + \varphi\epsilon - \beta\gamma\phi - \beta\lambda - \beta\varphi - \beta\epsilon,$$

$$a_0 = 1,$$

$$a_1 = \frac{-\beta(\gamma\phi + \lambda + \varphi + \delta)}{\gamma} + \frac{\lambda b_1(\beta + \lambda b_2)}{a} + 2\varphi + \epsilon,$$

$$a_2 = \left(\frac{-\delta b_1(\beta + \lambda b_2) - \varphi a}{a}\right)\left(-\frac{\beta(\gamma\phi + \lambda + \varphi + \delta)}{\gamma}\right),$$

$$a_3 = \left(\frac{\lambda^2 b_1 b_2 \frac{-\beta(\gamma\phi+\lambda+\varphi+\delta)}{\gamma} - \varphi - \epsilon}{a} - \frac{\beta\lambda^3 b_1^2 b_2}{a^2}\right) - (\lambda + \varphi + \epsilon) - \frac{\beta\lambda^3 b_1 b_2(\gamma\phi + \lambda + \varphi + \delta)}{a\gamma}.$$

From the characteristics equation above, the stability can be determined using the Routh-Hurwitz stability criterion which states that, if the conditions $a_1 > 0, a_2 > 0, a_1.a_2 - a_3 > 0$ are satisfied, then characteristic polynomial have negative real roots which shows that the endemic equilibrium point is asymptotically stable locally.

By Routh Hurwitz criteria therefore, the endemic equilibrium point is stable when $R_0 > 1$, otherwise unstable.

3.3.2. Global stability of the endemic equilibrium

Oludoun et al. [34] demonstrated the global asymptotic stability of the endemic equilibrium using the Lyapunouv method.

Theorem. *The equations of the model have a positive distinctive endemic equilibrium whenever $R_0 > 1$, which is said to be globally asymptotically stable.*

Proof. Considering the Lyapunov function defined as:

$$L(P_w^*, S_w^*, I_w^*, T_w^*) = \left(P_w - P_w^* ln\left(\frac{P_w}{P_w^*}\right)\right) + \left(S_w - S_w^* ln\left(\frac{S_w}{S_w^*}\right)\right) + \left(I_w - I_w^* ln\left(\frac{I_w}{I_w^*}\right)\right) + \left(T_w - T_w^* ln\left(\frac{T_w}{T_w^*}\right)\right), \tag{28}$$

Taking the derivative of L in equation (28) along the system as:

$$\frac{dL}{dt} = \left(1 - \frac{P_w^*}{P_w}\right)\frac{dP_w}{dt} + \left(1 - \frac{S_w^*}{S_w}\right)\frac{dS_w}{dt} + \left(1 - \frac{I_w^*}{I_w}\right)\frac{dI_w}{dt} + \left(1 - \frac{T_w^*}{T_w}\right)\frac{dT_w}{dt}, \tag{29}$$

It would be interesting to point out that, at this point, the partial derivatives in equation (29) are replaced by their expressions following equations (1) to (4).

$$\frac{dL}{dt} = \left(1 - \frac{P_w^*}{P_w}\right)\left(\alpha - \beta P_w S_w - \gamma P_w I_w + \phi\beta I_w - \varphi P_w\right) + \left(1 - \frac{S_w^*}{S_w}\right)\left(\beta P_w S_w + \lambda I_w - (\epsilon + \varphi)S_w\right) + \left(1 - \frac{I_w^*}{I_w}\right)\left(\gamma P_w I_w - \phi\beta I_w - (\lambda + \delta + \varphi)I_w\right) + \left(1 - \frac{T_w^*}{T_w}\right)\left(\epsilon S_w + \delta I_w - \varphi T_w\right), \tag{30}$$

At equilibrium, equation (30) gives:

$$\alpha = \beta P_w^* S_w^* - \gamma P_w^* I_w^* + \phi\beta I_w^* - \varphi P_w^*,$$

$$\epsilon + \varphi = \frac{\beta P_w^* S_w^* + \lambda I_w^*}{S_w^*},$$

$$\phi\beta + \lambda + \delta + \varphi = \frac{\gamma P_w^* I_w^*}{I_w^*},$$

$$\varphi = \frac{\epsilon S_w^* + \delta I_w^*}{T_w^*},$$

$$\begin{aligned}
\frac{dL}{dt} &= \left(1 - \frac{P_w^*}{P_w}\right)(\beta P_w^* S_w^* - \gamma P_w^* I_w^* + \phi\beta I_w^* - \varphi P_w^* - \beta P_w S_w - \gamma P_w I_w + \phi\beta I_w - \varphi P_w) \\
&+ \left(1 - \frac{S_w^*}{S_w}\right)\left(\beta P_w S_w + \lambda I_w - (\frac{\beta P_w^* S_w^* + \lambda I_w^*}{S_w^*})S_w\right) + \left(1 - \frac{I_w^*}{I_w}\right)\left(\gamma P_w I_w - \frac{\gamma P_w^* I_w^*}{I_w^*} I_w\right) + \\
&\left(1 - \frac{T_w^*}{T_w}\right)\left(\epsilon S_w + \delta I_w - \frac{\epsilon S_w^* + \delta I_w^*}{T_w}\right),
\end{aligned} \tag{31}$$

$$\begin{aligned}
\frac{dL}{dt} &= \left(1 - \frac{P_w^*}{P_w}\right)\left[\beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right) - \gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{P_w I_w}\right) + \phi\beta I_w \left(1 - \frac{I_w^*}{I_w}\right) - \varphi P_w \left(1 - \frac{P_w^*}{P_w}\right)\right] \\
&+ \left(1 - \frac{S_w^*}{S_w}\right)\left[\beta P_w S_w + \lambda I_w - \left(\frac{\beta P_w^* S_w^* S_w + \lambda I_w^* S_w}{S_w^*}\right)\right] + \left(1 - \frac{I_w^*}{I_w}\right)\left(\gamma P_w I_w - \frac{\gamma P_w^* I_w^* I_w}{I_w^*}\right) + \\
&\left(1 - \frac{T_w^*}{T_w}\right)\left((\epsilon S_w + \delta I_w) - \frac{\epsilon S_w^* + \delta I_w^*}{T_w}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{dL}{dt} &= \left(1 - \frac{P_w^*}{P_w}\right)\left[\beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right) - \gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{P_w I_w}\right) + \phi\beta I_w \left(1 - \frac{I_w^*}{I_w}\right) - \varphi P_w \left(1 - \frac{P_w^*}{P_w}\right)\right] \\
&+ \left(1 - \frac{S_w^*}{S_w}\right)\left[\beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right) + \lambda I_w \left(1 - \frac{I_w^* S_w}{S_w^*}\right)\right] + \left(1 - \frac{I_w^*}{I_w}\right)\left[\gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{I_w^*}\right)\right] + \\
&\left(1 - \frac{T_w^*}{T_w}\right)\left[\epsilon S_w \left(1 - \frac{S_w^*}{S_w T_w^*}\right) + \delta I_w - \left(1 - \frac{I_w^*}{I_w T_w^*}\right)\right],
\end{aligned} \tag{32}$$

$$\begin{aligned}
\frac{dL}{dt} &= -\varphi P_w \left(1 - \frac{P_w^*}{P_w}\right)^2 - \beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right)\left(1 - \frac{P_w^*}{P_w}\right) - \gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{P_w I_w}\right)\left(1 - \frac{P_w^*}{P_w}\right) + \\
&\phi\beta I_w \left(1 - \frac{I_w^*}{I_w}\right)\left(1 - \frac{P_w^*}{P_w}\right) + \beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right)\left(1 - \frac{S_w^*}{S_w}\right) + \lambda I_w \left(1 - \frac{I_w^* S_w}{S_w^*}\right)\left(1 - \frac{S_w^*}{S_w}\right) + \\
&\gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{I_w^*}\right)\left(1 - \frac{I_w^*}{I_w}\right) + \epsilon S_w \left(1 - \frac{S_w^*}{S_w T_w^*}\right)\left(1 - \frac{T_w^*}{T_w}\right) - \delta I_w - \left(1 - \frac{I_w^*}{I_w T_w^*}\right)\left(1 - \frac{T_w^*}{T_w}\right),
\end{aligned} \tag{33}$$

Simplifying equations (31)- (33) gives:

$$= -\varphi P_w \left(1 - \frac{P_w^*}{P_w}\right)^2 + P_1(P_w, S_w, I_w, T_w) + P_2(P_w, S_w, I_w, T_w).$$

Where,

$$\begin{aligned}
P_1(P_w, S_w, I_w, T_w) &= \phi\beta I_w \left(1 - \frac{I_w^*}{I_w}\right) + \beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right)\left(1 - \frac{S_w^*}{S_w}\right) + \lambda I_w \left(1 - \frac{I_w^* S_w}{S_w^*}\right)\left(1 - \frac{S_w^*}{S_w}\right) + \\
&\gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{I_w^*}\right)\left(1 - \frac{I_w^*}{I_w}\right) + \epsilon S_w \left(1 - \frac{S_w^*}{S_w T_w^*}\right)\left(1 - \frac{T_w^*}{T_w}\right), \\
P_2(P_w, S_w, I_w, T_w) &= -\beta P_w S_w \left(1 - \frac{P_w^* S_w^*}{P_w S_w}\right)\left(1 - \frac{P_w^*}{P_w}\right) - \gamma P_w I_w \left(1 - \frac{P_w^* I_w^*}{P_w I_w}\right)\left(1 - \frac{P_w^*}{P_w}\right) \\
&- \delta I_w - \left(1 - \frac{I_w^*}{I_w T_w^*}\right)\left(1 - \frac{T_w^*}{T_w}\right),
\end{aligned}$$

$P_1 \leq 0$, whenever,

$$I_w \geq I_w^*, P_w S_w \geq P_w^* S_w^*, S_w \geq S_w^*, I_w S_w \geq I_w^* S_w^*, \tag{34}$$

$P_2 \leq 0$, whenever,

$$P_w \geq P_w^*, P_w S_w \geq P_w^* S_w^*, T_w \geq T_w^*, I_w T_w \geq I_w^* T_w^*. \tag{35}$$

Thus, $\frac{dL}{dt} \leq 0$ if the conditions in (34) and (35) hold.

Therefore, by Oludoun et al. [34], the positive equilibrium state is globally asymptotically stable in the positive region R_+^4.

Table 1
Sensitivity Index of R_0 for some estimated parameters.

Parameters	Sensitivity index
α	1
β	1
φ	-0.66667
ϵ	-0.33333

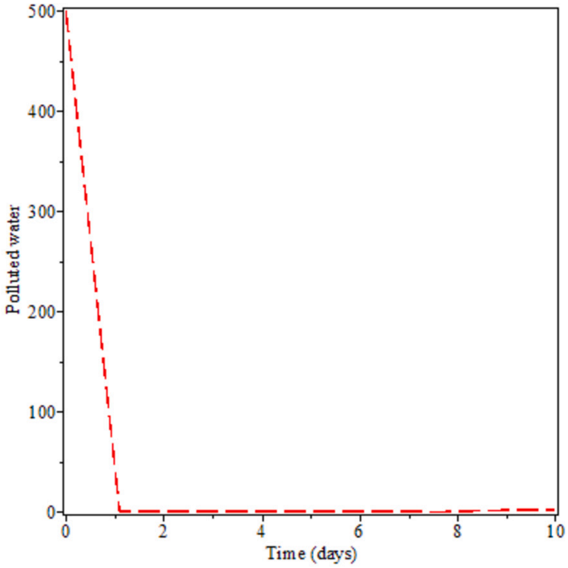

Fig. 2. Transmission dynamics for P_w.

4. Sensitivity analysis

This section outlines the method Oludoun et al. [34] used to identify the parameter values that significantly affect the fundamental reproduction number. Sensitivity analysis is important not only for exploratory research design only but also for understanding how important each parameter is to the spread of disease. Sensitivity analysis is mostly used to assess how well model predictions hold up to various parameter values. This will establish a direct link between the prevalence of water pollutants and the endemic equilibrium point, as well as between the early transmission of water pollutants and basic reproduction numbers. This aids in our understanding of the relative importance of each characteristic for the prevalence and transmission of contaminants in water. Extremely sensitive parameters need to be evaluated carefully because even a small deviation in one of these parameters can have a significant quantitative impact on the quantity of interest and, in turn, affect the qualitative outcomes. Conversely, determining the sensitivity index of an insensitive parameter doesn't need a lot of effort because even a small divergence from the parameter won't significantly alter the number of interests. To reduce the amount of water pollutants spreading into water sources, these factors ought to be the main consideration while figuring out how to regulate water pollution.

4.1. Explanation of sensitivity indices

Sensitivity indices are estimated to determine how a state variable varies in response to parameter changes. A normalized forward can be used to determine R_0 sensitivity index to its dependent parameter. The normalized forward sensitivity index is the ratio of a variable's relative change to that of a parameter. Partial derivatives can be used to define the sensitivity index for variables that are differentiable functions of the parameter.

Definition. The normalized forward sensitivity index of a differentiable variable R_0 is defined as follows:

$$\hbar_{\ell}^{R_0} = \frac{\partial R_0}{\partial \ell} \frac{\ell}{R_0}.$$

The sensitivity indices of R_0 with reference to $\alpha;\beta;\varphi$ and ϵ denoted by $\hbar_{\alpha}^{R_0};\hbar_{\beta}^{R_0};\hbar_{\varphi}^{R_0}$ and $\hbar_{\epsilon}^{R_0}$ respectively, are shown below. These indices denote the usefulness of each parameter in determining pollutant water transmission and incidence.

$$\hbar_{\alpha}^{R_0} = \frac{\partial R_0}{\partial \alpha} \times \frac{\alpha}{R_0} = 1, \hbar_{\beta}^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = 1,$$

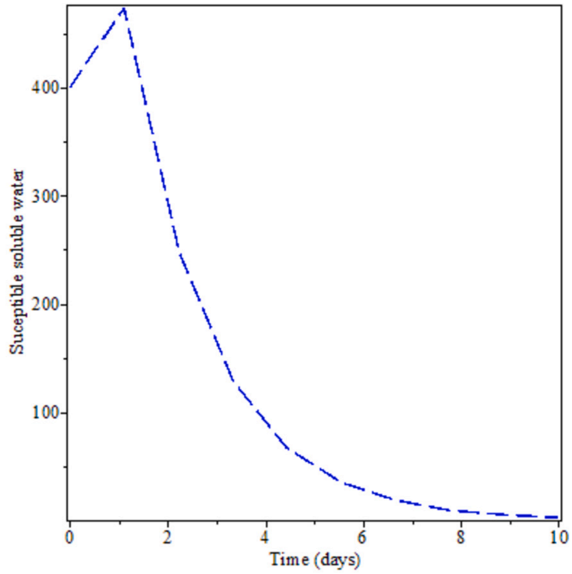

Fig. 3. Propagation dynamics for S_w.

$$\hbar_{\varphi}^{R_0} = \frac{\partial R_0}{\partial \varphi} \times \frac{\varphi}{R_0} = \varphi^2(\varphi+\epsilon)\left(\frac{-\alpha\beta}{\varphi(\varphi+\epsilon)^2} - \frac{\alpha\beta}{\varphi^2(\varphi+\epsilon)}\right),$$

$$\hbar_{\epsilon}^{R_0} = \frac{\partial R_0}{\partial \epsilon} \times \frac{\epsilon}{R_0} = \frac{-\epsilon}{\varphi+\epsilon},$$

Table 1 described the sensitivity index of R_0, which measures how changes in various parameters affect the basic reproduction number R_0. A positive sensitivity index means that an increase in the parameter will increase R_0, while a negative index indicates that an increase in parameter will decrease R_0. Sensitivity analysis helps prioritize which parameters have the most significant influence on controlling the spread of an infection, guiding public health interventions.

5. Numerical simulation

The following parameter values $\alpha = 0.8$, $\beta = 0.18$, $\gamma = 0.02$, $\phi = 0.25$, $\varphi = 0.4$, $\lambda = 0.3$, $\epsilon = 0.2$, and $\delta = 0.5$ as taken from Shah et al. [14] while some where assumed will be used for the numerical analysis and simulation to investigate the dynamics of the system of equations (1)-(4). The maple software is used for the simulation at the initial conditions $P_{w0} = 500$, $S_{w0} = 400$, $I_{w0} = 100$, and $T_{w0} = 0$.

Fig. 2 illustrates the transmission dynamics of polluted water. It can be seen from the figure that the increasing water pollutants do not have any significant effect as the days go by, this signifies that polluted water sources keeps increasing if measures are not put in place. An increasing water pollutants do not show significant immediate effects over time due to factors like pathogen decay, population immunity, or natural dilution, the underlying long-term risks remain significant. If left unchecked, increasing pollution could lead to chronic health issues, environmental degradation, and even delayed disease outbreaks under the right conditions. Fig. 3 shows the transmission dynamics of the susceptible soluble water, the plot indicates a decreasing soluble water pollutants as the time increases which may be as a result of the availability of treatment present in the soluble water pollutants. An initial rise in pollutant levels as contamination enters the water system (e.g., from runoff, industrial waste, or improper waste disposal). However, as time progresses, treatment processes reduce the concentration of soluble pollutants, leading to a gradual decline in the amount of contamination, which suggests that treatment or other mitigation efforts are effectively reducing contamination over time. Also a similar dynamics can be noticed in Fig. 4, i.e. a decreasing insoluble water pollutant was seen. This could be as a result of effective treatment and awareness on not dumping insoluble waste to the water ways which could be the best management practice or insoluble water pollutants. Likewise, some insoluble organic pollutants may be broken down by microorganisms over time. However, this process is slower and more dependent on environmental conditions such as temperature, oxygen levels, and microbial presence. The key challenge is ensuring these pollutants are not just removed from the water column but also effectively managed in the long term to avoid environmental buildup. It is displayed from Fig. 5 an upward spike in the plot at the point $0 \le t \le 2$ that which may be as a result of positive adherence to treatment but a continuous decrease in the plot as the time increases was shown which could be as a result of negligence to treatment. Meanwhile, treatment negligence of water pollutants, whether soluble or insoluble, can lead to severe long-term consequences for public health, the environment, and even the economy. If treatment efforts are neglected, especially in managing waterborne pollutants, the impacts could include increasing levels of contaminants, a resurgence of waterborne diseases, and irreversible damage to ecosystems.

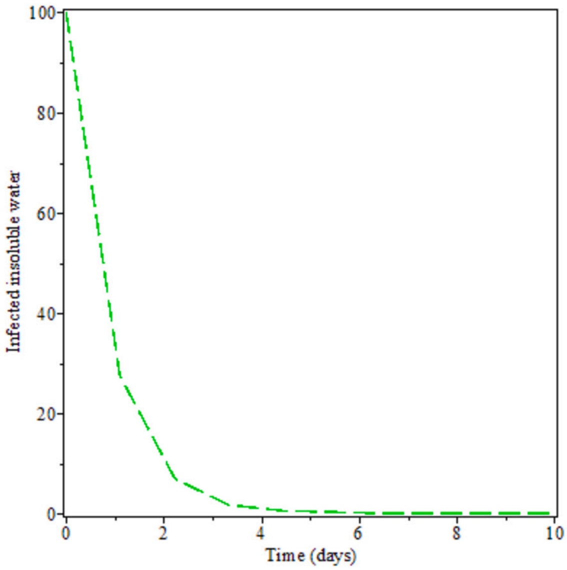

Fig. 4. Proliferation dynamics for I_w.

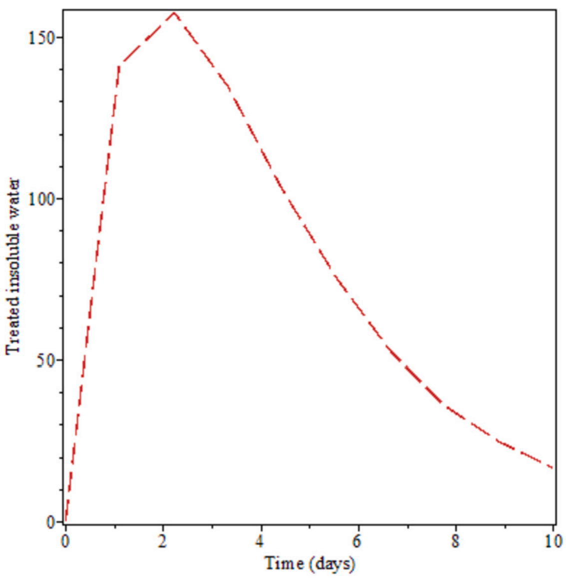

Fig. 5. Transmission dynamics for T_w.

Fig. 6 depicts the effect of water pollution transmission β on the soluble water compartments. It is shown from figure that as the rate of soluble water pollutants β increases, the soluble water pollutants decrease. The increasing rate of β increases the viability of soluble water pollutants in the water sources which poses threat to the water bodies and makes the water harmful for consumption and domestic uses. Fig. 7 reveals the effect of water pollution transmission γ on the insoluble water compartments, it can be seen that an increase in the rate of insoluble water pollutants increases the rate of the insoluble water pollutants γ. As γ increases, the water ways becomes dirty as a result of an increase in the insoluble water pollutants and this causes environmental degradation and consequential puts the lives of both humans and animals at risk of germs and probably extinction of some aquatic animals.

Fig. 8 illustrates the effect of treatments and controls on the soluble pollutants. It can be seen from plot that pollutants take longer to vanish without control unlike the control plot which treatment eradicates the pollutants just after 4 months. Also, Fig. 9 revealed the effect of treatments and controls on the insoluble water pollutants, the figure depicts that insoluble water pollutants diminished and got eradicated at 6 months but the control plot the pollutants were eradicated shortly after 2 months. This suggests that improving water quality should be encouraged in order to lower the prevalence of water pollution and consequential reduce the emerging of water borne diseases.

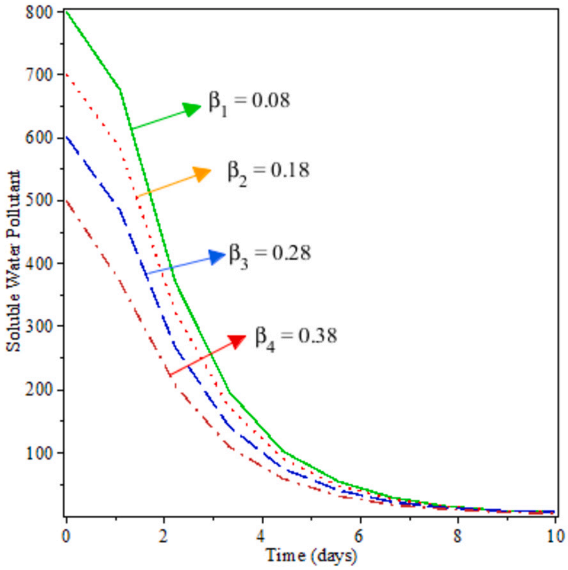

Fig. 6. Soluble pollutant spread for β.

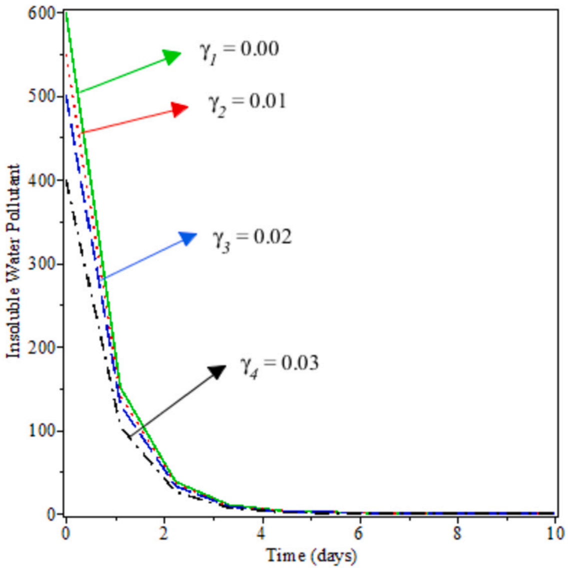

Fig. 7. Impact of γ on insoluble pollutant.

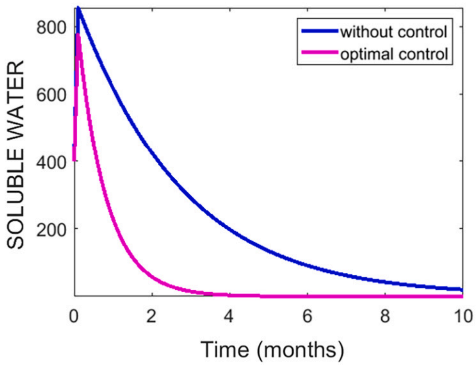

Fig. 8. Optimal control effect on S_w.

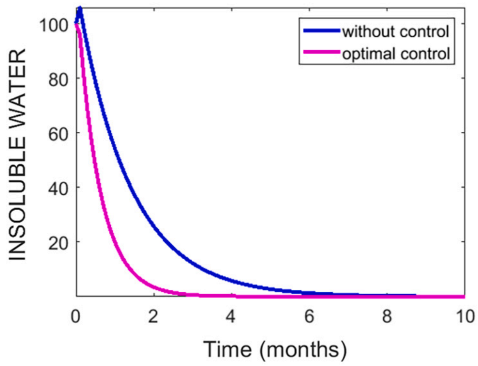

Fig. 9. Insoluble pollutant I_w control.

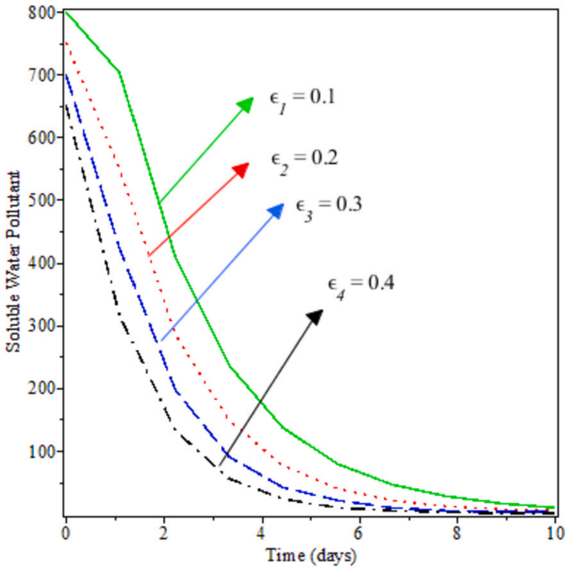

Fig. 10. Impact of ϵ on soluble pollutant.

Fig. 10 shows the effect of varying treatment of soluble water pollutants ϵ on the soluble water pollutants compartment. It can be seen clearly at as the treatment rate of soluble water pollutant is increased, it decreases the water pollutants and this will eventually fizzle out over time. Also Fig. 11 depicts the effect of varying treatment of insoluble water pollutants δ on the insoluble water compartment. The plots illustrate that as the treatment rate of insoluble water pollutant is increased, the pollutants decrease over time. This shows that improving treatment will help improves the water from both soluble and insoluble water pollutants and this eventually gives safe water for drinking and aquatic animals will thrive in safe water.

6. Conclusion

The study focuses on the causative factors of water pollution which have been identified in this study as the soluble water pollutants and the insoluble water pollutants arising from natural occurrences due to anthropogenic that resulted in eutrophication needs to be guided. Indiscriminate harmful waste discharge and dispersion into the water bodies leads to water pollution. As such, constant management and monitoring of water resources for an enhanced quality and quantity is essential. Therefore, this study employed a coupled derivative mathematical model to investigate soluble and insoluble water pollutants. The insoluble pollutants are converted to soluble pollutants through an applied control or treatment. To optimize the water quality, parametric sensitivities in relation to reproduction number are examined. From the computational analysis carried out, it was revealed that treatment still remains the only remedy for safe water. This means that, in order to minimize pollution, treatment of water pollutants before it is discharged into water bodies should be strictly adhered to. Sensitivity analysis conducted on the advection-diffusion parameters indicated that the dispersion of soluble pollutants was most sensitive to changes in flow velocity, while insoluble pollutants were highly affected by variations in sedimentation rates and particle size distribution.

Future research could expand on this study by developing more sophisticated multiphase flow models that capture a wider range of pollutant behaviors, including gas-phase pollutants or pollutants with more complex chemical reactions in water. This could further

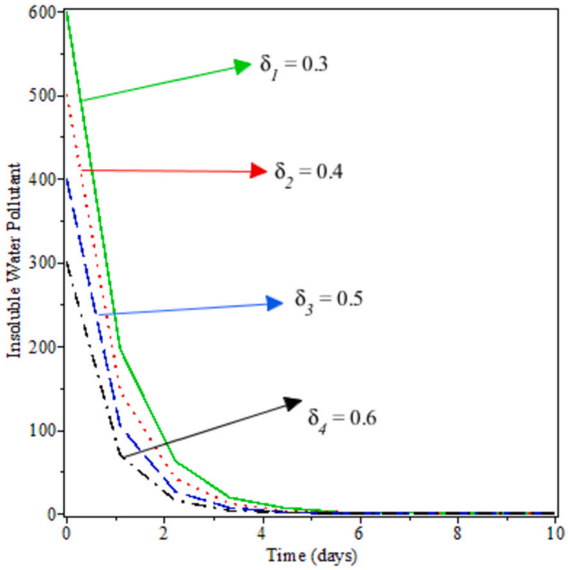

Fig. 11. Rising effect of δ on insoluble.

improve the prediction and control of pollutant dispersion in diverse environments, such as industrial waterways or areas affected by chemical spills.

CRediT authorship contribution statement

O.Y. Oludoun: Writing – original draft. **S.O. Salawu:** Conceptualization. **S.O. Adesanya:** Writing – review & editing. **O.E. Abiodun:** Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data and code availability

The authors declare that no data was used for the research described in the article.

References

[1] E. Bonyah, A. Atangana, M.A. Khan, Modeling the spread of computer virus via Caputo fractional derivative and the beta-derivative, Asia Pac. J. Comput. Eng. 4 (1) (2017) 1–15.
[2] E. Bonyah, Fractional optimal control for a corruption model, J. Prime Res. Math. 16 (1) (2020) 11–29.
[3] E. Bonyah, S. Ogunlade, S.D. Purohit, J. Singh, Modelling cultural hereditary transmission: insight through optimal control, Ecol. Complex. 45 (2021) 100890.
[4] M. Yavuz, N. Ozdemir, Analysis of an epidemic spreading model with exponential decay law, Math. Sci. Appl. E-Notes 8 (1) (2020) 142–154.
[5] A.M. Okedoye, S.O. Salawu, S.I. Oke, N.K. Oladejo, Mathematical analysis of affinity hemodialysis on T-cell depletion, Sci. Afr. 8 (2020) e00427.
[6] S. Qureshi, E. Bonyah, A.A. Shaikh, Classical and contemporary fractional operators for modeling diarrhea transmission dynamics under real statistical data, Physica A, Stat. Mech. Appl. 535 (2019) 122496.
[7] P.A. Naik, M. Yavuz, I. Zu, The role of prostitution on HIV transmission with memory: a modeling approach, Alex. Eng. J. 59 (4) (2020) 2513–2531.
[8] M. Yavuz, E. Bonyah, New approaches to the fractional dynamics of schistosomiasis disease model, Physica A, Stat. Mech. Appl. 525 (2019) 373–393.
[9] E. Bonyah, S.K. Asiedu, Analysis of a lymphatic filariasis-schistosomiasis coinfection with public health dynamics: model obtained through Mittag-Leffler function, Discrete Contin. Dyn. Syst., Ser. S 13 (3) (2020) 519.
[10] J. Crank, The Mathematics of Diffusion, Oxford University Press, 1979.
[11] G. Sposito, Chemical Kinetics in Contaminant Hydrology: An Introduction, Cambridge University Press, 2017.
[12] L. Baroni, L. Cenci, M. Tettamanti, M. Berati, Evaluating the environmental impact of various dietary patterns combined with different food production systems, Eur. J. Clin. Nutr. 61 (2) (2007) 279–286.
[13] D. Zocchi, G. Wennemuth, Y. Oka, The cellular mechanism for water detection in the mammalian taste system, Nat. Neurosci. 20 (7) (2017) 927–933.
[14] N.H. Shah, S.N. Patel, M.H. Satia, F.A. Thakkar, Optimal control for transmission of water pollutants, Int. J. Math. Eng. Manag. Sci. 3 (4) (2018) 381–391.
[15] S. Sharma, A. Bhattacharya, Drinking water contamination and treatment techniques, Appl. Water Sci. 7 (2017) 1043–1067.
[16] J. Singh, P. Yadav, A.K. Pal, V. Mishra, Water Pollutants: Origin and Status, Sensors in Water Pollutants Monitoring: Role of Material, vol. 12, Springer, Singapore, 2020, pp. 5–20.
[17] S.M.H. Tabatabaie, J.P. Bolte, G.S. Murthy, A regional scale modeling framework combining biogeochemical model with life cycle and economic analysis for integrated assessment of cropping systems, Sci. Total Environ. 625 (2018) 428–439.

[18] A. Saravanan, P.S. Kumar, S. Jeevanantham, S. Karishma, B. Tajsabreen, P.R. Yaashikaa, B. Reshma, Effective water/wastewater treatment methodologies for toxic pollutants removal: processes and applications towards sustainable development, Chemosphere 280 (2021) 130595.

[19] N. Abdullah, N. Yusof, W.J. Lau, J. Jaafar, A.F. Ismail, Recent trends of heavy metal removal from water/wastewater by membrane technologies, J. Ind. Eng. Chem. 76 (2019) 17e38.

[20] P.K. Pandey, P.H. Kass, M.L. Soupir, S. Biswas, V.P. Singh, Contamination of water resources by pathogenic bacteria, AMB Express 4 (1) (2014) 1–16.

[21] E. Boelee, G. Geerling, B. Van der Zaan, A. Blauw, A.D. Vethaak, Water and health: from environmental pressures to integrated responses, Acta Trop. 193 (2019) 217–226.

[22] F.B. Agusto, A. Bamigbola, Numerical treatment of the mathematical models for water pollution, J. Math. Stat. 3 (4) (2007) 172–180.

[23] N.H. Shah, M.H. Satia, B.M. Yeolekar, Optimum control for spread of pollutants through forest resources, Appl. Math. 8 (5) (2017) 607–620.

[24] A. Parsaie, A.H. Haghiabi, Computational modeling of pollution transmission in rivers, Appl. Water Sci. 7 (3) (2017) 1213–1222.

[25] B. Pimpunchat, W.L. Sweatman, G.C. Wake, W. Triampo, A. Parshotam, A mathematical model for pollution in a river and its remediation by aeration, Appl. Math. Lett. 22 (3) (2009) 304–308.

[26] N. Pochai, S. Tangmanee, L.I. Crane, J.J.H. Miller, A mathematical model of water pollution control using the finite element method, PAMM, Proc. Appl. Math. Mech. 6 (2006) 755–756.

[27] A. Bermdez, Mathematical modelling and optimal control methods in water pollution, in: The Mathematics of Models for Climatology and Environment, Springer, Berlin, Heidelberg, 1997, pp. 3–37.

[28] A. Bermdez, C. Rodrguez, M.E. Vzquez-Mndez, A. Martnez, Mathematical modelling and optimal control methods in waste water discharges, in: Ocean Circulation and Pollution ControlA Mathematical and Numerical Investigation, Springer, Berlin, Heidelberg, 2004, pp. 3–15.

[29] E. Bonyah, P. Agbekprnu, C. Unlu, Mathematical modeling of transmission of water pollution, J. Prime Res. Math. 17 (2) (2021) 20–38.

[30] M.O. Adeniyi, A.A. Amalare, S.I. Oke, S.O. Salawu, Bifurcation analysis and global sensitivity index for malaria disease transmission dynamics: information and treated bed-nets control, Int. J. Biomath. 23 (2023) 2350060.

[31] S.I. Oke, M.I. Ekum, O.J. Akintande, M.O. Adeniyi, T.A. Adekiya, O.J. Achadu, M.B. Matadi, O.S. Iyiola, S.O. Salawu, Optimal control of the coronavirus pandemic with both pharmaceutical and nonpharmaceutical interventions, Int. J. Dyn. Control 11 (5) (2023) 2295–2319.

[32] P.K. Pandey, P.H. Kass, M.L. Soupir, S. Biswas, V.P. Singh, Contamination of water resources by pathogenic bacteria, AMB Express 4 (1) (2014) 1–16.

[33] C. Chavez, Z. Feng, W. Huang, On the computation of R0 and its role on global stability, in: Mathematical Approaches for Emerging and Re-Emerging Infection Diseases: An Introduction, vol. 125, 2002, pp. 31–65.

[34] O. Oludoun, O. Adebimpe, J. Ndako, O. Abiodun, B. Gbadamosi, B. Aladeitan, Global stability analysis of hepatitis B virus dynamics, F1000Res. 10 (429) (2021) 429.