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Research article

Stochastic resonance noise modified decision solution for binary hypothesis-testing under minimax criterion ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, on the premise that the prior probability is unknown, a noise enhanced binary hypothesis-testing is investigated under the Minimax criterion for a general nonlinear system. Firstly, for lowering the decision risk, an additive noise is intentionally injected to the input and a decision is made under Minimax criterion based on the noise modified output. Then an optimization problem for minimizing the maximum of Bayesian conditional risk under an equality constraint is formulated via analyzing the relationship between the additive noise and the optimal noise modified Minimax decision rule. Furthermore, lemma and theorem are proposed to prove that the optimal noise is a constant vector, which simplifies the optimization problem greatly. An algorithm is also developed to search the optimal constant and the key parameter of detector, and further to determine the decision rule and the Bayes risk. Finally, simulation results about the original (in the absence of additive noise) and the noise-modified optimal decision solutions under Minimax criterion for a sine transform system are provided to illustrate the theoretical results.

1. Introduction

Although it seems counterintuitive, noise is not only harmless to the system but also beneficial for optimizing the performance under certain nonlinear conditions. Stochastic Resonance (SR) is a fairly typical theory to describe the beneficial effect of noise as an external driving force on the nonlinear system. Since the concept of SR was first given out by Benzi et al. in [1], the positive cooperative effect among the nonlinear system, signal and appropriate amount of noise has attracted intensive attention of researchers and experts in multi-fields [2–6], such as physical, chemical, optical, biomedical and information processing. With the exploration on the active effect of noise, SR has been investigated and applied in various fields, and it is more frequently called noise enhanced [7]. Compared to SR, the connotation of noise enhanced is more universal and easier to understand.

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In signal and information processing, SR phenomenon has also been widely focused and researched, especially in signal detection and parameter estimation. Through injecting additional noise or adjusting the level of background noise, the performance of a nonlinear system can be improved in forms of the increase of detection probability [7,8], output signal-to-noise ratio (SNR) [9] or mutual information (MI) [10], or the decrease of mean square error (MSE) [11–13], error probability [14] or Cramer-Rao Lower bound (CRLB) [15], etc.

To improve the SNR gain via SR mechanism was the earliest research hot spot in signal detection, and it was generally believed that a more accurate detect result could be achieved if SNR is improved. Nevertheless, there is not always positive correlation between SNR and the detection performance, a better detectability doesn't necessarily mean a higher SNR. For instance, for the detection of a dc level embedded in a Gaussian mixture noise studied by S. Kay [16], the detection probability of a suboptimal detector is increased via adding noise to the observed data, in effect decreasing the SNR. Thus, instead of SNR, the metric that directly reflects the detectability such as detection probability, error probability or Bayes risk has been focused by the noise enhanced detection in recent years, the corresponding research could be explored according to Neyman-Pearson, Maximum a posteriori (MAP), Bayesian or Minimax criterion.

In [7], a mathematical framework is developed for a fixed suboptimal detector to explore the mechanism of SR noise enhanced signal detection under the Neyman-Pearson criterion. The maximum achievable detection probability via adding an independent additive noise to the observation is studied with a constraint on false alarm probability, and the form of the optimal SR noise pdf is determined as a randomization of no more than two discrete signals. The study is further extended to variable detectors [8]. A. Patel and B. Kosko also investigated the noise benefits for Neyman-Pearson hypothesis testing from a different perspective [17], the result is consistent with [7] and an effective algorithm is presented to search the optimal noise. The SR effects are researched in [18] for binary composite hypothesis-testing problems in the Neyman-Pearson framework, in which the optimal noise enhanced detection performance is studied successively under Max-sum, Max-min and Max-max criteria with false alarm probability constraint. Noise enhanced M-ary hypothesis-testing problem is investigated in [19] under the restricted Bayesian framework, the optimal additive noise that minimizes the Bayes risk under certain constraints on the conditional risks is derived as a randomization of no more than M mass points. Furthermore, the noise enhanced detection under Bayesian and Minimax criteria could be viewed as two special cases under the restricted Bayesian framework. The exploration of reducing the error probability via adding noise is made in [14] for a suboptimal receiver. Quantum state discrimination and enhancement by noise is studied in [20], with the detectability measured in terms of the total error probability. A noise enhanced detection model which increases the detection probability and decreases the false-alarm probability simultaneously is discussed in [21], and divisional search algorithms are developed to solve the optimal SR noise under different conditions [22].

Almost all the studies mentioned above are considered to improve the performance of a suboptimal detector via adding additional noise to the observation or receive data. Since adding noise to the observation of a fixed detector can be equivalent to adjusting the detector without adding noise, the best detectability of the suboptimal detector achieved via adding SR noise will not be prior to the optimal detector, and also no noise exists to improve the optimal detector. In addition, it should be noted that the so-called optimal is a relative concept. Considering a general detection system including a nonlinear transform system, the decision result is made based on the output of nonlinear system. The optimal detector is usually defined according to the output and its performance cannot be enhanced via adding noise to the output. Nevertheless, if the additive noise is purposely introduced to the input of the nonlinear system instead of output, a new optimal detector can be achieved based on the noise modified output and the new optimal detectability may be superior to the original one (in absence of additive noise). Studies in [23] and [24] have proved this point, specifically, the optimal detection performance under the MAP and NP criteria can be improved significantly by adding additive noise to the input of the nonlinear system and adjusting the detector under certain conditions. The MAP case is applied under the condition that the conditional and prior probabilities are known, while the NP case is considered when only the prior probability is known. So far, few studies focus on the optimal noise enhanced decision solution for the case where the conditional probability and decision cost are known but the prior probability is unknown.

To fill the gap and enrich the noise enhanced signal detection theory, a binary hypothesis-testing problem with unknown prior probability for a nonlinear system studied under the Minimax criterion is considered in this paper. To explore the optimal performance obtained via the SR mechanism, a mathematical framework is established under the Minimax criterion with adding an additive noise to the input and making an optimal decision based on the noise modified output according to the Minimax criterion. Firstly, the form of the noise modified Minimax detector, i.e., the optimal detector achieved under Minimax criterion, is determined. Then an optimization problem for minimizing the maximum Bayes risk is formulated, and the corresponding optimal additive noise and the noise modified Minimax detector are derived. Meanwhile, the improvability of detection performance is also discussed. Furthermore, an algorithm for the optimal SR noise modified decision solution is developed. Finally, comparisons between the maximum conditional Bayes risks achieved in the original and SR cases are made in simulation results to show the practicality of theoretical results. The main contributions of this paper are summarized below:

•Noise enhanced Mathematical decision model under the Minimax criterion is established.

•Forms of the optimal additive noise and the noise modified Minimax detector and are derived.

•Algorithm for the optimal SR noise modified decision solution is developed.

•Simulations for the comparisons of the maximum conditional Bayes risks achieved in the original and SR cases are made.

The remainder of this paper is organized as follows. In Section 2, an optimization problem is formulated under Minimax criterion for a noise modified nonlinear detection system. The optimal SR noise and the corresponding noise modified Minimax decision rule are studied in Section 3. Finally, numerical examples are given in Section 4 and conclusions are presented in Section 5.

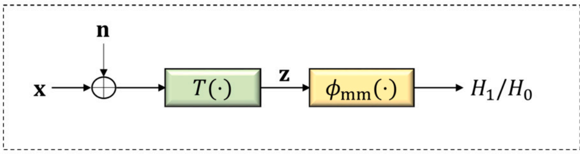

Fig. 1. A general noise modified nonlinear detection system under Minimax Framework.

2. Problem formulation

In this paper, we consider a binary hypothesis-testing problem under Minimax criterion for a general nonlinear system as described in Fig. 1. In such detection system, $\mathbf{x} \in \mathbb{R}^N$ is the original input of the nonlinear system with probability density functions (pdf) $p_i(\mathbf{x})$ under the hypothesis H_i, $i = 0, 1$, $\mathbf{n}$ is an independent additive noise purposely added to $\mathbf{x}$, $T(\cdot)$ represents the transform function of the nonlinear system, and $\mathbf{z} = T(\mathbf{x}+\mathbf{n}) \in \mathbb{R}^M$ denotes the noise modified nonlinear system output. Then a binary hypothesis-testing problem for $\mathbf{z}$ is given in Equation (1),

$$H_i : p_{\mathbf{z}}(\mathbf{z}|H_i), i = 0, 1 \tag{1}$$

where $p_{\mathbf{z}}(\mathbf{z}|H_i)$ denotes the pdf of $\mathbf{z}$ under the hypothesis H_i. Owing to the independence between $\mathbf{n}$ and $\mathbf{x}$, $p_{\mathbf{z}}(\mathbf{z}|H_i)$ can be expressed in Equation (2),

$$\begin{aligned} &p_{\mathbf{z}}(\mathbf{z}|H_i) = f_i(\mathbf{z}|p_{\mathbf{n}}) \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \delta(\mathbf{z} - T(\mathbf{x}+\mathbf{n})) p_i(\mathbf{x}) p_{\mathbf{n}}(\mathbf{n}) d\mathbf{x} d\mathbf{n} \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \delta(\mathbf{z} - T(\mathbf{x}+\mathbf{n})) p_i(\mathbf{x}) d\mathbf{x} p_{\mathbf{n}}(\mathbf{n}) d\mathbf{n} \\ &= \int_{\mathbb{R}^N} f_i(\mathbf{z}|\mathbf{n}) p_{\mathbf{n}}(\mathbf{n}) d\mathbf{n} \end{aligned} \tag{2}$$

where $\delta(\cdot)$ and $p_{\mathbf{n}}(\mathbf{n})$ respectively denote the Dirac delta function and the pdf of $\mathbf{n}$, while $f_i(\mathbf{z}|\mathbf{n}) = \int_{\mathbb{R}^N} \delta(\mathbf{z} - T(\mathbf{x}+\mathbf{n})) p_i(\mathbf{x}) d\mathbf{x}$ denotes the pdf of $\mathbf{z}$ for the case $\mathbf{n}$ is a constant vector.

Since the prior probability is unknown, the final decision is made based on the Minimax criterion, and the corresponding detector is denoted by ϕ_{mm}. From [25], the optimal test under Bayesian criterion is shown in Equation (3) below

$$\frac{p_{\mathbf{z}}(\mathbf{z}|H_1)}{p_{\mathbf{z}}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 (C_{10} - C_{00})}{(1-\pi_0)(C_{01} - C_{11})} \tag{3}$$

where π_0 denotes the prior probability of H_0, and $C_{ji} \geq 0$ denotes the cost of choosing H_j when H_i is true, $i, j = 0, 1$. Under the premise that π_0 is unknown, we first suppose that $\pi_0 = a$ and $0 \leq a \leq 1$, then a decision rule of selecting H_1 can be expressed in Equation (4),

$$\phi_{a,p_{\mathbf{n}}}(\mathbf{z}) = \begin{cases} 1, L(\mathbf{z}|p_{\mathbf{n}}) \geq \tau_a \\ 0, L(\mathbf{z}|p_{\mathbf{n}}) < \tau_a \end{cases} \tag{4}$$

where $L(\mathbf{z}|p_{\mathbf{n}}) = \frac{p_{\mathbf{z}}(\mathbf{z}|H_1)}{p_{\mathbf{z}}(\mathbf{z}|H_0)} = \frac{f_1(\mathbf{z}|p_{\mathbf{n}})}{f_0(\mathbf{z}|p_{\mathbf{n}})}$ is the Likelihood ratio function, $\tau_a = \frac{a(C_{10}-C_{00})}{(1-a)(C_{01}-C_{11})}$ is the decision threshold. Accordingly, the real Bayes risk for given a and $p_{\mathbf{n}}(\mathbf{n})$ is shown in Equation (5),

$$\begin{aligned} &R^{\mathbf{z}}(\pi_0|a, p_{\mathbf{n}}) = \pi_0 R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + (1-\pi_0) R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ &= \pi_0 \left(R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) - R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \right) + R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \end{aligned} \tag{5}$$

where $R_i(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ denotes the conditional risk under H_i. According to the definition of conditional risk, $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ can be calculated by Equations (6) and (7),

$$\begin{aligned} &R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = C_{10} P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + C_{00} P_{00}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ &= C_{10} P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + C_{00}(1 - P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})) \\ &= C_{00} + (C_{10} - C_{00}) P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \end{aligned} \tag{6}$$

$$\begin{aligned} &R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = C_{11} P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + C_{01} P_{01}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ &= C_{01} + (C_{11} - C_{01}) P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \end{aligned} \tag{7}$$

where $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ denote the probabilities of choosing H_1 when H_1 and H_0 are true, respectively.

Based on (4), the greater the τ_a, the smaller the region of selecting H_1, so $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ decrease with increasing τ_a. Furthermore, since τ_a is a monotonically increasing function of a, $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ decrease with a. In general, the cost of an error decision is greater than that of a right one, i.e., $C_{10} > C_{00}$ and $C_{11} < C_{01}$, thereby the conditional risk $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ is proportion to $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ while $R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ is in inverse proportion to $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ according to (6) and (7). In a word, $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ decreases and $R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ increases with a. When a increases from 0 to 1, $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ decreases from C_{10} to C_{00} while $R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ increases from C_{11} to C_{01}.

From (5), $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) - R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ is the slope of the function $R^{\mathbf{z}}(\pi_0|a, p_{\mathbf{n}})$ with respect to (w.r.t.) π_0, it decreases from $C_{10} - C_{11} > 0$ to $C_{00} - C_{01} < 0$ as a increases. If $R^{\mathbf{z}}(\pi_0|a, p_{\mathbf{n}})$ is differentiable w.r.t. π_0, the maximum Bayesian risk is achieved when $R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$, otherwise it is obtained at the point $a = \hat{a}$ such that $R_0^{\mathbf{z}}(\phi_{\hat{a}^-,p_{\mathbf{n}}}, p_{\mathbf{n}}) - R_1^{\mathbf{z}}(\phi_{\hat{a}^-,p_{\mathbf{n}}}, p_{\mathbf{n}}) > 0$ and $R_0^{\mathbf{z}}(\phi_{\hat{a}^+,p_{\mathbf{n}}}, p_{\mathbf{n}}) - R_1^{\mathbf{z}}(\phi_{\hat{a}^+,p_{\mathbf{n}}}, p_{\mathbf{n}}) < 0$ where $\hat{a}^- = \lim_{\varepsilon\to 0} a - \varepsilon$ and $\hat{a}^+ = \lim_{\varepsilon\to 0} a + \varepsilon$.

In this paper, we just discuss the differentiable case and the aim is to study the optimal pair of noise $\mathbf{n}$ and decision rule ϕ_{mm} in Fig. 1 under Minimax criterion, more specifically, to find the exact form of $p_{\mathbf{n}}(\mathbf{n})$ and the value of a that minimize the maximum Bayesian risk. According to the Minimax criterion, the value of a in the optimum decision rule ϕ_{mm} for a given $p_{\mathbf{n}}(\mathbf{n})$ is determined by $a_{p_{\mathbf{n}}} = \arg\min_a \left\{\max R^{\mathbf{z}}(\pi_0|a, p_{\mathbf{n}})\right\}$, and in differentiable case it can be substituted by (8) below

$$\begin{aligned} & a_{p_{\mathbf{n}}} = \arg\min_a R_i^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ & s.t. R_i^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = R_j^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \end{aligned} \tag{8}$$

where $i, j = 0, 1,\ i \neq j$. The solutions are the same for $i = 1$ and $i = 0$ owing to the equality constraint, and we take $i = 1$ for convenience in the subsequent discussion. Furthermore, the optimization problem for minimizing the maximum Bayesian risk with unknown prior probability can be formulated in (9) as below:

$$\begin{aligned} & \min_{p_{\mathbf{n}}} R_1^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, p_{\mathbf{n}}) = \min_{p_{\mathbf{n}}} \min_a R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ & s.t. R_0^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = R_1^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \end{aligned} \tag{9}$$

where $\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}$ is the noise modified Minimax decision rule achieved when an additive noise with pdf $p_{\mathbf{n}}(\mathbf{n})$ is added to $\mathbf{x}$, i.e., $\phi_{mm} = \phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}$ for a given $p_{\mathbf{n}}(\mathbf{n})$.

3. Optimal SR noise and decision rule

Combined (9) with (6) and (7), the optimization problem in (9) can be equivalently expressed by (10) below

$$\begin{aligned} & (a_{p_{\mathbf{n}}^{opt}}, p_{\mathbf{n}}^{opt}) = \arg\max_{p_{\mathbf{n}}} \max_a P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ & s.t. (C_{10} - C_{00}) P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + (C_{01} - C_{11}) P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) \\ & = C_{01} - C_{00} \end{aligned} \tag{10}$$

According to the definition of P_{11}^{z} and P_{10}^{z}, we can calculate them as shown in (11) and (12),

$$\begin{aligned} & P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = \int_{\mathbb{R}^M} \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) p_{\mathbf{z}}(\mathbf{z}|H_1) d\mathbf{z} \\ & = \int_{\mathbb{R}^M} \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) \int_{\mathbb{R}^N} f_1(\mathbf{z}|\mathbf{n}) p_{\mathbf{n}}(\mathbf{n}) dn d\mathbf{z} \\ & = \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} f_1(\mathbf{z}|\mathbf{n}) \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) d\mathbf{z} p_{\mathbf{n}}(\mathbf{n}) d\mathbf{n} \\ & = \int_{\mathbb{R}^N} P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, \mathbf{n}) p_{\mathbf{n}}(\mathbf{n}) dn \end{aligned} \tag{11}$$

$$\begin{aligned} & P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) = \int_{\mathbb{R}^M} \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) p_{\mathbf{z}}(\mathbf{z}|H_0) d\mathbf{z} \\ & = \int_{\mathbb{R}^N} \int_{\mathbb{R}^M} f_0(\mathbf{z}|\mathbf{n}) \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) d\mathbf{z} p_{\mathbf{n}}(\mathbf{n}) d\mathbf{n} \\ & = \int_{\mathbb{R}^N} P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, \mathbf{n}) p_{\mathbf{n}}(\mathbf{n}) d\mathbf{n} \end{aligned} \tag{12}$$

where $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, \mathbf{n}) = \int_{\mathbb{R}^M} f_1(\mathbf{z}|\mathbf{n}) \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) d\mathbf{z}$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, \mathbf{n}) = \int_{\mathbb{R}^M} f_0(\mathbf{z}|\mathbf{n}) \phi_{a,p_{\mathbf{n}}}(\mathbf{z}) d\mathbf{z}$ are obtained by adding a constant vector $\mathbf{n}$ to $\mathbf{x}$ and selecting $\phi_{a,p_{\mathbf{n}}}$ as the detector.

Obviously, (10) is a multi-parameter optimization problem with one equality constraint and it seems very different to directly solve the problem because it needs search all possible a and $p_{\mathbf{n}}(\mathbf{n})$. In addition, the complex connection between the detector $\phi_{a,p_{\mathbf{n}}}$ and the additive noise $\mathbf{n}$ also increases the difficulty. Fortunately, the form of $\phi_{a,p_{\mathbf{n}}}$ is known as (4) and the parameter a is uniquely determined by $p_{\mathbf{n}}(\mathbf{n})$ according to (8), so the key is still to determine the exact form of $p_{\mathbf{n}}(\mathbf{n})$. To achieve this, Lemma 1 is introduced first to discuss the relationship of a and the additive noise $\mathbf{n}$ under the equality constraint in (10).

Lemma 1. *Define a function $Y = \alpha_1 P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + \alpha_2 P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) - \alpha_3$, where $\alpha_1 = C_{10} - C_{00}$, $\alpha_2 = C_{01} - C_{11}$ and $\alpha_3 = C_{01} - C_{00}$. Then there must exist at least one value of a to make $Y = 0$ for any given $p_{\mathbf{n}}(\mathbf{n})$.*

Proof. From the definitions of $\phi_{a,p_{\mathbf{n}}}$, $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$, we have $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})\big|_{a=0} = P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})\big|_{a=0} = 1$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})\big|_{a=1} = P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})\big|_{a=1} = 0$. Then $Y\big|_{a=0} = \alpha_1 \cdot 1 + \alpha_2 \cdot 1 - \alpha_3 = \alpha_2 > 0$ and $Y\big|_{a=1} = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 - \alpha_3 = -\alpha_3 < 0$. In addition, since $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ are decreasing functions w.r.t. a for any given $p_{\mathbf{n}}(\mathbf{n})$, $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ decrease from 1 to 0 when a increases from 0 to 1. So Y is a decrease function of a. Given the above, there exists at least one value of $a \in (0, 1)$ to enable $Y = 0$.

According to Lemma 1, there is one or more values of a to satisfy the equality constraint in (10) or (8) for any given $p_{\mathbf{n}}(\mathbf{n})$, and let $G_{p_{\mathbf{n}}} = \{a|\alpha_1 P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) + \alpha_2 P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}}) - \alpha_3 = 0\}$ for ease of understanding. When $p_{\mathbf{n}}(\mathbf{n})$ is fixed, $P_{11}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ and $P_{10}^{\mathbf{z}}(\phi_{a,p_{\mathbf{n}}}, p_{\mathbf{n}})$ for $a \in G_{p_{\mathbf{n}}}$ are constants due to the monotonicity, otherwise the equality constraint cannot be established. In conclusion, $G_{p_{\mathbf{n}}}$ is the set of the solutions for the optimization problem in (8), i.e., $a_{p_{\mathbf{n}}} \in G_{p_{\mathbf{n}}}$. That also means $a_{p_{\mathbf{n}}}$ and the noise modified Minimax decision rule $\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}$ may not be unique. Once any $p_{\mathbf{n}}(\mathbf{n})$ and $a_{p_{\mathbf{n}}}$ make $\max_i R_i^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, p_{\mathbf{n}})$ less than the original maximum of conditional risk, it means the detectability can be improved via SR mechanism.

Based on the conclusions above, a theorem is shown below to present the optimal solution for the optimization problem in (10).

Theorem 1. *The minimum of the maximum Bayesian risk is achieved by adding the constant vector* $\mathbf{n}_o$ *to* $\mathbf{x}$ *and choosing* $\phi_{a_{\mathbf{n}_o},\mathbf{n}_o}$ *as decision rule, where*

$$\mathbf{n}_o = \arg\max_{\mathbf{n}} P_{11}^{\mathbf{z}}(\phi_{a_{\mathbf{n}},\mathbf{n}}, \mathbf{n}) \tag{13}$$

and $\phi_{a_{\mathbf{n}},\mathbf{n}}$ *denotes the noise modified Minimax detector corresponding to a constant* $\mathbf{n}$.

Proof. Based on the conclusion of Lemma 1, the noise modified Minimax detector $\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}$ can be achieved for any given additive noise with pdf $p_{\mathbf{n}}(\mathbf{n})$. In addition, the pdf of any random variable can approximately be expressed with Dirac delta function $\delta(\cdot)$, thereby $p_{\mathbf{n}}(\mathbf{n})$ can be formulated by $p_{\mathbf{n}} = \sum_{i=0}^{\infty} \lambda_i \delta(\mathbf{n} - \mathbf{n}_i)$ with $0 \le \lambda \le 1$ and $\sum_{i=0}^{\infty} \lambda_i = 1$. Then $P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, p_{\mathbf{n}})$ can be calculated as (14),

$$\begin{aligned}
&P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, p_{\mathbf{n}}) = \int_{\mathbb{R}^N} P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, \mathbf{n}) \sum_{i=0}^{\infty} \lambda_i \delta(\mathbf{n} - \mathbf{n}_i) d\mathbf{n} \\
&= \sum_{i=0}^{\infty} \lambda_i \int_{\mathbb{R}^N} P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, \mathbf{n}) \delta(\mathbf{n} - \mathbf{n}_i) d\mathbf{n} \\
&= \sum_{i=0}^{\infty} \lambda_i P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, \mathbf{n}_i) \\
&\le \max_{\mathbf{n}_i} P_{11}^{\mathbf{z}}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, \mathbf{n}_i) \\
&\le \max_{\mathbf{n}_i} P_{11}^{\mathbf{z}}(\phi_{a_{\mathbf{n}_i},\mathbf{n}_i}, \mathbf{n}_i)
\end{aligned} \tag{14}$$

where the last inequality holds since $\phi_{a_{\mathbf{n}_i},\mathbf{n}_i}$ denotes the optimal noise modified Minimax decision rule obtained when $\mathbf{n}_i$ is added to $\mathbf{x}$. Furthermore, we have

$$\max_{p_{\mathbf{n}}} P_{11}^{z}(\phi_{a_{p_{\mathbf{n}}},p_{\mathbf{n}}}, p_{\mathbf{n}}) \le \max_{\mathbf{n}} P_{11}^{z}(\phi_{a_{\mathbf{n}},\mathbf{n}}, \mathbf{n}) \tag{15}$$

and the equality constraint in (10) is satisfied for the two different noise modified decision solutions applied on both sides of the inequality. As a result, the optimal additive noise for the optimization problem in (10) is a constant vector, and it is denoted by $\mathbf{n}_o = \arg\max_{\mathbf{n}} P_{11}^{z}(\phi_{a_{\mathbf{n}},\mathbf{n}}, \mathbf{n})$. Furthermore, the maximum Bayesian risk is achieved by adding $\mathbf{n}_o$ to $\mathbf{x}$ and choosing $\phi_{a_{\mathbf{n}_o},\mathbf{n}_o}$ as decision rule.

According to Theorem 1, the optimization problem in (10) is simplified in (16) as below:

$$\begin{aligned}
&(a_{\mathbf{n}_o}, \mathbf{n}_o) = \arg\max_{\mathbf{n}} \max_{a} P_{11}^{\mathbf{z}}(\phi_{a,\mathbf{n}}, \mathbf{n}) \\
&s.t. (C_{10} - C_{00}) P_{10}^{\mathbf{z}}(\phi_{a,\mathbf{n}}, \mathbf{n}) + (C_{01} - C_{11}) P_{11}^{\mathbf{z}}(\phi_{a,\mathbf{n}}, \mathbf{n}) = \\
&C_{01} - C_{00}
\end{aligned} \tag{16}$$

where $\phi_{a,\mathbf{n}}(\mathbf{z})$=0, $L(\mathbf{z}|\mathbf{n}) < \tau_a$; 1, $L(\mathbf{z}|\mathbf{n}) \ge \tau_a$ and $L(\mathbf{z}|\mathbf{n}) = f_1(\mathbf{z}|\mathbf{n})/f_0(\mathbf{z}|\mathbf{n})$. Naturally, $\mathbf{n}_o$ in (16) is same with (13), and $\phi_{a_{\mathbf{n}_o},\mathbf{n}_o}$ is obtained by substituting $a_{\mathbf{n}_o}$ and $\mathbf{n}_o$ into $\phi_{a,\mathbf{n}}(\mathbf{z})$. Furthermore, inspired by Lemma 1, Algorithm 1 is provided for searching $a_{\mathbf{n}_o}$ and $\mathbf{n}_o$ by utilizing an auxiliary function $Y = \alpha_1 P_{10}^{\mathbf{z}}(\phi_{a,\mathbf{n}}, \mathbf{n}) + \alpha_2 P_{11}^{\mathbf{z}}(\phi_{a,\mathbf{n}}, \mathbf{n}) - \alpha_3$ with $\alpha_1 = C_{10} - C_{00}$, $\alpha_2 = C_{01} - C_{11}$ and $\alpha_3 = C_{01} - C_{00}$.

In step 1) of Algorithm 1, ε equals zero theoretically, but the algorithm is almost unable to converge. So ε is usually set as slightly greater than 0, such as 10^{-4} or 10^{-6}, in practical application to make the algorithm converge quickly. The smaller ε the slower the convergence speed, and the computing accuracy of a also hinges on the parameter ε. The existing optimization algorithms such as particle swarm optimization (PSO), differential evolution, and Genetic Algorithms (GA) are also available to search $a_{\mathbf{n}_o}$ and $\mathbf{n}_o$. A significant benefit of Algorithm 1 is that it can always secure a global optimal solution while the existing optimization algorithms are easy to fall into local optimum. Besides, in step 2) of Algorithm 1, although all the $\mathbf{n} \in \mathbb{R}^N$ should be searched in theory so that the work is very tremendous, the search scope of $\mathbf{n}$ can be narrowed according to the search results of adjacent data, which greatly reduces the workload and saves time. Thus, the convergence speed of Algorithm 1 is not slower than the existing in many cases.

4. Simulation results

In this section, a simple binary hypothesis-testing problem for sine transform is studied to illustrate the theoretical results, which is given in (17) below

$$\begin{cases} H_0 : x = v \\ H_1 : x = \mathrm{A} + v \end{cases} \tag{17}$$

Algorithm 1 Optimal SR noise modified decision solution.

1) Define a function to search the value of a that makes $Y = \alpha_1 P^{\mathbf{z}}_{10}(\phi_{a,\mathbf{n}}, \mathbf{n}) + \alpha_2 P^{\mathbf{z}}_{11}(\phi_{a,\mathbf{n}}, \mathbf{n}) - \alpha_3$ equal to zero for a given constant $\mathbf{n}$, the main part can be shown as below:

Let $a_s = 0$ and $a_e = 1$;
While $|a_s - a_e| > \varepsilon$
  Let $a_r = (a_s + a_e)/2$;
  If $Y\big|_{a=a_r} = 0$
    Break;
  Elseif $Y\big|_{a=a_r} > 0$
    $a_s = a_r$;
  Elseif $Y\big|_{a=a_r} < 0$
    $a_e = a_r$;
  End If
End While
The value of a_r at the end of the cycle is the solution.

2) Search all the pairs of $(a, \mathbf{n})$ that makes $Y = 0$ via call the function of 1) and let $Q = \left\{(a, \mathbf{n}) \middle| Y = \alpha_1 P^{\mathbf{z}}_{10}(\phi_{a,\mathbf{n}}, \mathbf{n}) + \alpha_2 P^{\mathbf{z}}_{11}(\phi_{a,\mathbf{n}}, \mathbf{n}) - \alpha_3 = 0, \quad n \in D\right\}$, where D denotes the search scope of $\mathbf{n}$ and it is a subset of $\mathbb{R}^N$.

3) Calculate $(a_{\mathbf{n}_o}, \mathbf{n}_o) = \arg\max_{(a,\mathbf{n})\in Q} P^{\mathbf{z}}_{11}(\phi_{a,\mathbf{n}}, \mathbf{n})$, then $(a_{\mathbf{n}_o}, \mathbf{n}_o)$ is the optimal solution of the optimization problem in (16), and determinate the corresponding decision rule is $\phi_{a_{\mathbf{n}_o}, \mathbf{n}_o}$.

4) Calculate the noise modified Bayse risk $R^{\mathbf{z}} = R^{\mathbf{z}}_i(\phi_{a_{\mathbf{n}_o}, \mathbf{n}_o}, \mathbf{n}_o)$, i =0 or 1.

where x denotes the input signal of sine transform system, A is the dc level and $A > 0$. In addition, υ denotes a Guassian background noise with pdf $p_\upsilon(\upsilon) = G(\upsilon; 0, \sigma)$, where $G(\upsilon; \mu, \sigma) = 1/\sqrt{2\pi\sigma^2}\exp(-(\upsilon - \mu)^2/2\sigma^2)$. According to Theorem 1, the optimal SR additive noise under Minimax criterion is a constant, so here a constant n is injected to the input x and the corresponding noise modified nonlinear system output is obtained by $z = \sin 2\pi(x + n)$. Then the pdf of z under H_0 and H_1 are computed respectively by (18) and (19).

$$f_0(z|n) = \frac{1}{2\pi\sqrt{1-z^2}} \sum_{k=-\infty}^{\infty} \Big[p_\upsilon(k + \frac{1}{2\pi}\arcsin z - n) + p_\upsilon(k + \frac{1}{2} - \frac{1}{2\pi}\arcsin z - n\Big] \tag{18}$$

$$f_1(z|n) = \frac{1}{2\pi\sqrt{1-z^2}} \sum_{k=-\infty}^{\infty} \Big[p_\upsilon(k + \frac{1}{2\pi}\arcsin z - n - A) + p_\upsilon(k + \frac{1}{2} - \frac{1}{2\pi}\arcsin z - n - A)\Big] \tag{19}$$

Under uniform cost assignment (UCA), namely $C_{10} = C_{01} = 1$ and $C_{00} = C_{11} = 0$, the decision rule obtained based on (4) includes an unknown parameter a on the premise that the probability of H_0 is unknown. The expressions of P^z_{11} and P^z_{10} are determined based on (11) and (12). Then Algorithm 1 is utilized to search the optimal constant n_o and the corresponding parameter a_{n_o}, and to determine the noise modified Minimax decision rule $\phi_{a_{n_o}, n_o}$. For explanatory purpose, the form of decision rule is redescribed in (20),

$$\phi(z) = \begin{cases} 1, z \in \Lambda_1 \\ 0, otherwise \end{cases} \tag{20}$$

where Λ_1 denotes the observation space of choosing H_1.

Take $A = 0.3$ and $\sigma = 0.1$ for instance, the maximums of conditional risk are 0.1691 and 0.0670 obtained respectively in the original (in the absence of additive noise) and the noise modified Minimax decision solutions. The decision risk is decreased by 60% via adding $n_o = -0.15$ to the input and adjusting observation space Λ_1 from $(-\infty, -1] \cup [0.5399, +\infty)$ to $(-\infty, -1] \cup [0, +\infty)$. Figs. 2 and 3 plot the Maximums of conditional risk for different σ and A achieved by the optimal decision rules under the Minimax criterion for the original and SR cases.

It is observed that from Fig. 2, the Maximums of conditional risk in the original and SR cases gradually increase from zero to 0.5 as σ increases. When σ closes to zero, the original Maximum of conditional risk equals to zero and it cannot be decreased by any methods. With the increase of σ, the SR approach plays a positive role in decreasing the decision risk, the decline degree increases first and then decreases to 0. When $\sigma > 0.5$, few SR phenomenon occurs.

As illustrated in Fig. 3, for $A \in (0, 0.5)$, the maximum of conditional risk obtained in SR case decreases monotonously from 0.5 to 0.0124 as A increases from 0 to 0.5, while the original maximum conditional risk decreases first from 0.5 to 0.1469 and then increases to 0.4980. When $A > 0.15$, the maximum of conditional risk obtained in SR case is lower than that in the original case, and the difference between them increases gradually with A and reaches the maximum 0.4856 when $A = 0.5$. Owing to the cyclicity of sine transform function, the maximum of conditional risk is a cycle function of A. In addition, considering the symmetry of sine function and Guassian background noise, it is also a symmetric function of A. In the original case, the cycle and the axis of symmetry are 0.5 and $A = 0.25 + 0.25k$, and they become 1 and $A = 0.5 + 0.5k$ in SR case, where $k = \pm 0, 1, 2, \ldots$. It implies that the decision risk can be decreased significantly via SR approach for any $0.15 + k < A < 0.85 + k$, namely the detection performance under Minimax criterion is improved greatly via SR mechanism.

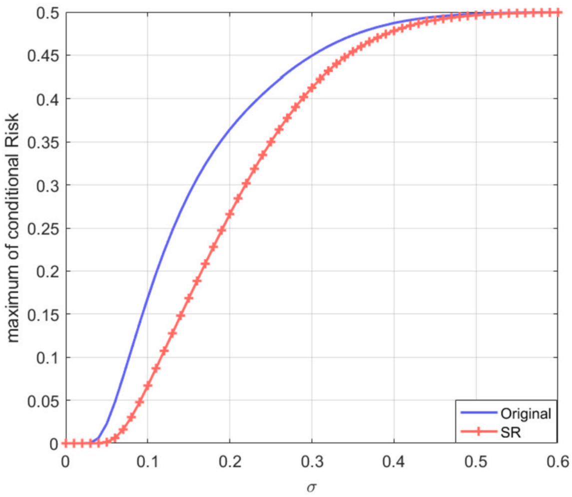

Fig. 2. Maximums of conditional risk versus σ achieved by different Minimax decision rules in the original and SR cases when $A = 0.3$. 'Original' denotes the optimal Minimax decision solution in the absence of additive noise, 'SR' denotes the optimal SR noise modified Minimax decision solution.

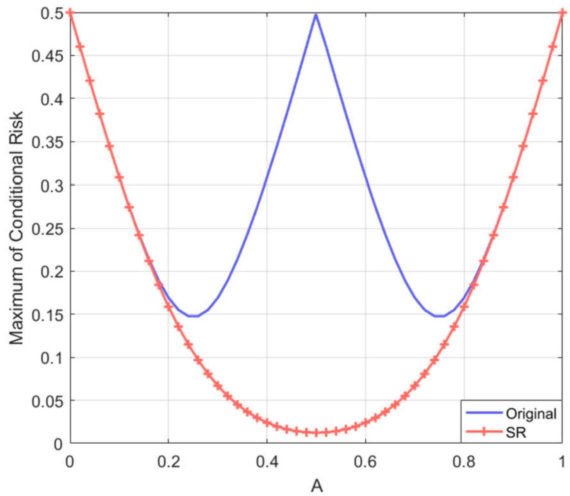

Fig. 3. Maximums of conditional risk as a function of A achieved by the Minimax decision rules in the original and SR cases when $\sigma = 0.1$.

Tables 1-4 present the original and the optimal SR noise modified Minimax decision solutions for different values of σ and A respectively so as to study the results in Figs. 2 and 3 further. From Tables 1 and 3, different values of a and observation spaces Λ_1 are applied in the original Minimax decision solution for different σ and A. As a contrast, it is observed from Tables 2 and 4, the optimal SR noise n_o is always $-A/2$ and the corresponding parameter a_{n_o} always equals to 0.5 in the noise modified Minimax decision solutions. Moreover, the noise modified observation spaces Λ_1 are also same when σ is lower than a certain value. It is obvious that detection performance is mainly depended on A when σ is small enough. The distribution of noise modified input signal under H_1 and H_0 are Gaussian distributions with mean 0 and $-A/2$, respectively, so they can match the sine transform system better to improve detection performance further. It should be noted that the optimal SR noise modified solution is not unique; for instance, when $A = 0.375$ and $\sigma = 0.1$, the Bayesian risk can also equal to 0.0313 via adding $n_o = -0.188$ to x and selecting $a_{n_o} = 0.5$ to adjust the observation space Λ_1 as $(-\infty, -1] \cup [-0.0027, +\infty)$. In addition, Tables 1-4 exhibit an interesting result that all the values of a_{n_o} are 0.5.

In the following sections, an asymmetric Gaussian mixture background noise with zero mean is considered for the same hypothesis-testing problem in (17) and its pdf is denoted by

$$p_v(v) = tG(v; -(1-t)\mu, \sigma) + (1-t)G(v; t\mu, \sigma) \tag{21}$$

where $0 \le t \le 1$. Then the pdf of z under H_i, $i = 0, 1$, can be obtained by substituting the $p_v(\cdot)$ of (18) and (19) with (21). The noise modified Minimax decision rule can also be searched by Algorithm 1 with UCA.

Fig. 4 shows the different maximums of conditional risk w.r.t. the weight t when $\sigma = 0.1$, $A = 0.3$ and $\mu = 0.5$ obtained in the original and SR cases. The SR phenomenon can be observed for almost all the values of t. As t increases from 0, the original

Table 1
Original Minimax decision solutions and Bayes risks for various values of σ when $A = 0.3$.

σ	a	Λ_1	R
0.05	0.5003	$(-\infty, -1] \cup [0.5878, +\infty)$	0.0228
0.15	0.5512	$(-\infty, -1] \cup [0.4938, +\infty)$	0.2893
0.25	0.5288	$(-\infty, -1] \cup [0.2639, +\infty)$	0.4139
0.35	0.5035	$\ldots \cup [-1.027, -1] \cup [0.0846, +\infty)$	0.4739
0.45	0.5002	$\ldots \cup [-1.074, -1] \cup [0.023, +\infty)$	0.4944

Table 2
Optimal SR noise modified Minimax decision solutions and Bayes risks for various values of σ when $A = 0.3$.

σ	n_o	a_{n_o}	Λ_1	R
0.05	-0.15	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.0013
0.15	-0.15	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.1685
0.25	-0.15	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.3500
0.35	-0.15	0.5	$\ldots \cup [-1.026, -1] \cup [0, +\infty)$	0.4541
0.45	-0.15	0.5	$\ldots \cup [-1.072, -1] \cup [0, +\infty)$	0.4905

Table 3
Original Minimax decision solutions and Bayes risks for various values of A when $\sigma = 0.1$.

A	a	Λ_1	R
0.100	0.5005	$(-\infty, -1] \cup [0.3088, +\infty)$	0.3087
0.175	0.5136	$(-\infty, -1] \cup [0.5142, +\infty)$	0.1950
0.250	0.5477	$(-\infty, -1] \cup [0.6127, +\infty)$	0.1469
0.375	0.5018	$(-\infty, -1] \cup [0.3819, +\infty)$	0.2664
0.500	0.5000	$(-\infty, -1] \cup [0, +\infty)$	0.5000

Table 4
Optimal SR noise modified Minimax decision solutions and Bayes risks for various values of A when $\sigma = 0.1$.

A	n_o	a_{n_o}	Λ_1	R
0.100	-0.0500	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.3085
0.175	-0.0875	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.1908
0.250	-0.1250	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.1057
0.375	-0.1875	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.0313
0.500	-0.2500	0.5	$(-\infty, -1] \cup [0, +\infty)$	0.0241

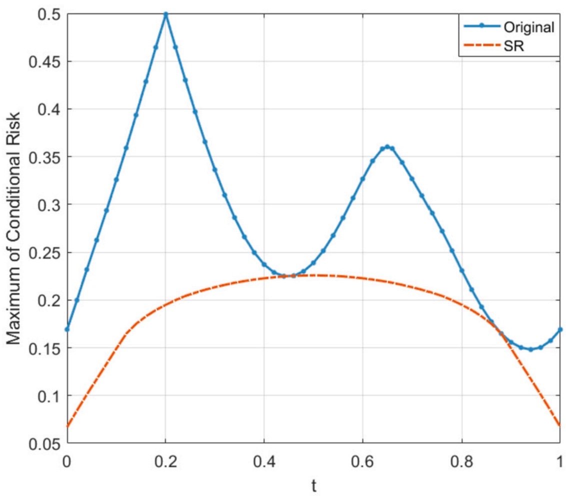

Fig. 4. Maximums of conditional risk versus t achieved by the Minimax decision rules in the original and SR cases when $\sigma = 0.1$, $A = 0.3$ and $\mu = 0.5$.

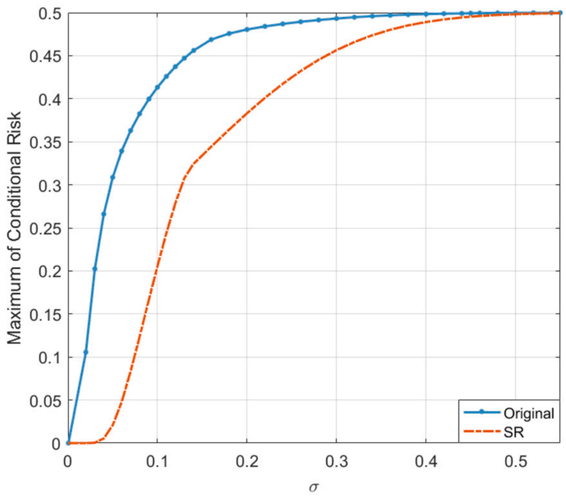

Fig. 5. Maximums of conditional risk versus σ achieved by the Minimax decision rules in the original and SR cases when $t = 0.25$, $A = 0.3$ and $\mu = 0.5$.

Table 5
Original Minimax decision solutions and Bayes risk for different t when $\sigma = 0.1$, $A = 0.3$ and $\mu = 0.5$.

t	a	Λ_1	R
0.10	0.4877	$(-\infty, -0.8790] \cup [0.5596, +\infty)$	0.3259
0.25	0.5141	$(-\infty, -1] \cup [-0.7322, 0.5614] \cup [1, +\infty)$	0.4133
0.50	0.4760	$(-\infty, -1] \cup [-0.7385, 0.7385] \cup [1, +\infty)$	0.2388
0.65	0.5394	$(-\infty, -1] \cup [-0.3192, 0.9846] \cup [1, +\infty)$	0.3603
0.94	0.4425	$(-\infty, -1] \cup [0.4769, +\infty)$	0.1481

Table 6
Optimal SR noise modified Minimax decision solutions and Bayes risks for different t when $\sigma = 0.1$, $A = 0.3$ and $\mu = 0.5$.

A	n_o	a_{n_o}	Λ_1	R
0.10	-0.14	0.3748	$(-\infty, -0.7531] \cup [0.4882, +\infty)$	0.1491
0.25	-0.16	0.4959	$(-\infty, -0.8655] \cup [0.5833, +\infty)$	0.2057
0.50	-0.27	0.4947	$(-\infty, -0.7005] \cup [0.7005, +\infty)$	0.2258
0.65	-0.35	0.4967	$(-\infty, -0.6357] \cup [0.7865, +\infty)$	0.2190
0.94	-0.07	0.4324	$(-\infty, -1] \cup [0.2576, +\infty)$	0.1171

conditional risk first linearly increases from 0.1691 to the maximum 0.5 till $t = 0.2$, then decreases to 0.2253 till $t = 0.46$, then increases to a local maximum 0.3603 till $t = 0.65$, then decreases to the minimum 0.1481 till $t = 0.94$, and then increases to 0.1691 till $t = 1$. In contrast, the SR noise modified conditional risk increases from 0.0670 to 0.2258 with t for $t \in (0, 0.5)$, and it is symmetric with $t = 0.5$. Specially, the original conditional risk equals to the noise modified one when $t = 0.46$ and $t = 0.88$. For investigating the results in Fig. 4, Tables 5 and 6 are given below to show different decision solutions in the original and SR cases. The decision risk can be reduced greatly by adding the constant noise and applying the noise modified decision rule. Special to note is that different from Tables 2 and 4, the values of a_{n_o} in Table 6 are not identically equal to 0.5.

Fig. 5 plots different maximums of conditional risk w.r.t. σ when $t = 0.25$, $A = 0.3$ and $\mu = 0.5$. The change trend is similar with that in Fig. 2. With the increase of σ, the original and the noise modified maximums of conditional risk increase gradually from 0 to 0.5, while the difference between them first increases and then decreases, the maximum is achieved at $\sigma = 0.06$ nearly. Namely, the detection performance under Minimax criterion is improved via SR effect for $\sigma < 0.5$. Maximums of conditional risk obtained by different decision rules versus A are compared in Fig. 6. The original conditional risk is symmetric with $A = 0.125$ and for $A \in (0, 0.25)$ and $A \in (0.25, 1)$, respectively. The change trends in the two intervals are similar. Specifically, the original conditional risk decreases first from 0.5 to the local or global minimum and then increases to 0.5 as A increases. In contrast, the noise modified conditional risk is symmetric with $A = 0.5$, it decreases first from 0.5 to the minimum 0.1998 at $A = 0.26$ and gradually increases to a certain degree, then decreases to a local minimum 0.2563 at $A = 0.5$. In addition, the maximum of conditional risk achieved in the SR case is lower than that achieved by the original Minimax decision rule for almost all the values of A. The maximum improvement is achieved at $A = 0.25$, where the noise enhanced Bayes risk $R^Z = 0.2005$ and it has been lowered by 59.9% compared to the original one.

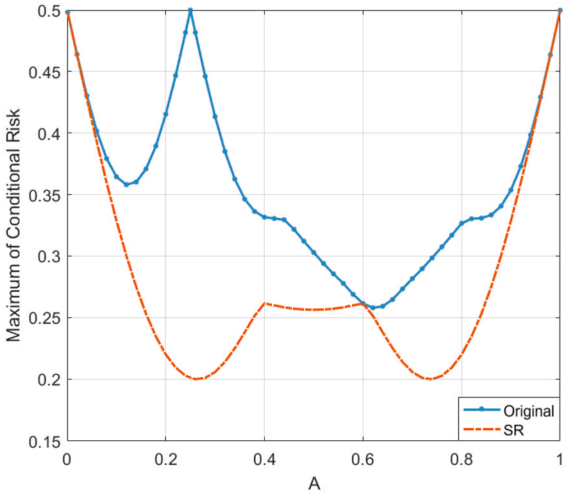

Fig. 6. Maximums of conditional risk versus A achieved by the Minimax decision rules in the original and SR cases when $t = 0.25$, $\sigma = 0.1$ and $\mu = 0.5$.

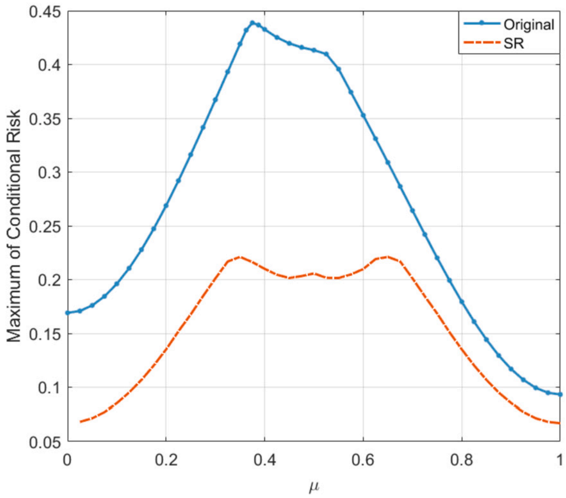

Fig. 7. Maximums of conditional risk versus μ achieved by the Minimax decision rules in the original and SR cases when $t = 0.25$, $\sigma = 0.1$ and $A = 0.3$.

Fig. 7 illustrates different maximums of conditional risk versus μ when $t = 0.25$, $\sigma = 0.1$ and $A = 0.3$. The original conditional risk increases from 0.1691 to 0.5565 as μ increases from 0 to 0.3625, and then gradually decreases to 0.0932 as μ increases to 1. As a comparison, the noise modified conditional risk is symmetric with $\mu = 0.5$; with the increase of μ in $\mu \in (0.0.5)$, it increases from 0.0670 to the maximum 0.2211 when $t = 0.35$, and then decreases to a local minimum 0.2015 when $t = 0.45$, and finally increases to a local maximum 0.2057 when $t = 0.5$. For any $\mu \in (0, 1)$, the detection performance under Minimax criterion can be greatly enhanced by adding a constant to the nonlinear system input and choosing the noise modified Minimax decision rule.

5. Conclusion

This paper focuses on a binary hypothesis-testing problem for nonlinear detection system with unknown prior probability. In order to improve the detection performance, a noise enhanced detection framework has been established under Minimax criterion. In detail, an additive noise has been purposely added to the input of system and a new optimal detector has been constructed based on the noise modified out according to Minimax criterion. Firstly, an optimization problem has been formulated to search the optimal additive noise and the corresponding noise modified minimax decision rule that minimize the maximum of Bayesian conditional risk under an equality constraint. An auxiliary function has been introduced in Lemma 1 to prove that the equality constraint can be satisfied for any additive noise. Based on this conclusion, the optimal additive noise has been derived as a constant vector in Theorem 1. Furthermore, an algorithm has been developed for searching the constant and the corresponding noise modified Minimax decision rule. An advantage of this algorithm compared with the traditional global search algorithms is that it can avoid falling into local optima. Additionally, the search range can be narrowed by utilizing the search results of neighboring data, thereby greatly saving reducing workload and increasing the convergence speed. It is worth mentioning that the optimal noise enhanced solution is not unique. Finally, a detection example for sine transform system has been researched under Gaussian and asymmetric Gaussian mixture background noises successively to show the practicality of the noise enhanced decision solution proposed in this paper.

CRediT authorship contribution statement

Ting Yang: Writing – review & editing, Methodology, Conceptualization. **Lin Liu:** Writing – original draft, Investigation. **You Xiang:** Software, Data curation. **Shujun Liu:** Supervision, Project administration, Methodology. **Wenli Zhang:** Writing – review & editing, Validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data available on request from the authors.

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