MethodsX 10 (2023) 102035

Contents lists available at ScienceDirect

MethodsX

journal homepage: www.elsevier.com/locate/mex

Method article

On numerical solution of boundary layer flow of viscous incompressible fluid past an inclined stretching sheet in porous medium and heat transfer using spline technique

Tahera Begum[a], Geetan Manchanda[b], Arshad Khan[a,*], Naseem Ahmad[a]

[a] Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
[b] Department of Mathematics, Maitreyi College, University of Delhi, New Delhi 110021, India

ARTICLE INFO

Method name:
Non polynomial Quintic Spline Method and Finite Difference Method

MSC:
65D07
65L10
65L12
65L20
80A20
76D05

Keywords:
Inclined stretching sheet
Boundary layer flow
Quasilinearization
Non polynomial quintic spline
Heat transfer
Convergence

ABSTRACT

In this paper, the boundary layer flow of viscous incompressible fluid over an inclined stretching plate in porous media with body force and heat transfer has been studied. To solve this problem, we develop a suitable spline method which is used to calculate the velocity function of the flow problem. We proceed as follows:

- With a suitable stream function, the concerned boundary layer equation is converted into non-linear third order ordinary differential equation together with appropriate boundary conditions in an infinite domain which has been further linearized by using quasi-linearization method.
- Then, we develop a non polynomial quintic spline technique which has been used to find the numerical values of the velocity function of the flow problem. The convergence analysis of the developed spline technique has been discussed.
- Later, the method developed so far has been applied to solve nonlinear boundary value problem for different angles of inclination and Froude number. The values obtained so far have been used to study heat flow problem. Finally, skin friction has been discussed.

Specification table

Subject area:	Mathematics
More specific subject area:	Boundary Layer Flow
Name of your Method:	Non polynomial Quintic Spline Method and Finite Difference Method
Name and reference of original method:	• A. Khan & T. Aziz, The numerical solution of third-orderboundary-value problems using quintic splines, Applied Mathematics & and Computation, 137(2–3) (2003) 253–260. • P. Khandelwal & A. Khan, Singularly perturbed convectiondiffusion boundary value problems with two small parameters using & nonpolynomial spline technique, Mathematical Sciences, 11(2) (2017) 119–126.
Resource availability:	MATLAB

* Corresponding author.
E-mail addresses: taherabegum765@gmail.com (T. Begum), gmanchanda@maitreyi.du.ac.in (G. Manchanda), akhan2@jmi.ac.in (A. Khan), nahmad4@jmi.ac.in (N. Ahmad).

https://doi.org/10.1016/j.mex.2023.102035
Available online 24 January 2023

Introduction

In the recent years, the study of boundary layer flow due to stretching plate has received a good attention among researchers due to the ever-increasing demand in industrial applications. Some of the applications lie in geo-mechanics, insulation engineering and in contemporary technology. The study of laminar flow over a stretching plate in porus media and heat transfer has been of tremendous interest due to its highly applicable aspect in industrial problems. Sakiadis [1] introduced the study of laminar flow over a solid surface which is continuously moving. Crane[2] acquired the closed form solution by extending this concept of studying the steady laminar flow due to stretching sheet. The pursuance of finding the exact solutions of the boundary layer equations is of great interest since decades. In hydrodynamics, the exact solution is of great importance because of the nonlinear term in the governing differential equations of the fluid motion. In such cases, to obtain the closed form solution of the governing differential equations become difficult and in some cases it becomes impossible. Therefore, most of the researchers obtain the similarity solution. Due to emergence of numerical techniques, many researchers have investigated numerical methods and solved these type of problems in different variants with or without heat transfer. Moreover, a few studies are available in literature (see [3–28]). Natural porous media fluxes include processes such as the movement of oil, gas, or moisture in hydrocarbons, as well as the potential deployment of methane in gas hydrates,the movement of Waterless Phase Liquids in polluted groundwater, the movement of fluids and soluble compounds in living organisms materials, and the dissolving metamorphism of ice are all examples of natural porous media and mechanisms although some illustrations of industrial porous media and the procedures that correspond to them are: drying of paper pulp, the soaking of liquids in nappies and other items with a same uptake capacity, the treatment of gas and water in fuel cells, and the drying of foods. In present work, we propose to study the boundary layer flow past a linear stretching plate at an angle with body force in porous media and heat transfer. The governing equations are nonlinear partial differential equation which cannot be solved exactly. Hence, we develop the numerical technique based on non polynomial quintic spline. This method is used to find the approximate numerical solutions in such type of situations.The details of splines can be seen in [29–32]. Recently, there has been a considerable interest for solving the nonlinear differential equations by using quasilinearisation technique [33,34]. This method has been applied by numerous researchers for solving the nonlinear differential equations. Our objective is to find a solution for the unidirectional movement of a viscous and incompressible fluid over an inclined stretched sheet in porous media while simultaneously taking into account body force. In addition to this, we conducted an investigation into the heat transfer problem. First, we transform the partial differential equations governing the motion of the fluid into third order nonlinear ordinary differential equation by applying a similarity transformations. Our target is to find the approximate solution of this transformed problem with the help of the proposed non polynomial quintic spline technique. Further, we discuss the convergence of the scheme developed along with its truncation error. The approximate numerical values of the velocity function obtained are discussed by using tables and graphs. Finally with the help of data obtained, we find the solution of heat transfer problem using incomplete gamma function. Nusselt number has also been discussed.

This paper is organised as follows: Section 2 portrays the mathematical modelling for the flow problem. Quasilinearization technique has been described in section 3 and in section 4 we have deduced the non polynomial quintic spline method. The truncation error is given in section 5 and in Section 6,we give the application of the method. The convergence analysis is carried out in section 7. In section 8, numerical experiments of the flow problem are given by displaying the numerical values of velocity functions through tables and figures. In Section 9, we discuss the Nusselt number and heat transfer assessment. Section 10 presents findings, discussion and conclusion.

Problem construction

Referring the Fig. 1, where we consider flow of viscous incompressible fluid past an infinite length inclined flat stretching plate which has been put in a porous medium of porosity k'. The body weight has been taken into consideration. The equations governing

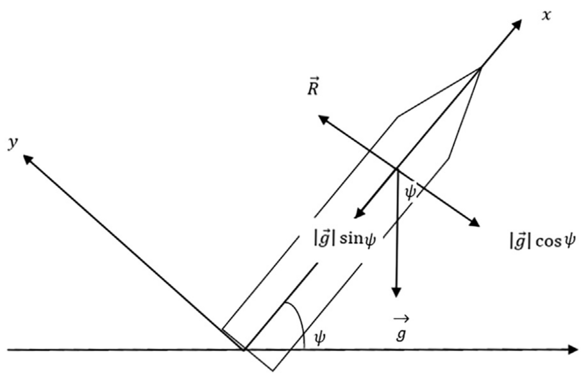

Fig. 1. Inclined Stretching Plate.

the flow are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{k'}u-|\vec{g}|\sin\psi, \tag{2}$$

with boundary conditions:

$$y=0, u=mx, v=0, \tag{3}$$

$$y\to\infty, u=0. \tag{4}$$

Define the dimentionless variable $\zeta=\left(\frac{m}{v}\right)^{\frac{1}{2}}y$ and $\psi(x,y)=(mv)^{\frac{1}{2}}xH(\zeta)$ so that $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$ satisfying the continuity Eq. (1). Thus, we have $u=mxH'(\zeta)$ and $v=-(mv)^{\frac{1}{2}}H(\zeta)$. Substituting the obtained form of u and v in Eqs. (2)-(4), we have

$$H'''(\zeta)+H(\zeta)H''(\zeta)-H'^2(\zeta)-\frac{\nu}{mk'}H'(\zeta)=\frac{|\vec{g}|}{m^2x}\sin\psi, \tag{5}$$

with relevant boundary conditions

$$\zeta=0, H(\zeta)=0, H'(\zeta)=1, \tag{6}$$

$$\zeta\to\infty, H'(\zeta)=0. \tag{7}$$

Here, the problem posed by the Eqs. (5), (6) and (7) is a non-linear boundary value problem of order three. Hence, we solve this problem by non polynomial quintic spline method. We develop the method in the following way:

Quasilinearization technique

To linearize the non linear differential Eq. (5), we use quasilinearization technique. Equations (5)-(7) are linearized with the help of quasilinearization technique, as

$$\begin{aligned}&H'''_{t+1}(\zeta)+a_t(\zeta)H''_{t+1}(\zeta)+b_t(\zeta)H'_{t+1}(\zeta)+c_t(\zeta)H_{t+1}(\zeta)\\&=G(\zeta,H_t(\zeta),H'_t(\zeta),H''_t(\zeta))+a_t(\zeta)H''_t(\zeta)+b_t(\zeta)H'_t(\zeta)+c_t(\zeta)H_t(\zeta), a\le\zeta\le b\end{aligned} \tag{8}$$

which can be written as

$$H'''_{t+1}(\zeta)+a_t(\zeta)H''_{t+1}(\zeta)+b_t(\zeta)H'_{t+1}(\zeta)+c_t(\zeta)H_{t+1}(\zeta)=k_t(\zeta),\quad a\le\zeta\le b \tag{9}$$

subject to boundary conditions

$$H_{t+1}(a)=\alpha_1,\quad H'_{t+1}(a)=\alpha_2\quad\text{and}\quad H'_{t+1}(b)=\alpha_3. \tag{10}$$

Here in Eq. (9)

$k_t(\zeta)=G(\zeta,H_t(\zeta),H'_t(\zeta),H''_t(\zeta))+a_t(\zeta)H''_t(\zeta)+b_t(\zeta)H'_t(\zeta)+c_t(\zeta)H_t(\zeta)$, where, $a_t(\zeta)=\left(\frac{\partial G}{\partial H''}\right)_{H=H_t}$, $b_t(\zeta)=\left(\frac{\partial G}{\partial H'}\right)_{H=H_t}$ and $c_t(\zeta)=\left(\frac{\partial G}{\partial H}\right)_{H=H_t}$.

For our convenience, we write $H^m_{t+1}(\zeta)=H^m(\zeta)$, $m=0,1,2,3$, $k_t(\zeta)=k(\zeta)$, $a_t(\zeta)=a(\zeta)$, $b_t(\zeta)=b(\zeta)$ and $c_t(\zeta)=c(\zeta)$, so that Eqs. (9)-(10) become

$$H'''(\zeta)+a(\zeta)H''(\zeta)+b(\zeta)H'(\zeta)+c(\zeta)H(\zeta)=k(\zeta), a\le\zeta\le b \tag{11}$$

subject to boundary conditions

$$H(a)=\alpha_1,\quad H'(a)=\alpha_2\quad \textit{and}\quad H'(b)=\alpha_3. \tag{12}$$

Derivation of the method

For achieving the approximate numerical solution of Eqs. (11)-(12), the interval $[a,b]$ is divided into N equal subintervals, i.e. $a=\zeta_0<\zeta_1<\zeta_2<\ldots<\zeta_N=b$, where the mesh points are defined as:

$$\zeta_i = a+ih, i=0(1)N \quad \text{and} \quad h=\frac{b-a}{N}.$$

Now, we construct the algorithm using the non polynomial spline G_Δ which interpolates $H(\zeta)$ at the mesh points ζ_i and has the following form:

$$G_\Delta(\zeta)=\alpha_{1i}\sin\kappa(\zeta-\zeta_i)+\alpha_{2i}\cos\kappa(\zeta-\zeta_i)+\alpha_{3i}e^{\kappa(\zeta-\zeta_i)}+\alpha_{4i}e^{-\kappa(\zeta-\zeta_i)}+\alpha_{5i}(\zeta-\zeta_i)+\alpha_{6i}, \tag{13}$$

where $\alpha_{1i},\alpha_{2i},\alpha_{3i},\alpha_{4i}$, α_{5i} and α_{6i} are real finite constants and $G_\Delta\in C^4(\Delta)$ is interpolated at the mesh points ζ_i which depends on the parameter κ. The coefficients $\alpha_{1i},\alpha_{2i},\alpha_{3i},\alpha_{4i},\alpha_{5i}$ and α_{6i} are obtained by using the interpolatory conditions. Creating consistency connections between the spline's value and its derivatives at mesh points, let

$$G_\Delta(\zeta_i)=H_i, \quad G_\Delta(\zeta_{i+1})=H_{i+1},$$

$$G''_\Delta(\zeta_i)=D_i, \quad G''_\Delta(\zeta_{i+1})=D_{i+1},$$

$$G'''_\Delta(\zeta_i)=T_i, \quad G'''_\Delta(\zeta_{i+1})=T_{i+1},$$

$$G''''_\Delta(\zeta_i)=F_i, \quad G''''_\Delta(\zeta_{i+1})=F_{i+1}.$$

Putting these values and solving, we get

$$\begin{aligned}
\alpha_{1i} &= \frac{1}{2\kappa^4\text{sinkh}}(F_{i+1}-\kappa^2D_{i+1})-\frac{\text{coskh}}{2\kappa^4\text{sinkh}}(F_i-\kappa^2D_i),\\
\alpha_{2i} &= \frac{1}{2\kappa^4}(F_i-\kappa^2D_i),\\
\alpha_{3i} &= \frac{1}{e^{2kh}-1}\Big(\frac{1}{2\kappa^4}(e^{kh}F_{i+1}-F_i)+\frac{1}{2\kappa^2}(e^{kh}D_{i+1}-D_i)\Big),\\
\alpha_{4i} &= \frac{1}{2\kappa^4}(F_i+\kappa^2D_i)-\alpha_{3i},\\
\alpha_{5i} &= \frac{1}{h}(H_{i+1}-H_i)-\frac{1}{h\kappa^4}(F_{i+1}-F_i),\\
\alpha_{6i} &= y_i-\frac{1}{\kappa^4}F_i.
\end{aligned}$$

Applying the continuity of the first and second derivatives at the mesh point $\zeta=\zeta_i$, we obtain the following relations:

$$D_{i-1}+4D_i+D_{i+1}=\frac{6}{h^2}(H_{i-1}-2H_i+H_{i+1})-6h^2(p_1F_{i-1}+2q_1F_i+p_1F_{i+1}), \tag{14}$$

$$D_{i-1}-2D_i+D_{i+1}=h^2(r_1F_{i-1}+2s_1F_i+r_1F_{i+1}), \tag{15}$$

where, $p_1=\frac{-h}{6k^2}+\frac{e^{kh}-e^{-kh}}{e^{kh}k^2(2\text{sinkh}+1)}$, $q_1=\frac{h}{k^2}-\frac{e^{kh}-e^{-kh}}{k^2}$, $r_1=\frac{2hk\text{coskh}-2\text{sinkh}}{k^4\text{sinkh}}$ and $s_1=\frac{\text{sinkh}-hk}{k^4\text{sinkh}}$.

Solving the Eqs. (14) and (15) for D_i, we get

$$D_i=\frac{1}{h^2}\big(H_{i-1}-2H_i+H_{i+1}\big)-h^2\Big(\Big(p_1+\frac{q_1}{6}\Big)F_{i+1}+2\Big(r_1+\frac{s_1}{6}\Big)F_i+\Big(p_1+\frac{q_1}{6}\Big)F_{i-1}\Big). \tag{16}$$

Now, using the continuity of third derivative at the point $\zeta=\zeta_i$ and using Eq. (16), we get

$$T_i=\frac{1}{h^3}\big(H_{i+2}-3H_{i+1}+3H_i-H_{i-1}\big)-h\big(pF_{i+2}+(\alpha-p+r_1)F_{i+1}+(p-\alpha+s_1)F_i-pF_{i-1}\big), \tag{17}$$

where, $p=p_1+\frac{q_1}{6}$ and $\alpha=2(r_1+\frac{s_1}{6})$.

We define the operator Δ by

$$\Delta T=pT_{i-2}+qT_{i-1}+sT_i+qT_{i+1}+pT_{i+2}. \tag{18}$$

Using Eqs. (17) and (18), we get

$$\Delta T=\frac{\chi}{h^3}\big(H_{i+2}-2H_{i+1}+2H_{i-1}-H_{i-2}\big). \tag{19}$$

Hence, we get

$$-\chi H_{i-2} + 2\chi H_{i-1} - 2\chi H_{i+1} + \chi H_{i+2} = h^3\left[pT_{i-2} + qT_{i-1} + sT_i + qT_{i+1} + pT_{i+2}\right] + O(h^9),\ i = 2(1)N-2, \tag{20}$$

where

$\chi = (r_1 + s_1)$, $p = p_1 + \frac{q_1}{6}$, $q = 2(\frac{2r_1+s_1}{6} - (p_1 + q_1))$ and $s = 2(\frac{r_1+4s_1}{6} + (p_1 - 2q_1))$.

The relation (20) gives $(N-2)$ linear equations in $(N-2)$ unknowns $H_i, i = 2, 3, \dots, N-2$.

Remark. As $\kappa \to 0$, then $(r_1, s_1, p_1, q_1) \to (\frac{1}{6}, \frac{1}{3}, \frac{-7}{360}, \frac{-8}{360})$ and $(\alpha, p, q, s) \to (\frac{1}{15}, \frac{1}{120}, \frac{13}{60}, \frac{33}{60})$, the spline relations defined in (20) is reduced to the corresponding ordinary quintic spline relations [29,30].

Construction of boundary equations

For unique solution of the system (20), we need to compute two additional equations, one at each end of the range of integration to find H_i. To discretize the boundary conditions, we define

$$\sum_{j=0}^{3} b_j H_j + c_1 h H_0' + h^3 \sum_{j=0}^{3} d_j H_j''' = t_1, \quad j = 1 \tag{21}$$

$$\sum_{j=N-3}^{N-1} a_j H_j + c_2 h H_N' + h^3 \sum_{j=N-4}^{N} c_j H_j''' = t_{N-1}, \quad j = N-1 \tag{22}$$

where b_j, c_1, c_2, a_j, c_j and d_j are arbitrary parameters to be determined.

Truncation error

By expanding Eq. (20) in Taylor's series about ζ_i, we obtain the following truncation error(TE):

$$\begin{aligned} t_i &= [2\chi - 2p - 2q - s]h^3 H_i''' + \left[\frac{1}{2}\chi - 4p - q\right]h^5 H_i^{(5)} \\ &\quad + \left[\frac{1}{20}\chi - \frac{4p}{3} - \frac{q}{12}\right]h^7 H_i^{(7)} + \left[\frac{17}{6048}\chi - \frac{8p}{45} - \frac{q}{360}\right]h^9 H_i^{(9)} + O(h^{11}), \\ i &= 2(1)N-2. \end{aligned} \tag{23}$$

With the help of the above equation, we eliminate the components of different powers of h for various selections of χ, p, q and s which allows us to derive the class of techniques that is described as follows:

Second-order methods

Taking $\chi = \frac{1}{2}$, $(b_0, b_1, b_2, b_3, c_1, d_0, d_1, d_2, d_3) = (1, \frac{-4}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{-1}{18}, \frac{-1}{6}, 0, 0)$, and $(a_{N-3}, a_{N-2}, a_{N-1}, c_2, c_{N-4}, c_{N-3}, c_{N-2}, c_{N-1}, c_N) = (-3, 8, -5, 2, 0, 0, 0, -5, \frac{4}{3})$, we have the local TE:

$$\begin{aligned} t_1 &= -\frac{1}{180}h^5 H_i^{(5)} + O(h^6), i = 1, \\ t_{N-1} &= \frac{178}{120}h^5 H_i^{(5)} + O(h^6), i = N-1. \end{aligned} \tag{24}$$

Case 1: When $(p, q, s) = (\frac{1}{4}, 0, \frac{1}{2})$, the TE is provided by

$$t_i = -\frac{3}{4}h^5 H_i^{(5)} + O(h^7), i = 2(1)N-2.$$

Case 2: When $(p, q, s) = (\frac{1}{4}, \frac{1}{4}, 0)$, the TE is provided by

$$t_i = (-1)h^5 H_i^{(5)} + O(h^7), i = 2(1)N-2.$$

Case 3: When $(p, q, s) = (\frac{1}{2}, \frac{1}{2}, -1)$, the TE is provided by

$$t_i = -\frac{9}{4}h^5 H_i^{(5)} + O(h^7), i = 2(1)N-2.$$

Fourth-order methods

Taking $\chi = \frac{1}{2}$, $(b_0, b_1, b_2, b_3, c_1, d_0, d_1, d_2, d_3) = (320, -360, 0, 40, 240, -16, -90, -12, -2)$, and $(a_{N-3}, a_{N-2}, a_{N-1}, c_2, c_{N-4}, c_{N-3}, c_{N-2}, c_{N-1}, c_N) = (-\frac{5}{2}, 4, -\frac{3}{2}, 1, \frac{59}{315}, -\frac{107}{168}, \frac{19}{420}, -\frac{3601}{2520}, 0)$, then we have the local TE:

$$
\begin{aligned}
t_1 &= \frac{2161}{140} h^7 H_i^{(7)} + O(h^8), i = 1,\\
t_{N-1} &= \frac{2465}{420} h^7 H_i^{(7)} + O(h^8), i = N-1.
\end{aligned} \tag{25}
$$

Case 1: When $(p, q, s) = (-\frac{1}{720}, \frac{11}{45}, \frac{183}{360})$, the TE is provided by

$$t_i = \frac{1}{369} h^7 H_i^{(7)} + O(h^8), i = 2(1)N-2.$$

Case 2: When $(p, q, s) = (-\frac{1}{720}, \frac{161}{630}, \frac{413}{840})$, the TE is provided by

$$t_i = \frac{1}{180} h^7 H_i^{(7)} + O(h^8), i = 2(1)N-2.$$

Case 3: When $(p, q, s) = (-\frac{10}{7199}, \frac{7359}{28796}, \frac{7079}{14398})$, the TE is provided by

$$t_i = -\frac{64791}{287960} h^7 H_i^{(7)} + O(h^8), i = 2(1)N-2.$$

Application of the method

Using $T_i = -k(\zeta_i)H_i'' - l(\zeta_i)H_i' - m(\zeta_i)\tau_i$ in Eq. (20), and with the help of higher order approximations to derivatives given as follows:

$$
\begin{aligned}
H'_{i-2} &= \frac{-5H_{i-1} + 8H_i - 3H_{i+1}}{2h},\\
H'_{i-1} &= \frac{-3H_{i-1} + 4H_i - H_{i+1}}{2h},\\
H'_i &= \frac{H_{i+1} - H_{i-1}}{2h},\\
H'_{i+1} &= \frac{H_{i-1} - 4H_i + 3H_{i+1}}{2h},\\
H''_{i-2} &= \frac{H_{i-1} - 2H_i + H_{i+1}}{h^2} - 2hH_i^{(3)},\\
H''_{i-1} &= \frac{H_{i-1} - 2H_i + H_{i+1}}{h^2} - hH_i^{(3)},\\
H''_i &= \frac{H_{i-1} - 2H_i + H_{i+1}}{h^2},\\
H''_{i+1} &= \frac{H_{i-1} - 2H_i + H_{i+1}}{h^2} + hH_i^{(3)},
\end{aligned}
$$

we obtain the following linear system:

$$\tilde{k}_i H_{i-2} + \tilde{l}_i H_{i-1} + \tilde{m}_i H_{i+1} + \tilde{n}_i H_{i+2} = \frac{h^3}{12}\left[\tau_{i-2} + 5k_{i-1} + (2hk_{i-2} + 5hk_{i-1} - hk_{i+1} + 5)\tau_i + \tau_{i+1}\right] + O(h^5), \quad i = 2(1)N-2, \tag{26}$$

where $k(\zeta_i) = k_i$, $l(\zeta_i) = l_i$, $m(\zeta_i) = m_i$, $\tau(\zeta_i) = \tau_i$ and

$$\tilde{k}_i = -1 + \frac{h^3}{12} m_{i-2},$$

$$
\begin{aligned}
\tilde{l}_i = 3 + \frac{h^3}{12}\Bigg[&\frac{1}{h^2}\left(k_{i-2} + 5k_{i-1} + k_i(2hk_{i-2} + 5hk_{i-1} - hk_{i+1} + 5) + k_{i+1}\right)\\
&- \frac{1}{2h}\left(8l_{i-2} + 20l_{i-1} + l_i(2hk_{i-2} + 5hk_{i-1} - hk_{i-1} + 5) + l_{i+1}\right) - 5m_{i-1}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{m}_i = -3 - \frac{h^3}{12}\Bigg[&\frac{1}{h^2}\left(2k_{i-2} - 10k_{i-1} + 2k_i(2hk_{i-2} + 5hk_{i-1} - hk_{i+1} + 5) + 2k_{i+1}\right)\\
&- \frac{1}{2h}\left(3l_{i-2} + 5l_{i-1} - l_i(2hk_{i-2} + 5hk_{i-1} - hk_{i-1} + 5) - 3l_{i+1}\right) - m_{i+1}\Bigg],
\end{aligned}
$$

$$\tilde{n}_i = 1 + \frac{h^3}{12}\left[\frac{1}{h^2}\left(k_{i-2} + 5k_{i-1} + k_i(2hk_{i-2} + 5hk_{i-1} - hk_{i+1} + 5) + k_{i+1}\right) - \frac{1}{2h}\left(8l_{i-2} + 20l_{i-1} + l_i(2hk_{i-2} + 5hk_{i-1} - hk_{i-1} + 5) + l_{i+1}\right) - 5m_{i-1}\right],$$

where $i = 2(1)N-2$.

The boundary equations for the second-order algorithm have been determined as:

$$\tilde{h}_0 H_0 + \tilde{h}_1 H_2 + \tilde{h}_2 H_3 = \frac{h^3}{12}\left[3\tau_0 + (3hk_0 - hk_2 + 4)\tau_1 + \tau_2\right] - 2h\alpha_2 + O(h^5), \qquad i = 1 \tag{27}$$

and

$$\tilde{h}_3 H_{N-3} + \tilde{h}_4 H_{N-2} + \tilde{h}_5 H_{N-1} = \frac{h^3}{12}\left[3\tau_{N-2} + 10\tau_{N-1} + (10hk_{N-1} + 6hk_{N-2} + 31)\tau_N\right] - 2h\alpha_3 + O(h^5), \quad i = N-1, \tag{28}$$

$$\begin{aligned}
\tilde{h}_0 &= 3 - \frac{h^3}{12}\left[\frac{1}{h^2}\left(-3k_0 - k_1\left(3hk_0 - hk_2 + 4\right) - k_2\right) + \frac{1}{2h}\left(9l_0 + l_1(3hk_0 - hk_2 + 4) - l_2\right) + 3m_0,\right. \\
\tilde{h}_1 &= -4 - \frac{h^3}{12}\left[\frac{1}{h^2}\left(6k_0 + 2k_1(3hk_0 - hk_2 + 4) + 2k_2\right) + \frac{1}{2h}\left(-12l_0 + 4l_2\right) - m_1\left(3hk_0 - hk_2 + 4\right)\right], \\
\tilde{h}_2 &= 1 - \frac{h^3}{12}\left[\frac{1}{h^2}\left(-3k_0 - k_1\left(3hk_0 - hk_2 + 4\right) - k_2\right) + \frac{1}{2h}\left(3l_0 - l_1(3hk_0 - hk_2 + 4) - 3l_2\right) - m_2\right], \\
\tilde{h}_3 &= -3 + \frac{h^3}{4} m_{N-2}, \\
\tilde{h}_4 &= 8 - \frac{h^3}{12}\left[\frac{1}{h^2}\left(-6k_{N-2} - 20k_{N-1} - 2k_N\left(10hk_{N-1} + 6hk_{N-2} + 31\right)\right)\right. \\
&\quad \left. + \frac{1}{2h}\left(24l_{N-2} + 40l_{N-1} + 2l_N\left(10hk_{N-1} + 6hk_{N-2} + 31\right)\right) - 3m_{N-1}\right], \\
\tilde{h}_5 &= -5 - \frac{h^3}{12}\left[\frac{1}{h^2}\left(6k_{N-2} + 20k_{N-1} + 2k_N\left(10hk_{N-1} + 6hk_{N-2} + 31\right)\right)\right. \\
&\quad \left. + \frac{1}{2h}\left(-24l_{N-2} - 40l_{N-1}\right) - m_N\left(10hk_{N-1} + 6hk_{N-2} + 31\right)\right].
\end{aligned}$$

The system (26), (27) and (28) has been solved by Newton Raphson method to calculate the approximate solution of Eqs. (11)-(12).

Convergence analysis

For studying the convergence of the scheme formulated so far, we assume $\hat{H} = [H_1, H_2, H_3, \dots, H_{N-1}, H_N]^T$, $R = [r_1, r_2, r_3, \dots, r_{N-1}, r_N]^T$ be known vectors and , $\omega_1 = -[t_1, t_2, t_3, \dots, t_{N-1}, t_N]^T$ and $\omega_2 = [e_1, e_2, e_3, \dots, e_{N-1}, e_N]^T$ be the local TE and discretization error, respectively. Now, let $\hat{H} = [H_1, H_2, H_3, \dots, H_{N-1}, H_N]^T$ be the approximate solution vector and $\tilde{H} = [H(\zeta_1), H(\zeta_2), \dots, H(\zeta_N)]^T$ of Eq. (11) be the exact solution vector subject to conditions (12). We represent the system of N Eqs. (26), (27) and (28) with N unknowns as:

$$A\hat{H} + \frac{1}{12}h^3 B\tau = R, \tag{29}$$

where $\tau = (\tau_i)$, $i = 1(1)N$ and

$$A = \begin{bmatrix}
\tilde{h}_0 & \tilde{h}_1 & \tilde{h}_2 & & & & & & \\
\tilde{b}_2 & \tilde{c}_2 & \tilde{d}_2 & \tilde{e}_2 & & & & & \\
\tilde{a}_3 & \tilde{b}_3 & \tilde{c}_3 & \tilde{d}_3 & \tilde{e}_3 & & & & \\
 & & & \ddots & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & a_{\tilde{N-1}} & b_{\tilde{N-1}} & c_{\tilde{N-1}} & d_{\tilde{N-1}} \\
 & & & & & & \tilde{h}_3 & \tilde{h}_4 & \tilde{h}_5
\end{bmatrix},$$

$$B = \begin{bmatrix}
\xi_1 & -d_2 & -d_3 & & & & & & \\
q & \xi_2 & q & p & & & & & \\
p & q & \xi_3 & q & p & & & & \\
 & & & \ddots & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & p & q & \xi_{N-1} & q \\
 & & & & & & -5 & \frac{4}{3} & \xi_N
\end{bmatrix},$$

$$\tau = \begin{bmatrix} \tau_1 & & & & & \\ & \tau_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \tau_{N-1} & \\ & & & & & \tau_N \end{bmatrix}$$

and $R = [r_1,\ r_2, \cdots, r_N]^T$ with

$$r_i = \begin{cases} -2h\alpha_2 - \tilde{h}_0\alpha_1, & i = 1, \\ \tilde{k}_2\alpha_1, & i = 2, \\ 0, & i = 3(1)N-1, \\ -2h\alpha_N, & i = N, \end{cases}$$

where $\alpha_1 = (3hk_0 - hk_2 + 4)$, $\alpha_i = (2hk_{i-2} + 5hk_{i-1} - hk_{i+1} + 5)$, $i = 2(1)N-1$ and $\alpha_N = (10hk_{N-1} + 6hk_{N-2} + 31)$.

Since $\tilde{H} = (H(\zeta_1), H(\zeta_2), \ldots, H(\zeta_N))^T$ is the exact solution vector of Eqs. (11) and (12), we represented the system as:

$$A\tilde{H} + \frac{1}{12}h^3 B\tau = \omega_1(h) + R, \tag{30}$$

where $\omega_1(h) = [t_1(h),\ t_2(h), \cdots, t_N(h)]^T$ is defined as follows:

$$t_i = \begin{cases} -\frac{1}{10}h^5 H_0^5 + O(h^6), & i = 1 \\ -\frac{1}{6}h^5 H_i^5 + O(h^6), & i = 2(1)N-1 \\ -\frac{1}{10}h^5 H_N^5 + O(h^6), & i = N. \end{cases} \tag{31}$$

From Eqs. (29) and (30), the error equation is found to be

$$A(\tilde{H} - \hat{H}) = \omega_1(h),$$

or

$$A\omega_2 = \omega_1(h), \tag{32}$$

where $\omega_2 = \tilde{H} - \hat{H} = [e_1,\ e_2,\ \cdots, e_N]^T$.

The row sums $S_1,\ S_2, \cdots, S_N$ of A are

$$S_i = \begin{cases} \tilde{h}_1 + \tilde{h}_2, & i = 1, \\ \tilde{l}_2 + \tilde{m}_2 + \tilde{n}_2, & i = 2, \\ \tilde{k}_i + \tilde{l}_i + \tilde{m}_i + \tilde{n}_i, & i = 3(1)N-1, \\ \tilde{h}_3 + \tilde{h}_4 + \tilde{h}_5, & i = N. \end{cases} \tag{33}$$

Because the matrix A is irreducible and monotone when h is taken to a suitably small [35], therefore A^{-1} exists and

$$\omega_2 = A^{-1}\omega_1(h),$$

or,

$$\|\omega_2\|_\infty \le \|A^{-1}\|_\infty \cdot \|\omega_1(h)\|_\infty. \tag{34}$$

Hence, we obtain $\|\omega_2\| = O(h^2)$, i.e. second order convergence.

Similarly we can prove the fourth order convergence.

Numerical experiments

We consider a finite domain [0,1] so that it becomes easy to study the velocity components. We have used the derived scheme to compute the approximate numerical values of the function. These values will give an approximate idea of how the function varies in the domain. We compute the values using MATLAB by developing a significant code which satisfies our scientific need. When Froude number F_{r_x} is fixed, the values of $H(\zeta)$ and $H'(\zeta)$ are obtained and given in tables 1 – 5 from where we read the effect of $H(\zeta)$ and $H'(\zeta)$ on changing the angle of inclination ψ. Also, graphs are represented to show how $H(\zeta)$ and $H'(\zeta)$ varies for different values of F_{r_x} and ψ.

Table 1
$H'(\zeta)$ at various inclination angles with $F_{r_x} = 1$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0.1	0.833881771	0.8356603464	0.8461455576	0.8555678902
0.2	0.677774937	0.694343333	0.7155785651	0.7259677887
0.3	0.546552409	0.568321009	0.580984556	0.5946787722
0.4	0.434446761	0.4522445853	0.47580266525	0.4966727414
0.5	0.327443486	0.3523444579	0.3727056548	0.3996665036
0.6	0.235547614	0.2648655994	0.2822345677	0.3157663459
0.7	0.173217578	0.1826677532	0.206724419	0.2329766984
0.8	0.1122325322	0.1266676347	0.1421466213	0.166673498
0.9	0.0635623877	0.0738766632	0.0855555081	0.1056653217
1.0	0.0217845974	0.025179800	0.0298455238	0.0355562907

Table 2
The values of $H'(\zeta)$ at various inclination angles when $F_{r_x} = 1.5$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0.1	0.857568900	0.85902734567	0.8699941123	0.8666920721
0.2	0.7279334567	0.7344442677	0.7374567899	0.7442328765
0.3	0.6108342567	0.6126532134	0.62322451552	0.6324985432
0.4	0.5037444456	0.5104357907	0.5186247689	0.5289045433
0.5	0.4034456788	0.4137216778	0.4222423778	0.4341678400
0.6	0.3154711234	0.32602349801	0.3311476543	0.346399719
0.7	0.2316623455	0.2466742345	0.254182213	0.2652355657
0.8	0.1685042234	0.1747112567	0.1833456687	0.1902547892
0.9	0.101842345	0.10983534509	0.1138777623	0.12967655431
1.0	0.0378344567	0.0391143321	0.04122533308	0.0438711296

Table 3
Values of $H'(\zeta)$ at various inclination angles when $F_{r_x} = 2$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0.1	0.8670368432	0.8677888256	0.8655571467	0.8707728967
0.2	0.74240554311	0.7453875432	0.7490545555	0.75263888801
0.3	0.63084567544	0.6334528456	0.6361001111	0.6433569402
0.4	0.5262333467	0.53270904287	0.5373889023	0.5415670921
0.5	0.4301278911	0.4395067654	0.4437844448	0.4457833547
0.6	0.3437343899	0.3439978905	0.3594305557	0.3578944865
0.7	0.2628522678	0.2642653321	0.2713502234	0.2735345998
0.8	0.1894356221	0.1930857890	0.1960263245	0.1996931124
0.9	0.1103634590	0.1293245677	0.1261445678	0.1278763487
1.0	0.0437132458	0.0449015678	0.0459623345	0.0460020122

Table 4
Observed $H(\zeta)$ for $F_{r_x} = 1$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0	0	0	0	0
0.1	0.0912458234	0.09189603285	0.09199295906	0.09222319091
0.2	0.1663547890	0.16778909914	0.16985525181	0.1714339809
0.3	0.2209647152	0.23068692670	0.2334332327	0.2379001043
0.4	0.2783781508	0.2812561466	0.2857540875	0.2913100069
0.5	0.3153719717	0.3219152293	0.3272822661	0.3361875437
0.6	0.3440188078	0.3518055172	0.3604745269	0.3711412618
0.7	0.3644169429	0.3748597062	0.3847363114	0.39807161381
0.8	0.3787911477	0.3895030872	0.4029790695	0.41867213381
0.9	0.3878918811	0.3999754537	0.4139980352	0.43104277711
1.0	0.3918437983	0.4042272992	0.4194523027	0.43828999823

Now to check the efficiency of this method when porosity is absent, we compare the results obtained so far for $H'(\zeta)$ with the results of T. Begum et al.[25] for fixed $F_{r_x} = 1.5$ and $\psi = \frac{\pi}{2}$, given in Table 6.

Now we study the heat transfer by fitting the data for $h(\zeta)$ in a suitable function, so that

$$h(\zeta) = \frac{P}{Q}\left(1 - e^{-Q\zeta}\right), \tag{35}$$

Table 5
Observed $H(\zeta)$ for $F_{r_x} = 1.5$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0	0	0	0	0
0.1	0.09254593518	0.0926009315	0.09210099089	0.0930656041
0.2	0.1714245346	0.17155986296	0.1721882013	0.1731828582
0.3	0.2386473725	0.23952979580	0.2402578599	0.2425135761
0.4	0.2936086584	0.2951142226	0.2973392144	0.2966501740
0.5	0.3384926651	0.34142230336	0.3441271609	0.347699349
0.6	0.37447278668	0.3771204825	0.3817369367	0.3869948411
0.7	0.4027748064	0.4063599141	0.4113383030	0.41745819773
0.8	0.4228440777	0.4272419187	0.4321275597	0.4390241625
0.9	0.4368635067	0.4410266956	0.4410748112	0.4552870121
1.0	0.4434109867	0.4493264387	0.4561736341	0.46409287285

Table 6
Comparing the values of the velocity function $H'(\zeta)$ taking $F_{r_x} = 1.5$ and $\psi = \frac{\pi}{2}$ with [25].

ζ	T. Begum et al. [25]	Present Method
0.1	0.8573686273	0.88859368228
0.2	0.7278102499	0.73960883852
0.3	0.6100947008	0.61769998720
0.4	0.5032064107	0.47905840100
0.5	0.4062769943	0.38188982974
0.6	0.3185645547	0.28570266819
0.7	0.2394350166	0.19014187690
0.8	0.1683454504	0.09495756214
0.9	0.1048293184	0.04745601762

where the values of P and Q have been computed by using data fitting technique and stated in Table 7.

We compare $H(\zeta)$ and $h(\zeta)$ in order to validate the function computed in Eq. (35) and given in Table 8.

We observe that the function $H(\zeta)$ and $h(\zeta)$ agree with each other with third order accuracy. Now, we use the values of $h(\zeta)$ to analyze the heat transfer.

Heat transfer flow

Now we study the heat transfer within the boundary layer. We consider the following heat flow problem,

$$\rho S_p \left(u \frac{\partial V}{\partial x} + v \frac{\partial V}{\partial y} \right) = K' \frac{\partial^2 V}{\partial y^2}, \tag{36}$$

with boundary conditions

$$y = 0, V = V_p, \tag{37}$$

$$y \to \infty, V = V_\infty, \tag{38}$$

where the density of the fluid is defined as ρ, the temperature of the plate is defined as V_p, the temperature of the fluid that surrounds the plate is written as V_∞, the surface temperature at relentless pressure is represented as S_p, and the thermal efficiency is written as K'.

The boundary conditions imply that the heat transfer depends only on y, so the Eq. (36) becomes

$$\rho S_p \left(v \frac{\partial V}{\partial y} \right) = K' \frac{\partial^2 V}{\partial y^2}. \tag{39}$$

Defining dimensionless temperature by

$$\theta(\zeta) = \frac{V - V_\infty}{V_p - V_\infty}. \tag{40}$$

Using the expression $v = -(mv)^{\frac{1}{2}} H(\zeta)$ and Eq. (40) in Eq. (39), we get

$$\theta'' + P_r H(\zeta)\theta' + \theta = 0, \tag{41}$$

Table 8
Observed $H(\zeta)$ and $h(\zeta)$ for $\psi = \frac{\pi}{6}$, $\psi = \frac{\pi}{4}$ and $F_{r_x} = 1.5$.

ζ	$H(\zeta)$	$h(\zeta)$	*Error*	$H(\zeta)$	$h(\zeta)$	**Error**
		$\psi = \frac{\pi}{6}$			$\psi = \frac{\pi}{4}$	
0	0	0	0	0	0	0
0.1	0.09363894	0.095667289	$2.513445e-03$	0.09282013	0.09494420	$2.124071e-03$
0.2	0.17328581	0.174553328	$1.423222e-03$	0.17257859	0.17330304	$7.244438e-04$
0.3	0.24357610	0.240869756	$1.512345e-03$	0.24039214	0.23797373	$2.418400e-03$
0.4	0.29620740	0.295324607	$4.926788e-03$	0.29727160	0.29134742	$5.924188e-03$
0.5	0.34199349	0.34070250	$7.7402345e-03$	0.34413693	0.33539749	$8.739439e-03$
0.6	0.38669484	0.377082000	$9.124322e-03$	0.38183030	0.37175267	$1.007763e-03$
0.7	0.41197771	0.40895452	$8.4615456e-03$	0.41112755	0.40175713	$9.370429e-03$
0.8	0.43246256	0.43785236	$5.3077776e-03$	0.43274811	0.42652024	$6.227862e-03$
0.9	0.45587027	0.45156930	$6.658448e-04$	0.44736341	0.44695761	$4.057967e-04$
1.0	0.46487289	0.47353134	$9.632914e-03$	0.45560417	0.46382487	$8.220695e-03$

Table 9
Variation in $\theta(\zeta)$ taking $F_{r_x} = 1.5$.

ζ	$\psi = \frac{\pi}{2}$	$\psi = \frac{\pi}{3}$	$\psi = \frac{\pi}{4}$	$\psi = \frac{\pi}{6}$
0	1	1	1	1
0.4	0.8767484944	0.8726768876	0.8786766566	0.8677777799
0.8	0.7523455566	0.7546767888	0.7503566566	0.7445789913
1.2	0.6544556665	0.6497879978	0.6467676777	0.6355657755
1.6	0.5634565777	0.5534343434	0.5567766666	0.5434467667
2.0	0.4856287647	0.4793344333	0.4743455677	0.4656565645
2.4	0.4123345767	0.4224324445	0.4335677786	0.39564577453
2.8	0.3567985545	0.3535466777	0.3466896444	0.3456777545
3.2	0.3093446776	0.3032457677	0.2567886443	0.2856767776
3.6	0.2663444565	0.2645566677	0.2534567788	0.2454467776
4.0	0.2293565767	0.2235667877	0.2163345677	0.2344676777

subject to the conditions

$$\theta(0) = 1, \tag{42}$$

$$\theta(\infty) = 0. \tag{43}$$

The solution of the Eq. (41), using Eqs. (42) and (43) is

$$\theta(\zeta) = \frac{\int_{\zeta}^{\infty} e^{-P_r \int \zeta H(\zeta) d\zeta} d\zeta}{\int_{0}^{\infty} e^{-P_r \int \zeta H(\zeta) d\zeta} d\zeta}, \tag{44}$$

or,

$$\theta(\zeta) = \frac{\gamma\left(\frac{P_r P}{Q^2}, \frac{P_r A}{Q^2} e^{-Q\zeta}\right)}{\gamma\left(\frac{P_r P}{Q^2}, \frac{P_r P}{Q^2}\right)}, \tag{45}$$

where P and Q are taken from the Table 7, and $\gamma(\tilde{w}, z) = \int_0^z e^{-t} t^{\tilde{w}-1} dt$ is the incomplete gamma function.

Theorem. For $\tilde{w} > 0$ and $z > 0$,

$$\gamma(\tilde{w}, z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+\tilde{w}}}{n!(\tilde{w} + n)}. \tag{46}$$

When we fix $z > 0$, the series (46) converges for all $\tilde{w} > 0$.

Now, we apply the above theorem in Eq. (45) so that $\theta(\zeta)$ becomes

$$\theta(\zeta) = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{P_r P}{Q^2}\right)^n e^{-B\zeta\left(\frac{P_r P}{Q^2} - n\right)}}{n!\left(\frac{P_r P}{Q^2} + n\right)}}{\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{P_r P}{Q^2}\right)^n}{n!\left(\frac{P_r P}{Q^2} + n\right)}}.$$

The numerical values of θ are determined for $F_{r_x} = 1.5$, which have been presented in Table 9.

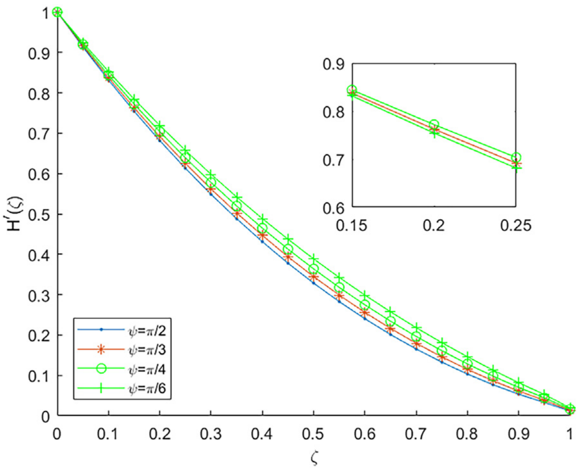

Fig. 2. Variation in $H'(\zeta)$ at various inclination angles when $F_{r_x} = 1$.

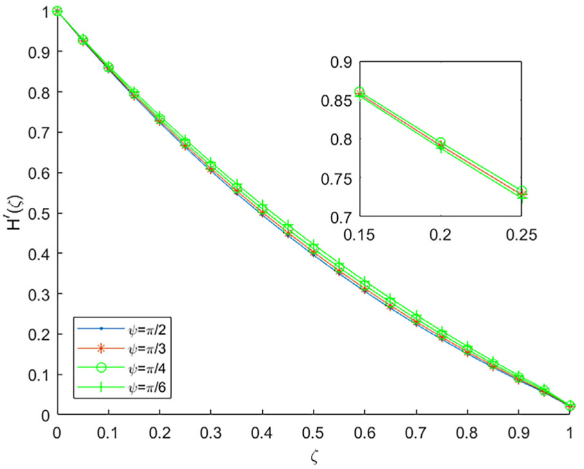

Fig. 3. Fluctuation in $H'(\zeta)$ at various inclination angles when $F_{r_x} = 1.5$.

Nusselt number

The coefficient of Heat Transfer is defined as

$$q_w = -K'\left(\frac{\partial V}{\partial y}\right)_{y=0} \tag{47}$$

or,

$$q_w = -K'(V_p - V_\infty)\left(\frac{m}{\nu}\right)^{\frac{1}{2}}. \tag{48}$$

Now, we calculate the Nusselt number as

$$F = \frac{q_w}{K'(V_p - V_\infty)}, \tag{49}$$

which gives

$$F = -\left(\frac{m}{\nu}\right)^{\frac{1}{2}}. \tag{50}$$

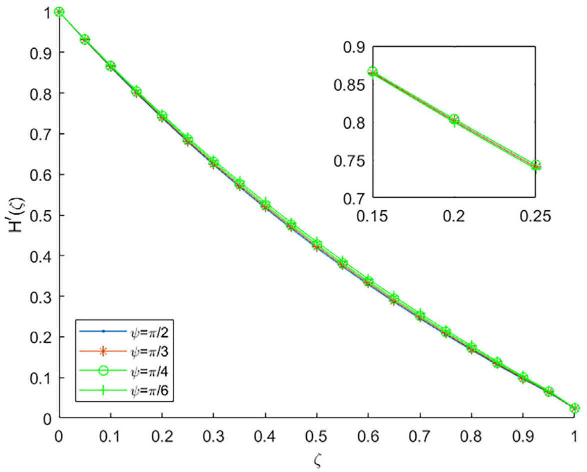

Fig. 4. Fluctuation in $H'(\zeta)$ for various inclination angles when $F_{r_x} = 2$.

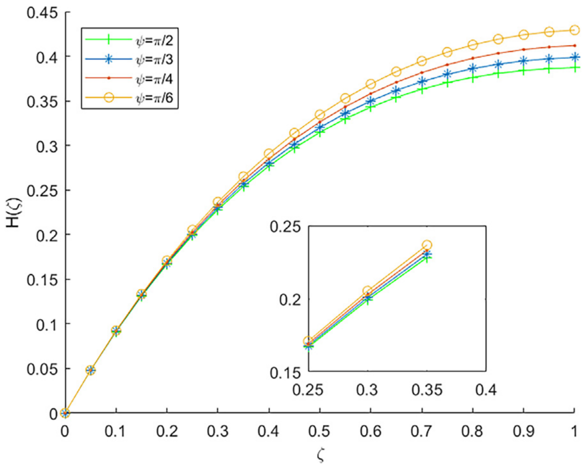

Fig. 5. Fluctuation in $H(\zeta)$ at various inclination angles when $F_{r_x} = 1$.

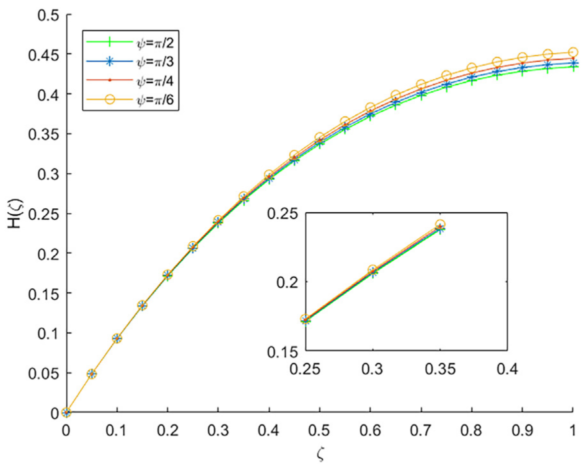

Fig. 6. Fluctuation in $H(\zeta)$ at various inclination angles when $F_{r_x} = 1.5$.

Table 7
$h(\zeta)$ for different ψ and $F_{r_x} = 1.5$.

ψ	P	Q	$h(\zeta)$
$\frac{\pi}{2}$	1.0456753643	2.00724178	$0.52095137(1-e^{-2.00724178\zeta})$
$\frac{\pi}{3}$	1.044612625	1.96658697	$0.53118049(1-e^{-1.96658697\zeta})$
$\frac{\pi}{4}$	1.04349858	1.91990566	$0.54351555(1-e^{-1.91990566\zeta})$
$\frac{\pi}{6}$	1.04764382	1.87122063	$0.55987188(1-e^{-1.87122063\zeta})$

Discussions, results and conclusion

We studied a physical situation where the flow of viscous incompressible fluid occurs over an inclined plate in porus media and heat transfer. Several authors solved this problem in various variants but here we developed a numerical technique called non polynomial quintic spline. On the basis of our solution, we summarise the findings as follows:

We solved the obtained non linear ordinary differential equation which has been further linearised by the quasilinearisation technique. With the application of our method, finally we get a solution in discrete form.

Tables 1, 2, 3 and Figs. 2, 3, 4 show that the variation in velocity component $\frac{u}{mx}$ depending upon $H'(\zeta)$ linearly when angle of inclination changes. Moreover we observe that as angle of inclination increases the velocity component decreases. At greater inclination the downward force mgh is more because of height h is more.

When we observe the Tables 4–5 and Fig. 5–6, we note that the magnitude of vertical component of velocity decreases as angle of inclination increases because the gravity is having more influence here. Table 6 shows the comparative result of $H'(\zeta)$ with [25] when F_{r_x} is fixed at 1.5 and angle of inclination is fixed at 90 °. Reading the Table 9 for temperature distribution $\theta(\zeta)$, we notice that flow of heat occurs in the form of wave. As we go away the inclined plate, the intensity of heat is in reducing pattern for some particular value of F_{r_x}.

Finally, we derive the formula for Nusselt number $_F$ which comes out to be $-\sqrt{\frac{m}{\nu}}$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

CRediT authorship contribution statement

Tahera Begum: Methodology, Formal analysis, Software, Validation, Visualization, Writing – original draft. **Geetan Manchanda:** Validation. **Arshad Khan:** Conceptualization, Supervision, Writing – review & editing. **Naseem Ahmad:** Validation.

Acknowledgement

The authors are thankful to the anonymous reviewer/editor for their careful reading and valuable comments/suggestions which have greatly improved the quality and presentation of the manuscript. First author is also thankful to University Grant Commission for providing Maulana Azad National Senior Research Fellowship.

References

[1] B.C. Sakiadis, Boundary layer behavior on continuous solid surfaces: i. boundary layer equations for two dimensional and axisymmetric flow, Am. Inst. Chem. Eng. J 7 (2) (1961) 221–225.
[2] L.J. Crane, Flow past a stretching plate, J. Appl. Math. Phys. (ZAMP) 21 (4) (1970) 645–647.
[3] K. Rajagopal, T.Y. Na, A.S. Gupta, Flow of a viscoelastic fluid over a stretching sheet, Rheol. Acta 23 (2) (1984) 213–215.
[4] C.K. Chen, M.I. Char, Heat transfer of a continuous, stretching surface with suction or blowing, J. Math. Anal. Appl. 135 (2) (1988) 568–580.
[5] C. Wang, Free convection on a vertical stretching surface, ZAMM-J. Appl. Math. Mech./Zeitschrift für Angewandte Mathematik und Mechanik 69 (11) (1989) 418–420.
[6] B. Dandapat, A.S. Gupta, Flow and heat transfer in a viscoelastic fluid over a stretching sheet, Int. J. Non-Linear Mech. 24 (3) (1989) 215–219.
[7] N. Ahmad, G.S. Patel, B. Siddappa, Visco-elastic boundary layer flow past a stretching plate and heat transfer, J. Appl. Math. Phys.(ZAMP) 41 (3) (1990) 294–299.
[8] H.I. Andersson, B.S. Dandapat, Flow of a power-law fluid over a stretching sheet, Stability Appl. Anal. Contin. Media 1 (6) (1991) 339–347.
[9] N. Ahmad, K. Marwah, Visco-elastic boundary layer flow past a stretching plate with suction and heat transfer with variable thermal conductivity, Indian. J. Eng. Mater. S. 7 (3) (2000) 54–56.
[10] E. Magyari, B. Keller, Exact solutions for self-similar boundary-layer flows induced by permeable stretching walls, Eur. J. Mech. B Fluids 19 (1) (2000) 109–122.
[11] A. Ishak, R. Nazar, I. Pop, Unsteady mixed convection boundary layer flow due to a stretching vertical surface, Arabian J. Sci. & Engi. (Springer Sci. Bus. Media BV) 31 (2006) 165–182.
[12] A. Ishak, R. Nazar, I. Pop, Boundary layer flow and heat transfer over an unsteady stretching vertical surface, Meccanica 44 (4) (2009) 369–375.

[13] N. Ahmad, Z.U. Siddiqui, M.K. Mishra, Boundary layer flow and heat transfer past a stretching plate with variable thermal conductivity, Int. J. Non-Linear Mech. 45 (3) (2010) 306–309.
[14] N. Ahmad, K. Ahmad, Radiation effects on unsteady boundary layer flow of nanofluids Cu-water and Ag-water over a stretching plate and heat transfer with convective boundary condition, Int. J. Dyn. Fluids 13 (2017) 1–12.
[15] K. Vajravelu, J.R. Cannon, Fluid flow over a nonlinearly stretching sheet, Appl. Math. Comput. 181 (1) (2006) 609–618.
[16] J. Alinejad, S. Samarbakhsh, Viscous flow over nonlinearly stretching sheet with effects of viscous dissipation, J. Appl. Math. (2012) 1–10.
[17] K. Zaimi, A. Ishak, I. Pop, Boundary layer flow and heat transfer over a nonlinearly permeable stretching/shrinking sheet in a nanofluid with radiation effect, Adv. Mech. Eng. 340354 (2012) 1–7.
[18] S. Khan, Boundary layer viscoelastic fluid flow over an exponentially stretching sheet, Int. J. Appl. Mech. Eng. 11 (2) (2006) 321–335.
[19] N. Bachok, A. Ishak, I. Pop, Boundary layer stagnation-point flow and heat transfer over an exponentially stretching/shrinking sheet in a nanofluid, Int. J. of Heat Mass Transf. 55 (25–26) (2012) 8122–8128.
[20] E.H. Hafidzuddin, R. Nazar, N.M. Arifin, I. Pop, Boundary layer flow and heat transfer over a permeable exponentially stretching/shrinking sheet with generalised slip velocity, J. Appl. Fluid Mech. 9 (2016) 2025–2036.
[21] G.K. Ramesh, B.J. Gireesha, C.S. Bagewadi, Heat transfer in MHD dusty boundary layer flow over an inclined stretching sheet with non-uniform heat source/sink, Adv. Math. Phys. (2012) 1–13.
[22] G.K. Ramesh, A.J. Chamkha, B.J. Gireesha, Boundary layer flow past an inclined stationary/moving flat plate with convective boundary condition, Afrika Matematika 27 (1) (2016) 87–95.
[23] M. Ali, M.A. Alim, M.S. Alam, Heat transfer boundary layer flow past an inclined stretching sheet in presence of magnetic field, IJOART 5 (3) (2014) 34–40.
[24] M. Misra, R. Dohare, N. Ahmad, Boundary layer flow of incompressible fluid past an inclined stretching sheet and heat transfer, Int. J. Trend Res. Dev. 26 (5) (2018) 1–5.
[25] T. Begum, A. Khan, N. Ahmad, A numerical study of boundary layer flow of viscous incompressible fluid past an inclined stretching sheet and heat transfer using nonpolynomial spline method, Math. Methods Appl. Sci. 43 (17) (2020) 9948–9967.
[26] M.P. Alam, D. Kumar, A. Khan, Trigonometric quintic B-spline collocation method for singularly perturbed turning point boundary value problems, Int. J. Comput. Math. 98 (5) (2021) 1029–1048.
[27] M.P. Alam, T. Begum, A. Khan, A new spline algorithm for solving non-isothermal reaction diffusion model equations in a spherical catalyst and spherical biocatalyst, Chem. Phys. Lett. 754 (2020) 137651.
[28] M.P. Alam, T. Begum, A. Khan, A high-order numerical algorithm for solving Lane–Emden equations with various types of boundary conditions, Comput. Appl. Math. 40 (6) (2021) 1–28.
[29] J.H. Ahlberg, E.N. Nilson, J.L. Walsh, The theory of splines and their applications, Academic Press, New York, 1967.
[30] A. Khan, T. Aziz, The numerical solution of third-order boundary-value problems using quintic splines, Appl. Math. Comput. 137 (2–3) (2003) 253–260.
[31] P. Khandelwal, A. Khan, Singularly perturbed convection-diffusion boundary value problems with two small parameters using nonpolynomial spline technique, Math. Sci. 11 (2) (2017) 119–126.
[32] A. Khan, I. Khan, T. Aziz, A survey on parametric spline function approximation, Appl. Math. Comput. 171 (2) (2005) 983–1003.
[33] R.E. Bellman, R.E. Kalaba, Quasilinearization and nonlinear boundary-value problems, American Elsevier Publishing Co., Inc., New York, 1965.
[34] V.B. Mandelzweig, F. Tabakin, Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs, Comput. Phys. Commun. 141 (2) (2001) 268–281.
[35] P. Henrici, Discrete variable methods in ordinary differential equations, Wiley, New York, Springer, 1962.