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Power modified XLindley distribution: Statistical properties and applications

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This research introduces a novel two-parameter distribution, the power-modified XLindley distribution, developed through the application of power transformation techniques to the existing modified XLindley distribution. This new distribution enhances flexibility and adaptability in statistical modeling. We conduct a thorough examination of its statistical properties, exploring its potential to improve data fitting and modeling accuracy. To assess the effectiveness of the model, we employ multiple estimation techniques and evaluate their performance through extensive simulation experiments. Our findings indicate that the maximum product of the spacings method is particularly effective for parameter estimation. To demonstrate the practical utility of the proposed model, we apply it to two real-world datasets: one related to flood data and the other to reliability engineering. The results underscore the distribution's superior ability to capture the characteristics of these datasets compared to existing models, highlighting its significance for applications in natural disaster analysis and reliability studies.

Keywords Modified XLindley distribution, Moments, Estimation methods, Application

Modifications of the well-known Lindley distribution can be found in the literature; one is XLindley. Currently, some authors have studied the modifications of the XLindley distribution also. In recent years, the XLindley distribution (XLD) has emerged as a versatile statistical model, blending characteristics of the exponential and Lindley distributions, thereby offering enhanced flexibility in modeling various data types. Originating from the seminal work introduced by Chouia and Zeghdoudi[1], the XLindley distribution has sparked significant interest in the research community. This distribution, formed as a special mixture of exponential and Lindley distributions, exhibits notable statistical properties such as stochastic ordering, quantile function, maximum likelihood method, and method of moments.

Motivated by the simplicity and ease of application of the XLindley distribution, several researchers have extended and explored its applications in diverse domains. This introduction provides a comprehensive overview of notable contributions in the field, highlighting key advancements and applications.

The foundational work by Chouia and Zeghdoudi[1] laid the groundwork for subsequent studies by proposing the XLindley distribution and demonstrating its applicability. The simplicity of the distribution, as evidenced by straightforward formulas for mean, variance, etc., has made it a convenient choice for various statistical analyses.

The research landscape expanded with studies addressing estimation issues and computational analyses of the XLindley distribution. Alotaibi et al.[2] employed classical and Bayesian approaches to estimate parameters, reliability, and hazard rate functions, focusing on adaptive Type-II progressive hybrid censoring schemes. Meriem et al.[3] introduced the Power XLindley (PXL) distribution, utilizing power transformation $X = Y^{1/\lambda}$ to enhance the distribution's capabilities.

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In the realm of survival analysis, Khodja et al.[4] introduced the truncated XLindley distribution, emphasizing parameter estimation under censored data. A dual approach, combining traditional maximum likelihood and Bayesian techniques, was employed to estimate the underlying parameters. Ahsan-ul-Haq et al.[5] proposed the Poisson XLindley distribution, expanding the scope to count data and investigating various statistical and reliability properties. Metiri et al.[6] characterized the XLindley distribution, drawing connections between truncated moments and failure rate/reverse failure rate functions. Applications of this distribution to real data, such as survival times of individuals infected with coronavirus, demonstrated its practical utility. By compounding the exponential and XLindley distribution[7] have defined a new model having two parameters.

The research frontier continued to evolve with the introduction of variants like the Generalized XLindley distribution by mixing two models (two-parameter Lindley and exponential)[8], Wrapped XLindley distribution[9], Inverse XLindley distribution[10], and Discrete XLindley distribution[11]. These extensions explore diverse domains, from circular distributions to count datasets, broadening the applicability of the XLindley framework.

Despite these advancements, existing modifications of the XLindley distribution encounter specific limitations, such as handling highly skewed data, accommodating diverse censoring mechanisms in survival analysis, and providing flexible hazard rate functions for complex reliability data. These gaps underscore the necessity for new distribution models that can offer enhanced adaptability and improved fit for a wider range of data types and structures.

Gemeay et al.[12] have presented a new extension of the XLindley distribution called the modified XLindley distribution (MXLD), demonstrating its potential to fit lifetime data. Building upon this extension, the present research introduces a novel model, the power-modified XLindley distribution, obtained through power transformation techniques applied to the MXLD.

Power transformation is a widely used method in statistical modeling to stabilize variance, make the data more closely approximate a normal distribution, and improve the performance of statistical models[13]. The rationale behind choosing power transformation techniques to modify the existing modified XLindley distribution is grounded in its ability to:

- Enhance Flexibility: Power transformations can produce a family of distributions with varying degrees of skewness and kurtosis, making the model more adaptable to different types of data.
- Improve Data Fit: By adjusting the shape parameters through power transformation, the new distribution can better capture the underlying patterns in the data, especially for datasets with extreme values or heavy tails.

The power-modified XLindley distribution aims to address specific gaps in existing models by:

1. **Enhanced Flexibility in Data Fitting:** Providing more adaptable parameterization to better fit highly skewed and kurtotic data.
2. **Improved Handling of Censored Data:** Offering advanced methods for dealing with various types of censoring, particularly in survival analysis.
3. **Versatile Hazard Rate Functions:** Allowing for a broader range of hazard rate shapes, crucial for modeling complex reliability and life data.
4. **Robustness in Parameter Estimation:** Strengthening the accuracy and reliability of parameter estimation methods under diverse conditions and data types.

By addressing these gaps, the power-modified XLindley distribution not only extends the versatility of the XLindley family but also enhances its practical utility in statistical modeling and data analysis across numerous applications.

Formulation of the proposed distribution

Gemeay et al.[12] presented a new extension of the XLindley distribution called MXLD. Its PDF is

$$r(e) = \frac{\rho e^{-\frac{\rho}{e}}(\rho + e)}{2e^3},\ e,\ \rho > 0, \tag{1}$$

and its CDF is

$$R(e) = e^{-\frac{\rho}{e}}\left(\frac{\rho}{2e} + 1\right). \tag{2}$$

We have used the power transformation method to add a new shape parameter to the MXLD for a new statistical model called power MXLD (PMXLD). We obtain this model by taking $X = E^{\frac{1}{\xi}}, E \sim$ MXLD. The PDF of the PMXLD is

$$g(x) = \frac{1}{2}\xi\rho x^{-2\xi-1}e^{\rho\left(-x^{-\xi}\right)}\left(\rho + x^{\xi}\right),\ x,\ \rho,\ \xi > 0, \tag{3}$$

and its CDF is

$$G(x) = e^{-\frac{\rho}{x^{\xi}}}\left(\frac{\rho}{2x^{\xi}} + 1\right). \tag{4}$$

Survival function of PMXLD

The survival function is one of the important functions in survival or reliability analysis. For PMXLD it can be expressed as

$$G(x) = 1 - e^{-\frac{\rho}{x^{\xi}}}\left(\frac{\rho}{2x^{\xi}} + 1\right).$$

Hazard rate function (HRF) of PMXLD

The HRF is also an important function in statistical theory and model analysis. For PMXLD it can be expressed as

$$h(x) = \frac{\frac{1}{2}\xi\rho x^{-2\xi-1} e^{\rho\left(-x^{-\xi}\right)}\left(\rho + x^{\xi}\right)}{1 - e^{-\frac{\rho}{x^{\xi}}}\left(\frac{\rho}{2x^{\xi}} + 1\right)}.$$

For the different values of the parameter combination, we have plotted the graphs of PDF and HRF in Fig. 1. PDFs have different shapes with right skewed while HRF can have reverse-j and inverted bathtub-shaped curves.

Statistical properties of PMXLD

Mode

In probability theory and statistics, "mode" refers to the value or values in a probability distribution where the PDF reaches its maximum. For a continuous probability distribution with PDF denoted by $f(x;\, \rho, \xi)$, where X is the random variable and ρ and ξ are parameters, the mode corresponds to the values of X at which the PDF is maximized.

$$M = 2^{-1/\xi}\left(\frac{\sqrt{5\xi^2 + 6\xi + 1}\rho - \xi\rho - \rho}{\xi + 1}\right)^{1/\xi}. \tag{5}$$

Quantile

The quantile function of PMXLD can be obtained by inverting its CDF defined in (4) and it can be expressed as

$$Q = \left(-\frac{\rho}{W\left(-\frac{2p}{e^2}\right) + 2}\right)^{1/\xi}, \tag{6}$$

where $W(.)$ is Lambert function.

Various moments of PMXLD

Raw moments

Let μ_k' represent the K^{th} raw moment of PMXLD and mathematically it can be computed as

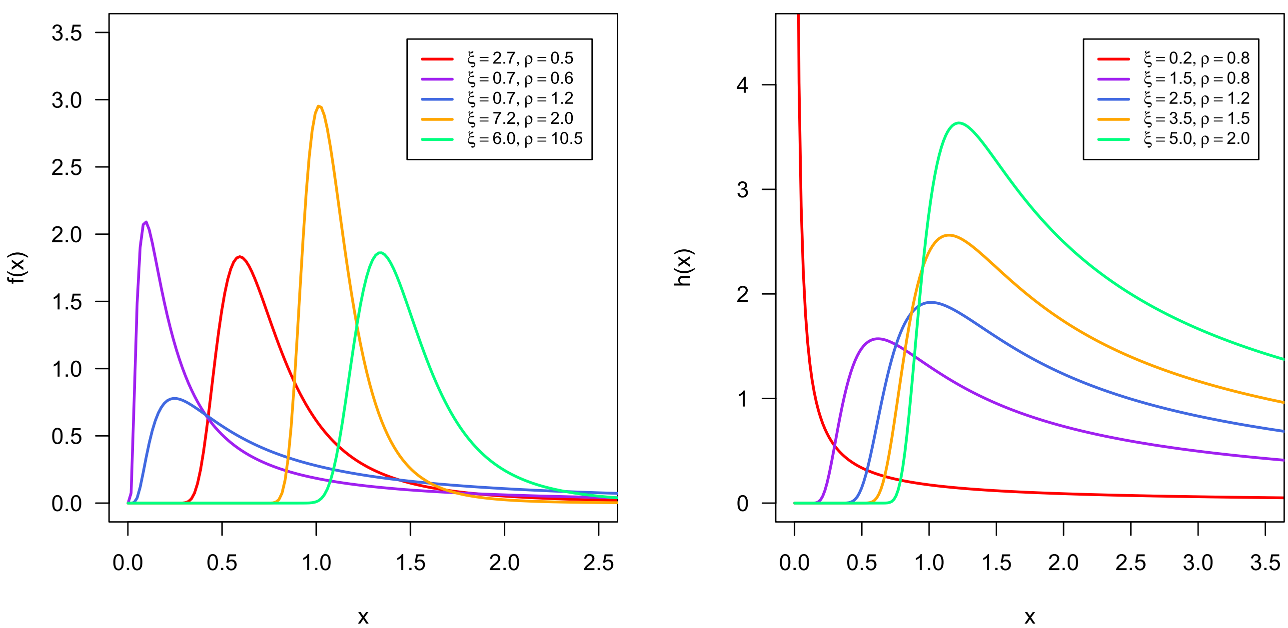

Figure 1. Various shapes of PDF and HRF of PMXLD.

$$\mu'_k = \int_0^\infty \frac{1}{2}\xi\rho x^{k-2\xi-1}(\rho + x^\xi)e^{-\rho x^{-\xi}}dx
= \frac{\rho^{k/\xi}\left(\xi(\rho+1)^2 - k\right)\Gamma\left(1-\frac{k}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > k. \tag{7}$$

where $\Gamma(.,.)$ is the upper incomplete gamma function.

Mean and Variance of PMXLD

Using the Equation defined in (7) mean and variance can be presented as follows:

$$\mu = \frac{\rho^{1/\xi}\left(\xi(\rho+1)^2 - 1\right)\Gamma\left(1-\frac{1}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > 1.$$

$$V(X) = \frac{\rho^{2/\xi}\left(\xi(\rho+1)^2 - 2\right)\Gamma\left(1-\frac{2}{\xi}\right)}{\xi(\rho+1)^2} - \left[\frac{\rho^{1/\xi}\left(\xi(\rho+1)^2 - 1\right)\Gamma\left(1-\frac{1}{\xi}\right)}{\xi(\rho+1)^2}\right]^2, \ \xi > 2.$$

The first four raw moments can be obtained by putting $k = 1, 2, 3 \ and \ , 4$ in Equation (7) and expressed as follows

$$\mu'_1 = \frac{\rho^{1/\xi}\left(\xi(\rho+1)^2 - 1\right)\Gamma\left(1-\frac{1}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > 1.$$

$$\mu'_2 = \frac{\rho^{2/\xi}\left(\xi(\rho+1)^2 - 3\right)\Gamma\left(1-\frac{2}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > 2.$$

$$\mu'_3 = \frac{\rho^{3/\xi}\left(\xi(\rho+1)^2 - 3\right)\Gamma\left(1-\frac{3}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > 3.$$

$$\mu'_4 = \frac{\rho^{4/\xi}\left(\xi(\rho+1)^2 - 4\right)\Gamma\left(1-\frac{4}{\xi}\right)}{\xi(\rho+1)^2}, \ \xi > 4.$$

To compute central moments we can employ the following mathematical relation $\mu_1 = \mu'_1, \mu_2 = \mu'_2 - (\mu'_1)^2$, $\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$, and $\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$ and compute $skewness = \frac{\mu_3^2}{\mu_2^3}$ and $kurtosis = \frac{\mu_4}{\mu_2^2}$. The numerical values of the first four central moments, skewness and kurtosis are presented in Table 1.

Central moments (CM)

CM are the moments taken from mean $\bar{X}$ and h^{th} CM can also be computed as

ρ	ξ	μ_1	μ_2	μ_3	μ_4	skewness	kurtosis
1.0	4.5	1.1240	0.1594	0.2747	2.5887	18.6335	101.8702
1.0	5.0	1.1060	0.1170	0.1443	0.6793	12.9963	49.6267
1.0	5.5	1.0924	0.0897	0.0851	0.2678	10.0293	33.2713
1.0	6.0	1.0818	0.0711	0.0544	0.1288	8.2261	25.4955
1.3	4.5	1.2169	0.1896	0.3535	3.6106	18.3277	100.4245
1.3	5.0	1.1879	0.1370	0.1812	0.9179	12.7683	48.9065
1.3	5.5	1.1656	0.1037	0.1048	0.3526	9.843	32.7803
1.3	6.0	1.1480	0.0813	0.0658	0.1660	8.0655	25.1142
1.7	4.5	1.2916	0.2150	0.4251	4.6119	18.1792	99.7420
1.7	5.0	1.2532	0.1535	0.2140	1.1449	12.6568	48.5649
1.7	5.5	1.2237	0.1151	0.1220	0.4314	9.7514	32.5465
1.7	6.0	1.2003	0.0895	0.0757	0.1998	7.9863	24.9322
2.0	4.5	1.3541	0.2372	0.4914	5.5907	18.1000	99.3845
2.0	5.0	1.3076	0.1678	0.2439	1.3619	12.5971	48.3855
2.0	5.5	1.2719	0.1248	0.1374	0.5053	9.7022	32.4234
2.0	6.0	1.2436	0.0965	0.0844	0.2311	7.9436	24.8362

Table 1. First four central moments, skewness and kurtosis of PMXLD.

$$
\begin{aligned}
\mu_h &= \sum_{j=0}^{h}(-1)^j\binom{h}{j}(\mu_1')^j\mu_{h-j}' \\
&= \sum_{j=0}^{h}(-1)^j\binom{h}{j}\left[\frac{\rho^{1/\xi}\left(\xi(\rho+1)^2-1\right)\Gamma\left(1-\frac{1}{\xi}\right)}{\xi(\rho+1)^2}\right]^j\frac{\rho^{(h-k)/\xi}\left(\xi(\rho+1)^2-h+k\right)\Gamma\left(1-\frac{h-k}{\xi}\right)}{\xi(\rho+1)^2}.
\end{aligned}
$$

hence we can compute the first four central moments and skewness and kurtosis as for h=1, 2, 3, 4 in μ_h compute $\mu_1, \mu_2, \mu_3, \mu_4$ and also compute skewness ($\beta_1 = \frac{\mu_3^2}{\mu_2^3}$) and kurtosis ($\beta_2 = \frac{\mu_4}{\mu_2^2}$)

Mean residual life (MRL) function

Let $Z(t)$ denote the MRL time function of an item or device, and mathematically, it can be presented as

$$
\begin{aligned}
Z(t) &= \frac{\int_t^{\infty} xf(x)dx}{1-F(t)} - t \\
&= \frac{1}{2}\rho^{\frac{1}{\xi}}\left[\Gamma(2-\frac{1}{\xi},\ \rho x) + \Gamma(1-\frac{1}{\xi},\ \rho x)\right].
\end{aligned}
$$

Estimation

In this estimation section, we have used twelve estimation techniques to estimate the parameter of the model PMXLD, and their detailed information is provided in the simulation section.

1. Method of maximum likelihood (MLE)
   The MLE method is one of the good methods for estimating model parameters. It provides estimates that are asymptotically efficient and often have desirable statistical properties where we optimized the log-likelihood function $\log L$. Here L represents the likelihood function of the PDF of PMXLD.

$$
\log L = \log(\frac{1}{2}\xi\rho) - (2\xi+1)\sum_{i=0}^{n}\log(x_i) + \rho\left(-x^{-\xi}\right) + \log(\left(\rho + x^{\xi}\right)).
$$

2. Methods of Anderson_Darling (ADE)
   The ADE is selected for its sensitivity to deviations from the assumed distribution. For the PMXLD distribution, the ADE can be obtained by optimizing the following function with respect to the parameters of the model.

$$
A(x_i) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}) + \log S(x_{n-i-1:n})].
$$

3. Method of Cramer_von_Mises (CVME)
   In parameter estimation using the CVME method, the goal is to find the parameter values of a theoretical distribution that minimizes the Cramér-von Mises statistic. It is chosen for its sensitivity to differences in the cumulative distribution function (CDF) and its ability to detect departure from the assumed distribution. This involves optimizing the parameters to achieve the best possible fit between the observed data and the theoretical distribution.

$$
C(x_i) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2.
$$

4. Method of Maximum Product of Spacings (MPS)
   The MPS is a nonparametric estimation method used in statistics to estimate the parameters of the probability model and is suitable when the underlying distribution is not known . The Estimation Method of MPS involves maximizing the product of the observed spacings between ordered data points. This method is particularly employed when estimating parameters for continuous probability distributions.

$$
\begin{aligned}
\delta(x_i) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log \nu_i(x_i), \\
\nu_i(x_i) &= F(x_i) - F(x_{i-1}).
\end{aligned}
$$

5. Methods of least squares (LSE)
   The Method of LSE is a statistical technique used for estimating the parameters of a mathematical model by minimizing the sum of the squares of the differences between observed and predicted values and is selected for its simplicity and ease of interpretation.

$$V(x_i)=\sum_{i=1}^{n}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2.$$

6. Methods of Right_tail Anderson_Darling
   The Estimation Method of Right-Tail Anderson-Darling involves determining the parameters of a statistical distribution, This method is chosen to specifically assess the fit of the model in the right tail of the distribution typically a CDF, by optimizing the right-tail version of the Anderson-Darling statistic.

$$R(x_i)=\frac{n}{2}-2\sum_{i=1}^{n}F(x_{i:n})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{i:n}).$$

7. Methods of weighted least squares (WLSE)
   WLSE is a statistical method used for estimating the parameters of a probability model where varying weights are assigned to each observation. WLSE is used when certain data points are considered more important or reliable than others, allowing for a more nuanced estimation.

$$W(x_i)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n})-\frac{i}{n+1}\right]^2.$$

8. Methods of left_tail Anderson_Darling
   The left-tail Anderson-Darling estimation methods refer to approaches used to estimate the parameters of a distribution, particularly focusing on the left tail, based on the Anderson-Darling statistic. The Anderson-Darling statistic is a measure of how well a given distribution fits a set of data. By minimizing the CDF of PMXLD for ordered random variables, we can estimate the parameters of the suggested model. It complements the right-tail method by examining deviations in the opposite tail, providing a more comprehensive evaluation of model fit

$$L(x_i)=-\frac{3}{2}n+2\sum_{i=1}^{n}F(x_{i:n})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n}).$$

9. Minimum spacing absolute distance
   For its simplicity and ability to capture the spacing between data points, which can be informative for parameter estimation. If we have an ordered sample denoted as $X_{1:n},\ldots,X_{n:n}$ from PMXLD, the minimum spacing absolute distance estimates can be derived by minimizing the following function

$$\zeta(x_i)=\sum_{i=1}^{n+1}|\nu_i-\frac{1}{n+1}|.$$

10. Minimum spacing absolute-log distance
   This method extends the minimum spacing approach by considering the logarithm of the distances, which can be useful for handling data with varying scales.Using this estimation method, we can estimate the model parameters by minimizing the following expression

$$\Upsilon_2(x_i)=\sum_{i=1}^{n+1}|\log\nu_i-\log\frac{1}{n+1}|.$$

11. Anderson Darling Method-2LT (Left-Tail Second-Order)
   This method is chosen to explore second-order effects in the left tail of the distribution, providing a more detailed analysis. The parameters of the PMXLD distribution can be derived for Left-Tail Second-Order estimation using the Anderson-Darling Method-2LT, achieved by minimizing the following function.

$$LTS=2\sum_{i=1}^{n}\log F(x_i)+\frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{F(x_i)}.$$

12. Kolmogorov Method
   For its simplicity and effectiveness in assessing the maximum deviations between empirical and theoretical distributions. The Kolmogorov estimators for the parameters of the PMXLD distribution can be obtained through the optimization of the following expression.

$$KoLM=\underset{1\le i\le n}{MAX}\left[\frac{i}{n}-F(x_i),F(x_i)-\frac{i-1}{n}\right].$$

Simulation

In this segment of our paper, our objective is to evaluate the effectiveness of different estimation methods in determining the distribution parameters associated with the PMXLD. To accomplish this, we employ a simulation-based approach, systematically examining the performance of these estimation techniques. Our analysis encompasses a range of sample sizes denoted as $n=30, 80, 130, 180, 220, 275, 350, 500$, and we investigate various parameter configurations. To facilitate our assessment, we have generated a total of 1,000 random samples drawn from the PMXLD. Subsequently, we proceed to assess the performance of these estimation methods by calculating three fundamental statistical metrics: the Average Absolute Bias (BIAS), the Mean Square Error (MSE), and the Mean Relative Estimate (MRE). These results are executed using the statistical software package R.

Through the systematic variation of both sample size and parameter values, we can provide valuable insights into the performance of these methods across various scenarios. These insights hold great significance for researchers and practitioners seeking dependable and robust estimation techniques in the domain of PIXL distribution modeling. The expressions for the Average Absolute Bias (BIAS), Mean Square Error (MSE), and Mean Relative Estimate (MRE) are as follows:

$$BIAS = \frac{1}{B}\sum_{i=1}^{B}|\widehat{\Upsilon}_i - \Upsilon_i|, \quad MSE = \frac{1}{B}\sum_{i=1}^{B}(\widehat{\Upsilon}_i - \Upsilon_i)^2, \quad MRE = \frac{1}{B}\sum_{i=1}^{B}|\widehat{\Upsilon}_i - \Upsilon_i|/\Upsilon_i,$$

where Υ_i can represent ρ and ζ.

The Simulation algorithm can described as:

1. Generate standards uniform random numbers.
2. Use the quantile function on the generated uniform random numbers to generate PMXLD random numbers.
3. Derive the estimators for all methods.
4. Repeat steps 1-3 1000 times to create a Monte Carlo simulation with 1000 replications.
5. Compute BIAS, MSE, and MRE.

Tables 2, 3, 4, 5, 6 and 7 present the outcomes of our simulation. These tables contain the computed values of three fundamental performance metrics, which were determined using twelve distinct estimation methods. Furthermore, graphical representations of these values can be found in Figs. 2, 3, 4, 5, 6 and 7. It is important to highlight that the data presented in both the tables and figures consistently demonstrate relatively small values across various parameter combinations. This observation underscores the tendency of the estimators to yield results with minimal deviations from the true values. Significantly, this phenomenon reflects the property of consistency in the estimators, wherein the magnitudes of these metrics decrease as the sample size increases. This pattern is consistently observed across all experimental scenarios. Based on the information provided in Table 8, we can deduce that the most effective methods for estimating our proposed model parameters are EM_1 and EM_4, as they consistently attain a high rank in the simulation tables.

Application

In this context, we present two real datasets to illustrate the practical application and adaptability of the proposed PMXLD distribution. We utilize these same datasets for a comparative analysis between the PMXLD distribution and other well-known distributions renowned for their ability to fit diverse datasets. The probability density functions (PDFs) for these compared distributions are as follows:

- the modified XLindley distribution (MXLD). Its PDF is

$$f(x) = \frac{\rho e^{-\frac{\rho}{x}}(\rho + x)}{2x^3}.$$

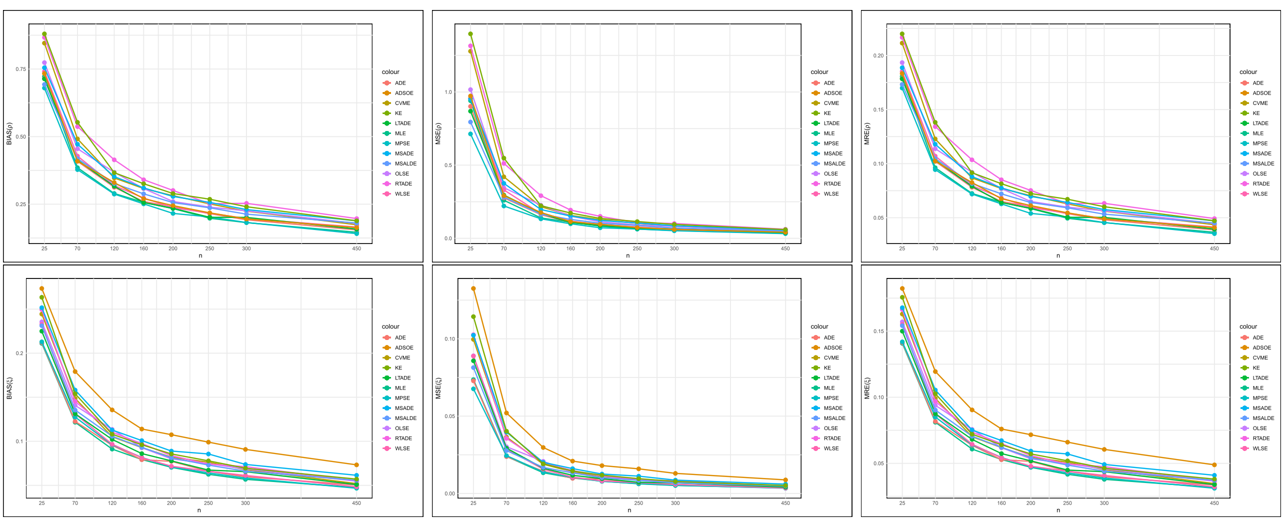

Figure 2. Plots for simulation measures presented in Table 2.

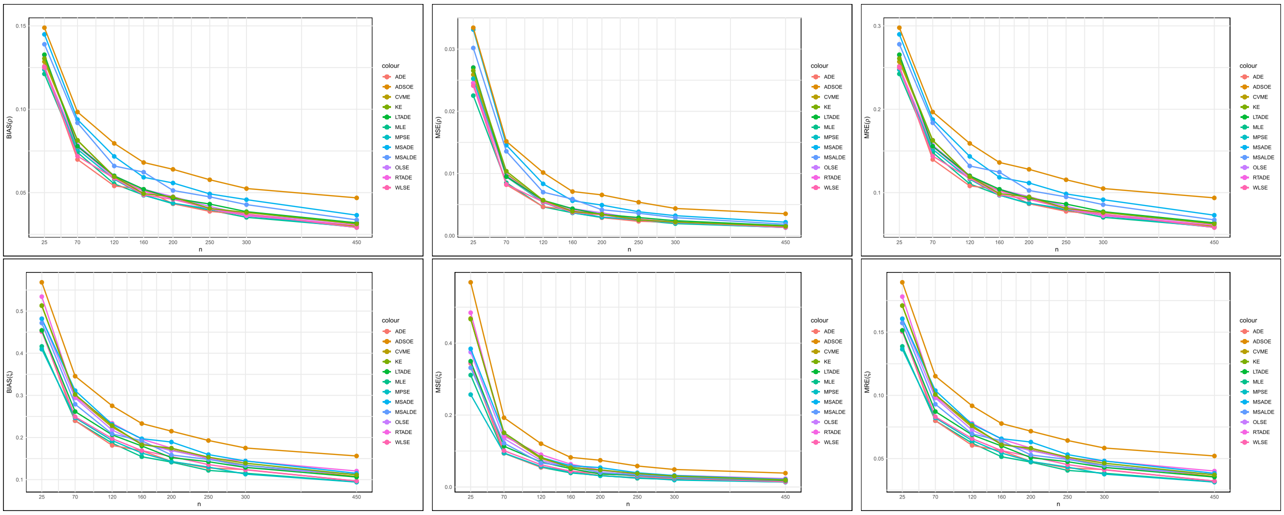

Figure 3. Plots for simulation measures presented in Table 3.

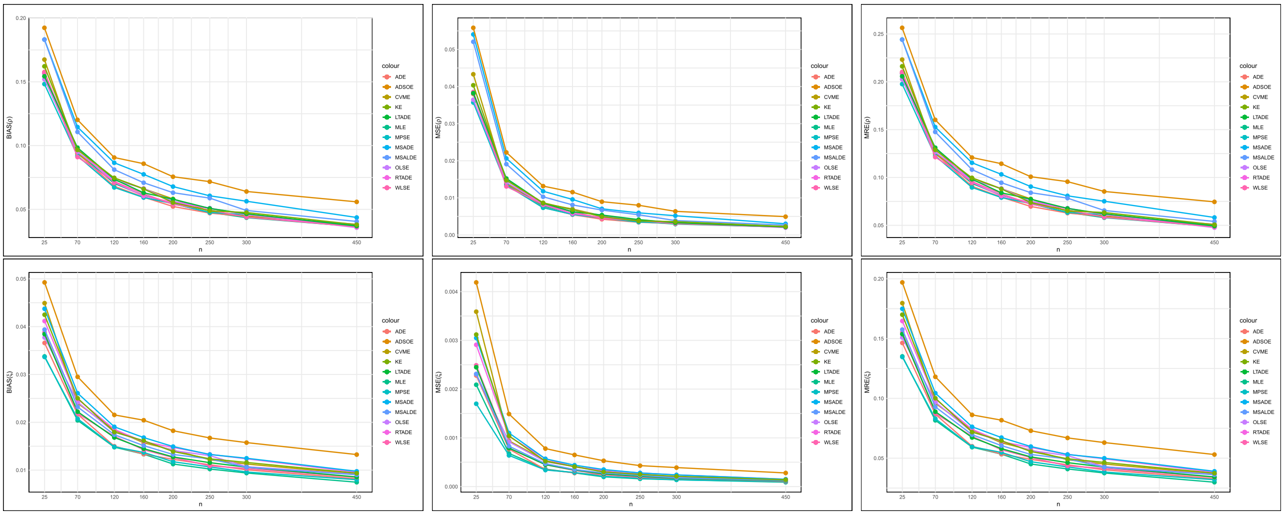

Figure 4. Plots for simulation measures presented in Table 4.

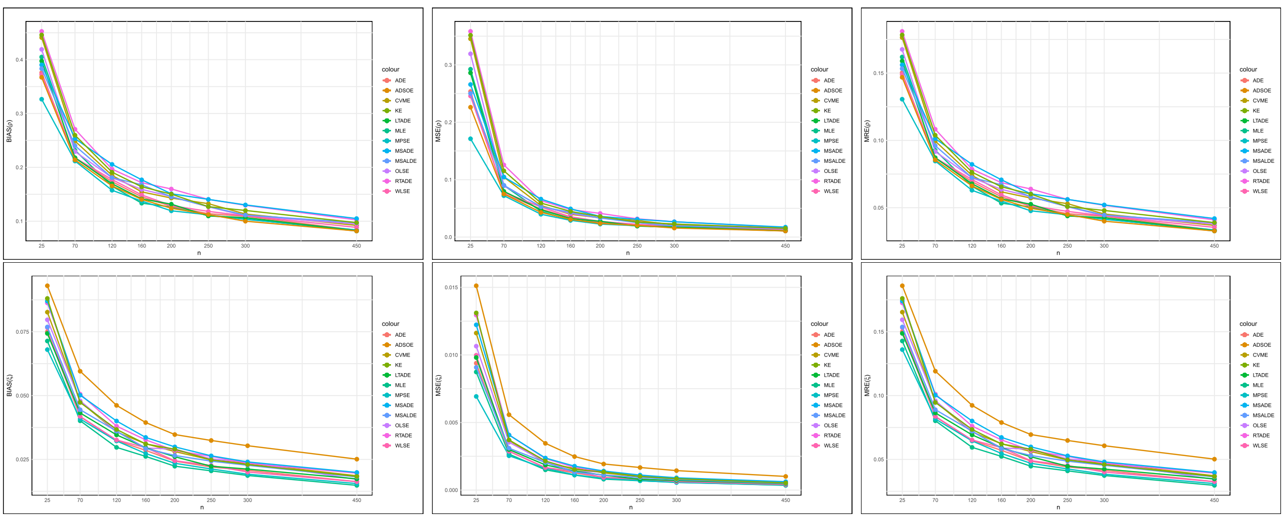

Figure 5. Plots for simulation measures presented in Table 5.

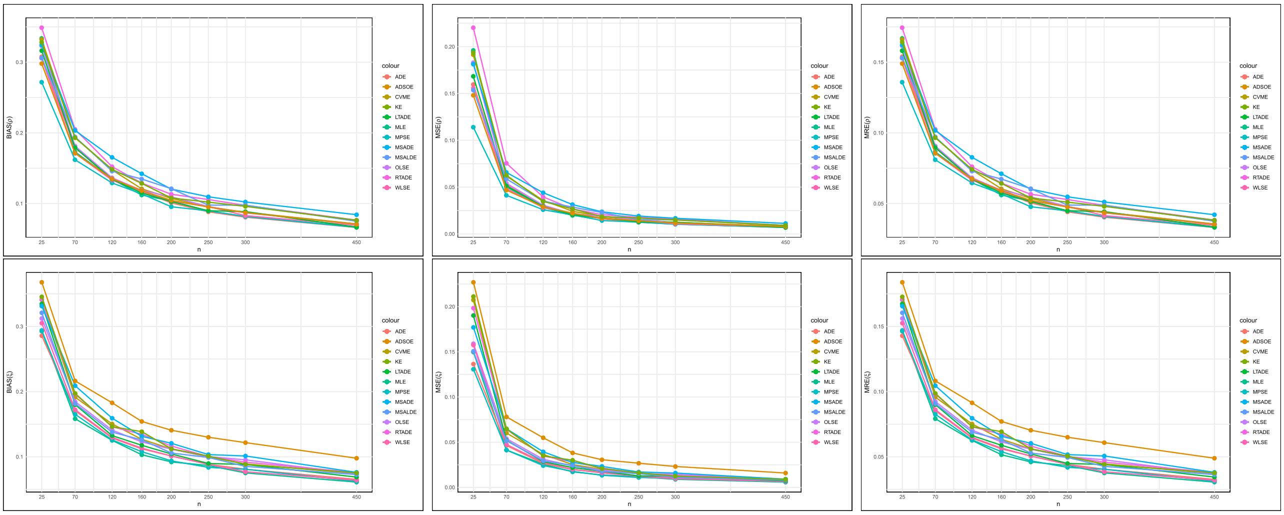

Figure 6. Plots for simulation measures presented in Table 6.

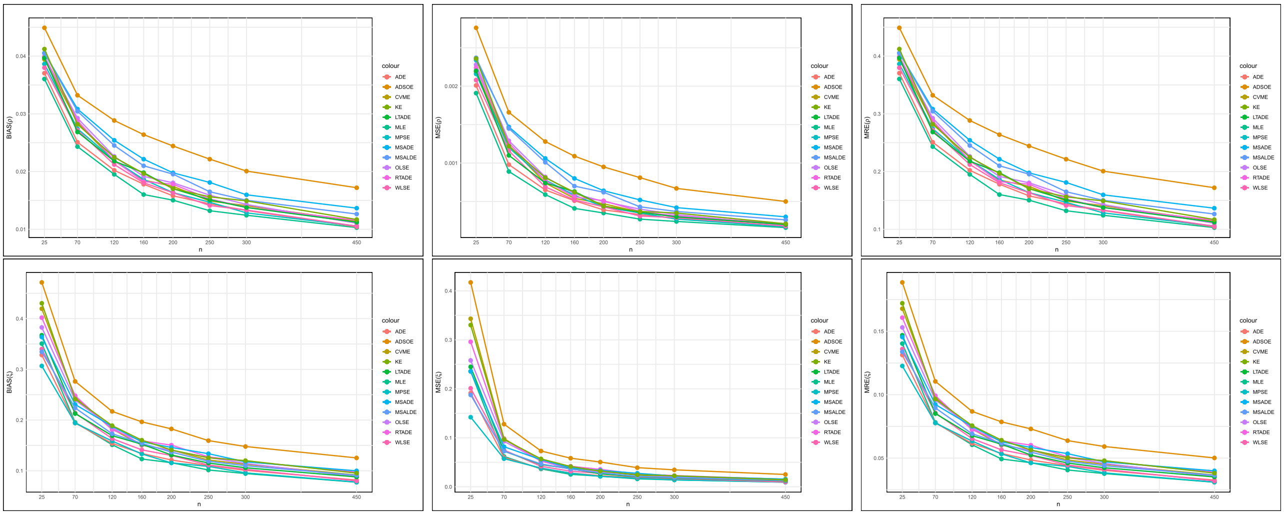

Figure 7. Plots for simulation measures presented in Table 7.

- the quasi Lindley distribution[14]

$$f(x) = \frac{\theta(\alpha + x\theta)}{\alpha + 1} e^{-\theta x}$$

- the Weibull distribution[15]

$$f(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} e^{-(x/\theta)^{\alpha}}$$

- the xgamma distribution[16]

$$f(x) = \frac{\theta^2 \cdot \left(\alpha\theta x^2 + 2\right) e^{-\theta x}}{2(\theta + \alpha)}$$

First data set: The initial dataset showcases right-skewed data,[17] illustrating the exceedances of flood peaks (measured in m^3/s) recorded at the Wheaton River near Carcross in Yukon Territory, Canada. This dataset encompasses 72 recorded exceedances spanning the years 1958 to 1984, with values rounded to one decimal place. Please refer to Table 9 for the specific data points.

Second data set: This dataset pertains to the failure times of 24 mechanical components, as documented in[18]. The data values are itemized in Table 10.

Table 11 includes information on the datasets and characteristics. For a visual representation, refer to Fig. 8, which portrays the TTT (time-to-event), PP (probability plot), and box plots for the first dataset. Similarly, Fig. 9 offers a corresponding visualization for the second dataset.

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	0.72256$^{\{5\}}$	0.71542$^{\{4\}}$	0.84596$^{\{10\}}$	0.67989$^{\{1\}}$	0.77395$^{\{9\}}$	0.86693$^{\{11\}}$	0.74106$^{\{7\}}$	0.71379$^{\{3\}}$	0.75501$^{\{8\}}$	0.69304$^{\{2\}}$	0.73348$^{\{6\}}$	0.88019$^{\{12\}}$
		$\hat{\xi}$	0.21121$^{\{1\}}$	0.21209$^{\{2\}}$	0.24447$^{\{8\}}$	0.21293$^{\{3\}}$	0.23559$^{\{7\}}$	0.24955$^{\{9\}}$	0.23266$^{\{6\}}$	0.22503$^{\{4\}}$	0.2517$^{\{10\}}$	0.23129$^{\{5\}}$	0.2735$^{\{12\}}$	0.26348$^{\{11\}}$
	MSE	$\hat{\rho}$	0.94057$^{\{5\}}$	0.90183$^{\{4\}}$	1.27864$^{\{10\}}$	0.71364$^{\{1\}}$	1.01665$^{\{9\}}$	1.31544$^{\{11\}}$	0.95787$^{\{7\}}$	0.86905$^{\{3\}}$	0.95264$^{\{6\}}$	0.79515$^{\{2\}}$	0.97276$^{\{8\}}$	1.39747$^{\{12\}}$
		$\hat{\xi}$	0.07363$^{\{3\}}$	0.07283$^{\{2\}}$	0.09964$^{\{8\}}$	0.06766$^{\{1\}}$	0.08904$^{\{7\}}$	0.10268$^{\{10\}}$	0.08885$^{\{6\}}$	0.08581$^{\{5\}}$	0.10234$^{\{9\}}$	0.08143$^{\{4\}}$	0.13257$^{\{12\}}$	0.11437$^{\{11\}}$
	MRE	$\hat{\rho}$	0.18064$^{\{5\}}$	0.17886$^{\{4\}}$	0.21149$^{\{11\}}$	0.16997$^{\{1\}}$	0.19349$^{\{10\}}$	0.21673$^{\{12\}}$	0.18527$^{\{8\}}$	0.17845$^{\{3\}}$	0.18875$^{\{9\}}$	0.17326$^{\{2\}}$	0.18337$^{\{6.5\}}$	0.18337$^{\{6.5\}}$
		$\hat{\xi}$	0.14081$^{\{1\}}$	0.1414$^{\{2\}}$	0.16298$^{\{8\}}$	0.14196$^{\{3\}}$	0.15706$^{\{7\}}$	0.16636$^{\{9\}}$	0.15511$^{\{6\}}$	0.15002$^{\{4\}}$	0.1678$^{\{10\}}$	0.15419$^{\{5\}}$	4.34039$^{\{12\}}$	0.22005$^{\{11\}}$
	$\sum$ *Ranks*		20$^{\{3.5\}}$	18$^{\{2\}}$	55$^{\{9\}}$	10$^{\{1\}}$	49$^{\{7\}}$	62$^{\{11\}}$	40$^{\{6\}}$	22$^{\{5\}}$	52$^{\{8\}}$	20$^{\{3.5\}}$	56.5$^{\{10\}}$	63.5$^{\{12\}}$
70	BIAS	$\hat{\rho}$	0.38509$^{\{2\}}$	0.40874$^{\{3\}}$	0.49156$^{\{10\}}$	0.37804$^{\{1\}}$	0.45528$^{\{8\}}$	0.53722$^{\{11\}}$	0.42814$^{\{7\}}$	0.41302$^{\{5\}}$	0.47199$^{\{9\}}$	0.41862$^{\{6\}}$	0.40929$^{\{4\}}$	0.5529$^{\{12\}}$
		$\hat{\xi}$	0.12164$^{\{1\}}$	0.12261$^{\{2\}}$	0.14864$^{\{9\}}$	0.12755$^{\{3\}}$	0.14079$^{\{7\}}$	0.1454$^{\{8\}}$	0.1312$^{\{4\}}$	0.13125$^{\{5\}}$	0.15821$^{\{11\}}$	0.1355$^{\{6\}}$	0.17918$^{\{12\}}$	0.15395$^{\{10\}}$
	MSE	$\hat{\rho}$	0.26002$^{\{2\}}$	0.27658$^{\{3\}}$	0.42047$^{\{10\}}$	0.22113$^{\{1\}}$	0.3459$^{\{8\}}$	0.51202$^{\{11\}}$	0.33105$^{\{7\}}$	0.29581$^{\{6\}}$	0.37564$^{\{9\}}$	0.28158$^{\{4\}}$	0.2916$^{\{5\}}$	0.54828$^{\{12\}}$
		$\hat{\xi}$	0.02395$^{\{2\}}$	0.02385$^{\{1\}}$	0.03653$^{\{9\}}$	0.02448$^{\{3\}}$	0.03023$^{\{7\}}$	0.03563$^{\{8\}}$	0.02873$^{\{5\}}$	0.02932$^{\{6\}}$	0.0397$^{\{10\}}$	0.02776$^{\{4\}}$	0.05206$^{\{12\}}$	0.04028$^{\{11\}}$
	MRE	$\hat{\rho}$	0.09627$^{\{2\}}$	0.10218$^{\{3\}}$	0.12289$^{\{11\}}$	0.09451$^{\{1\}}$	0.11382$^{\{9\}}$	0.1343$^{\{12\}}$	0.10704$^{\{8\}}$	0.10326$^{\{6\}}$	0.118$^{\{10\}}$	0.10466$^{\{7\}}$	0.10232$^{\{4.5\}}$	0.10232$^{\{4.5\}}$
		$\hat{\xi}$	0.08109$^{\{1\}}$	0.08174$^{\{2\}}$	0.0991$^{\{9\}}$	0.08503$^{\{3\}}$	0.09386$^{\{7\}}$	0.09693$^{\{8\}}$	0.08747$^{\{4\}}$	0.0875$^{\{5\}}$	0.10547$^{\{10\}}$	0.09033$^{\{6\}}$	4.206$^{\{12\}}$	0.13822$^{\{11\}}$
	$\sum$ *Ranks*		10$^{\{1\}}$	14$^{\{3\}}$	58$^{\{9.5\}}$	12$^{\{2\}}$	46$^{\{7\}}$	58$^{\{9.5\}}$	35$^{\{6\}}$	33$^{\{4.5\}}$	59$^{\{11\}}$	33$^{\{4.5\}}$	49.5$^{\{8\}}$	60.5$^{\{12\}}$
120	BIAS	$\hat{\rho}$	0.28974$^{\{2\}}$	0.3128$^{\{3\}}$	0.34722$^{\{8\}}$	0.28753$^{\{1\}}$	0.36823$^{\{11\}}$	0.41428$^{\{12\}}$	0.31886$^{\{5\}}$	0.31636$^{\{4\}}$	0.35206$^{\{9\}}$	0.3264$^{\{6\}}$	0.33019$^{\{7\}}$	0.36652$^{\{10\}}$
		$\hat{\xi}$	0.09117$^{\{1\}}$	0.09545$^{\{2\}}$	0.10516$^{\{6\}}$	0.09598$^{\{3\}}$	0.11355$^{\{11\}}$	0.11023$^{\{9\}}$	0.09668$^{\{4\}}$	0.10215$^{\{5\}}$	0.11279$^{\{10\}}$	0.10572$^{\{7\}}$	0.13565$^{\{12\}}$	0.10803$^{\{8\}}$
	MSE	$\hat{\rho}$	0.13929$^{\{2\}}$	0.16171$^{\{3\}}$	0.21593$^{\{9\}}$	0.13309$^{\{1\}}$	0.22634$^{\{11\}}$	0.29154$^{\{12\}}$	0.16809$^{\{5\}}$	0.16924$^{\{6\}}$	0.19661$^{\{8\}}$	0.16517$^{\{4\}}$	0.17636$^{\{7\}}$	0.2227$^{\{10\}}$
		$\hat{\xi}$	0.01341$^{\{1\}}$	0.0144$^{\{3\}}$	0.01893$^{\{7\}}$	0.01387$^{\{2\}}$	0.0208$^{\{11\}}$	0.01968$^{\{9\}}$	0.01571$^{\{4\}}$	0.01631$^{\{5\}}$	0.02012$^{\{10\}}$	0.01685$^{\{6\}}$	0.02974$^{\{12\}}$	0.01917$^{\{8\}}$
	MRE	$\hat{\rho}$	0.07243$^{\{2\}}$	0.0782$^{\{3\}}$	0.08681$^{\{9\}}$	0.07188$^{\{1\}}$	0.09206$^{\{11\}}$	0.10357$^{\{12\}}$	0.07971$^{\{5\}}$	0.07909$^{\{4\}}$	0.08801$^{\{10\}}$	0.0816$^{\{6\}}$	0.08255$^{\{7.5\}}$	0.08255$^{\{7.5\}}$
		$\hat{\xi}$	0.06078$^{\{1\}}$	0.06364$^{\{2\}}$	0.07011$^{\{6\}}$	0.06399$^{\{3\}}$	0.0757$^{\{10\}}$	0.07348$^{\{8\}}$	0.06445$^{\{4\}}$	0.0681$^{\{5\}}$	0.07519$^{\{9\}}$	0.07048$^{\{7\}}$	4.06459$^{\{12\}}$	0.09163$^{\{11\}}$
	$\sum$ *Ranks*		9$^{\{1\}}$	16$^{\{3\}}$	45$^{\{7\}}$	11$^{\{2\}}$	65$^{\{12\}}$	62$^{\{11\}}$	27$^{\{4\}}$	29$^{\{5\}}$	56$^{\{9\}}$	36$^{\{6\}}$	57.5$^{\{10\}}$	54.5$^{\{8\}}$
160	BIAS	$\hat{\rho}$	0.25684$^{\{3\}}$	0.25986$^{\{4\}}$	0.30799$^{\{8\}}$	0.25131$^{\{1\}}$	0.308$^{\{9\}}$	0.34061$^{\{12\}}$	0.27312$^{\{6\}}$	0.25617$^{\{2\}}$	0.3103$^{\{10\}}$	0.28834$^{\{7\}}$	0.27048$^{\{5\}}$	0.32585$^{\{11\}}$
		$\hat{\xi}$	0.07934$^{\{2\}}$	0.07913$^{\{1\}}$	0.09285$^{\{7\}}$	0.08173$^{\{4\}}$	0.09276$^{\{6\}}$	0.09743$^{\{10\}}$	0.08137$^{\{3\}}$	0.08605$^{\{5\}}$	0.10082$^{\{11\}}$	0.09317$^{\{8\}}$	0.11397$^{\{12\}}$	0.09606$^{\{9\}}$
	MSE	$\hat{\rho}$	0.11008$^{\{4\}}$	0.10854$^{\{3\}}$	0.15656$^{\{10\}}$	0.09934$^{\{1\}}$	0.15164$^{\{8\}}$	0.19317$^{\{12\}}$	0.11775$^{\{6\}}$	0.10773$^{\{2\}}$	0.15207$^{\{9\}}$	0.12917$^{\{7\}}$	0.11445$^{\{5\}}$	0.1727$^{\{11\}}$
		$\hat{\xi}$	0.01002$^{\{2\}}$	0.00979$^{\{1\}}$	0.01396$^{\{8\}}$	0.01041$^{\{3\}}$	0.01354$^{\{7\}}$	0.01484$^{\{10\}}$	0.01058$^{\{4\}}$	0.01174$^{\{5\}}$	0.01608$^{\{11\}}$	0.0133$^{\{6\}}$	0.02095$^{\{12\}}$	0.01449$^{\{9\}}$
	MRE	$\hat{\rho}$	0.06421$^{\{3\}}$	0.06497$^{\{4\}}$	0.077$^{\{9.5\}}$	0.06283$^{\{1\}}$	0.077$^{\{9.5\}}$	0.08515$^{\{12\}}$	0.06828$^{\{7\}}$	0.06404$^{\{2\}}$	0.07758$^{\{11\}}$	0.07209$^{\{8\}}$	0.06762$^{\{5.5\}}$	0.06762$^{\{5.5\}}$
		$\hat{\xi}$	0.05289$^{\{2\}}$	0.05275$^{\{1\}}$	0.0619$^{\{7\}}$	0.05449$^{\{4\}}$	0.06184$^{\{6\}}$	0.06495$^{\{9\}}$	0.05424$^{\{3\}}$	0.05737$^{\{5\}}$	0.06722$^{\{10\}}$	0.06211$^{\{8\}}$	4.06518$^{\{12\}}$	0.08146$^{\{11\}}$
	$\sum$ *Ranks*		16$^{\{3\}}$	14$^{\{1.5\}}$	49.5$^{\{8\}}$	14$^{\{1.5\}}$	45.5$^{\{7\}}$	65$^{\{12\}}$	29$^{\{5\}}$	21$^{\{4\}}$	62$^{\{11\}}$	44$^{\{6\}}$	51.5$^{\{9\}}$	56.5$^{\{10\}}$
200	BIAS	$\hat{\rho}$	0.23391$^{\{2\}}$	0.2404$^{\{5\}}$	0.28155$^{\{10\}}$	0.21603$^{\{1\}}$	0.26007$^{\{8\}}$	0.30091$^{\{12\}}$	0.23713$^{\{4\}}$	0.2348$^{\{3\}}$	0.2787$^{\{9\}}$	0.25608$^{\{7\}}$	0.24485$^{\{6\}}$	0.28952$^{\{11\}}$
		$\hat{\xi}$	0.07046$^{\{1\}}$	0.07723$^{\{4\}}$	0.08199$^{\{8\}}$	0.07092$^{\{2\}}$	0.08009$^{\{6\}}$	0.08375$^{\{9\}}$	0.07172$^{\{3\}}$	0.07777$^{\{5\}}$	0.08893$^{\{11\}}$	0.08128$^{\{7\}}$	0.1075$^{\{12\}}$	0.08565$^{\{10\}}$
	MSE	$\hat{\rho}$	0.08349$^{\{2\}}$	0.09393$^{\{5\}}$	0.12734$^{\{10\}}$	0.07215$^{\{1\}}$	0.11266$^{\{8\}}$	0.15037$^{\{12\}}$	0.09347$^{\{4\}}$	0.08655$^{\{3\}}$	0.12229$^{\{9\}}$	0.10309$^{\{7\}}$	0.09579$^{\{6\}}$	0.13605$^{\{11\}}$
		$\hat{\xi}$	0.00796$^{\{2\}}$	0.00931$^{\{4\}}$	0.01097$^{\{8\}}$	0.00784$^{\{1\}}$	0.01009$^{\{6\}}$	0.01132$^{\{9\}}$	0.00815$^{\{3\}}$	0.00957$^{\{5\}}$	0.01258$^{\{11\}}$	0.01022$^{\{7\}}$	0.01803$^{\{12\}}$	0.01183$^{\{10\}}$
	MRE	$\hat{\rho}$	0.05848$^{\{2\}}$	0.0601$^{\{5\}}$	0.07039$^{\{11\}}$	0.05401$^{\{1\}}$	0.06502$^{\{9\}}$	0.07523$^{\{12\}}$	0.05928$^{\{4\}}$	0.0587$^{\{3\}}$	0.06967$^{\{10\}}$	0.06402$^{\{8\}}$	0.06121$^{\{6.5\}}$	0.06121$^{\{6.5\}}$
		$\hat{\xi}$	0.04697$^{\{1\}}$	0.05148$^{\{4\}}$	0.05466$^{\{8\}}$	0.04728$^{\{2\}}$	0.0534$^{\{6\}}$	0.05584$^{\{9\}}$	0.04781$^{\{3\}}$	0.05185$^{\{5\}}$	0.05929$^{\{10\}}$	0.05419$^{\{7\}}$	4.06863$^{\{12\}}$	0.07238$^{\{11\}}$
	$\sum$ *Ranks*		10$^{\{2\}}$	27$^{\{5\}}$	55$^{\{9\}}$	8$^{\{1\}}$	43$^{\{6.5\}}$	63$^{\{12\}}$	21$^{\{3\}}$	24$^{\{4\}}$	60$^{\{11\}}$	43$^{\{6.5\}}$	54.5$^{\{8\}}$	59.5$^{\{10\}}$
250	BIAS	$\hat{\rho}$	0.19803$^{\{1\}}$	0.21518$^{\{4\}}$	0.25114$^{\{9\}}$	0.20291$^{\{3\}}$	0.23918$^{\{8\}}$	0.25191$^{\{10\}}$	0.21692$^{\{5\}}$	0.20157$^{\{2\}}$	0.25664$^{\{11\}}$	0.23667$^{\{7\}}$	0.21805$^{\{6\}}$	0.26848$^{\{12\}}$
		$\hat{\xi}$	0.06254$^{\{1\}}$	0.066$^{\{3\}}$	0.07631$^{\{9\}}$	0.06413$^{\{2\}}$	0.07459$^{\{8\}}$	0.07407$^{\{7\}}$	0.06601$^{\{4\}}$	0.06732$^{\{5\}}$	0.08563$^{\{11\}}$	0.07299$^{\{6\}}$	0.09906$^{\{12\}}$	0.07782$^{\{10\}}$
	MSE	$\hat{\rho}$	0.06333$^{\{2\}}$	0.07505$^{\{5\}}$	0.10348$^{\{9\}}$	0.06331$^{\{1\}}$	0.09197$^{\{8\}}$	0.10695$^{\{11\}}$	0.07769$^{\{6\}}$	0.06866$^{\{3\}}$	0.10445$^{\{10\}}$	0.08813$^{\{7\}}$	0.07368$^{\{4\}}$	0.11452$^{\{12\}}$
		$\hat{\xi}$	0.00622$^{\{1\}}$	0.00678$^{\{3\}}$	0.00947$^{\{9\}}$	0.00638$^{\{2\}}$	0.00885$^{\{7\}}$	0.00903$^{\{8\}}$	0.00698$^{\{4\}}$	0.00723$^{\{5\}}$	0.0112$^{\{11\}}$	0.00847$^{\{6\}}$	0.01593$^{\{12\}}$	0.00958$^{\{10\}}$
	MRE	$\hat{\rho}$	0.04951$^{\{1\}}$	0.05379$^{\{4\}}$	0.06279$^{\{10\}}$	0.05073$^{\{3\}}$	0.05979$^{\{9\}}$	0.06298$^{\{11\}}$	0.05423$^{\{5\}}$	0.05039$^{\{2\}}$	0.06416$^{\{12\}}$	0.05917$^{\{8\}}$	0.05451$^{\{6.5\}}$	0.05451$^{\{6.5\}}$
		$\hat{\xi}$	0.0417$^{\{1\}}$	0.044$^{\{3\}}$	0.05087$^{\{9\}}$	0.04275$^{\{2\}}$	0.04972$^{\{8\}}$	0.04938$^{\{7\}}$	0.04401$^{\{4\}}$	0.04488$^{\{5\}}$	0.05708$^{\{10\}}$	0.04866$^{\{6\}}$	4.03705$^{\{12\}}$	0.06712$^{\{11\}}$
	$\sum$ *Ranks*		7$^{\{1\}}$	22$^{\{3.5\}}$	55$^{\{10\}}$	13$^{\{2\}}$	48$^{\{7\}}$	54$^{\{9\}}$	28$^{\{5\}}$	22$^{\{3.5\}}$	65$^{\{12\}}$	40$^{\{6\}}$	52.5$^{\{8\}}$	61.5$^{\{11\}}$

Continued

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.18194^{\{1\}}$	$0.1924^{\{3\}}$	$0.22423^{\{8\}}$	$0.18228^{\{2\}}$	$0.22547^{\{9\}}$	$0.25333^{\{12\}}$	$0.197^{\{5\}}$	$0.20124^{\{6\}}$	$0.2302^{\{10\}}$	$0.21315^{\{7\}}$	$0.19557^{\{4\}}$	$0.24067^{\{11\}}$
		$\hat{\zeta}$	$0.05683^{\{1\}}$	$0.05976^{\{3\}}$	$0.06777^{\{7\}}$	$0.0583^{\{2\}}$	$0.06859^{\{8\}}$	$0.07125^{\{10\}}$	$0.06127^{\{4\}}$	$0.0655^{\{5\}}$	$0.07373^{\{11\}}$	$0.06603^{\{6\}}$	$0.09081^{\{12\}}$	$0.06944^{\{9\}}$
	MSE	$\hat{\rho}$	$0.05366^{\{2\}}$	$0.05873^{\{3\}}$	$0.08242^{\{9\}}$	$0.05172^{\{1\}}$	$0.08116^{\{8\}}$	$0.10135^{\{12\}}$	$0.06225^{\{5\}}$	$0.06453^{\{6\}}$	$0.08573^{\{10\}}$	$0.06982^{\{7\}}$	$0.06116^{\{4\}}$	$0.09499^{\{11\}}$
		$\hat{\zeta}$	$0.00517^{\{1\}}$	$0.00566^{\{3\}}$	$0.00704^{\{7\}}$	$0.00533^{\{2\}}$	$0.00725^{\{8\}}$	$0.00812^{\{10\}}$	$0.00583^{\{4\}}$	$0.00686^{\{5\}}$	$0.00863^{\{11\}}$	$0.0069^{\{6\}}$	$0.013^{\{12\}}$	$0.00773^{\{9\}}$
	MRE	$\hat{\rho}$	$0.04548^{\{1\}}$	$0.0481^{\{3\}}$	$0.05606^{\{9\}}$	$0.04557^{\{2\}}$	$0.05637^{\{10\}}$	$0.06333^{\{12\}}$	$0.04925^{\{6\}}$	$0.05031^{\{7\}}$	$0.05755^{\{11\}}$	$0.05329^{\{8\}}$	$0.04889^{\{4.5\}}$	$0.04889^{\{4.5\}}$
		$\hat{\zeta}$	$0.03789^{\{1\}}$	$0.03984^{\{3\}}$	$0.04518^{\{7\}}$	$0.03887^{\{2\}}$	$0.04573^{\{8\}}$	$0.0475^{\{9\}}$	$0.04085^{\{4\}}$	$0.04366^{\{5\}}$	$0.04915^{\{10\}}$	$0.04402^{\{6\}}$	$4.03672^{\{12\}}$	$0.06017^{\{11\}}$
	$\sum$ *Ranks*		$7^{\{1\}}$	$18^{\{3\}}$	$47^{\{7\}}$	$11^{\{2\}}$	$51^{\{9\}}$	$65^{\{12\}}$	$28^{\{4\}}$	$34^{\{5\}}$	$63^{\{11\}}$	$40^{\{6\}}$	$48.5^{\{8\}}$	$55.5^{\{10\}}$
450	BIAS	$\hat{\rho}$	$0.14627^{\{2\}}$	$0.15501^{\{3\}}$	$0.17399^{\{7\}}$	$0.14118^{\{1\}}$	$0.17875^{\{8\}}$	$0.19689^{\{12\}}$	$0.16558^{\{6\}}$	$0.15783^{\{4\}}$	$0.18843^{\{10\}}$	$0.18028^{\{9\}}$	$0.16345^{\{5\}}$	$0.18867^{\{11\}}$
		$\hat{\zeta}$	$0.04753^{\{2\}}$	$0.05039^{\{4\}}$	$0.05201^{\{6\}}$	$0.04671^{\{1\}}$	$0.05475^{\{7\}}$	$0.05509^{\{8\}}$	$0.04826^{\{3\}}$	$0.05087^{\{5\}}$	$0.06143^{\{11\}}$	$0.05579^{\{9\}}$	$0.07327^{\{12\}}$	$0.05708^{\{10\}}$
	MSE	$\hat{\rho}$	$0.03401^{\{2\}}$	$0.03855^{\{3\}}$	$0.04775^{\{7\}}$	$0.03199^{\{1\}}$	$0.05014^{\{9\}}$	$0.06214^{\{12\}}$	$0.04272^{\{6\}}$	$0.03868^{\{4\}}$	$0.05673^{\{10\}}$	$0.04919^{\{8\}}$	$0.04177^{\{5\}}$	$0.05938^{\{11\}}$
		$\hat{\zeta}$	$0.00357^{\{3\}}$	$0.00396^{\{4\}}$	$0.00437^{\{6\}}$	$0.00338^{\{1\}}$	$0.00465^{\{7\}}$	$0.00482^{\{8.5\}}$	$0.00354^{\{2\}}$	$0.0041^{\{5\}}$	$0.00606^{\{11\}}$	$0.00482^{\{8.5\}}$	$0.00879^{\{12\}}$	$0.00515^{\{10\}}$
	MRE	$\hat{\rho}$	$0.03657^{\{2\}}$	$0.03875^{\{3\}}$	$0.0435^{\{8\}}$	$0.03529^{\{1\}}$	$0.04469^{\{9\}}$	$0.04922^{\{12\}}$	$0.0414^{\{7\}}$	$0.03946^{\{4\}}$	$0.04711^{\{11\}}$	$0.04507^{\{10\}}$	$0.04086^{\{5.5\}}$	$0.04086^{\{5.5\}}$
		$\hat{\zeta}$	$0.03169^{\{2\}}$	$0.03359^{\{4\}}$	$0.03468^{\{6\}}$	$0.03114^{\{1\}}$	$0.0365^{\{7\}}$	$0.03673^{\{8\}}$	$0.03217^{\{3\}}$	$0.03391^{\{5\}}$	$0.04096^{\{10\}}$	$0.03719^{\{9\}}$	$4.02931^{\{12\}}$	$0.04717^{\{11\}}$
	$\sum$ *Ranks*		$13^{\{2\}}$	$21^{\{3\}}$	$40^{\{6\}}$	$6^{\{1\}}$	$47^{\{7\}}$	$60.5^{\{11\}}$	$27^{\{4.5\}}$	$27^{\{4.5\}}$	$63^{\{12\}}$	$53.5^{\{9\}}$	$51.5^{\{8\}}$	$58.5^{\{10\}}$

Table 2. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 4.0,\ \zeta = 1.5$.

The Tables contain documented information on the estimated unknown parameters and their corresponding log-likelihood (ll) for two distinct data sets, showcasing the outcomes of fitting distributions. To identify the most suitable model among the competing ones, various statistical measures are computed. These measures

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	$0.12121^{\{1\}}$	$0.12584^{\{5\}}$	$0.12853^{\{6\}}$	$0.12376^{\{2\}}$	$0.13239^{\{8\}}$	$0.12529^{\{4\}}$	$0.12494^{\{3\}}$	$0.13275^{\{9\}}$	$0.14492^{\{11\}}$	$0.13899^{\{10\}}$	$0.14892^{\{12\}}$	$0.13047^{\{7\}}$
		$\hat{\xi}$	$0.41634^{\{2\}}$	$0.45394^{\{4\}}$	$0.51301^{\{9\}}$	$0.40955^{\{1\}}$	$0.48214^{\{8\}}$	$0.53446^{\{11\}}$	$0.45133^{\{3\}}$	$0.45432^{\{5\}}$	$0.48179^{\{7\}}$	$0.47185^{\{6\}}$	$0.56845^{\{12\}}$	$0.51322^{\{10\}}$
	MSE	$\hat{\rho}$	$0.02253^{\{1\}}$	$0.02455^{\{4\}}$	$0.02589^{\{6\}}$	$0.02525^{\{5\}}$	$0.02706^{\{9\}}$	$0.02452^{\{3\}}$	$0.02412^{\{2\}}$	$0.02703^{\{8\}}$	$0.03315^{\{11\}}$	$0.03019^{\{10\}}$	$0.03345^{\{12\}}$	$0.02655^{\{7\}}$
		$\hat{\xi}$	$0.31141^{\{2\}}$	$0.34386^{\{4\}}$	$0.46878^{\{10\}}$	$0.25714^{\{1\}}$	$0.37531^{\{7\}}$	$0.48487^{\{11\}}$	$0.34592^{\{5\}}$	$0.34986^{\{6\}}$	$0.38444^{\{8\}}$	$0.33172^{\{3\}}$	$0.56933^{\{12\}}$	$0.46705^{\{9\}}$
	MRE	$\hat{\rho}$	$0.24243^{\{1\}}$	$0.25168^{\{5\}}$	$0.25706^{\{6\}}$	$0.24752^{\{2\}}$	$0.26478^{\{7\}}$	$0.25058^{\{4\}}$	$0.24989^{\{3\}}$	$0.2655^{\{8\}}$	$0.28985^{\{10\}}$	$0.27798^{\{9\}}$	$0.29785^{\{11.5\}}$	$0.29785^{\{11.5\}}$
		$\hat{\xi}$	$0.13878^{\{2\}}$	$0.15131^{\{4\}}$	$0.171^{\{9\}}$	$0.13652^{\{1\}}$	$0.16071^{\{8\}}$	$0.17815^{\{10\}}$	$0.15044^{\{3\}}$	$0.15144^{\{5\}}$	$0.1606^{\{7\}}$	$0.15728^{\{6\}}$	$0.50067^{\{12\}}$	$0.26094^{\{11\}}$
	$\sum Ranks$		$9^{\{1\}}$	$26^{\{4\}}$	$46^{\{8\}}$	$12^{\{2\}}$	$47^{\{9\}}$	$43^{\{6\}}$	$19^{\{3\}}$	$41^{\{5\}}$	$54^{\{10\}}$	$44^{\{7\}}$	$71.5^{\{12\}}$	$55.5^{\{11\}}$
70	BIAS	$\hat{\rho}$	$0.07411^{\{3\}}$	$0.06991^{\{1\}}$	$0.08126^{\{8\}}$	$0.07601^{\{4\}}$	$0.07843^{\{7\}}$	$0.07763^{\{5\}}$	$0.07192^{\{2\}}$	$0.07783^{\{6\}}$	$0.09385^{\{11\}}$	$0.09188^{\{10\}}$	$0.09839^{\{12\}}$	$0.08144^{\{9\}}$
		$\hat{\xi}$	$0.23997^{\{2\}}$	$0.23994^{\{1\}}$	$0.29867^{\{9\}}$	$0.24669^{\{3\}}$	$0.29395^{\{7\}}$	$0.29598^{\{8\}}$	$0.24927^{\{4\}}$	$0.26167^{\{5\}}$	$0.31145^{\{11\}}$	$0.27908^{\{6\}}$	$0.34563^{\{12\}}$	$0.3022^{\{10\}}$
	MSE	$\hat{\rho}$	$0.00851^{\{3\}}$	$0.0082^{\{1\}}$	$0.01001^{\{8\}}$	$0.00947^{\{4\}}$	$0.00972^{\{7\}}$	$0.00954^{\{6\}}$	$0.00826^{\{2\}}$	$0.00951^{\{5\}}$	$0.01456^{\{11\}}$	$0.01357^{\{10\}}$	$0.01517^{\{12\}}$	$0.01037^{\{9\}}$
		$\hat{\xi}$	$0.09381^{\{1\}}$	$0.09403^{\{2\}}$	$0.14551^{\{9\}}$	$0.09458^{\{3\}}$	$0.13471^{\{7\}}$	$0.14132^{\{8\}}$	$0.10111^{\{4\}}$	$0.11336^{\{5\}}$	$0.15045^{\{10\}}$	$0.122^{\{6\}}$	$0.19277^{\{12\}}$	$0.15117^{\{11\}}$
	MRE	$\hat{\rho}$	$0.14822^{\{3\}}$	$0.13982^{\{1\}}$	$0.16252^{\{8\}}$	$0.15203^{\{4\}}$	$0.15686^{\{7\}}$	$0.15526^{\{5\}}$	$0.14383^{\{2\}}$	$0.15567^{\{6\}}$	$0.18769^{\{10\}}$	$0.18376^{\{9\}}$	$0.19678^{\{11.5\}}$	$0.19678^{\{11.5\}}$
		$\hat{\xi}$	$0.07999^{\{2\}}$	$0.07998^{\{1\}}$	$0.09956^{\{9\}}$	$0.08223^{\{3\}}$	$0.09798^{\{7\}}$	$0.09866^{\{8\}}$	$0.08309^{\{4\}}$	$0.08722^{\{5\}}$	$0.10382^{\{10\}}$	$0.09303^{\{6\}}$	$0.49295^{\{12\}}$	$0.16288^{\{11\}}$
	$\sum Ranks$		$14^{\{2\}}$	$7^{\{1\}}$	$51^{\{9\}}$	$21^{\{4\}}$	$42^{\{7\}}$	$40^{\{6\}}$	$18^{\{3\}}$	$32^{\{5\}}$	$63^{\{11\}}$	$47^{\{8\}}$	$71.5^{\{12\}}$	$61.5^{\{10\}}$
120	BIAS	$\hat{\rho}$	$0.05526^{\{2\}}$	$0.05404^{\{1\}}$	$0.05967^{\{7\}}$	$0.05805^{\{3\}}$	$0.06014^{\{9\}}$	$0.05906^{\{5\}}$	$0.0586^{\{4\}}$	$0.06009^{\{8\}}$	$0.07175^{\{11\}}$	$0.06601^{\{10\}}$	$0.07955^{\{12\}}$	$0.05961^{\{6\}}$
		$\hat{\xi}$	$0.18672^{\{2\}}$	$0.18129^{\{1\}}$	$0.2245^{\{8\}}$	$0.19284^{\{3\}}$	$0.21764^{\{7\}}$	$0.23385^{\{11\}}$	$0.19744^{\{4\}}$	$0.20634^{\{5\}}$	$0.23136^{\{10\}}$	$0.20851^{\{6\}}$	$0.27526^{\{12\}}$	$0.22853^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00464^{\{1\}}$	$0.00466^{\{2\}}$	$0.00552^{\{5\}}$	$0.00536^{\{3\}}$	$0.0056^{\{7\}}$	$0.00553^{\{6\}}$	$0.00542^{\{4\}}$	$0.00569^{\{8\}}$	$0.00834^{\{11\}}$	$0.007^{\{10\}}$	$0.01016^{\{12\}}$	$0.00571^{\{9\}}$
		$\hat{\xi}$	$0.05479^{\{2\}}$	$0.05474^{\{1\}}$	$0.07913^{\{8\}}$	$0.05702^{\{3\}}$	$0.07372^{\{7\}}$	$0.09009^{\{11\}}$	$0.0619^{\{4\}}$	$0.06945^{\{6\}}$	$0.08228^{\{10\}}$	$0.06705^{\{5\}}$	$0.12058^{\{12\}}$	$0.08221^{\{9\}}$
	MRE	$\hat{\rho}$	$0.11051^{\{2\}}$	$0.10808^{\{1\}}$	$0.11933^{\{6\}}$	$0.1161^{\{3\}}$	$0.12028^{\{8\}}$	$0.11812^{\{5\}}$	$0.1172^{\{4\}}$	$0.12019^{\{7\}}$	$0.14349^{\{10\}}$	$0.13201^{\{9\}}$	$0.15909^{\{11.5\}}$	$0.15909^{\{11.5\}}$
		$\hat{\xi}$	$0.06224^{\{2\}}$	$0.06043^{\{1\}}$	$0.07483^{\{8\}}$	$0.06428^{\{3\}}$	$0.07255^{\{7\}}$	$0.07795^{\{10\}}$	$0.06581^{\{4\}}$	$0.06878^{\{5\}}$	$0.07712^{\{9\}}$	$0.0695^{\{6\}}$	$0.49844^{\{12\}}$	$0.11922^{\{11\}}$
	$\sum Ranks$		$11^{\{2\}}$	$7^{\{1\}}$	$42^{\{6\}}$	$18^{\{3\}}$	$45^{\{7\}}$	$48^{\{9\}}$	$24^{\{4\}}$	$39^{\{5\}}$	$61^{\{11\}}$	$46^{\{8\}}$	$71.5^{\{12\}}$	$55.5^{\{10\}}$
160	BIAS	$\hat{\rho}$	$0.04856^{\{2\}}$	$0.05202^{\{8\}}$	$0.05086^{\{6\}}$	$0.04845^{\{1\}}$	$0.05201^{\{7\}}$	$0.05081^{\{5\}}$	$0.0487^{\{3\}}$	$0.05205^{\{9\}}$	$0.05924^{\{10\}}$	$0.06231^{\{11\}}$	$0.06811^{\{12\}}$	$0.04945^{\{4\}}$
		$\hat{\xi}$	$0.15446^{\{1\}}$	$0.16862^{\{3\}}$	$0.18556^{\{8\}}$	$0.16318^{\{2\}}$	$0.18497^{\{7\}}$	$0.19609^{\{10\}}$	$0.169^{\{4\}}$	$0.1799^{\{5\}}$	$0.19718^{\{11\}}$	$0.19238^{\{9\}}$	$0.23322^{\{12\}}$	$0.18277^{\{6\}}$
	MSE	$\hat{\rho}$	$0.00368^{\{1.5\}}$	$0.00416^{\{7\}}$	$0.00398^{\{5\}}$	$0.00368^{\{1.5\}}$	$0.00435^{\{9\}}$	$0.00408^{\{6\}}$	$0.00376^{\{3\}}$	$0.00434^{\{8\}}$	$0.00562^{\{10\}}$	$0.00589^{\{11\}}$	$0.0071^{\{12\}}$	$0.0039^{\{4\}}$
		$\hat{\xi}$	$0.03922^{\{1\}}$	$0.0451^{\{4\}}$	$0.05619^{\{7\}}$	$0.0408^{\{2\}}$	$0.05628^{\{8\}}$	$0.06345^{\{11\}}$	$0.04374^{\{3\}}$	$0.05119^{\{5\}}$	$0.05931^{\{10\}}$	$0.05685^{\{9\}}$	$0.08237^{\{12\}}$	$0.05389^{\{6\}}$
	MRE	$\hat{\rho}$	$0.09712^{\{2\}}$	$0.10403^{\{7\}}$	$0.10171^{\{5\}}$	$0.0969^{\{1\}}$	$0.10402^{\{6\}}$	$0.10161^{\{4\}}$	$0.0974^{\{3\}}$	$0.1041^{\{8\}}$	$0.11847^{\{9\}}$	$0.12463^{\{10\}}$	$0.13623^{\{11.5\}}$	$0.13623^{\{11.5\}}$
		$\hat{\xi}$	$0.05149^{\{1\}}$	$0.05621^{\{3\}}$	$0.06185^{\{7\}}$	$0.05439^{\{2\}}$	$0.06166^{\{6\}}$	$0.06536^{\{9\}}$	$0.05633^{\{4\}}$	$0.05997^{\{5\}}$	$0.06573^{\{10\}}$	$0.06413^{\{8\}}$	$0.50104^{\{12\}}$	$0.09891^{\{11\}}$
	$\sum Ranks$		$8.5^{\{1\}}$	$32^{\{4\}}$	$38^{\{5\}}$	$9.5^{\{2\}}$	$43^{\{8\}}$	$45^{\{9\}}$	$20^{\{3\}}$	$40^{\{6\}}$	$60^{\{11\}}$	$58^{\{10\}}$	$71.5^{\{12\}}$	$42.5^{\{7\}}$
200	BIAS	$\hat{\rho}$	$0.04336^{\{1\}}$	$0.04338^{\{2\}}$	$0.04731^{\{7\}}$	$0.04371^{\{3\}}$	$0.04761^{\{9\}}$	$0.0464^{\{5\}}$	$0.046^{\{4\}}$	$0.04667^{\{6\}}$	$0.05581^{\{11\}}$	$0.05132^{\{10\}}$	$0.064^{\{12\}}$	$0.04745^{\{8\}}$
		$\hat{\xi}$	$0.14157^{\{1\}}$	$0.1433^{\{3\}}$	$0.17085^{\{8\}}$	$0.14313^{\{2\}}$	$0.16893^{\{7\}}$	$0.17479^{\{10\}}$	$0.1549^{\{5\}}$	$0.15229^{\{4\}}$	$0.18931^{\{11\}}$	$0.1586^{\{6\}}$	$0.21512^{\{12\}}$	$0.17476^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00291^{\{2\}}$	$0.0029^{\{1\}}$	$0.00347^{\{8\}}$	$0.00299^{\{3\}}$	$0.00368^{\{9\}}$	$0.00333^{\{4\}}$	$0.00336^{\{5\}}$	$0.00342^{\{6\}}$	$0.00492^{\{11\}}$	$0.00417^{\{10\}}$	$0.00656^{\{12\}}$	$0.00344^{\{7\}}$
		$\hat{\xi}$	$0.03221^{\{3\}}$	$0.03192^{\{2\}}$	$0.04676^{\{8\}}$	$0.03119^{\{1\}}$	$0.04522^{\{7\}}$	$0.04773^{\{10\}}$	$0.03914^{\{5\}}$	$0.03614^{\{4\}}$	$0.05399^{\{11\}}$	$0.04011^{\{6\}}$	$0.07442^{\{12\}}$	$0.04749^{\{9\}}$
	MRE	$\hat{\rho}$	$0.08672^{\{1\}}$	$0.08676^{\{2\}}$	$0.09462^{\{7\}}$	$0.08742^{\{3\}}$	$0.09522^{\{8\}}$	$0.0928^{\{5\}}$	$0.09199^{\{4\}}$	$0.09334^{\{6\}}$	$0.11163^{\{10\}}$	$0.10263^{\{9\}}$	$0.128^{\{11.5\}}$	$0.128^{\{11.5\}}$
		$\hat{\xi}$	$0.04719^{\{1\}}$	$0.04777^{\{3\}}$	$0.05695^{\{8\}}$	$0.04771^{\{2\}}$	$0.05631^{\{7\}}$	$0.05826^{\{9\}}$	$0.05163^{\{5\}}$	$0.05076^{\{4\}}$	$0.0631^{\{10\}}$	$0.05287^{\{6\}}$	$0.50049^{\{12\}}$	$0.0949^{\{11\}}$
	$\sum Ranks$		$9^{\{1\}}$	$13^{\{2\}}$	$46^{\{7\}}$	$14^{\{3\}}$	$47^{\{8.5\}}$	$43^{\{6\}}$	$28^{\{4\}}$	$30^{\{5\}}$	$64^{\{11\}}$	$47^{\{8.5\}}$	$71.5^{\{12\}}$	$55.5^{\{10\}}$
250	BIAS	$\hat{\rho}$	$0.03953^{\{2\}}$	$0.03879^{\{1\}}$	$0.04053^{\{6\}}$	$0.04023^{\{3\}}$	$0.04171^{\{8\}}$	$0.04032^{\{4\}}$	$0.04052^{\{5\}}$	$0.04318^{\{9\}}$	$0.04926^{\{11\}}$	$0.04748^{\{10\}}$	$0.05775^{\{12\}}$	$0.04055^{\{7\}}$
		$\hat{\xi}$	$0.12204^{\{1\}}$	$0.1276^{\{2\}}$	$0.15^{\{7\}}$	$0.12986^{\{3\}}$	$0.15078^{\{9\}}$	$0.15068^{\{8\}}$	$0.1357^{\{4\}}$	$0.14255^{\{5\}}$	$0.15945^{\{11\}}$	$0.14794^{\{6\}}$	$0.19297^{\{12\}}$	$0.15334^{\{10\}}$
	MSE	$\hat{\rho}$	$0.00246^{\{2\}}$	$0.00231^{\{1\}}$	$0.0026^{\{6\}}$	$0.00255^{\{4\}}$	$0.00269^{\{8\}}$	$0.00255^{\{4\}}$	$0.00266^{\{7\}}$	$0.00292^{\{9\}}$	$0.00385^{\{11\}}$	$0.00361^{\{10\}}$	$0.00539^{\{12\}}$	$0.00255^{\{4\}}$
		$\hat{\xi}$	$0.02428^{\{1\}}$	$0.02561^{\{2\}}$	$0.03494^{\{7\}}$	$0.02593^{\{3\}}$	$0.03691^{\{9\}}$	$0.03591^{\{8\}}$	$0.03009^{\{4\}}$	$0.03264^{\{5\}}$	$0.03964^{\{11\}}$	$0.03379^{\{6\}}$	$0.05846^{\{12\}}$	$0.03745^{\{10\}}$
	MRE	$\hat{\rho}$	$0.07906^{\{2\}}$	$0.07757^{\{1\}}$	$0.08107^{\{6\}}$	$0.08045^{\{3\}}$	$0.08343^{\{7\}}$	$0.08063^{\{4\}}$	$0.08104^{\{5\}}$	$0.08635^{\{8\}}$	$0.09852^{\{10\}}$	$0.09497^{\{9\}}$	$0.11551^{\{11.5\}}$	$0.11551^{\{11.5\}}$
		$\hat{\xi}$	$0.04068^{\{1\}}$	$0.04253^{\{2\}}$	$0.05^{\{7\}}$	$0.04329^{\{3\}}$	$0.05026^{\{9\}}$	$0.05023^{\{8\}}$	$0.04523^{\{4\}}$	$0.04752^{\{5\}}$	$0.05315^{\{10\}}$	$0.04931^{\{6\}}$	$0.50142^{\{12\}}$	$0.0811^{\{11\}}$
	$\sum Ranks$		$9^{\{1.5\}}$	$9^{\{1.5\}}$	$39^{\{6\}}$	$19^{\{3\}}$	$50^{\{9\}}$	$36^{\{5\}}$	$29^{\{4\}}$	$41^{\{7\}}$	$64^{\{11\}}$	$47^{\{8\}}$	$71.5^{\{12\}}$	$53.5^{\{10\}}$
Continued														

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.03521^{\{1\}}$	$0.03684^{\{5\}}$	$0.03649^{\{4\}}$	$0.03611^{\{2\}}$	$0.03689^{\{6\}}$	$0.03617^{\{3\}}$	$0.03726^{\{7\}}$	$0.03861^{\{9\}}$	$0.04581^{\{11\}}$	$0.04283^{\{10\}}$	$0.05248^{\{12\}}$	$0.03814^{\{8\}}$
		$\hat{\zeta}$	$0.11528^{\{2\}}$	$0.12386^{\{4\}}$	$0.13469^{\{8\}}$	$0.11311^{\{1\}}$	$0.12886^{\{5\}}$	$0.14411^{\{10\}}$	$0.12324^{\{3\}}$	$0.12905^{\{6\}}$	$0.14434^{\{11\}}$	$0.13355^{\{7\}}$	$0.17511^{\{12\}}$	$0.13911^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00194^{\{1\}}$	$0.00217^{\{6\}}$	$0.00209^{\{3\}}$	$0.00204^{\{2\}}$	$0.00215^{\{5\}}$	$0.00214^{\{4\}}$	$0.0022^{\{7.5\}}$	$0.00241^{\{9\}}$	$0.00322^{\{11\}}$	$0.0029^{\{10\}}$	$0.00438^{\{12\}}$	$0.0022^{\{7.5\}}$
		$\hat{\zeta}$	$0.0211^{\{2\}}$	$0.0242^{\{3\}}$	$0.02969^{\{8\}}$	$0.01942^{\{1\}}$	$0.02542^{\{5\}}$	$0.03232^{\{11\}}$	$0.02443^{\{4\}}$	$0.02638^{\{6\}}$	$0.03206^{\{10\}}$	$0.02715^{\{7\}}$	$0.04879^{\{12\}}$	$0.0303^{\{9\}}$
	MRE	$\hat{\rho}$	$0.07043^{\{1\}}$	$0.07368^{\{5\}}$	$0.07297^{\{4\}}$	$0.07221^{\{2\}}$	$0.07378^{\{6\}}$	$0.07235^{\{3\}}$	$0.07453^{\{7\}}$	$0.07722^{\{8\}}$	$0.09161^{\{10\}}$	$0.08565^{\{9\}}$	$0.10497^{\{11.5\}}$	$0.10497^{\{11.5\}}$
		$\hat{\zeta}$	$0.03843^{\{2\}}$	$0.04129^{\{4\}}$	$0.0449^{\{8\}}$	$0.0377^{\{1\}}$	$0.04295^{\{5\}}$	$0.04804^{\{9\}}$	$0.04108^{\{3\}}$	$0.04302^{\{6\}}$	$0.04811^{\{10\}}$	$0.04452^{\{7\}}$	$0.50075^{\{12\}}$	$0.07629^{\{11\}}$
	$\sum Ranks$		$9^{\{1.5\}}$	$27^{\{3\}}$	$35^{\{6\}}$	$9^{\{1.5\}}$	$32^{\{5\}}$	$40^{\{7\}}$	$31.5^{\{4\}}$	$44^{\{8\}}$	$63^{\{11\}}$	$50^{\{9\}}$	$71.5^{\{12\}}$	$56^{\{10\}}$
450	BIAS	$\hat{\rho}$	$0.02951^{\{4\}}$	$0.02919^{\{2.5\}}$	$0.02991^{\{5\}}$	$0.02919^{\{2.5\}}$	$0.03164^{\{8\}}$	$0.03064^{\{6\}}$	$0.02913^{\{1\}}$	$0.03193^{\{9\}}$	$0.0365^{\{11\}}$	$0.03368^{\{10\}}$	$0.04689^{\{12\}}$	$0.03128^{\{7\}}$
		$\hat{\zeta}$	$0.09475^{\{2\}}$	$0.09647^{\{3\}}$	$0.10753^{\{6\}}$	$0.0942^{\{1\}}$	$0.11258^{\{9\}}$	$0.12043^{\{11\}}$	$0.09677^{\{4\}}$	$0.10617^{\{5\}}$	$0.11507^{\{10\}}$	$0.1122^{\{8\}}$	$0.15623^{\{12\}}$	$0.1114^{\{7\}}$
	MSE	$\hat{\rho}$	$0.00134^{\{4\}}$	$0.00133^{\{2.5\}}$	$0.00143^{\{5\}}$	$0.00132^{\{1\}}$	$0.00159^{\{8\}}$	$0.00149^{\{6\}}$	$0.00133^{\{2.5\}}$	$0.00162^{\{9\}}$	$0.00217^{\{11\}}$	$0.00184^{\{10\}}$	$0.00353^{\{12\}}$	$0.00151^{\{7\}}$
		$\hat{\zeta}$	$0.01383^{\{2\}}$	$0.01474^{\{4\}}$	$0.01816^{\{6\}}$	$0.01369^{\{1\}}$	$0.0202^{\{9\}}$	$0.02328^{\{11\}}$	$0.01462^{\{3\}}$	$0.01802^{\{5\}}$	$0.02122^{\{10\}}$	$0.01946^{\{7\}}$	$0.03868^{\{12\}}$	$0.01966^{\{8\}}$
	MRE	$\hat{\rho}$	$0.05901^{\{4\}}$	$0.05838^{\{2.5\}}$	$0.05982^{\{5\}}$	$0.05838^{\{2.5\}}$	$0.06328^{\{7\}}$	$0.06128^{\{6\}}$	$0.05826^{\{1\}}$	$0.06385^{\{8\}}$	$0.073^{\{10\}}$	$0.06736^{\{9\}}$	$0.09377^{\{11.5\}}$	$0.09377^{\{11.5\}}$
		$\hat{\zeta}$	$0.03158^{\{2\}}$	$0.03216^{\{3\}}$	$0.03584^{\{6\}}$	$0.0314^{\{1\}}$	$0.03753^{\{8\}}$	$0.04014^{\{10\}}$	$0.03226^{\{4\}}$	$0.03539^{\{5\}}$	$0.03836^{\{9\}}$	$0.0374^{\{7\}}$	$0.49898^{\{12\}}$	$0.06256^{\{11\}}$
	$\sum Ranks$		$18^{\{4\}}$	$17.5^{\{3\}}$	$33^{\{5\}}$	$9^{\{1\}}$	$49^{\{7\}}$	$50^{\{8\}}$	$15.5^{\{2\}}$	$41^{\{6\}}$	$61^{\{11\}}$	$51^{\{9\}}$	$71.5^{\{12\}}$	$51.5^{\{10\}}$

Table 3. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 0.5$, $\zeta = 3.0$.

encompass the Akaike Information Criterion (AIC), Correction Akaike Information Criterion (CAIC), Hannan Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (K-S) statistics accompanied by its associated p-value.

The formulas for these measures are as follows:

$$AIC = -2\log L + 2n, \qquad CAIC = AIC + \frac{2k^2 + 2k}{n-k-1}, \qquad BIC = -2\log L + k(log(n))$$

$$HQIC = -2\log L + 2klog(log(n)) \qquad \text{K.S} = \max_{1\le j\le p}\left(F(x_j) - \frac{j-1}{p}, \frac{j}{n} - F(x_j)\right),$$

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	$0.15648^{\{4\}}$	$0.15775^{\{7\}}$	$0.16745^{\{9\}}$	$0.14827^{\{1\}}$	$0.15201^{\{2\}}$	$0.15656^{\{5\}}$	$0.15737^{\{6\}}$	$0.15434^{\{3\}}$	$0.18315^{\{11\}}$	$0.18312^{\{10\}}$	$0.19236^{\{12\}}$	$0.16217^{\{8\}}$
		$\hat{\zeta}$	$0.03378^{\{2\}}$	$0.0366^{\{3\}}$	$0.04492^{\{11\}}$	$0.03367^{\{1\}}$	$0.03773^{\{4\}}$	$0.04118^{\{8\}}$	$0.03882^{\{6\}}$	$0.03845^{\{5\}}$	$0.04374^{\{10\}}$	$0.03939^{\{7\}}$	$0.04925^{\{12\}}$	$0.0425^{\{9\}}$
	MSE	$\hat{\rho}$	$0.03806^{\{4\}}$	$0.038^{\{3\}}$	$0.04332^{\{9\}}$	$0.03572^{\{1\}}$	$0.03645^{\{2\}}$	$0.03854^{\{7\}}$	$0.03845^{\{6\}}$	$0.03823^{\{5\}}$	$0.05406^{\{11\}}$	$0.05208^{\{10\}}$	$0.05584^{\{12\}}$	$0.04036^{\{8\}}$
		$\hat{\zeta}$	$0.00209^{\{2\}}$	$0.00228^{\{3\}}$	$0.00359^{\{11\}}$	$0.0017^{\{1\}}$	$0.00231^{\{4.5\}}$	$0.00291^{\{8\}}$	$0.00249^{\{7\}}$	$0.00245^{\{6\}}$	$0.00305^{\{9\}}$	$0.00231^{\{4.5\}}$	$0.00419^{\{12\}}$	$0.00312^{\{10\}}$
	MRE	$\hat{\rho}$	$0.20864^{\{4\}}$	$0.21033^{\{7\}}$	$0.22327^{\{8\}}$	$0.19769^{\{1\}}$	$0.20268^{\{2\}}$	$0.20875^{\{5\}}$	$0.20982^{\{6\}}$	$0.20579^{\{3\}}$	$0.2442^{\{10\}}$	$0.24416^{\{9\}}$	$0.25648^{\{11.5\}}$	$0.25648^{\{11.5\}}$
		$\hat{\zeta}$	$0.13513^{\{2\}}$	$0.14639^{\{3\}}$	$0.17968^{\{10\}}$	$0.13468^{\{1\}}$	$0.15091^{\{4\}}$	$0.16472^{\{8\}}$	$0.15527^{\{6\}}$	$0.1538^{\{5\}}$	$0.17494^{\{9\}}$	$0.15756^{\{7\}}$	$0.75057^{\{12\}}$	$0.21622^{\{11\}}$
	$\sum$ *Ranks*		$18^{\{2\}}$	$26^{\{4\}}$	$58^{\{10\}}$	$6^{\{1\}}$	$18.5^{\{3\}}$	$41^{\{7\}}$	$37^{\{6\}}$	$27^{\{5\}}$	$60^{\{11\}}$	$47.5^{\{8\}}$	$71.5^{\{12\}}$	$57.5^{\{9\}}$
70	BIAS	$\hat{\rho}$	$0.09479^{\{5\}}$	$0.09394^{\{4\}}$	$0.09344^{\{3\}}$	$0.09195^{\{2\}}$	$0.09506^{\{6\}}$	$0.09735^{\{8\}}$	$0.09104^{\{1\}}$	$0.09836^{\{9\}}$	$0.11455^{\{11\}}$	$0.11073^{\{10\}}$	$0.12022^{\{12\}}$	$0.09665^{\{7\}}$
		$\hat{\zeta}$	$0.02041^{\{1\}}$	$0.02169^{\{3\}}$	$0.02398^{\{7\}}$	$0.02075^{\{2\}}$	$0.02421^{\{8\}}$	$0.02485^{\{9\}}$	$0.02186^{\{4\}}$	$0.02222^{\{5\}}$	$0.02609^{\{11\}}$	$0.02322^{\{6\}}$	$0.02951^{\{12\}}$	$0.02504^{\{10\}}$
	MSE	$\hat{\rho}$	$0.01404^{\{5\}}$	$0.01365^{\{3\}}$	$0.01378^{\{4\}}$	$0.01345^{\{2\}}$	$0.0142^{\{6\}}$	$0.01489^{\{8\}}$	$0.01308^{\{1\}}$	$0.01522^{\{9\}}$	$0.02064^{\{11\}}$	$0.0191^{\{10\}}$	$0.02225^{\{12\}}$	$0.01473^{\{7\}}$
		$\hat{\zeta}$	$0.00068^{\{2\}}$	$0.00077^{\{3\}}$	$0.00094^{\{8\}}$	$0.00064^{\{1\}}$	$0.00092^{\{7\}}$	$0.00103^{\{9\}}$	$0.00079^{\{4.5\}}$	$0.00079^{\{4.5\}}$	$0.0011^{\{11\}}$	$0.00083^{\{6\}}$	$0.00149^{\{12\}}$	$0.00104^{\{10\}}$
	MRE	$\hat{\rho}$	$0.12638^{\{5\}}$	$0.12526^{\{4\}}$	$0.12459^{\{3\}}$	$0.1226^{\{2\}}$	$0.12674^{\{6\}}$	$0.1298^{\{7\}}$	$0.12138^{\{1\}}$	$0.13114^{\{8\}}$	$0.15273^{\{10\}}$	$0.14764^{\{9\}}$	$0.16029^{\{11.5\}}$	$0.16029^{\{11.5\}}$
		$\hat{\zeta}$	$0.08165^{\{1\}}$	$0.08676^{\{3\}}$	$0.09592^{\{7\}}$	$0.08301^{\{2\}}$	$0.09685^{\{8\}}$	$0.09938^{\{9\}}$	$0.08742^{\{4\}}$	$0.08889^{\{5\}}$	$0.10437^{\{10\}}$	$0.09288^{\{6\}}$	$0.74319^{\{12\}}$	$0.12887^{\{11\}}$
	$\sum$ *Ranks*		$19^{\{3\}}$	$20^{\{4\}}$	$32^{\{5\}}$	$11^{\{1\}}$	$41^{\{7\}}$	$50^{\{9\}}$	$15.5^{\{2\}}$	$40.5^{\{6\}}$	$64^{\{11\}}$	$47^{\{8\}}$	$71.5^{\{12\}}$	$56.5^{\{10\}}$
120	BIAS	$\hat{\rho}$	$0.07036^{\{3\}}$	$0.06813^{\{2\}}$	$0.07469^{\{9\}}$	$0.0672^{\{1\}}$	$0.07297^{\{6\}}$	$0.07127^{\{4\}}$	$0.0718^{\{5\}}$	$0.07365^{\{7\}}$	$0.08655^{\{11\}}$	$0.08117^{\{10\}}$	$0.09066^{\{12\}}$	$0.07466^{\{8\}}$
		$\hat{\zeta}$	$0.01481^{\{1\}}$	$0.015^{\{3\}}$	$0.01838^{\{9\}}$	$0.01488^{\{2\}}$	$0.01798^{\{8\}}$	$0.01853^{\{10\}}$	$0.01708^{\{5\}}$	$0.01683^{\{4\}}$	$0.01906^{\{11\}}$	$0.01741^{\{6\}}$	$0.02155^{\{12\}}$	$0.01796^{\{7\}}$
	MSE	$\hat{\rho}$	$0.00778^{\{3\}}$	$0.00741^{\{2\}}$	$0.00849^{\{7\}}$	$0.00734^{\{1\}}$	$0.00832^{\{6\}}$	$0.00812^{\{4\}}$	$0.00821^{\{5\}}$	$0.00858^{\{8\}}$	$0.01179^{\{11\}}$	$0.0103^{\{10\}}$	$0.01317^{\{12\}}$	$0.00868^{\{9\}}$
		$\hat{\zeta}$	$0.00036^{\{3\}}$	$0.00035^{\{2\}}$	$0.00053^{\{9\}}$	$0.00034^{\{1\}}$	$0.00051^{\{7\}}$	$0.00057^{\{10.5\}}$	$0.00045^{\{4.5\}}$	$0.00045^{\{4.5\}}$	$0.00057^{\{10.5\}}$	$0.00047^{\{6\}}$	$0.00078^{\{12\}}$	$0.00052^{\{8\}}$
	MRE	$\hat{\rho}$	$0.09381^{\{3\}}$	$0.09083^{\{2\}}$	$0.09959^{\{8\}}$	$0.0896^{\{1\}}$	$0.09729^{\{6\}}$	$0.09503^{\{4\}}$	$0.09573^{\{5\}}$	$0.0982^{\{7\}}$	$0.1154^{\{10\}}$	$0.10823^{\{9\}}$	$0.12087^{\{11.5\}}$	$0.12087^{\{11.5\}}$
		$\hat{\zeta}$	$0.05924^{\{1\}}$	$0.05998^{\{3\}}$	$0.07354^{\{8\}}$	$0.05952^{\{2\}}$	$0.07191^{\{7\}}$	$0.07412^{\{9\}}$	$0.06831^{\{5\}}$	$0.06734^{\{4\}}$	$0.07626^{\{10\}}$	$0.06964^{\{6\}}$	$0.75673^{\{12\}}$	$0.09954^{\{11\}}$
	$\sum$ *Ranks*		$14^{\{2.5\}}$	$14^{\{2.5\}}$	$50^{\{9\}}$	$8^{\{1\}}$	$40^{\{6\}}$	$41.5^{\{7\}}$	$29.5^{\{4\}}$	$34.5^{\{5\}}$	$63.5^{\{11\}}$	$47^{\{8\}}$	$71.5^{\{12\}}$	$54.5^{\{10\}}$
160	BIAS	$\hat{\rho}$	$0.06093^{\{4\}}$	$0.05957^{\{2\}}$	$0.06622^{\{8\}}$	$0.0593^{\{1\}}$	$0.06645^{\{9\}}$	$0.06046^{\{3\}}$	$0.06225^{\{5\}}$	$0.06298^{\{6\}}$	$0.07747^{\{11\}}$	$0.0709^{\{10\}}$	$0.08582^{\{12\}}$	$0.06602^{\{7\}}$
		$\hat{\zeta}$	$0.01334^{\{1\}}$	$0.01336^{\{2\}}$	$0.01572^{\{7\}}$	$0.0137^{\{3\}}$	$0.01606^{\{9\}}$	$0.01594^{\{8\}}$	$0.01426^{\{4\}}$	$0.01449^{\{5\}}$	$0.01681^{\{11\}}$	$0.01512^{\{6\}}$	$0.02044^{\{12\}}$	$0.01615^{\{10\}}$
	MSE	$\hat{\rho}$	$0.00581^{\{4\}}$	$0.0055^{\{1\}}$	$0.007^{\{9\}}$	$0.00555^{\{2\}}$	$0.00684^{\{8\}}$	$0.00576^{\{3\}}$	$0.00623^{\{5\}}$	$0.00626^{\{6\}}$	$0.00958^{\{11\}}$	$0.0081^{\{10\}}$	$0.01156^{\{12\}}$	$0.00682^{\{7\}}$
		$\hat{\zeta}$	$0.00028^{\{1.5\}}$	$0.00028^{\{1.5\}}$	$0.00041^{\{9\}}$	$0.00029^{\{3\}}$	$0.00041^{\{9\}}$	$4e-04^{\{7\}}$	$0.00033^{\{4\}}$	$0.00034^{\{5\}}$	$0.00044^{\{11\}}$	$0.00035^{\{6\}}$	$0.00065^{\{12\}}$	$0.00041^{\{9\}}$
	MRE	$\hat{\rho}$	$0.08124^{\{4\}}$	$0.07943^{\{2\}}$	$0.0883^{\{7\}}$	$0.07906^{\{1\}}$	$0.0886^{\{8\}}$	$0.08061^{\{3\}}$	$0.083^{\{5\}}$	$0.08398^{\{6\}}$	$0.10329^{\{10\}}$	$0.09454^{\{9\}}$	$0.11442^{\{11.5\}}$	$0.11442^{\{11.5\}}$
		$\hat{\zeta}$	$0.05336^{\{1\}}$	$0.05345^{\{2\}}$	$0.06288^{\{7\}}$	$0.05481^{\{3\}}$	$0.06423^{\{9\}}$	$0.06376^{\{8\}}$	$0.05703^{\{4\}}$	$0.05798^{\{5\}}$	$0.06725^{\{10\}}$	$0.06049^{\{6\}}$	$0.75057^{\{12\}}$	$0.08803^{\{11\}}$
	$\sum$ *Ranks*		$15.5^{\{3\}}$	$10.5^{\{1\}}$	$46^{\{7\}}$	$13^{\{2\}}$	$51^{\{9\}}$	$37^{\{6\}}$	$27^{\{4\}}$	$33^{\{5\}}$	$63^{\{11\}}$	$47^{\{8\}}$	$70.5^{\{12\}}$	$54.5^{\{10\}}$
200	BIAS	$\hat{\rho}$	$0.05444^{\{3\}}$	$0.05227^{\{1\}}$	$0.0582^{\{9\}}$	$0.05449^{\{4\}}$	$0.05639^{\{7\}}$	$0.0555^{\{6\}}$	$0.05394^{\{2\}}$	$0.05798^{\{8\}}$	$0.06789^{\{11\}}$	$0.06309^{\{10\}}$	$0.07561^{\{12\}}$	$0.05537^{\{5\}}$
		$\hat{\zeta}$	$0.01128^{\{1\}}$	$0.01225^{\{3\}}$	$0.01421^{\{9\}}$	$0.01176^{\{2\}}$	$0.0141^{\{8\}}$	$0.01475^{\{10\}}$	$0.01264^{\{4\}}$	$0.01276^{\{5\}}$	$0.01493^{\{11\}}$	$0.01327^{\{6\}}$	$0.01824^{\{12\}}$	$0.0139^{\{7\}}$
	MSE	$\hat{\rho}$	$0.00454^{\{3\}}$	$0.00424^{\{1\}}$	$0.00518^{\{8\}}$	$0.00457^{\{4\}}$	$0.00499^{\{7\}}$	$0.00473^{\{5\}}$	$0.00446^{\{2\}}$	$0.00538^{\{9\}}$	$0.00705^{\{11\}}$	$0.00671^{\{10\}}$	$0.00897^{\{12\}}$	$0.00493^{\{6\}}$
		$\hat{\zeta}$	$2e-04^{\{1\}}$	$0.00024^{\{3\}}$	$0.00032^{\{9\}}$	$0.00021^{\{2\}}$	$0.00031^{\{8\}}$	$0.00034^{\{10\}}$	$0.00025^{\{4\}}$	$0.00026^{\{5\}}$	$0.00035^{\{11\}}$	$0.00028^{\{6\}}$	$0.00053^{\{12\}}$	$3e-04^{\{7\}}$
	MRE	$\hat{\rho}$	$0.07258^{\{3\}}$	$0.0697^{\{1\}}$	$0.0776^{\{8\}}$	$0.07265^{\{4\}}$	$0.07518^{\{6\}}$	$0.074^{\{5\}}$	$0.07191^{\{2\}}$	$0.07731^{\{7\}}$	$0.09051^{\{10\}}$	$0.08411^{\{9\}}$	$0.10082^{\{11.5\}}$	$0.10082^{\{11.5\}}$
		$\hat{\zeta}$	$0.04512^{\{1\}}$	$0.04898^{\{3\}}$	$0.05682^{\{8\}}$	$0.04703^{\{2\}}$	$0.05639^{\{7\}}$	$0.05899^{\{9\}}$	$0.05056^{\{4\}}$	$0.05106^{\{5\}}$	$0.05973^{\{10\}}$	$0.05307^{\{6\}}$	$0.7488^{\{12\}}$	$0.07382^{\{11\}}$
	$\sum$ *Ranks*		$22^{\{4\}}$	$11^{\{1\}}$	$49^{\{9\}}$	$17^{\{2.5\}}$	$41^{\{6\}}$	$43^{\{7\}}$	$17^{\{2.5\}}$	$38^{\{5\}}$	$62^{\{11\}}$	$46^{\{8\}}$	$69.5^{\{12\}}$	$52.5^{\{10\}}$
250	BIAS	$\hat{\rho}$	$0.04763^{\{3\}}$	$0.04708^{\{1\}}$	$0.04874^{\{4\}}$	$0.04753^{\{2\}}$	$0.05097^{\{9\}}$	$0.0507^{\{7\}}$	$0.04952^{\{6\}}$	$0.05073^{\{8\}}$	$0.06062^{\{11\}}$	$0.05883^{\{10\}}$	$0.07174^{\{12\}}$	$0.04889^{\{5\}}$
		$\hat{\zeta}$	$0.01024^{\{1\}}$	$0.01112^{\{4\}}$	$0.01216^{\{6\}}$	$0.01067^{\{2\}}$	$0.01303^{\{9\}}$	$0.0133^{\{11\}}$	$0.0109^{\{3\}}$	$0.01156^{\{5\}}$	$0.01327^{\{10\}}$	$0.01235^{\{7\}}$	$0.01673^{\{12\}}$	$0.01236^{\{8\}}$
	MSE	$\hat{\rho}$	$0.00358^{\{3\}}$	$0.00342^{\{1\}}$	$0.0038^{\{5\}}$	$0.0035^{\{2\}}$	$0.00407^{\{8\}}$	$0.00396^{\{7\}}$	$0.00392^{\{6\}}$	$0.00414^{\{9\}}$	$0.00596^{\{11\}}$	$0.00544^{\{10\}}$	$0.00801^{\{12\}}$	$0.00378^{\{4\}}$
		$\hat{\zeta}$	$0.00016^{\{1\}}$	$0.00019^{\{3.5\}}$	$0.00024^{\{7\}}$	$0.00017^{\{2\}}$	$0.00026^{\{9\}}$	$0.00028^{\{10.5\}}$	$0.00019^{\{3.5\}}$	$0.00021^{\{5\}}$	$0.00028^{\{10.5\}}$	$0.00023^{\{6\}}$	$0.00043^{\{12\}}$	$0.00025^{\{8\}}$
	MRE	$\hat{\rho}$	$0.06351^{\{3\}}$	$0.06278^{\{1\}}$	$0.06498^{\{4\}}$	$0.06338^{\{2\}}$	$0.06796^{\{8\}}$	$0.0676^{\{6\}}$	$0.06603^{\{5\}}$	$0.06764^{\{7\}}$	$0.08082^{\{10\}}$	$0.07844^{\{9\}}$	$0.09565^{\{11.5\}}$	$0.09565^{\{11.5\}}$
		$\hat{\zeta}$	$0.04097^{\{1\}}$	$0.04447^{\{4\}}$	$0.04865^{\{6\}}$	$0.04267^{\{2\}}$	$0.05214^{\{8\}}$	$0.05319^{\{10\}}$	$0.04359^{\{3\}}$	$0.04626^{\{5\}}$	$0.05308^{\{9\}}$	$0.0494^{\{7\}}$	$0.74794^{\{12\}}$	$0.06518^{\{11\}}$
	$\sum$ *Ranks*		$12^{\{1.5\}}$	$14.5^{\{3\}}$	$32^{\{5\}}$	$12^{\{1.5\}}$	$51^{\{9\}}$	$51.5^{\{10\}}$	$26.5^{\{4\}}$	$39^{\{6\}}$	$61.5^{\{11\}}$	$49^{\{8\}}$	$71.5^{\{12\}}$	$47.5^{\{7\}}$

Continued

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.04353^{\{1\}}$	$0.04453^{\{3\}}$	$0.04468^{\{4\}}$	$0.04591^{\{6\}}$	$0.04583^{\{5\}}$	$0.046^{\{7\}}$	$0.0439^{\{2\}}$	$0.0465^{\{8\}}$	$0.05638^{\{11\}}$	$0.04912^{\{10\}}$	$0.06403^{\{12\}}$	$0.04754^{\{9\}}$
		$\hat{\zeta}$	$0.0094^{\{1\}}$	$0.01007^{\{3\}}$	$0.01128^{\{8\}}$	$0.00961^{\{2\}}$	$0.01072^{\{7\}}$	$0.01237^{\{10\}}$	$0.01035^{\{4\}}$	$0.01063^{\{6\}}$	$0.01249^{\{11\}}$	$0.01061^{\{5\}}$	$0.01576^{\{12\}}$	$0.01161^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00294^{\{1\}}$	$0.00311^{\{3\}}$	$0.0032^{\{5\}}$	$0.00315^{\{4\}}$	$0.00333^{\{7\}}$	$0.00335^{\{8\}}$	$0.00306^{\{2\}}$	$0.0033^{\{6\}}$	$0.00519^{\{11\}}$	$0.00388^{\{10\}}$	$0.00637^{\{12\}}$	$0.00359^{\{9\}}$
		$\hat{\zeta}$	$0.00014^{\{1.5\}}$	$0.00016^{\{3\}}$	$2e-04^{\{8\}}$	$0.00014^{\{1.5\}}$	$0.00019^{\{7\}}$	$0.00024^{\{10.5\}}$	$0.00017^{\{4.5\}}$	$0.00017^{\{4.5\}}$	$0.00024^{\{10.5\}}$	$0.00018^{\{6\}}$	$0.00039^{\{12\}}$	$0.00021^{\{9\}}$
	MRE	$\hat{\rho}$	$0.05804^{\{1\}}$	$0.05938^{\{3\}}$	$0.05957^{\{4\}}$	$0.06122^{\{6\}}$	$0.06111^{\{5\}}$	$0.06134^{\{7\}}$	$0.05854^{\{2\}}$	$0.06199^{\{8\}}$	$0.07518^{\{10\}}$	$0.06549^{\{9\}}$	$0.08538^{\{11.5\}}$	$0.08538^{\{11.5\}}$
		$\hat{\zeta}$	$0.0376^{\{1\}}$	$0.04026^{\{3\}}$	$0.04512^{\{8\}}$	$0.03844^{\{2\}}$	$0.04287^{\{7\}}$	$0.04947^{\{9\}}$	$0.04139^{\{4\}}$	$0.04254^{\{6\}}$	$0.04996^{\{10\}}$	$0.04243^{\{5\}}$	$0.74912^{\{12\}}$	$0.06339^{\{11\}}$
	$\sum Ranks$		$6.5^{\{1\}}$	$18^{\{2\}}$	$41^{\{7\}}$	$21.5^{\{4\}}$	$38^{\{5\}}$	$50.5^{\{9\}}$	$18.5^{\{3\}}$	$38.5^{\{6\}}$	$62.5^{\{11\}}$	$45^{\{8\}}$	$70.5^{\{12\}}$	$57.5^{\{10\}}$
450	BIAS	$\hat{\rho}$	$0.03646^{\{4\}}$	$0.03831^{\{9\}}$	$0.03795^{\{7\}}$	$0.03738^{\{6\}}$	$0.03572^{\{1\}}$	$0.03605^{\{2\}}$	$0.03642^{\{3\}}$	$0.03695^{\{5\}}$	$0.04365^{\{11\}}$	$0.04054^{\{10\}}$	$0.0559^{\{12\}}$	$0.038^{\{8\}}$
		$\hat{\zeta}$	$0.00748^{\{1\}}$	$0.00819^{\{3\}}$	$0.00919^{\{8\}}$	$0.008^{\{2\}}$	$0.00913^{\{7\}}$	$0.00955^{\{10\}}$	$0.00851^{\{4\}}$	$0.00857^{\{5\}}$	$0.00976^{\{11\}}$	$0.00904^{\{6\}}$	$0.01326^{\{12\}}$	$0.00935^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00212^{\{3.5\}}$	$0.00229^{\{8.5\}}$	$0.00224^{\{7\}}$	$0.0022^{\{6\}}$	$0.00199^{\{1\}}$	$0.00212^{\{3.5\}}$	$0.00211^{\{2\}}$	$0.00216^{\{5\}}$	$0.00302^{\{11\}}$	$0.00268^{\{10\}}$	$0.00494^{\{12\}}$	$0.00229^{\{8.5\}}$
		$\hat{\zeta}$	$9e-05^{\{1\}}$	$1e-04^{\{2.5\}}$	$0.00013^{\{7\}}$	$1e-04^{\{2.5\}}$	$0.00013^{\{7\}}$	$0.00015^{\{10.5\}}$	$0.00011^{\{4\}}$	$0.00012^{\{5\}}$	$0.00015^{\{10.5\}}$	$0.00013^{\{7\}}$	$0.00028^{\{12\}}$	$0.00014^{\{9\}}$
	MRE	$\hat{\rho}$	$0.04861^{\{4\}}$	$0.05108^{\{8\}}$	$0.05061^{\{7\}}$	$0.04984^{\{6\}}$	$0.04762^{\{1\}}$	$0.04806^{\{2\}}$	$0.04856^{\{3\}}$	$0.04927^{\{5\}}$	$0.0582^{\{10\}}$	$0.05406^{\{9\}}$	$0.07453^{\{11.5\}}$	$0.07453^{\{11.5\}}$
		$\hat{\zeta}$	$0.02991^{\{1\}}$	$0.03276^{\{3\}}$	$0.03676^{\{8\}}$	$0.03199^{\{2\}}$	$0.03653^{\{7\}}$	$0.0382^{\{9\}}$	$0.03406^{\{4\}}$	$0.03428^{\{5\}}$	$0.03905^{\{10\}}$	$0.03617^{\{6\}}$	$0.75118^{\{12\}}$	$0.05067^{\{11\}}$
	$\sum Ranks$		$25.5^{\{3\}}$	$42^{\{8\}}$	$41^{\{7\}}$	$32.5^{\{5\}}$	$21^{\{2\}}$	$34^{\{6\}}$	$17^{\{1\}}$	$27^{\{4\}}$	$60.5^{\{11\}}$	$45^{\{9\}}$	$68.5^{\{12\}}$	$54^{\{10\}}$

Table 4. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 0.75,\ \zeta = 0.25$.

where L represents the maximum value of the likelihood function for the model, k denotes the number of unknown parameters, and n signifies the length of the data set. The computed results for these measures are then presented in Tables 12, 13, 14 and 15, corresponding to the two data sets under consideration.

It is crucial to emphasize that the model exhibiting the smallest value for the specified criterion is chosen, indicating the best fit for the dataset. Furthermore, compared with other fitted models, the findings in Tables 13 and 15 strongly indicate that the PMXLD distribution is a favorable alternative. This assertion is substantiated by Figs. 10 and 11, where the plotted estimated Probability Density Functions (PDFs) and Cumulative Distribution Functions (CDFs) provide visual support for our claims.

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	$0.40498^{\{8\}}$	$0.37589^{\{4\}}$	$0.44113^{\{10\}}$	$0.32655^{\{1\}}$	$0.41911^{\{9\}}$	$0.45256^{\{12\}}$	$0.37252^{\{3\}}$	$0.39754^{\{7\}}$	$0.39032^{\{6\}}$	$0.38342^{\{5\}}$	$0.36724^{\{2\}}$	$0.44599^{\{11\}}$
		$\hat{\xi}$	$0.0714^{\{2\}}$	$0.07503^{\{4\}}$	$0.08269^{\{8\}}$	$0.06801^{\{1\}}$	$0.0797^{\{7\}}$	$0.0863^{\{9\}}$	$0.07667^{\{5\}}$	$0.07437^{\{3\}}$	$0.08703^{\{10\}}$	$0.07688^{\{6\}}$	$0.09303^{\{12\}}$	$0.08806^{\{11\}}$
	MSE	$\hat{\rho}$	$0.29248^{\{8\}}$	$0.25374^{\{5\}}$	$0.34545^{\{10\}}$	$0.17133^{\{1\}}$	$0.31934^{\{9\}}$	$0.35841^{\{12\}}$	$0.24581^{\{3\}}$	$0.28615^{\{7\}}$	$0.26591^{\{6\}}$	$0.25026^{\{4\}}$	$0.22627^{\{2\}}$	$0.35174^{\{11\}}$
		$\hat{\xi}$	$0.00874^{\{2\}}$	$0.00939^{\{4\}}$	$0.01162^{\{8\}}$	$0.00693^{\{1\}}$	$0.01065^{\{7\}}$	$0.01295^{\{10\}}$	$0.00998^{\{6\}}$	$0.00981^{\{5\}}$	$0.01223^{\{9\}}$	$0.00908^{\{3\}}$	$0.01512^{\{12\}}$	$0.01311^{\{11\}}$
	MRE	$\hat{\rho}$	$0.16199^{\{9\}}$	$0.15036^{\{5\}}$	$0.17645^{\{11\}}$	$0.13062^{\{1\}}$	$0.16764^{\{10\}}$	$0.18102^{\{12\}}$	$0.14901^{\{4\}}$	$0.15901^{\{8\}}$	$0.15613^{\{7\}}$	$0.15337^{\{6\}}$	$0.1469^{\{2.5\}}$	$0.1469^{\{2.5\}}$
		$\hat{\xi}$	$0.14279^{\{2\}}$	$0.15007^{\{4\}}$	$0.16539^{\{8\}}$	$0.13603^{\{1\}}$	$0.15941^{\{7\}}$	$0.17261^{\{9\}}$	$0.15333^{\{5\}}$	$0.14874^{\{3\}}$	$0.17406^{\{10\}}$	$0.15376^{\{6\}}$	$2.63328^{\{12\}}$	$0.1784^{\{11\}}$
	$\sum Ranks$		$31^{\{5\}}$	$26^{\{2.5\}}$	$55^{\{10\}}$	$6^{\{1\}}$	$49^{\{9\}}$	$64^{\{12\}}$	$26^{\{2.5\}}$	$33^{\{6\}}$	$48^{\{8\}}$	$30^{\{4\}}$	$42.5^{\{7\}}$	$57.5^{\{11\}}$
70	BIAS	$\hat{\rho}$	$0.23182^{\{7\}}$	$0.21711^{\{3\}}$	$0.24874^{\{9\}}$	$0.21176^{\{1\}}$	$0.22933^{\{6\}}$	$0.27082^{\{12\}}$	$0.21764^{\{4\}}$	$0.21786^{\{5\}}$	$0.25337^{\{10\}}$	$0.2402^{\{8\}}$	$0.21386^{\{2\}}$	$0.25972^{\{11\}}$
		$\hat{\xi}$	$0.04014^{\{1\}}$	$0.04174^{\{3\}}$	$0.0476^{\{8\}}$	$0.04083^{\{2\}}$	$0.04796^{\{9\}}$	$0.05058^{\{11\}}$	$0.04185^{\{4\}}$	$0.0432^{\{5\}}$	$0.05017^{\{10\}}$	$0.04448^{\{6\}}$	$0.05956^{\{12\}}$	$0.04731^{\{7\}}$
	MSE	$\hat{\rho}$	$0.08987^{\{7\}}$	$0.07564^{\{3\}}$	$0.10453^{\{9\}}$	$0.07214^{\{1\}}$	$0.08933^{\{6\}}$	$0.12577^{\{12\}}$	$0.07806^{\{4\}}$	$0.07973^{\{5\}}$	$0.1047^{\{10\}}$	$0.09019^{\{8\}}$	$0.07497^{\{2\}}$	$0.11547^{\{11\}}$
		$\hat{\xi}$	$0.00266^{\{2\}}$	$0.00288^{\{4\}}$	$0.00368^{\{8\}}$	$0.00254^{\{1\}}$	$0.00354^{\{7\}}$	$0.00408^{\{10.5\}}$	$0.00287^{\{3\}}$	$0.003^{\{5\}}$	$0.00408^{\{10.5\}}$	$0.0031^{\{6\}}$	$0.00558^{\{12\}}$	$0.00371^{\{9\}}$
	MRE	$\hat{\rho}$	$0.09273^{\{8\}}$	$0.08684^{\{4\}}$	$0.0995^{\{10\}}$	$0.0847^{\{1\}}$	$0.09173^{\{7\}}$	$0.10833^{\{12\}}$	$0.08706^{\{5\}}$	$0.08715^{\{6\}}$	$0.10135^{\{11\}}$	$0.09608^{\{9\}}$	$0.08554^{\{2.5\}}$	$0.08554^{\{2.5\}}$
		$\hat{\xi}$	$0.08027^{\{1\}}$	$0.08348^{\{3\}}$	$0.09519^{\{7\}}$	$0.08165^{\{2\}}$	$0.09593^{\{8\}}$	$0.10116^{\{10\}}$	$0.08369^{\{4\}}$	$0.08641^{\{5\}}$	$0.10033^{\{9\}}$	$0.08896^{\{6\}}$	$2.55487^{\{12\}}$	$0.10389^{\{11\}}$
	$\sum Ranks$		$26^{\{4\}}$	$20^{\{2\}}$	$51^{\{9\}}$	$8^{\{1\}}$	$43^{\{7.5\}}$	$67.5^{\{12\}}$	$24^{\{3\}}$	$31^{\{5\}}$	$60.5^{\{11\}}$	$43^{\{7.5\}}$	$42.5^{\{6\}}$	$51.5^{\{10\}}$
120	BIAS	$\hat{\rho}$	$0.16491^{\{2\}}$	$0.17342^{\{5\}}$	$0.18521^{\{9\}}$	$0.15746^{\{1\}}$	$0.18083^{\{8\}}$	$0.19672^{\{11\}}$	$0.17734^{\{6\}}$	$0.16953^{\{4\}}$	$0.20555^{\{12\}}$	$0.18081^{\{7\}}$	$0.16557^{\{3\}}$	$0.19103^{\{10\}}$
		$\hat{\xi}$	$0.02972^{\{1\}}$	$0.03238^{\{3\}}$	$0.03574^{\{6\}}$	$0.03224^{\{2\}}$	$0.03596^{\{7\}}$	$0.03804^{\{10\}}$	$0.0326^{\{4\}}$	$0.03461^{\{5\}}$	$0.04005^{\{11\}}$	$0.03608^{\{8\}}$	$0.04617^{\{12\}}$	$0.03674^{\{9\}}$
	MSE	$\hat{\rho}$	$0.045^{\{3\}}$	$0.04868^{\{5\}}$	$0.05502^{\{9\}}$	$0.0398^{\{1\}}$	$0.05306^{\{8\}}$	$0.06353^{\{11\}}$	$0.05243^{\{6\}}$	$0.0476^{\{4\}}$	$0.06624^{\{12\}}$	$0.0528^{\{7\}}$	$0.04306^{\{2\}}$	$0.05929^{\{10\}}$
		$\hat{\xi}$	$0.0015^{\{1\}}$	$0.00165^{\{3\}}$	$0.00201^{\{6\}}$	$0.00161^{\{2\}}$	$0.00209^{\{8\}}$	$0.00233^{\{10\}}$	$0.00167^{\{4\}}$	$0.00186^{\{5\}}$	$0.00237^{\{11\}}$	$0.00204^{\{7\}}$	$0.00346^{\{12\}}$	$0.00215^{\{9\}}$
	MRE	$\hat{\rho}$	$0.06596^{\{2\}}$	$0.06937^{\{6\}}$	$0.07408^{\{10\}}$	$0.06298^{\{1\}}$	$0.07233^{\{9\}}$	$0.07869^{\{11\}}$	$0.07094^{\{7\}}$	$0.06781^{\{5\}}$	$0.08222^{\{12\}}$	$0.07232^{\{8\}}$	$0.06623^{\{3.5\}}$	$0.06623^{\{3.5\}}$
		$\hat{\xi}$	$0.05944^{\{1\}}$	$0.06476^{\{3\}}$	$0.07148^{\{6\}}$	$0.06449^{\{2\}}$	$0.07191^{\{7\}}$	$0.07608^{\{9\}}$	$0.06521^{\{4\}}$	$0.06921^{\{5\}}$	$0.08011^{\{11\}}$	$0.07216^{\{8\}}$	$2.51839^{\{12\}}$	$0.07641^{\{10\}}$
	$\sum Ranks$		$10^{\{2\}}$	$25^{\{3\}}$	$46^{\{8\}}$	$9^{\{1\}}$	$47^{\{9\}}$	$62^{\{11\}}$	$31^{\{5\}}$	$28^{\{4\}}$	$69^{\{12\}}$	$45^{\{7\}}$	$44.5^{\{6\}}$	$51.5^{\{10\}}$
160	BIAS	$\hat{\rho}$	$0.13346^{\{1\}}$	$0.14528^{\{5\}}$	$0.15444^{\{7\}}$	$0.13737^{\{2\}}$	$0.1591^{\{8\}}$	$0.1715^{\{11\}}$	$0.14881^{\{6\}}$	$0.1414^{\{4\}}$	$0.17687^{\{12\}}$	$0.16765^{\{10\}}$	$0.14045^{\{3\}}$	$0.16389^{\{9\}}$
		$\hat{\xi}$	$0.02618^{\{1\}}$	$0.02855^{\{3\}}$	$0.03113^{\{9\}}$	$0.0273^{\{2\}}$	$0.02944^{\{5\}}$	$0.03266^{\{10\}}$	$0.02961^{\{7\}}$	$0.02958^{\{6\}}$	$0.03362^{\{11\}}$	$0.02925^{\{4\}}$	$0.03945^{\{12\}}$	$0.03106^{\{8\}}$
	MSE	$\hat{\rho}$	$0.02951^{\{2\}}$	$0.03373^{\{5\}}$	$0.03957^{\{7\}}$	$0.02915^{\{1\}}$	$0.04135^{\{8\}}$	$0.04796^{\{11\}}$	$0.03533^{\{6\}}$	$0.03224^{\{4\}}$	$0.04922^{\{12\}}$	$0.0431^{\{9\}}$	$0.03086^{\{3\}}$	$0.04469^{\{10\}}$
		$\hat{\xi}$	$0.00111^{\{1\}}$	$0.00127^{\{3\}}$	$0.0016^{\{9\}}$	$0.00113^{\{2\}}$	$0.00138^{\{6\}}$	$0.00168^{\{10\}}$	$0.00137^{\{5\}}$	$0.00141^{\{7\}}$	$0.00177^{\{11\}}$	$0.00133^{\{4\}}$	$0.00248^{\{12\}}$	$0.00152^{\{8\}}$
	MRE	$\hat{\rho}$	$0.05338^{\{1\}}$	$0.05811^{\{6\}}$	$0.06178^{\{8\}}$	$0.05495^{\{2\}}$	$0.06364^{\{9\}}$	$0.0686^{\{11\}}$	$0.05953^{\{7\}}$	$0.05656^{\{5\}}$	$0.07075^{\{12\}}$	$0.06706^{\{10\}}$	$0.05618^{\{3.5\}}$	$0.05618^{\{3.5\}}$
		$\hat{\xi}$	$0.05236^{\{1\}}$	$0.05711^{\{3\}}$	$0.06226^{\{8\}}$	$0.05461^{\{2\}}$	$0.05889^{\{5\}}$	$0.06531^{\{9\}}$	$0.05922^{\{7\}}$	$0.05916^{\{6\}}$	$0.06724^{\{11\}}$	$0.05851^{\{4\}}$	$2.51807^{\{12\}}$	$0.06556^{\{10\}}$
	$\sum Ranks$		$7^{\{1\}}$	$25^{\{3\}}$	$48^{\{9\}}$	$11^{\{2\}}$	$41^{\{6.5\}}$	$62^{\{11\}}$	$38^{\{5\}}$	$32^{\{4\}}$	$69^{\{12\}}$	$41^{\{6.5\}}$	$45.5^{\{8\}}$	$48.5^{\{10\}}$
200	BIAS	$\hat{\rho}$	$0.12638^{\{3\}}$	$0.12765^{\{4\}}$	$0.14299^{\{7\}}$	$0.1193^{\{1\}}$	$0.14575^{\{9\}}$	$0.15984^{\{12\}}$	$0.12893^{\{5\}}$	$0.13161^{\{6\}}$	$0.15093^{\{11\}}$	$0.14528^{\{8\}}$	$0.12397^{\{2\}}$	$0.15032^{\{10\}}$
		$\hat{\xi}$	$0.02235^{\{1\}}$	$0.02433^{\{3\}}$	$0.02807^{\{7\}}$	$0.02348^{\{2\}}$	$0.02904^{\{10\}}$	$0.02848^{\{8\}}$	$0.02448^{\{4\}}$	$0.02605^{\{5\}}$	$0.02991^{\{11\}}$	$0.02654^{\{6\}}$	$0.03474^{\{12\}}$	$0.02899^{\{9\}}$
	MSE	$\hat{\rho}$	$0.02596^{\{3\}}$	$0.02601^{\{4\}}$	$0.03348^{\{9\}}$	$0.0226^{\{1\}}$	$0.03325^{\{8\}}$	$0.04123^{\{12\}}$	$0.02733^{\{5\}}$	$0.02754^{\{6\}}$	$0.036^{\{11\}}$	$0.0332^{\{7\}}$	$0.02378^{\{2\}}$	$0.03574^{\{10\}}$
		$\hat{\xi}$	$0.00081^{\{1\}}$	$0.00094^{\{3.5\}}$	$0.00127^{\{7\}}$	$0.00086^{\{2\}}$	$0.00135^{\{9\}}$	$0.00137^{\{10\}}$	$0.00094^{\{3.5\}}$	$0.0011^{\{5\}}$	$0.00141^{\{11\}}$	$0.00112^{\{6\}}$	$0.00193^{\{12\}}$	$0.00134^{\{8\}}$
	MRE	$\hat{\rho}$	$0.05055^{\{4\}}$	$0.05106^{\{5\}}$	$0.05719^{\{8\}}$	$0.04772^{\{1\}}$	$0.0583^{\{10\}}$	$0.06394^{\{12\}}$	$0.05157^{\{6\}}$	$0.05264^{\{7\}}$	$0.06037^{\{11\}}$	$0.05811^{\{9\}}$	$0.04959^{\{2.5\}}$	$0.04959^{\{2.5\}}$
		$\hat{\xi}$	$0.04469^{\{1\}}$	$0.04865^{\{3\}}$	$0.05614^{\{7\}}$	$0.04696^{\{2\}}$	$0.05808^{\{9\}}$	$0.05695^{\{8\}}$	$0.04897^{\{4\}}$	$0.05209^{\{5\}}$	$0.05982^{\{10\}}$	$0.05307^{\{6\}}$	$2.53167^{\{12\}}$	$0.06013^{\{11\}}$
	$\sum Ranks$		$13^{\{2\}}$	$22.5^{\{3\}}$	$45^{\{8\}}$	$9^{\{1\}}$	$55^{\{10\}}$	$62^{\{11\}}$	$27.5^{\{4\}}$	$34^{\{5\}}$	$65^{\{12\}}$	$42^{\{6\}}$	$42.5^{\{7\}}$	$50.5^{\{9\}}$
250	BIAS	$\hat{\rho}$	$0.11055^{\{2\}}$	$0.11387^{\{5\}}$	$0.13249^{\{10\}}$	$0.11186^{\{3\}}$	$0.12669^{\{7\}}$	$0.1404^{\{11\}}$	$0.11816^{\{6\}}$	$0.1097^{\{1\}}$	$0.1405^{\{12\}}$	$0.12715^{\{9\}}$	$0.11227^{\{4\}}$	$0.12696^{\{8\}}$
		$\hat{\xi}$	$0.02052^{\{1\}}$	$0.02203^{\{3\}}$	$0.02478^{\{7\}}$	$0.02125^{\{2\}}$	$0.02519^{\{9\}}$	$0.02619^{\{10\}}$	$0.02262^{\{5\}}$	$0.02232^{\{4\}}$	$0.0264^{\{11\}}$	$0.02431^{\{6\}}$	$0.03242^{\{12\}}$	$0.02479^{\{8\}}$
	MSE	$\hat{\rho}$	$0.01925^{\{2\}}$	$0.02059^{\{5\}}$	$0.02898^{\{10\}}$	$0.01979^{\{3\}}$	$0.02694^{\{9\}}$	$0.03179^{\{12\}}$	$0.02216^{\{6\}}$	$0.01915^{\{1\}}$	$0.03097^{\{11\}}$	$0.0251^{\{7\}}$	$0.02039^{\{4\}}$	$0.02621^{\{8\}}$
		$\hat{\xi}$	$0.00069^{\{1\}}$	$0.00077^{\{3\}}$	$0.00099^{\{7\}}$	$7e-04^{\{2\}}$	$0.00105^{\{9\}}$	$0.00107^{\{10\}}$	$0.00083^{\{5\}}$	$8e-04^{\{4\}}$	$0.00111^{\{11\}}$	$0.00093^{\{6\}}$	$0.00167^{\{12\}}$	$0.00101^{\{8\}}$
	MRE	$\hat{\rho}$	$0.04422^{\{2\}}$	$0.04555^{\{6\}}$	$0.053^{\{10\}}$	$0.04474^{\{3\}}$	$0.05068^{\{8\}}$	$0.05616^{\{11\}}$	$0.04726^{\{7\}}$	$0.04388^{\{1\}}$	$0.0562^{\{12\}}$	$0.05086^{\{9\}}$	$0.04491^{\{4.5\}}$	$0.04491^{\{4.5\}}$
		$\hat{\xi}$	$0.04104^{\{1\}}$	$0.04407^{\{3\}}$	$0.04956^{\{7\}}$	$0.04251^{\{2\}}$	$0.05038^{\{8\}}$	$0.05239^{\{10\}}$	$0.04524^{\{5\}}$	$0.04464^{\{4\}}$	$0.05279^{\{11\}}$	$0.04863^{\{6\}}$	$2.50979^{\{12\}}$	$0.05078^{\{9\}}$
	$\sum Ranks$		$9^{\{1\}}$	$24^{\{3.5\}}$	$49^{\{10\}}$	$24^{\{3.5\}}$	$48^{\{9\}}$	$62^{\{11\}}$	$32^{\{5\}}$	$23^{\{2\}}$	$66^{\{12\}}$	$41^{\{6\}}$	$46.5^{\{8\}}$	$43.5^{\{7\}}$
Continued														

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.10412^{\{3\}}$	$0.10914^{\{5\}}$	$0.1133^{\{9\}}$	$0.10339^{\{2\}}$	$0.10992^{\{6.5\}}$	$0.12951^{\{11\}}$	$0.10992^{\{6.5\}}$	$0.10654^{\{4\}}$	$0.13026^{\{12\}}$	$0.11133^{\{8\}}$	$0.1^{\{1\}}$	$0.11991^{\{10\}}$
		$\hat{\zeta}$	$0.0187^{\{1\}}$	$0.02072^{\{4\}}$	$0.02343^{\{8\}}$	$0.01921^{\{2\}}$	$0.02391^{\{10\}}$	$0.0235^{\{9\}}$	$0.02005^{\{3\}}$	$0.02122^{\{5\}}$	$0.02401^{\{11\}}$	$0.0227^{\{6\}}$	$0.03036^{\{12\}}$	$0.02299^{\{7\}}$
	MSE	$\hat{\rho}$	$0.01717^{\{3\}}$	$0.0186^{\{5\}}$	$0.0211^{\{9\}}$	$0.01704^{\{2\}}$	$0.01904^{\{6\}}$	$0.02671^{\{12\}}$	$0.01956^{\{8\}}$	$0.01775^{\{4\}}$	$0.02666^{\{11\}}$	$0.0195^{\{7\}}$	$0.01577^{\{1\}}$	$0.02281^{\{10\}}$
		$\hat{\zeta}$	$0.00058^{\{2\}}$	$0.00068^{\{4\}}$	$0.00087^{\{8.5\}}$	$0.00056^{\{1\}}$	$0.00088^{\{10\}}$	$0.00087^{\{8.5\}}$	$0.00063^{\{3\}}$	$0.00071^{\{5\}}$	$0.00091^{\{11\}}$	$0.00078^{\{6\}}$	$0.00144^{\{12\}}$	$0.00084^{\{7\}}$
	MRE	$\hat{\rho}$	$0.04165^{\{4\}}$	$0.04366^{\{6\}}$	$0.04532^{\{10\}}$	$0.04136^{\{3\}}$	$0.04397^{\{7.5\}}$	$0.0518^{\{11\}}$	$0.04397^{\{7.5\}}$	$0.04262^{\{5\}}$	$0.0521^{\{12\}}$	$0.04453^{\{9\}}$	$0.04^{\{1.5\}}$	$0.04^{\{1.5\}}$
		$\hat{\zeta}$	$0.03739^{\{1\}}$	$0.04143^{\{4\}}$	$0.04686^{\{7\}}$	$0.03842^{\{2\}}$	$0.04782^{\{9\}}$	$0.047^{\{8\}}$	$0.0401^{\{3\}}$	$0.04244^{\{5\}}$	$0.04802^{\{11\}}$	$0.0454^{\{6\}}$	$2.51356^{\{12\}}$	$0.04796^{\{10\}}$
	$\sum Ranks$		$14^{\{2\}}$	$28^{\{3.5\}}$	$51.5^{\{10\}}$	$12^{\{1\}}$	$49^{\{9\}}$	$59.5^{\{11\}}$	$31^{\{5\}}$	$28^{\{3.5\}}$	$68^{\{12\}}$	$42^{\{7\}}$	$39.5^{\{6\}}$	$45.5^{\{8\}}$
450	BIAS	$\hat{\rho}$	$0.08181^{\{1\}}$	$0.08324^{\{4\}}$	$0.09216^{\{7\}}$	$0.08375^{\{5\}}$	$0.09491^{\{8\}}$	$0.10245^{\{11\}}$	$0.08852^{\{6\}}$	$0.08213^{\{3\}}$	$0.10491^{\{12\}}$	$0.09723^{\{9\}}$	$0.08183^{\{2\}}$	$0.09731^{\{10\}}$
		$\hat{\zeta}$	$0.01484^{\{1\}}$	$0.01627^{\{3.5\}}$	$0.01859^{\{9\}}$	$0.01553^{\{2\}}$	$0.01714^{\{5\}}$	$0.01961^{\{10\}}$	$0.01627^{\{3.5\}}$	$0.01737^{\{6\}}$	$0.01989^{\{11\}}$	$0.01814^{\{7\}}$	$0.02511^{\{12\}}$	$0.01833^{\{8\}}$
	MSE	$\hat{\rho}$	$0.01082^{\{3\}}$	$0.01062^{\{1\}}$	$0.01366^{\{7\}}$	$0.0109^{\{4\}}$	$0.01392^{\{8\}}$	$0.01661^{\{11\}}$	$0.01268^{\{6\}}$	$0.01101^{\{5\}}$	$0.01734^{\{12\}}$	$0.0151^{\{9\}}$	$0.01066^{\{2\}}$	$0.0153^{\{10\}}$
		$\hat{\zeta}$	$0.00035^{\{1\}}$	$0.00042^{\{4\}}$	$0.00056^{\{9\}}$	$0.00037^{\{2\}}$	$0.00048^{\{6\}}$	$6e-04^{\{10\}}$	$0.00041^{\{3\}}$	$0.00047^{\{5\}}$	$0.00062^{\{11\}}$	$5e-04^{\{7\}}$	$0.00101^{\{12\}}$	$0.00053^{\{8\}}$
	MRE	$\hat{\rho}$	$0.03272^{\{1\}}$	$0.0333^{\{5\}}$	$0.03686^{\{8\}}$	$0.0335^{\{6\}}$	$0.03796^{\{9\}}$	$0.04098^{\{11\}}$	$0.03541^{\{7\}}$	$0.03285^{\{4\}}$	$0.04196^{\{12\}}$	$0.03889^{\{10\}}$	$0.03273^{\{2.5\}}$	$0.03273^{\{2.5\}}$
		$\hat{\zeta}$	$0.02968^{\{1\}}$	$0.03255^{\{4\}}$	$0.03719^{\{8\}}$	$0.03105^{\{2\}}$	$0.03428^{\{5\}}$	$0.03922^{\{10\}}$	$0.03254^{\{3\}}$	$0.03473^{\{6\}}$	$0.03977^{\{11\}}$	$0.03628^{\{7\}}$	$2.50953^{\{12\}}$	$0.03892^{\{9\}}$
	$\sum Ranks$		$8^{\{1\}}$	$21.5^{\{3\}}$	$47^{\{9\}}$	$21^{\{2\}}$	$41^{\{7\}}$	$65^{\{11\}}$	$28.5^{\{4\}}$	$29^{\{5\}}$	$67^{\{12\}}$	$53^{\{10\}}$	$40.5^{\{6\}}$	$46.5^{\{8\}}$

Table 5. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 2.5,\ \zeta = 0.5$.

Conclusion

We have introduced a novel and flexible model called the "Power Modified XLindley distribution." This distribution is thoroughly examined for its mathematical properties. In this study, we emphasize various estimation techniques, employing twelve distinct methods. Furthermore, we have conducted an extensive simulation

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	$0.33366^{\{11\}}$	$0.30762^{\{5\}}$	$0.33226^{\{10\}}$	$0.27176^{\{1\}}$	$0.32312^{\{7\}}$	$0.34882^{\{12\}}$	$0.30584^{\{3.5\}}$	$0.31631^{\{6\}}$	$0.32438^{\{8\}}$	$0.30584^{\{3.5\}}$	$0.2981^{\{2\}}$	$0.32833^{\{9\}}$
		$\hat{\xi}$	$0.29286^{\{2\}}$	$0.28572^{\{1\}}$	$0.34112^{\{9\}}$	$0.29391^{\{3\}}$	$0.31227^{\{5\}}$	$0.3413^{\{10\}}$	$0.30509^{\{4\}}$	$0.3347^{\{8\}}$	$0.33132^{\{7\}}$	$0.32093^{\{6\}}$	$0.36777^{\{12\}}$	$0.34545^{\{11\}}$
	MSE	$\hat{\rho}$	$0.19606^{\{11\}}$	$0.15964^{\{5\}}$	$0.19141^{\{9\}}$	$0.11403^{\{1\}}$	$0.18291^{\{8\}}$	$0.2203^{\{12\}}$	$0.15556^{\{4\}}$	$0.16836^{\{6\}}$	$0.18137^{\{7\}}$	$0.15362^{\{3\}}$	$0.14808^{\{2\}}$	$0.19377^{\{10\}}$
		$\hat{\xi}$	$0.14979^{\{3\}}$	$0.13648^{\{2\}}$	$0.20735^{\{10\}}$	$0.13072^{\{1\}}$	$0.15903^{\{6\}}$	$0.19837^{\{9\}}$	$0.15752^{\{5\}}$	$0.19022^{\{8\}}$	$0.17704^{\{7\}}$	$0.15067^{\{4\}}$	$0.22694^{\{12\}}$	$0.21121^{\{11\}}$
	MRE	$\hat{\rho}$	$0.16683^{\{11\}}$	$0.15381^{\{6\}}$	$0.16613^{\{10\}}$	$0.13588^{\{1\}}$	$0.16156^{\{8\}}$	$0.17441^{\{12\}}$	$0.15292^{\{4.5\}}$	$0.15815^{\{7\}}$	$0.16219^{\{9\}}$	$0.15292^{\{4.5\}}$	$0.14905^{\{2.5\}}$	$0.14905^{\{2.5\}}$
		$\hat{\xi}$	$0.14643^{\{2\}}$	$0.14286^{\{1\}}$	$0.17056^{\{10\}}$	$0.14695^{\{3\}}$	$0.15614^{\{5\}}$	$0.17065^{\{11\}}$	$0.15255^{\{4\}}$	$0.16735^{\{9\}}$	$0.16566^{\{8\}}$	$0.16047^{\{6\}}$	$2.08719^{\{12\}}$	$0.16417^{\{7\}}$
	$\sum Ranks$		$40^{\{6\}}$	$20^{\{2\}}$	$58^{\{11\}}$	$10^{\{1\}}$	$39^{\{5\}}$	$66^{\{12\}}$	$25^{\{3\}}$	$44^{\{8\}}$	$46^{\{9\}}$	$27^{\{4\}}$	$42.5^{\{7\}}$	$50.5^{\{10\}}$
70	BIAS	$\hat{\rho}$	$0.17328^{\{4\}}$	$0.17896^{\{5\}}$	$0.19391^{\{9\}}$	$0.16199^{\{1\}}$	$0.18148^{\{7\}}$	$0.20514^{\{12\}}$	$0.17117^{\{2\}}$	$0.17934^{\{6\}}$	$0.20347^{\{11\}}$	$0.19445^{\{10\}}$	$0.1713^{\{3\}}$	$0.19315^{\{8\}}$
		$\hat{\xi}$	$0.15839^{\{1\}}$	$0.17081^{\{3\}}$	$0.19153^{\{8\}}$	$0.16522^{\{2\}}$	$0.18458^{\{7\}}$	$0.19641^{\{9\}}$	$0.17224^{\{4\}}$	$0.18004^{\{5\}}$	$0.20904^{\{11\}}$	$0.18152^{\{6\}}$	$0.21641^{\{12\}}$	$0.1972^{\{10\}}$
	MSE	$\hat{\rho}$	$0.05033^{\{4\}}$	$0.05276^{\{6\}}$	$0.06177^{\{9\}}$	$0.04123^{\{1\}}$	$0.0541^{\{7\}}$	$0.07553^{\{12\}}$	$0.04844^{\{3\}}$	$0.05135^{\{5\}}$	$0.0657^{\{11\}}$	$0.05882^{\{8\}}$	$0.0468^{\{2\}}$	$0.06266^{\{10\}}$
		$\hat{\xi}$	$0.04135^{\{2\}}$	$0.04636^{\{3\}}$	$0.06015^{\{8\}}$	$0.04129^{\{1\}}$	$0.05387^{\{7\}}$	$0.06433^{\{9\}}$	$0.04748^{\{4\}}$	$0.05217^{\{6\}}$	$0.06484^{\{11\}}$	$0.05161^{\{5\}}$	$0.07802^{\{12\}}$	$0.06472^{\{10\}}$
	MRE	$\hat{\rho}$	$0.08664^{\{5\}}$	$0.08948^{\{6\}}$	$0.09695^{\{9\}}$	$0.081^{\{1\}}$	$0.09074^{\{8\}}$	$0.10257^{\{12\}}$	$0.08558^{\{2\}}$	$0.08967^{\{7\}}$	$0.10174^{\{11\}}$	$0.09722^{\{10\}}$	$0.08565^{\{3.5\}}$	$0.08565^{\{3.5\}}$
		$\hat{\xi}$	$0.0792^{\{1\}}$	$0.0854^{\{3\}}$	$0.09577^{\{8\}}$	$0.08261^{\{2\}}$	$0.09229^{\{7\}}$	$0.0982^{\{10\}}$	$0.08612^{\{4\}}$	$0.09002^{\{5\}}$	$0.10452^{\{11\}}$	$0.09076^{\{6\}}$	$2.03882^{\{12\}}$	$0.09658^{\{9\}}$
	$\sum Ranks$		$17^{\{2\}}$	$26^{\{4\}}$	$51^{\{10\}}$	$8^{\{1\}}$	$43^{\{6\}}$	$64^{\{11\}}$	$19^{\{3\}}$	$34^{\{5\}}$	$66^{\{12\}}$	$45^{\{8\}}$	$44.5^{\{7\}}$	$50.5^{\{9\}}$
120	BIAS	$\hat{\rho}$	$0.13436^{\{3\}}$	$0.13448^{\{4\}}$	$0.14766^{\{9\}}$	$0.12919^{\{1\}}$	$0.13647^{\{7\}}$	$0.15218^{\{11\}}$	$0.13313^{\{2\}}$	$0.13455^{\{5\}}$	$0.16547^{\{12\}}$	$0.14597^{\{8\}}$	$0.13512^{\{6\}}$	$0.14843^{\{10\}}$
		$\hat{\xi}$	$0.12516^{\{1\}}$	$0.12868^{\{3\}}$	$0.15036^{\{10\}}$	$0.12589^{\{2\}}$	$0.14089^{\{7\}}$	$0.14605^{\{8\}}$	$0.12983^{\{4\}}$	$0.13247^{\{5\}}$	$0.15919^{\{11\}}$	$0.13782^{\{6\}}$	$0.1828^{\{12\}}$	$0.14617^{\{9\}}$
	MSE	$\hat{\rho}$	$0.02846^{\{3\}}$	$0.02902^{\{5\}}$	$0.03589^{\{10\}}$	$0.02589^{\{1\}}$	$0.03069^{\{7\}}$	$0.03953^{\{11\}}$	$0.02812^{\{2\}}$	$0.02947^{\{6\}}$	$0.04404^{\{12\}}$	$0.03507^{\{9\}}$	$0.02894^{\{4\}}$	$0.03468^{\{8\}}$
		$\hat{\xi}$	$0.02565^{\{2\}}$	$0.02647^{\{3\}}$	$0.03643^{\{10\}}$	$0.02413^{\{1\}}$	$0.03113^{\{7\}}$	$0.03476^{\{8\}}$	$0.02769^{\{4\}}$	$0.02844^{\{5\}}$	$0.03938^{\{11\}}$	$0.02934^{\{6\}}$	$0.05491^{\{12\}}$	$0.03491^{\{9\}}$
	MRE	$\hat{\rho}$	$0.06718^{\{3\}}$	$0.06724^{\{4\}}$	$0.07383^{\{10\}}$	$0.06459^{\{1\}}$	$0.06823^{\{8\}}$	$0.07609^{\{11\}}$	$0.06656^{\{2\}}$	$0.06727^{\{5\}}$	$0.08274^{\{12\}}$	$0.07298^{\{9\}}$	$0.06756^{\{6.5\}}$	$0.06756^{\{6.5\}}$
		$\hat{\xi}$	$0.06258^{\{1\}}$	$0.06434^{\{3\}}$	$0.07518^{\{10\}}$	$0.06295^{\{2\}}$	$0.07045^{\{7\}}$	$0.07302^{\{8\}}$	$0.06492^{\{4\}}$	$0.06624^{\{5\}}$	$0.0796^{\{11\}}$	$0.06891^{\{6\}}$	$2.02047^{\{12\}}$	$0.07421^{\{9\}}$
	$\sum Ranks$		$13^{\{2\}}$	$22^{\{4\}}$	$59^{\{11\}}$	$8^{\{1\}}$	$43^{\{6\}}$	$57^{\{10\}}$	$18^{\{3\}}$	$31^{\{5\}}$	$69^{\{12\}}$	$44^{\{7\}}$	$52.5^{\{9\}}$	$51.5^{\{8\}}$
160	BIAS	$\hat{\rho}$	$0.11208^{\{1\}}$	$0.11718^{\{4\}}$	$0.12092^{\{8\}}$	$0.11369^{\{2\}}$	$0.11871^{\{7\}}$	$0.12899^{\{10\}}$	$0.11744^{\{5\}}$	$0.11503^{\{3\}}$	$0.14222^{\{12\}}$	$0.13469^{\{11\}}$	$0.11827^{\{6\}}$	$0.12857^{\{9\}}$
		$\hat{\xi}$	$0.10312^{\{1\}}$	$0.11214^{\{3\}}$	$0.1268^{\{8\}}$	$0.10785^{\{2\}}$	$0.12283^{\{6\}}$	$0.13452^{\{10\}}$	$0.11329^{\{4\}}$	$0.11762^{\{5\}}$	$0.13143^{\{9\}}$	$0.12593^{\{7\}}$	$0.15422^{\{12\}}$	$0.13854^{\{11\}}$
	MSE	$\hat{\rho}$	$0.01978^{\{1\}}$	$0.02143^{\{4\}}$	$0.02422^{\{8\}}$	$0.02023^{\{2\}}$	$0.02151^{\{5\}}$	$0.0267^{\{9\}}$	$0.02204^{\{6\}}$	$0.0204^{\{3\}}$	$0.03135^{\{12\}}$	$0.0286^{\{11\}}$	$0.02217^{\{7\}}$	$0.02672^{\{10\}}$
		$\hat{\xi}$	$0.01719^{\{1\}}$	$0.01965^{\{3\}}$	$0.02524^{\{8\}}$	$0.01776^{\{2\}}$	$0.0237^{\{6\}}$	$0.02823^{\{10\}}$	$0.02001^{\{4\}}$	$0.0219^{\{5\}}$	$0.02755^{\{9\}}$	$0.02432^{\{7\}}$	$0.03817^{\{12\}}$	$0.03009^{\{11\}}$
	MRE	$\hat{\rho}$	$0.05604^{\{1\}}$	$0.05859^{\{4\}}$	$0.06046^{\{9\}}$	$0.05684^{\{2\}}$	$0.05935^{\{8\}}$	$0.0645^{\{10\}}$	$0.05872^{\{5\}}$	$0.05751^{\{3\}}$	$0.07111^{\{12\}}$	$0.06734^{\{11\}}$	$0.05914^{\{6.5\}}$	$0.05914^{\{6.5\}}$
		$\hat{\xi}$	$0.05156^{\{1\}}$	$0.05607^{\{3\}}$	$0.0634^{\{8\}}$	$0.05392^{\{2\}}$	$0.06141^{\{6\}}$	$0.06726^{\{11\}}$	$0.05665^{\{4\}}$	$0.05881^{\{5\}}$	$0.06571^{\{10\}}$	$0.06297^{\{7\}}$	$2.01991^{\{12\}}$	$0.06428^{\{9\}}$
	$\sum Ranks$		$6^{\{1\}}$	$21^{\{3\}}$	$49^{\{7\}}$	$12^{\{2\}}$	$38^{\{6\}}$	$60^{\{11\}}$	$28^{\{5\}}$	$24^{\{4\}}$	$64^{\{12\}}$	$54^{\{8\}}$	$55.5^{\{9\}}$	$56.5^{\{10\}}$
200	BIAS	$\hat{\rho}$	$0.10357^{\{5\}}$	$0.10098^{\{2\}}$	$0.10736^{\{7\}}$	$0.09548^{\{1\}}$	$0.10954^{\{9\}}$	$0.11321^{\{10\}}$	$0.10348^{\{4\}}$	$0.10284^{\{3\}}$	$0.12048^{\{11\}}$	$0.12104^{\{12\}}$	$0.10454^{\{6\}}$	$0.10788^{\{8\}}$
		$\hat{\xi}$	$0.09206^{\{1\}}$	$0.10127^{\{4\}}$	$0.11199^{\{8\}}$	$0.09377^{\{2\}}$	$0.11161^{\{7\}}$	$0.11651^{\{10\}}$	$0.10114^{\{3\}}$	$0.10432^{\{5\}}$	$0.12068^{\{11\}}$	$0.10576^{\{6\}}$	$0.14073^{\{12\}}$	$0.11225^{\{9\}}$
	MSE	$\hat{\rho}$	$0.01685^{\{5\}}$	$0.01616^{\{2\}}$	$0.0187^{\{8\}}$	$0.0143^{\{1\}}$	$0.01889^{\{9\}}$	$0.02059^{\{10\}}$	$0.01646^{\{3\}}$	$0.0171^{\{6\}}$	$0.02362^{\{12\}}$	$0.0229^{\{11\}}$	$0.01681^{\{4\}}$	$0.01852^{\{7\}}$
		$\hat{\xi}$	$0.01376^{\{2\}}$	$0.01652^{\{4\}}$	$0.0199^{\{8\}}$	$0.0134^{\{1\}}$	$0.0193^{\{7\}}$	$0.02143^{\{10\}}$	$0.01603^{\{3\}}$	$0.0178^{\{5\}}$	$0.02324^{\{11\}}$	$0.01788^{\{6\}}$	$0.03059^{\{12\}}$	$0.02076^{\{9\}}$
	MRE	$\hat{\rho}$	$0.05179^{\{5\}}$	$0.05049^{\{2\}}$	$0.05368^{\{8\}}$	$0.04774^{\{1\}}$	$0.05477^{\{9\}}$	$0.05661^{\{10\}}$	$0.05174^{\{4\}}$	$0.05142^{\{3\}}$	$0.06024^{\{11\}}$	$0.06052^{\{12\}}$	$0.05227^{\{6.5\}}$	$0.05227^{\{6.5\}}$
		$\hat{\xi}$	$0.04603^{\{1\}}$	$0.05063^{\{4\}}$	$0.056^{\{9\}}$	$0.04689^{\{2\}}$	$0.0558^{\{8\}}$	$0.05825^{\{10\}}$	$0.05057^{\{3\}}$	$0.05216^{\{5\}}$	$0.06034^{\{11\}}$	$0.05288^{\{6\}}$	$2.01049^{\{12\}}$	$0.05394^{\{7\}}$
	$\sum Ranks$		$19^{\{3\}}$	$18^{\{2\}}$	$48^{\{7\}}$	$8^{\{1\}}$	$49^{\{8\}}$	$60^{\{11\}}$	$20^{\{4\}}$	$27^{\{5\}}$	$67^{\{12\}}$	$53^{\{10\}}$	$52.5^{\{9\}}$	$46.5^{\{6\}}$
250	BIAS	$\hat{\rho}$	$0.08931^{\{2\}}$	$0.08835^{\{1\}}$	$0.0952^{\{6\}}$	$0.08972^{\{3\}}$	$0.09577^{\{8\}}$	$0.10569^{\{11\}}$	$0.09575^{\{7\}}$	$0.08991^{\{4\}}$	$0.10959^{\{12\}}$	$0.09778^{\{9\}}$	$0.09511^{\{5\}}$	$0.10206^{\{10\}}$
		$\hat{\xi}$	$0.08663^{\{2\}}$	$0.08838^{\{3\}}$	$0.0996^{\{7\}}$	$0.08372^{\{1\}}$	$0.1002^{\{8\}}$	$0.10036^{\{9\}}$	$0.0884^{\{4\}}$	$0.08969^{\{5\}}$	$0.10334^{\{11\}}$	$0.09897^{\{6\}}$	$0.12999^{\{12\}}$	$0.10099^{\{10\}}$
	MSE	$\hat{\rho}$	$0.01277^{\{3\}}$	$0.01241^{\{1\}}$	$0.01469^{\{7\}}$	$0.01256^{\{2\}}$	$0.0151^{\{8\}}$	$0.01779^{\{11\}}$	$0.01439^{\{6\}}$	$0.01294^{\{4\}}$	$0.0191^{\{12\}}$	$0.01521^{\{9\}}$	$0.01409^{\{5\}}$	$0.01699^{\{10\}}$
		$\hat{\xi}$	$0.01168^{\{2\}}$	$0.0127^{\{4\}}$	$0.0155^{\{8\}}$	$0.01095^{\{1\}}$	$0.01532^{\{7\}}$	$0.01574^{\{9\}}$	$0.01235^{\{3\}}$	$0.01272^{\{5\}}$	$0.01702^{\{11\}}$	$0.01517^{\{6\}}$	$0.02676^{\{12\}}$	$0.01615^{\{10\}}$
	MRE	$\hat{\rho}$	$0.04465^{\{2\}}$	$0.04417^{\{1\}}$	$0.0476^{\{7\}}$	$0.04486^{\{3\}}$	$0.04789^{\{9\}}$	$0.05285^{\{11\}}$	$0.04787^{\{8\}}$	$0.04496^{\{4\}}$	$0.05479^{\{12\}}$	$0.04889^{\{10\}}$	$0.04756^{\{5.5\}}$	$0.04756^{\{5.5\}}$
		$\hat{\xi}$	$0.04331^{\{2\}}$	$0.04419^{\{3\}}$	$0.0498^{\{7\}}$	$0.04186^{\{1\}}$	$0.0501^{\{8\}}$	$0.05018^{\{9\}}$	$0.0442^{\{4\}}$	$0.04485^{\{5\}}$	$0.05167^{\{11\}}$	$0.04949^{\{6\}}$	$2.01309^{\{12\}}$	$0.05103^{\{10\}}$
	$\sum Ranks$		$13^{\{2.5\}}$	$13^{\{2.5\}}$	$42^{\{6\}}$	$11^{\{1\}}$	$48^{\{8\}}$	$60^{\{11\}}$	$32^{\{5\}}$	$27^{\{4\}}$	$69^{\{12\}}$	$46^{\{7\}}$	$51.5^{\{9\}}$	$55.5^{\{10\}}$
Continued														

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.0817^{\{2\}}$	$0.08077^{\{1\}}$	$0.08773^{\{7\}}$	$0.08176^{\{3\}}$	$0.08683^{\{5\}}$	$0.09739^{\{10\}}$	$0.08299^{\{4\}}$	$0.08862^{\{8\}}$	$0.10228^{\{12\}}$	$0.09784^{\{11\}}$	$0.08721^{\{6\}}$	$0.09601^{\{9\}}$
		$\hat{\zeta}$	$0.07524^{\{1\}}$	$0.08088^{\{3.5\}}$	$0.08906^{\{8\}}$	$0.08088^{\{3.5\}}$	$0.09534^{\{10\}}$	$0.09173^{\{9\}}$	$0.07702^{\{2\}}$	$0.0882^{\{6\}}$	$0.10124^{\{11\}}$	$0.08527^{\{5\}}$	$0.12172^{\{12\}}$	$0.08843^{\{7\}}$
	MSE	$\hat{\rho}$	$0.01062^{\{3\}}$	$0.01054^{\{2\}}$	$0.01231^{\{7\}}$	$0.01031^{\{1\}}$	$0.01191^{\{6\}}$	$0.01586^{\{11\}}$	$0.01089^{\{4\}}$	$0.01246^{\{8\}}$	$0.01674^{\{12\}}$	$0.01507^{\{10\}}$	$0.01169^{\{5\}}$	$0.01494^{\{9\}}$
		$\hat{\zeta}$	$0.00894^{\{1\}}$	$0.01056^{\{4\}}$	$0.01284^{\{8\}}$	$0.01007^{\{3\}}$	$0.01447^{\{10\}}$	$0.01311^{\{9\}}$	$0.00971^{\{2\}}$	$0.01236^{\{6\}}$	$0.01587^{\{11\}}$	$0.01132^{\{5\}}$	$0.02312^{\{12\}}$	$0.01244^{\{7\}}$
	MRE	$\hat{\rho}$	$0.04085^{\{2\}}$	$0.04038^{\{1\}}$	$0.04386^{\{8\}}$	$0.04088^{\{3\}}$	$0.04342^{\{5\}}$	$0.04869^{\{10\}}$	$0.0415^{\{4\}}$	$0.04431^{\{9\}}$	$0.05114^{\{12\}}$	$0.04892^{\{11\}}$	$0.04361^{\{6.5\}}$	$0.04361^{\{6.5\}}$
		$\hat{\zeta}$	$0.03762^{\{1\}}$	$0.04044^{\{3.5\}}$	$0.04453^{\{7\}}$	$0.04044^{\{3.5\}}$	$0.04767^{\{9\}}$	$0.04587^{\{8\}}$	$0.03851^{\{2\}}$	$0.0441^{\{6\}}$	$0.05062^{\{11\}}$	$0.04264^{\{5\}}$	$2.00624^{\{12\}}$	$0.04801^{\{10\}}$
	$\sum Ranks$		$10^{\{1\}}$	$15^{\{2\}}$	$45^{\{6.5\}}$	$17^{\{3\}}$	$45^{\{6.5\}}$	$57^{\{11\}}$	$18^{\{4\}}$	$43^{\{5\}}$	$69^{\{12\}}$	$47^{\{8\}}$	$53.5^{\{10\}}$	$48.5^{\{9\}}$
450	BIAS	$\hat{\rho}$	$0.06696^{\{4\}}$	$0.06619^{\{1\}}$	$0.07127^{\{8\}}$	$0.06643^{\{2\}}$	$0.07083^{\{7\}}$	$0.07661^{\{11\}}$	$0.06924^{\{5\}}$	$0.06668^{\{3\}}$	$0.08425^{\{12\}}$	$0.07496^{\{9\}}$	$0.07072^{\{6\}}$	$0.07637^{\{10\}}$
		$\hat{\zeta}$	$0.06117^{\{1\}}$	$0.06527^{\{4\}}$	$0.07421^{\{8\}}$	$0.06235^{\{2\}}$	$0.07348^{\{7\}}$	$0.07578^{\{10\}}$	$0.06406^{\{3\}}$	$0.06884^{\{5\}}$	$0.07612^{\{11\}}$	$0.07273^{\{6\}}$	$0.09781^{\{12\}}$	$0.07523^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00703^{\{3\}}$	$0.00688^{\{1\}}$	$0.00796^{\{8\}}$	$0.00698^{\{2\}}$	$0.00777^{\{6\}}$	$0.00905^{\{10\}}$	$0.00757^{\{5\}}$	$0.00706^{\{4\}}$	$0.01121^{\{12\}}$	$0.00903^{\{9\}}$	$0.00788^{\{7\}}$	$0.00906^{\{11\}}$
		$\hat{\zeta}$	$0.00586^{\{1\}}$	$0.00676^{\{4\}}$	$0.00889^{\{8\}}$	$0.00601^{\{2\}}$	$0.00837^{\{7\}}$	$0.00893^{\{9\}}$	$0.00663^{\{3\}}$	$0.00735^{\{5\}}$	$0.00911^{\{11\}}$	$0.00818^{\{6\}}$	$0.0159^{\{12\}}$	$0.00896^{\{10\}}$
	MRE	$\hat{\rho}$	$0.03348^{\{4\}}$	$0.03309^{\{1\}}$	$0.03563^{\{9\}}$	$0.03321^{\{2\}}$	$0.03542^{\{8\}}$	$0.03831^{\{11\}}$	$0.03462^{\{5\}}$	$0.03334^{\{3\}}$	$0.04212^{\{12\}}$	$0.03748^{\{10\}}$	$0.03536^{\{6.5\}}$	$0.03536^{\{6.5\}}$
		$\hat{\zeta}$	$0.03058^{\{1\}}$	$0.03264^{\{4\}}$	$0.0371^{\{8\}}$	$0.03118^{\{2\}}$	$0.03674^{\{7\}}$	$0.03789^{\{9\}}$	$0.03203^{\{3\}}$	$0.03442^{\{5\}}$	$0.03806^{\{10\}}$	$0.03636^{\{6\}}$	$2.00128^{\{12\}}$	$0.03819^{\{11\}}$
	$\sum Ranks$		$14^{\{2\}}$	$15^{\{3\}}$	$49^{\{8\}}$	$12^{\{1\}}$	$42^{\{6\}}$	$60^{\{11\}}$	$24^{\{4\}}$	$25^{\{5\}}$	$68^{\{12\}}$	$46^{\{7\}}$	$55.5^{\{9\}}$	$57.5^{\{10\}}$

Table 6. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 2.0,\ \zeta = 2.0$.

experiment to evaluate the performance of these estimation techniques. Our findings consistently indicate that the maximum product of the spacings method excels in estimating the parameters of our proposed model, as it consistently achieves high rankings in the simulation results.

Additionally, our research delves into this model's practical applications, particularly in modeling real data related to phenomena such as flood and reliability engineering data. Through rigorous analysis, we have

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
25	BIAS	$\hat{\rho}$	$0.03604^{\{1\}}$	$0.03705^{\{2\}}$	$0.03944^{\{5\}}$	$0.03865^{\{4\}}$	$0.03992^{\{8\}}$	$0.03988^{\{7\}}$	$0.038^{\{3\}}$	$0.03969^{\{6\}}$	$0.04055^{\{10\}}$	$0.0405^{\{9\}}$	$0.04491^{\{12\}}$	$0.04123^{\{11\}}$
		$\hat{\zeta}$	$0.35084^{\{5\}}$	$0.32836^{\{2\}}$	$0.41968^{\{10\}}$	$0.30676^{\{1\}}$	$0.38269^{\{8\}}$	$0.40203^{\{9\}}$	$0.34016^{\{4\}}$	$0.36729^{\{7\}}$	$0.36379^{\{6\}}$	$0.3346^{\{3\}}$	$0.47143^{\{12\}}$	$0.43036^{\{11\}}$
	MSE	$\hat{\rho}$	$0.00191^{\{1\}}$	$0.00201^{\{2\}}$	$0.00219^{\{5\}}$	$0.00216^{\{4\}}$	$0.00228^{\{8\}}$	$0.00224^{\{7\}}$	$0.00208^{\{3\}}$	$0.0022^{\{6\}}$	$0.00234^{\{9\}}$	$0.00237^{\{11\}}$	$0.00276^{\{12\}}$	$0.00236^{\{10\}}$
		$\hat{\zeta}$	$0.23562^{\{5\}}$	$0.19135^{\{3\}}$	$0.34316^{\{11\}}$	$0.14217^{\{1\}}$	$0.25808^{\{8\}}$	$0.296^{\{9\}}$	$0.20118^{\{4\}}$	$0.24496^{\{7\}}$	$0.23585^{\{6\}}$	$0.18758^{\{2\}}$	$0.41725^{\{12\}}$	$0.33026^{\{10\}}$
	MRE	$\hat{\rho}$	$0.36043^{\{1\}}$	$0.37054^{\{2\}}$	$0.39445^{\{5\}}$	$0.38646^{\{4\}}$	$0.39917^{\{8\}}$	$0.39877^{\{7\}}$	$0.37997^{\{3\}}$	$0.39694^{\{6\}}$	$0.40551^{\{10\}}$	$0.40499^{\{9\}}$	$0.44905^{\{11.5\}}$	$0.44905^{\{11.5\}}$
		$\hat{\zeta}$	$0.14034^{\{6\}}$	$0.13135^{\{3\}}$	$0.16787^{\{11\}}$	$0.1227^{\{2\}}$	$0.15308^{\{9\}}$	$0.16081^{\{10\}}$	$0.13606^{\{5\}}$	$0.14692^{\{8\}}$	$0.14552^{\{7\}}$	$0.13384^{\{4\}}$	$0.09116^{\{1\}}$	$0.41229^{\{12\}}$
	$\sum Ranks$		$19^{\{3\}}$	$14^{\{1\}}$	$47^{\{7\}}$	$16^{\{2\}}$	$49^{\{9.5\}}$	$49^{\{9.5\}}$	$22^{\{4\}}$	$40^{\{6\}}$	$48^{\{8\}}$	$38^{\{5\}}$	$60.5^{\{11\}}$	$65.5^{\{12\}}$
70	BIAS	$\hat{\rho}$	$0.02432^{\{1\}}$	$0.02509^{\{2\}}$	$0.0285^{\{7\}}$	$0.02742^{\{5\}}$	$0.02925^{\{9\}}$	$0.0288^{\{8\}}$	$0.02718^{\{4\}}$	$0.02686^{\{3\}}$	$0.03083^{\{11\}}$	$0.03045^{\{10\}}$	$0.03322^{\{12\}}$	$0.02819^{\{6\}}$
		$\hat{\zeta}$	$0.19546^{\{3\}}$	$0.19429^{\{2\}}$	$0.24499^{\{10\}}$	$0.19422^{\{1\}}$	$0.23949^{\{8\}}$	$0.24815^{\{11\}}$	$0.2142^{\{5\}}$	$0.21285^{\{4\}}$	$0.23079^{\{7\}}$	$0.22391^{\{6\}}$	$0.27626^{\{12\}}$	$0.24154^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00089^{\{1\}}$	$0.00098^{\{2\}}$	$0.00123^{\{7\}}$	$0.0012^{\{5\}}$	$0.00129^{\{9\}}$	$0.00126^{\{8\}}$	$0.00116^{\{4\}}$	$0.0011^{\{3\}}$	$0.00147^{\{11\}}$	$0.00145^{\{10\}}$	$0.00166^{\{12\}}$	$0.00122^{\{6\}}$
		$\hat{\zeta}$	$0.06195^{\{3\}}$	$0.05985^{\{2\}}$	$0.09847^{\{11\}}$	$0.05745^{\{1\}}$	$0.09282^{\{8\}}$	$0.09748^{\{10\}}$	$0.07508^{\{6\}}$	$0.07281^{\{4\}}$	$0.08186^{\{7\}}$	$0.07282^{\{5\}}$	$0.12797^{\{12\}}$	$0.09663^{\{9\}}$
	MRE	$\hat{\rho}$	$0.24324^{\{1\}}$	$0.25088^{\{2\}}$	$0.28503^{\{6\}}$	$0.27424^{\{5\}}$	$0.29251^{\{8\}}$	$0.28797^{\{7\}}$	$0.27184^{\{4\}}$	$0.26856^{\{3\}}$	$0.30832^{\{10\}}$	$0.30453^{\{9\}}$	$0.33218^{\{11.5\}}$	$0.33218^{\{11.5\}}$
		$\hat{\zeta}$	$0.07818^{\{3\}}$	$0.07772^{\{2\}}$	$0.09799^{\{9\}}$	$0.07769^{\{1\}}$	$0.0958^{\{8\}}$	$0.09926^{\{10\}}$	$0.08568^{\{5\}}$	$0.08514^{\{4\}}$	$0.09232^{\{7\}}$	$0.08957^{\{6\}}$	$0.10001^{\{11\}}$	$0.28191^{\{12\}}$
	$\sum Ranks$		$12^{\{1.5\}}$	$12^{\{1.5\}}$	$50^{\{7.5\}}$	$18^{\{3\}}$	$50^{\{7.5\}}$	$54^{\{11\}}$	$28^{\{5\}}$	$21^{\{4\}}$	$53^{\{9\}}$	$46^{\{6\}}$	$70.5^{\{12\}}$	$53.5^{\{10\}}$
120	BIAS	$\hat{\rho}$	$0.01952^{\{1\}}$	$0.02022^{\{2\}}$	$0.02185^{\{5\}}$	$0.02194^{\{7\}}$	$0.02263^{\{9\}}$	$0.02192^{\{6\}}$	$0.02122^{\{3\}}$	$0.0218^{\{4\}}$	$0.02542^{\{11\}}$	$0.02452^{\{10\}}$	$0.02885^{\{12\}}$	$0.02248^{\{8\}}$
		$\hat{\zeta}$	$0.15129^{\{1\}}$	$0.15407^{\{2\}}$	$0.18243^{\{8\}}$	$0.1586^{\{3\}}$	$0.18513^{\{9\}}$	$0.18076^{\{7\}}$	$0.16341^{\{4\}}$	$0.16926^{\{5\}}$	$0.18763^{\{10\}}$	$0.17299^{\{6\}}$	$0.21713^{\{12\}}$	$0.18908^{\{11\}}$
	MSE	$\hat{\rho}$	$0.00059^{\{1\}}$	$0.00065^{\{2\}}$	$0.00074^{\{4.5\}}$	$0.00078^{\{7\}}$	$0.00082^{\{9\}}$	$0.00077^{\{6\}}$	$0.00069^{\{3\}}$	$0.00074^{\{4.5\}}$	$0.00106^{\{11\}}$	$0.00101^{\{10\}}$	$0.00128^{\{12\}}$	$0.00081^{\{8\}}$
		$\hat{\zeta}$	$0.03656^{\{1\}}$	$0.03753^{\{2\}}$	$0.05306^{\{9\}}$	$0.03818^{\{3\}}$	$0.05301^{\{8\}}$	$0.05193^{\{7\}}$	$0.04192^{\{4\}}$	$0.04553^{\{5\}}$	$0.05541^{\{10\}}$	$0.04662^{\{6\}}$	$0.07314^{\{12\}}$	$0.05701^{\{11\}}$
	MRE	$\hat{\rho}$	$0.19515^{\{1\}}$	$0.20223^{\{2\}}$	$0.21846^{\{5\}}$	$0.21938^{\{7\}}$	$0.2263^{\{8\}}$	$0.21916^{\{6\}}$	$0.21222^{\{3\}}$	$0.21796^{\{4\}}$	$0.25422^{\{10\}}$	$0.24521^{\{9\}}$	$0.28851^{\{11.5\}}$	$0.28851^{\{11.5\}}$
		$\hat{\zeta}$	$0.06052^{\{1\}}$	$0.06163^{\{2\}}$	$0.07297^{\{8\}}$	$0.06344^{\{3\}}$	$0.07405^{\{9\}}$	$0.07231^{\{7\}}$	$0.06537^{\{4\}}$	$0.06771^{\{5\}}$	$0.07505^{\{10\}}$	$0.06919^{\{6\}}$	$0.10114^{\{11\}}$	$0.22484^{\{12\}}$
	$\sum Ranks$		$6^{\{1\}}$	$12^{\{2\}}$	$39.5^{\{7\}}$	$30^{\{5\}}$	$52^{\{9\}}$	$39^{\{6\}}$	$21^{\{3\}}$	$27.5^{\{4\}}$	$62^{\{11\}}$	$47^{\{8\}}$	$70.5^{\{12\}}$	$61.5^{\{10\}}$
160	BIAS	$\hat{\rho}$	$0.01602^{\{1\}}$	$0.01781^{\{2\}}$	$0.01852^{\{5\}}$	$0.01864^{\{6\}}$	$0.01916^{\{7\}}$	$0.01826^{\{4\}}$	$0.01806^{\{3\}}$	$0.01981^{\{9\}}$	$0.02215^{\{11\}}$	$0.021^{\{10\}}$	$0.0264^{\{12\}}$	$0.01967^{\{8\}}$
		$\hat{\zeta}$	$0.12331^{\{1\}}$	$0.13416^{\{3\}}$	$0.15042^{\{5\}}$	$0.13325^{\{2\}}$	$0.15116^{\{6\}}$	$0.15903^{\{10\}}$	$0.14129^{\{4\}}$	$0.15248^{\{7\}}$	$0.15714^{\{9\}}$	$0.1541^{\{8\}}$	$0.19652^{\{12\}}$	$0.16042^{\{11\}}$
	MSE	$\hat{\rho}$	$0.00041^{\{1\}}$	$0.00051^{\{2\}}$	$0.00054^{\{4\}}$	$0.00058^{\{6\}}$	$6e-04^{\{7\}}$	$0.00056^{\{5\}}$	$0.00052^{\{3\}}$	$0.00063^{\{9\}}$	$8e-04^{\{11\}}$	$7e-04^{\{10\}}$	$0.00109^{\{12\}}$	$0.00062^{\{8\}}$
		$\hat{\zeta}$	$0.0254^{\{1\}}$	$0.02811^{\{3\}}$	$0.03684^{\{7\}}$	$0.02788^{\{2\}}$	$0.03577^{\{5\}}$	$0.04196^{\{11\}}$	$0.03185^{\{4\}}$	$0.03701^{\{8\}}$	$0.03869^{\{9\}}$	$0.03619^{\{6\}}$	$0.0584^{\{12\}}$	$0.04139^{\{10\}}$
	MRE	$\hat{\rho}$	$0.16018^{\{1\}}$	$0.17808^{\{2\}}$	$0.18522^{\{5\}}$	$0.18644^{\{6\}}$	$0.19161^{\{7\}}$	$0.18257^{\{4\}}$	$0.18062^{\{3\}}$	$0.19811^{\{8\}}$	$0.22154^{\{10\}}$	$0.21004^{\{9\}}$	$0.26398^{\{11.5\}}$	$0.26398^{\{11.5\}}$
		$\hat{\zeta}$	$0.04932^{\{1\}}$	$0.05366^{\{3\}}$	$0.06017^{\{5\}}$	$0.0533^{\{2\}}$	$0.06046^{\{6\}}$	$0.06361^{\{10\}}$	$0.05652^{\{4\}}$	$0.06099^{\{7\}}$	$0.06286^{\{9\}}$	$0.06164^{\{8\}}$	$0.10115^{\{11\}}$	$0.19672^{\{12\}}$
	$\sum Ranks$		$6^{\{1\}}$	$15^{\{2\}}$	$31^{\{5\}}$	$24^{\{4\}}$	$41^{\{6\}}$	$44^{\{7\}}$	$21^{\{3\}}$	$47^{\{8\}}$	$60^{\{11\}}$	$52^{\{9\}}$	$67.5^{\{12\}}$	$59.5^{\{10\}}$
200	BIAS	$\hat{\rho}$	$0.01504^{\{1\}}$	$0.01579^{\{2\}}$	$0.01744^{\{7\}}$	$0.01635^{\{4\}}$	$0.01805^{\{9\}}$	$0.01781^{\{8\}}$	$0.01633^{\{3\}}$	$0.01703^{\{5\}}$	$0.01978^{\{11\}}$	$0.01955^{\{10\}}$	$0.02443^{\{12\}}$	$0.01704^{\{6\}}$
		$\hat{\zeta}$	$0.11592^{\{2\}}$	$0.1213^{\{3\}}$	$0.14079^{\{8\}}$	$0.11562^{\{1\}}$	$0.13998^{\{7\}}$	$0.15059^{\{11\}}$	$0.13017^{\{4\}}$	$0.13116^{\{5\}}$	$0.14604^{\{10\}}$	$0.13685^{\{6\}}$	$0.18265^{\{12\}}$	$0.14103^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00035^{\{1\}}$	$0.00039^{\{2\}}$	$0.00047^{\{7\}}$	$0.00043^{\{3.5\}}$	$5e-04^{\{8\}}$	$0.00051^{\{9\}}$	$0.00043^{\{3.5\}}$	$0.00044^{\{5.5\}}$	$0.00064^{\{11\}}$	$0.00062^{\{10\}}$	$0.00095^{\{12\}}$	$0.00044^{\{5.5\}}$
		$\hat{\zeta}$	$0.02172^{\{2\}}$	$0.0228^{\{3\}}$	$0.03174^{\{8\}}$	$0.0213^{\{1\}}$	$0.03026^{\{7\}}$	$0.03563^{\{11\}}$	$0.02753^{\{5\}}$	$0.02646^{\{4\}}$	$0.03344^{\{10\}}$	$0.02869^{\{6\}}$	$0.05065^{\{12\}}$	$0.0321^{\{9\}}$
	MRE	$\hat{\rho}$	$0.15044^{\{1\}}$	$0.15794^{\{2\}}$	$0.17437^{\{6\}}$	$0.16355^{\{4\}}$	$0.18053^{\{8\}}$	$0.17811^{\{7\}}$	$0.16333^{\{3\}}$	$0.17031^{\{5\}}$	$0.19777^{\{10\}}$	$0.19553^{\{9\}}$	$0.24435^{\{11.5\}}$	$0.24435^{\{11.5\}}$
		$\hat{\zeta}$	$0.04637^{\{2\}}$	$0.04852^{\{3\}}$	$0.05632^{\{8\}}$	$0.04625^{\{1\}}$	$0.05599^{\{7\}}$	$0.06024^{\{10\}}$	$0.05207^{\{4\}}$	$0.05246^{\{5\}}$	$0.05841^{\{9\}}$	$0.05474^{\{6\}}$	$0.09965^{\{11\}}$	$0.17042^{\{12\}}$
	$\sum Ranks$		$9^{\{1\}}$	$15^{\{3\}}$	$44^{\{6\}}$	$14.5^{\{2\}}$	$50^{\{8\}}$	$55^{\{10\}}$	$22.5^{\{4\}}$	$29.5^{\{5\}}$	$60^{\{11\}}$	$46^{\{7\}}$	$69.5^{\{12\}}$	$53^{\{9\}}$
250	BIAS	$\hat{\rho}$	$0.01321^{\{1\}}$	$0.0144^{\{3\}}$	$0.0149^{\{5\}}$	$0.01468^{\{4\}}$	$0.0159^{\{9\}}$	$0.01524^{\{7\}}$	$0.01411^{\{2\}}$	$0.01511^{\{6\}}$	$0.01811^{\{11\}}$	$0.0165^{\{10\}}$	$0.02215^{\{12\}}$	$0.01562^{\{8\}}$
		$\hat{\zeta}$	$0.10162^{\{1\}}$	$0.11011^{\{3\}}$	$0.12125^{\{7\}}$	$0.10863^{\{2\}}$	$0.12635^{\{9\}}$	$0.12752^{\{10\}}$	$0.11143^{\{4\}}$	$0.11533^{\{5\}}$	$0.13374^{\{11\}}$	$0.11881^{\{6\}}$	$0.15952^{\{12\}}$	$0.12567^{\{8\}}$
	MSE	$\hat{\rho}$	$0.00027^{\{1\}}$	$0.00033^{\{3\}}$	$0.00036^{\{5.5\}}$	$0.00035^{\{4\}}$	$4e-04^{\{9\}}$	$0.00037^{\{7.5\}}$	$0.00031^{\{2\}}$	$0.00036^{\{5.5\}}$	$0.00052^{\{11\}}$	$0.00043^{\{10\}}$	$0.00081^{\{12\}}$	$0.00037^{\{7.5\}}$
		$\hat{\zeta}$	$0.01619^{\{1\}}$	$0.01864^{\{3\}}$	$0.02394^{\{7\}}$	$0.01774^{\{2\}}$	$0.02565^{\{9\}}$	$0.02613^{\{10\}}$	$0.01977^{\{4\}}$	$0.02093^{\{5\}}$	$0.02763^{\{11\}}$	$0.02158^{\{6\}}$	$0.03911^{\{12\}}$	$0.02453^{\{8\}}$
	MRE	$\hat{\rho}$	$0.13213^{\{1\}}$	$0.14399^{\{3\}}$	$0.149^{\{5\}}$	$0.14675^{\{4\}}$	$0.15899^{\{8\}}$	$0.15244^{\{7\}}$	$0.1411^{\{2\}}$	$0.15109^{\{6\}}$	$0.18113^{\{10\}}$	$0.16502^{\{9\}}$	$0.22148^{\{11.5\}}$	$0.22148^{\{11.5\}}$
		$\hat{\zeta}$	$0.04065^{\{1\}}$	$0.04404^{\{3\}}$	$0.0485^{\{7\}}$	$0.04345^{\{2\}}$	$0.05054^{\{8\}}$	$0.05101^{\{9\}}$	$0.04457^{\{4\}}$	$0.04613^{\{5\}}$	$0.0535^{\{10\}}$	$0.04752^{\{6\}}$	$0.10029^{\{11\}}$	$0.15625^{\{12\}}$
	$\sum Ranks$		$6^{\{1\}}$	$18^{\{3\}}$	$36.5^{\{6\}}$	$18^{\{3\}}$	$55^{\{9.5\}}$	$50.5^{\{8\}}$	$18^{\{3\}}$	$32.5^{\{5\}}$	$63^{\{11\}}$	$46^{\{7\}}$	$69.5^{\{12\}}$	$55^{\{9.5\}}$
Continued														

n	Est.	Est.Par.	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_8	EM_{10}	EM_{11}	EM_{12}
300	BIAS	$\hat{\rho}$	$0.01243^{\{1\}}$	$0.01326^{\{3\}}$	$0.01409^{\{7\}}$	$0.01284^{\{2\}}$	$0.01429^{\{8\}}$	$0.0138^{\{6\}}$	$0.01333^{\{4\}}$	$0.01377^{\{5\}}$	$0.01597^{\{11\}}$	$0.01502^{\{10\}}$	$0.02008^{\{12\}}$	$0.01492^{\{9\}}$
		$\hat{\zeta}$	$0.09456^{\{1\}}$	$0.10097^{\{3\}}$	$0.1128^{\{7\}}$	$0.09597^{\{2\}}$	$0.11449^{\{8\}}$	$0.11846^{\{10\}}$	$0.10292^{\{4\}}$	$0.10598^{\{5\}}$	$0.11765^{\{9\}}$	$0.11^{\{6\}}$	$0.14791^{\{12\}}$	$0.12009^{\{11\}}$
	MSE	$\hat{\rho}$	$0.00024^{\{1\}}$	$0.00028^{\{3\}}$	$0.00031^{\{6.5\}}$	$0.00027^{\{2\}}$	$0.00033^{\{8\}}$	$0.00031^{\{6.5\}}$	$0.00029^{\{4\}}$	$3e-04^{\{5\}}$	$0.00042^{\{11\}}$	$0.00037^{\{10\}}$	$0.00067^{\{12\}}$	$0.00035^{\{9\}}$
		$\hat{\zeta}$	$0.0141^{\{1\}}$	$0.01671^{\{3\}}$	$0.01996^{\{7\}}$	$0.01419^{\{2\}}$	$0.02043^{\{8\}}$	$0.02288^{\{11\}}$	$0.01717^{\{4\}}$	$0.01786^{\{5\}}$	$0.02187^{\{9\}}$	$0.01876^{\{6\}}$	$0.03463^{\{12\}}$	$0.02254^{\{10\}}$
	MRE	$\hat{\rho}$	$0.1243^{\{1\}}$	$0.13257^{\{3\}}$	$0.14089^{\{7\}}$	$0.12844^{\{2\}}$	$0.14289^{\{8\}}$	$0.13799^{\{6\}}$	$0.13325^{\{4\}}$	$0.13772^{\{5\}}$	$0.15969^{\{10\}}$	$0.15017^{\{9\}}$	$0.20078^{\{11.5\}}$	$0.20078^{\{11.5\}}$
		$\hat{\zeta}$	$0.03782^{\{1\}}$	$0.04039^{\{3\}}$	$0.04512^{\{7\}}$	$0.03839^{\{2\}}$	$0.0458^{\{8\}}$	$0.04739^{\{10\}}$	$0.04117^{\{4\}}$	$0.04239^{\{5\}}$	$0.04706^{\{9\}}$	$0.044^{\{6\}}$	$0.10077^{\{11\}}$	$0.14916^{\{12\}}$
	$\sum Ranks$		$6^{\{1\}}$	$18^{\{3\}}$	$40.5^{\{6\}}$	$12^{\{2\}}$	$47^{\{8\}}$	$48.5^{\{9\}}$	$24^{\{4\}}$	$37^{\{5\}}$	$58^{\{10\}}$	$46^{\{7\}}$	$69.5^{\{12\}}$	$61.5^{\{11\}}$
450	BIAS	$\hat{\rho}$	$0.01029^{\{1\}}$	$0.01042^{\{2\}}$	$0.01141^{\{7\}}$	$0.01056^{\{4\}}$	$0.01104^{\{5\}}$	$0.01157^{\{8\}}$	$0.01054^{\{3\}}$	$0.01123^{\{6\}}$	$0.01366^{\{11\}}$	$0.01264^{\{10\}}$	$0.0172^{\{12\}}$	$0.01169^{\{9\}}$
		$\hat{\zeta}$	$0.07752^{\{1\}}$	$0.08168^{\{4\}}$	$0.0917^{\{8\}}$	$0.07775^{\{2\}}$	$0.08849^{\{6\}}$	$0.09742^{\{10\}}$	$0.08004^{\{3\}}$	$0.08794^{\{5\}}$	$0.09992^{\{11\}}$	$0.09117^{\{7\}}$	$0.12541^{\{12\}}$	$0.09544^{\{9\}}$
	MSE	$\hat{\rho}$	$0.00016^{\{1\}}$	$0.00017^{\{2.5\}}$	$2e-04^{\{6.5\}}$	$0.00017^{\{2.5\}}$	$0.00019^{\{5\}}$	$0.00021^{\{8.5\}}$	$0.00018^{\{4\}}$	$2e-04^{\{6.5\}}$	$3e-04^{\{11\}}$	$0.00026^{\{10\}}$	$5e-04^{\{12\}}$	$0.00021^{\{8.5\}}$
		$\hat{\zeta}$	$0.00955^{\{2\}}$	$0.01044^{\{4\}}$	$0.01314^{\{8\}}$	$0.0094^{\{1\}}$	$0.0125^{\{6\}}$	$0.01526^{\{10\}}$	$0.00988^{\{3\}}$	$0.01235^{\{5\}}$	$0.01573^{\{11\}}$	$0.01305^{\{7\}}$	$0.02518^{\{12\}}$	$0.01429^{\{9\}}$
	MRE	$\hat{\rho}$	$0.10295^{\{1\}}$	$0.10416^{\{2\}}$	$0.11408^{\{7\}}$	$0.10564^{\{4\}}$	$0.11041^{\{5\}}$	$0.11572^{\{8\}}$	$0.10543^{\{3\}}$	$0.11231^{\{6\}}$	$0.13656^{\{10\}}$	$0.12645^{\{9\}}$	$0.172^{\{11.5\}}$	$0.172^{\{11.5\}}$
		$\hat{\zeta}$	$0.03101^{\{1\}}$	$0.03267^{\{4\}}$	$0.03668^{\{8\}}$	$0.0311^{\{2\}}$	$0.0354^{\{6\}}$	$0.03897^{\{9\}}$	$0.03201^{\{3\}}$	$0.03518^{\{5\}}$	$0.03997^{\{10\}}$	$0.03647^{\{7\}}$	$0.1001^{\{11\}}$	$0.11688^{\{12\}}$
	$\sum Ranks$		$7^{\{1\}}$	$18.5^{\{3\}}$	$47.5^{\{7\}}$	$15.5^{\{2\}}$	$33^{\{5\}}$	$51.5^{\{9\}}$	$19^{\{4\}}$	$36.5^{\{6\}}$	$64^{\{11\}}$	$48^{\{8\}}$	$70.5^{\{12\}}$	$57^{\{10\}}$

Table 7. Numerical values of simulation measures (BIAS, MSE and MRE) for $\rho = 0.1,\ \zeta = 2.5$.

demonstrated that our suggested model outperforms several competing models examined. As a result, our work stands to make a valuable contribution to the field of applied statistics, particularly in the domains of modeling for real data associated with natural calamities and reliability engineering, among others.

Parameter	n	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8	EM_9	EM_{10}	EM_{11}	EM_{12}
$\rho = 4.0,\ \zeta = 1.5$	25	3.5	2.0	9.0	1.0	7.0	11.0	6.0	5.0	8.0	3.5	10.0	12.0
	70	1.0	3.0	9.5	2.0	7.0	9.5	6.0	4.5	11.0	4.5	8.0	12.0
	120	1.0	3.0	7.0	2.0	12.0	11.0	4.0	5.0	9.0	6.0	10.0	8.0
	160	3.0	1.5	8.0	1.5	7.0	12.0	5.0	4.0	11.0	6.0	9.0	10.0
	200	2.0	5.0	9.0	1.0	6.5	12.0	3.0	4.0	11.0	6.5	8.0	10.0
	250	1.0	3.5	10.0	2.0	7.0	9.0	5.0	3.5	12.0	6.0	8.0	11.0
	300	1.0	3.0	7.0	2.0	9.0	12.0	4.0	5.0	11.0	6.0	8.0	10.0
	450	2.0	3.0	6.0	1.0	7.0	11.0	4.5	4.5	12.0	9.0	8.0	10.0
$\rho = 0.5,\ \zeta = 3.0$	25	1.0	4.0	8.0	2.0	9.0	6.0	3.0	5.0	10.0	7.0	12.0	11.0
	70	2.0	1.0	9.0	4.0	7.0	6.0	3.0	5.0	11.0	8.0	12.0	10.0
	120	2.0	1.0	6.0	3.0	7.0	9.0	4.0	5.0	11.0	8.0	12.0	10.0
	160	1.0	4.0	5.0	2.0	8.0	9.0	3.0	6.0	11.0	10.0	12.0	7.0
	200	1.0	2.0	7.0	3.0	8.5	6.0	4.0	5.0	11.0	8.5	12.0	10.0
	250	1.5	1.5	6.0	3.0	9.0	5.0	4.0	7.0	11.0	8.0	12.0	10.0
	300	1.5	3.0	6.0	1.5	5.0	7.0	4.0	8.0	11.0	9.0	12.0	10.0
	450	4.0	3.0	5.0	1.0	7.0	8.0	2.0	6.0	11.0	9.0	12.0	10.0
$\rho = 0.75,\ \zeta = 0.25$	25	2.0	4.0	10.0	1.0	3.0	7.0	6.0	5.0	11.0	8.0	12.0	9.0
	70	3.0	4.0	5.0	1.0	7.0	9.0	2.0	6.0	11.0	8.0	12.0	10.0
	120	2.5	2.5	9.0	1.0	6.0	7.0	4.0	5.0	11.0	8.0	12.0	10.0
	160	3.0	1.0	7.0	2.0	9.0	6.0	4.0	5.0	11.0	8.0	12.0	10.0
	200	4.0	1.0	9.0	2.5	6.0	7.0	2.5	5.0	11.0	8.0	12.0	10.0
	250	1.5	3.0	5.0	1.5	9.0	10.0	4.0	6.0	11.0	8.0	12.0	7.0
	300	1.0	2.0	7.0	4.0	5.0	9.0	3.0	6.0	11.0	8.0	12.0	10.0
	450	3.0	8.0	7.0	5.0	2.0	6.0	1.0	4.0	11.0	9.0	12.0	10.0
$\rho = 2.5,\ \zeta = 0.5$	25	5.0	2.5	10.0	1.0	9.0	12.0	2.5	6.0	8.0	4.0	7.0	11.0
	70	4.0	2.0	9.0	1.0	7.5	12.0	3.0	5.0	11.0	7.5	6.0	10.0
	120	2.0	3.0	8.0	1.0	9.0	11.0	5.0	4.0	12.0	7.0	6.0	10.0
	160	1.0	3.0	9.0	2.0	6.5	11.0	5.0	4.0	12.0	6.5	8.0	10.0
	200	2.0	3.0	8.0	1.0	10.0	11.0	4.0	5.0	12.0	6.0	7.0	9.0
	250	1.0	3.5	10.0	3.5	9.0	11.0	5.0	2.0	12.0	6.0	8.0	7.0
	300	2.0	3.5	10.0	1.0	9.0	11.0	5.0	3.5	12.0	7.0	6.0	8.0
	450	1.0	3.0	9.0	2.0	7.0	11.0	4.0	5.0	12.0	10.0	6.0	8.0
$\rho = 2.0,\ \zeta = 2.0$	25	6.0	2.0	11.0	1.0	5.0	12.0	3.0	8.0	9.0	4.0	7.0	10.0
	70	2.0	4.0	10.0	1.0	6.0	11.0	3.0	5.0	12.0	8.0	7.0	9.0
	120	2.0	4.0	11.0	1.0	6.0	10.0	3.0	5.0	12.0	7.0	9.0	8.0
	160	1.0	3.0	7.0	2.0	6.0	11.0	5.0	4.0	12.0	8.0	9.0	10.0
	200	3.0	2.0	7.0	1.0	8.0	11.0	4.0	5.0	12.0	10.0	9.0	6.0
	250	2.5	2.5	6.0	1.0	8.0	11.0	5.0	4.0	12.0	7.0	9.0	10.0
	300	1.0	2.0	6.5	3.0	6.5	11.0	4.0	5.0	12.0	8.0	10.0	9.0
	450	2.0	3.0	8.0	1.0	6.0	11.0	4.0	5.0	12.0	7.0	9.0	10.0
$\rho = 0.1,\ \zeta = 2.5$	25	3.0	1.0	7.0	2.0	9.5	9.5	4.0	6.0	8.0	5.0	11.0	12.0
	70	1.5	1.5	7.5	3.0	7.5	11.0	5.0	4.0	9.0	6.0	12.0	10.0
	120	1.0	2.0	7.0	5.0	9.0	6.0	3.0	4.0	11.0	8.0	12.0	10.0
	160	1.0	2.0	5.0	4.0	6.0	7.0	3.0	8.0	11.0	9.0	12.0	10.0
	200	1.0	3.0	6.0	2.0	8.0	10.0	4.0	5.0	11.0	7.0	12.0	9.0
	250	1.0	3.0	6.0	3.0	9.5	8.0	3.0	5.0	11.0	7.0	12.0	9.5
	300	1.0	3.0	6.0	2.0	8.0	9.0	4.0	5.0	10.0	7.0	12.0	11.0
	450	1.0	3.0	7.0	2.0	5.0	9.0	4.0	6.0	11.0	8.0	12.0	10.0
$\sum$ Ranks		96.5	133.5	367.5	96.5	351.0	452.0	185.5	243.0	526.0	350.0	479.0	463.5
Overall Rank		1.5	3.0	8.0	1.5	7.0	9.0	4.0	5.0	12.0	6.0	11.0	10.0

Table 8. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of model parameters.

19.885	20.940	21.820	23.700	24.888	25.460	25.760	26.720
27.500	28.100	28.600	30.200	30.380	31.500	32.600	32.680
34.400	35.347	35.700	38.100	39.020	39.200	40.000	40.400
40.400	42.250	44.020	44.730	44.900	46.300	50.330	51.442
57.220	58.700	58.800	61.200	61.740	65.440	65.597	66.000
74.100	75.800	84.100	106.600	109.700	121.970	121.970	185.560

Table 9. The exceedances of flood peaks (in m3/s) of the Wheaton River.

30.94	18.51	16.62	51.56	22.85	22.38	19.08	49.56	17.12	10.67
25.43	10.24	27.47	14.70	14.10	29.93	27.98	36.02	19.40	14.97
22.57	12.26	18.14	18.84						

Table 10. the failure times of 24 mechanical components.

	n	Mean	Median	Skewness	Kurtosis	Range	Minimum	Maximum	Sum
First data	48.0000	51.4952	40.4000	2.0686	7.9507	165.6750	185.5600	19.8850	2471.7690
Second data	24.0000	22.9725	19.2400	1.3454	4.3599	41.3200	51.5600	10.2400	551.3400

Table 11. Conducting descriptive statistics and visualizations on the dataset.

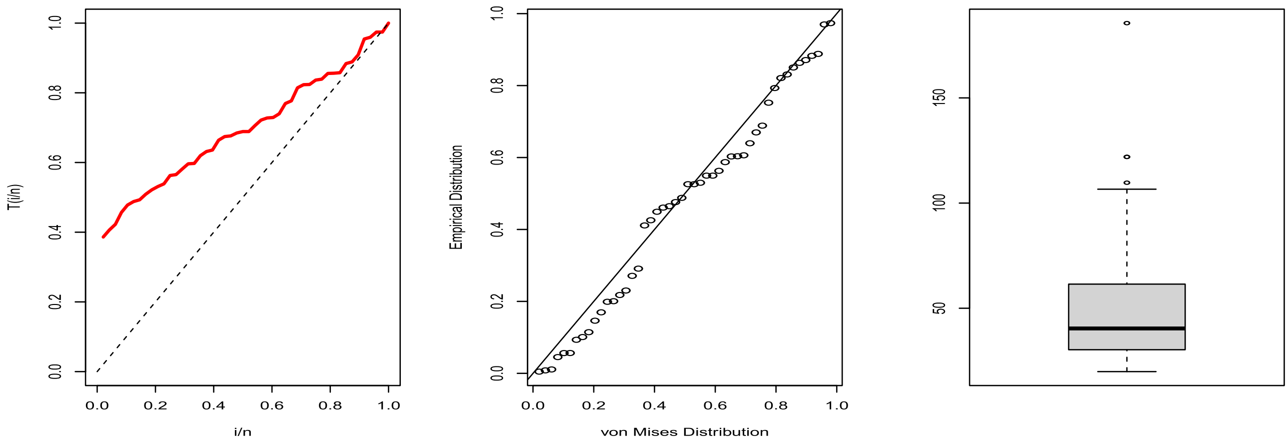

Figure 8. TTT, PP and box plots for the first data set.

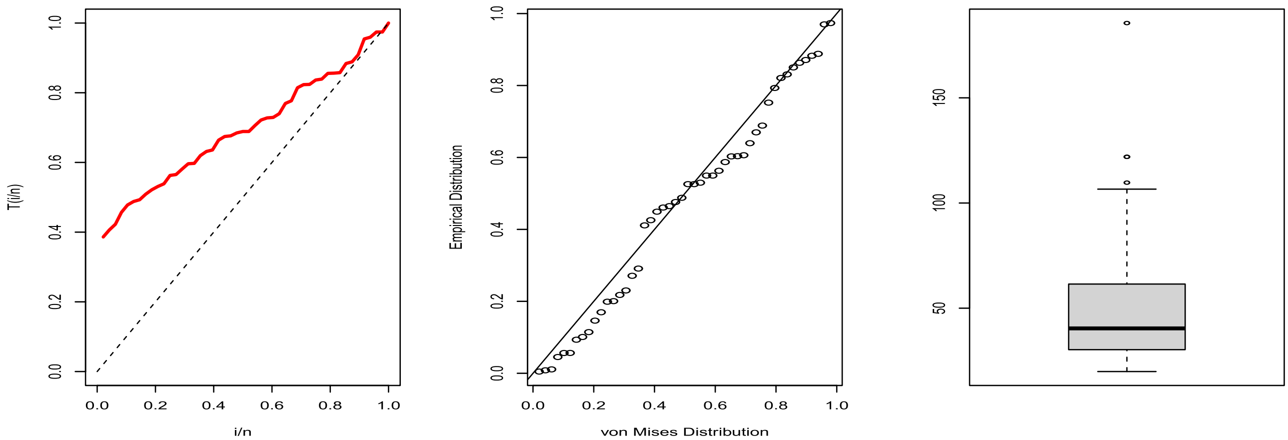

Figure 9. TTT, PP and box plots for the second data set.

Distribution	Parameters	ll
PMXLD	$\hat{\rho}$=3697.776 $\hat{\xi}$= 2.182	− 215.2514
MXLD	$\hat{\rho}$= 62.980	− 231.2627
QLD	$\hat{\theta}$=0.039 $\hat{\alpha}$= 0.007	− 225.6368
WD	$\hat{\theta}$= 1.772 $\hat{\alpha}$=58.370	− 225.7065
XGD	$\hat{\theta}$=0.058 $\hat{\alpha}$=17232	− 221.9946

Table 12. The estimated parameters and corresponding log-likelihood (ll) values for various fitting models using the first data set.

Model	AIC	CAIC	HQIC	BIC	K-S	*p*-value
PMXLD	434.5028	434.7695	438.2452	435.9171	0.0728	0.9609
MXLD	464.5253	464.6123	466.3965	465.2325	0.2645	0.0024
QLD	455.2736	455.5403	459.0160	456.6879	0.1828	0.0810
WD	455.4130	455.6797	459.1554	456.8273	0.1401	0.3025
XGD	447.9891	448.2558	451.7315	449.4034	0.1192	0.5031

Table 13. The goodness of fit tests for the first data set.

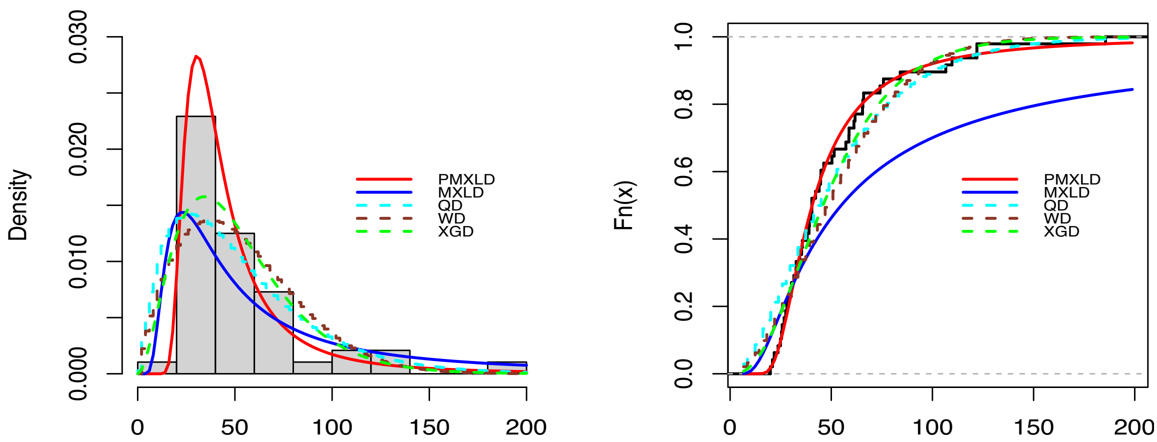

Figure 10. Plots of the estimated PDFs and CDFs for different fitting distributions using the first data set.

Distribution	Parameters	ll
PMXLD	$\hat{\rho}$=1533.369 $\hat{\xi}$= 2.421	−86.23283
MXLD	$\hat{\rho}$= 30.784	−96.5213
QLD	$\hat{\theta}$=0.086 $\hat{\alpha}$=0.016	−92.27737
WD	$\hat{\theta}$= 2.309 $\hat{\alpha}$=26.025	−88.89098
XGD	$\hat{\theta}$= 0.130 $\hat{\alpha}$=62.446	−89.13185

Table 14. The estimated parameters and corresponding log-likelihood (ll) values for various fitting models using the second data set.

Model	AIC	CAIC	HQIC	BIC	K-S	*p*-value
PMXLD	176.4657	177.0371	178.8218	177.0907	0.0928	0.9736
MXLD	195.0426	195.2244	196.2207	195.3551	0.3213	0.0105
QLD	188.5547	189.1262	190.9109	189.1798	0.2273	0.1430
WD	181.7820	182.3534	184.1381	182.4070	0.1437	0.6529
XGD	182.2637	182.8351	184.6198	182.8888	0.1559	0.5516

Table 15. The goodness of fit tests for the second data set.

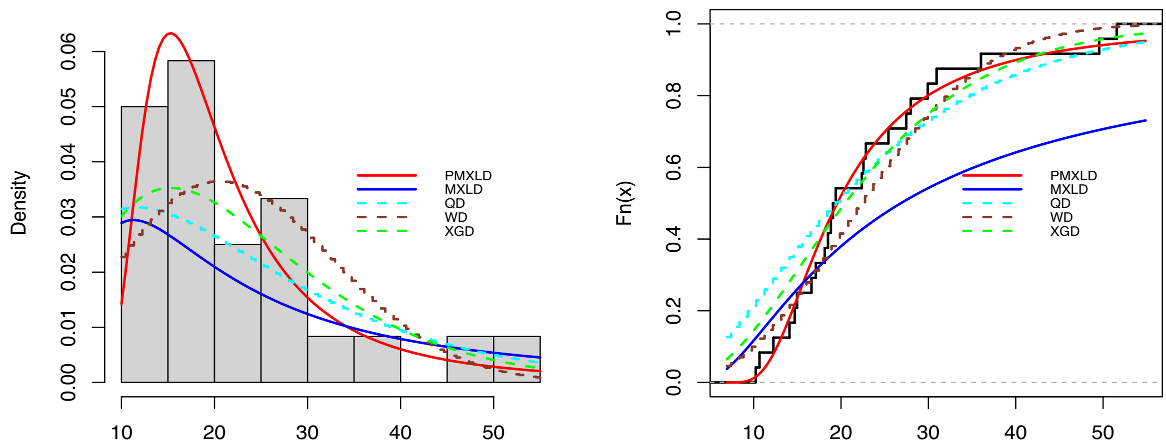

Figure 11. Plots of the estimated PDFs and CDFs for different fitting distributions using the second data set.

Data availibility

The data that supports the findings of this study are available within the article.

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Author contributions
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Competing interests
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