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Schramm-Loewner evolution and perimeter of percolation clusters of correlated random landscapes

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Motivated by the fact that many physical landscapes are characterized by long-range height-height correlations that are quantified by the Hurst exponent H, we investigate the statistical properties of the iso-height lines of correlated surfaces in the framework of Schramm-Loewner evolution (SLE). We show numerically that in the continuum limit the external perimeter of a percolating cluster of correlated surfaces with $H \in [-1, 0]$ is statistically equivalent to SLE curves. Our results suggest that the external perimeter also retains the Markovian properties, confirmed by the absence of time correlations in the driving function and the fact that the latter is Gaussian distributed for any specific time. We also confirm that for all H the variance of the winding angle grows logarithmically with size.

Random landscapes have been used as the basis for modeling a vast range of properties of different natural systems such as the sea surface temperature, ocean depth, height of land masses above sea level and plasma vorticity fields. Generally, such surfaces are correlated and in some cases they can have long range correlations that are characterized by the Hurst exponent, H. It has been shown recently that the iso-height lines taken at the percolation threshold of a long-range correlated random surface are scale-invariant with a fractal dimension d_f that depends on H. What still remains elusive is whether these curves have a richer symmetry in the form of conformal invariance.

As in most physical systems, symmetry plays an important role in classifying and understanding the nature of these iso-height lines and the random landscapes from which they are extracted. For this reason, conformally invariant random curves extracted at critical heights of random surfaces and their fractal properties have received a lot of attention in the last decades[1–4]. The interest in such curves was triggered by the seminal works of Schramm[5,6], who combined conformal mapping with stochastic processes into a process now known as Schramm-Loewner Evolution (*SLE*). *SLE* is a one-parameter family of non-intersecting paths exhibiting conformal invariance that can be generated from Brownian motion whose diffusivity corresponds to the *SLE* parameter, κ. It has been conjectured, and in a few cases proven, that SLE_κ is the scaling limit of a variety of discrete random processes in two-dimensional space[7]. Inversely, it provides us with an alternative way to validate existing conjectures regarding the dependence of critical exponents on the Hurst exponent in percolation[4,8]. Moreover, the SLE_κ approach allows us to generate directly such conformally invariant curves without the need to generating correlated surfaces or simulate growth models. Taking advantage of this feature, several numerical and empirical studies of correlated random systems such as turbulent vorticity fields[9,10], graphene sheets[11], topology landscapes[12], percolation in correlated surfaces[13] and accessible perimeters at fixed scale[14], have analyzed the corresponding two-dimensional random curves in the context of SLE_κ.

In what is known as *chordal SLE*, the random continuous non-intersecting curve under study is parametrized over time such that at $t = 0$ it starts from the origin located on the boundary of the upper half-plane $\mathbb{H}$ and tends to infinity as $t \to \infty$. Although such a curve does not intersect itself, in the continuum limit it might touch itself (although it still should not cross). The union of the space inside the loops formed when the trace touches itself, together with the curve up to time t is called the hull and denoted by $\mathbb{K}_t$[15]. Such a definition guarantees a simple connected domain, i.e., a domain without holes, $\mathbb{H}\backslash\mathbb{K}$, bounded by the upper half plane. According to the Riemann mapping theorem, there exists an analytical function $g_t(z)$ which maps $\mathbb{H}\backslash\mathbb{K}_t$ into $\mathbb{H}$[16]. This map satisfies the Loewner differential equation,

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$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \zeta_t}, \quad \zeta_t = \sqrt{\kappa} B_t,$$

(1)

where $g_0(z) = z$ and ζ_t is a continuum function called the *driving function*. Schramm[6] proved that if the curves are conformal invariant and follow Markov properties, then ζ_t must be a Brownian motion with a single parameter κ.

We present our study of iso-height lines of long-range correlated surfaces in the framework of chordal SLE_κ. More precisely, we investigated whether the complete perimeter of a percolating cluster obeys the SLE_κ statistics properties in the continuum limit. By taking the percolation threshold iso-height lines from correlated surfaces with $-1 \le H \le 0$[8], we find that the lines indeed do follow SLE statistics. For $H = -1$ and $H = 0$ we recover the analytical results that predict $\kappa = 6$ and $\kappa = 4$, respectively. Using the relationship between κ and d_f demonstrated by Beffara[7], we also show that the conjectured H-dependencies of the diffusivity κ and fractal dimension[4,8], d_f mutually corroborate each other. Finally, we also verify the Markov property of the curves by showing that the corresponding driving functions are uncorrelated in time and that they follow Gaussian statistics.

In this study we use the complete perimeter of the percolation clusters generated by random Gaussian surfaces, whereas previously published results were based either on different physical systems or different curves. For example, ref.[14] only deals with the accessible perimeter of percolation clusters. This is not a simple difference because an approach using the accessible perimeter does not include the effects of fjords in the calculations. Moreover, the exclusion of fjords is subject to the scale used to define the accessible perimeter, a problem that is not faced by the current study. Furthermore, our study presents a consistent positive answer to the proposed problem, based both on the evaluation of κ by the driving function within the zipper-algorithm, and by the evaluation of m using the winding angle (see below).

Method

We generate correlated random Gaussian surfaces on square lattices, with maximum lattice size $L = 4096$, by associating to each lattice site (x_1, x_2) the height $h(x) = h(x_1, x_2)$ and we use the Fourier Filtering Method (FFM)[17,18,19], in order to impose long-range correlations. Furthermore, we define the Hurst exponent associated with the correlation by choosing an appropriate power spectrum $S(\mathbf{q})$ in the form of a power law such that,

$$S(\mathbf{q}) \sim |\mathbf{q}|^{-\beta_c} = \left(\sqrt{q_1^2 + q_2^2} \right)^{-\beta_c},$$

(2)

where $\beta_c = 2(H + 1)$[17,20]. By multiplying a real-valued random variable $u(q)$ in two-dimensional Fourier space by the square-root of the power spectrum and subsequently applying the inverse Fourier transform, we obtain the correlated random Gaussian surface

$$h(\mathbf{x}) = \mathfrak{I}^{-1}[\sqrt{S(\mathbf{q})}\, u(\mathbf{q})].$$

(3)

Without loss of generality, the two-dimensional random variable $u(q)$ is taken to be Gaussian distributed with unit variance.

According to the definition above, if $H = -1$ and therefore $\beta_c = 0$, the power spectrum in Eq. (2) becomes independent of the frequency, giving rise to uncorrelated surfaces. As H is increased from -1, height-height correlations are introduced into the surface. It should be noted that, as a consequence of the extended Harris criterion[21-26], there are some critical exponents of 2D systems that are not influenced by correlation effects introduced by values of H in the interval $[-1, -1/\nu_{uncorr}]$, where ν_{uncorr} is the correlation length critical exponent for the uncorrelated percolation problem. As stated by the extended Harris criterion, if $H \le -1/\nu_{uncorr}$ then the correlations do not affect the critical exponents of the percolation transition. For 2D systems $\nu_{uncorr} = 4/3$ so that for $H \in [-1, -3/4]$ the exponents are expected to be the same as for the uncorrelated system[4,8], whereas for $H \in [-3/4, 0]$ the critical exponents are expected to depend on H.

After generating the discrete random Gaussian surfaces, we use the rank method[27] to reach the percolation threshold. One first ranks all sites of the landscapes according to height, from the smallest to the largest value. Subsequently, a ranked surface is constructed where each site has a number corresponding to its position in the ranking. Initially, all sites of the ranked surface are unoccupied. The sites are then occupied one by one, according to their rank. At each step, the fraction of occupied sites p increases by the inverse of the total number of sites, thereby changing the configuration of occupied sites. By continuing this procedure a critical height h_c is reached at which the occupied neighboring sites create a spanning cluster (percolation cluster) that connects two opposite borders of the system. At the critical height, the fraction of occupied sites reaches the percolation threshold p_c. From the percolation cluster we extract the fractal iso-height line that corresponds to the *complete perimeter*, which is the boundary between the percolating cluster and unoccupied sites. More precisely, the complete perimeter consists of all the lattice site edges that separate sites belonging to the percolating cluster from unoccupied sites that can be reached from the boundaries of the system without crossing the percolating cluster itself[4]. In the following study and analysis of the winding angle, direct SLE method and correlation time of the driving functions, we consider only the complete perimeter of the percolating cluster extracted from such correlated surfaces[8].

Winding Angle.

A simple and straight-forward necessary condition for conformal invariance is based on the statistical properties of the winding angle of the curve under study[12]. Although the presence of conformal invariance does not guarantee SLE, it is certainly a necessary condition. Since we are working with a discrete set of points that define the curve on the square lattice, we can consider the winding angle θ_i at a point z_i to be the sum of all the turning angles α_i along the curve, starting from a point z_0[28]. Therefore, the winding angle at a point z_N is given by
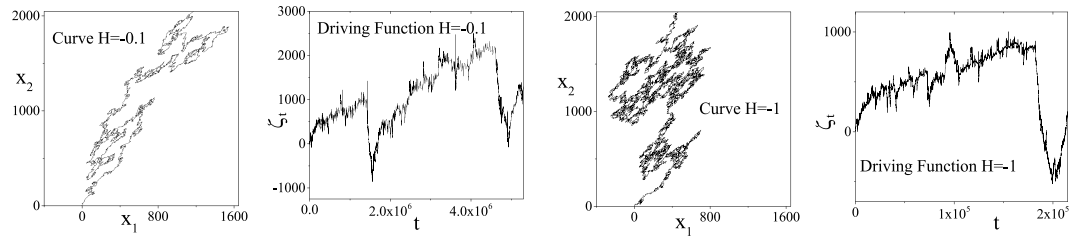
Figure 1. Examples of complete perimeters of percolating clusters for $H = -0.1$ and $H = -1$ and their respective driving functions, calculated by the zipper algorithm. A detailed scale analysis of the complete perimeter is found in the work by Janke and Weigel[26]. The jumps of the driving function reproduce the sinuosity of its respective curve.

$$\theta_N = \sum_{i=0}^{N} \alpha_i, \tag{4}$$

where α_i is the turning angle between two consecutive points on the curve. Curves that are conformally invariant have a probability distribution of the winding angle that is necessarily Gaussian with a variance that increases logarithmically with L so that

$$Var[\theta_L] = \langle \theta_L^2 \rangle - \langle \theta_L \rangle^2 = a + m \ln L, \tag{5}$$

where a is a constant. Furthermore, it has been shown that for SLE curves[6,28,29], the following relation holds:

$$m = \kappa/4. \tag{6}$$

Driving Function-Direct SLE. In order to determine whether a curve is indeed SLE_κ and estimate the value of κ, we use the zipper algorithm with a vertical slit discretization[30,31], to solve Eq. (1). So, given a discrete curve in the upper half complex plane $(0, \gamma_1, ..., \gamma_N)$, by using the inverse of $f_k(z) = g_k^{-1}(z)$[30], its driving function can be recovered by applying the relations

$$t_k = \frac{1}{4}\sum_{i=1}^{k} Im\{\omega_i\}^2 \quad \text{and} \quad \zeta_{t_k} = \sum_{i=1}^{k} Re\{\omega_i\}, \tag{7}$$

where the ω_k's are determined recursively by

$$\omega_k = f_{k-1} \times f_{k-2} \times ... \times f_1(\gamma_k), \qquad \omega_1 = \gamma_1 \tag{8}$$

and

$$f_k(z) = i\sqrt{-Im\{\omega_i\}^2 - (z - Re\{\omega_i\})^2}. \tag{9}$$

Given that even for curves with equal length and step sizes the discretized times t_k are not equally distributed, we linearly interpolate the measured driving function at equally spaced time intervals.

Results and Discussion

Our main goal is to study the properties and symmetries of the complete perimeter of the percolation cluster extracted from correlated landscapes with H in the interval $[-1, 0]$. In Fig. 1 we show examples of complete perimeters and their respective driving functions. So far, analytical results for the critical exponents have been obtained only in the cases where $H = -1$ (uncorrelated surface) and $H = 0$. Schrenk *et al.*[4] made the conjecture (see Fig. 2) that the H-dependence of the complete perimeter fractal dimension has the form $d_f(H) = 3/2 - H/3$ for $H \in [-3/4, 0]$. Moreover, the H-dependence was later also shown to be independent of the shape of the distribution of the random numbers, $u(\mathbf{q})$, used to generate the correlated landscapes[8].

It was conjectured by Rohde and Schramm[32] and demonstrated by Beffara[7] that the SLE_κ curves are fractals whose dimension, d_f, is related to the diffusion coefficient, κ, by the expression

$$d_f = min\left(1 + \frac{\kappa}{8}, 2\right). \tag{10}$$

Therefore, the accuracy of the value of κ estimated from a random curve can be verified by comparing the value of d_f obtained via Eq. (10) with the value of d_f determined directly using scale invariant methods such as the yardstick method.

For the complete perimeters we calculated the variance of the distribution of all the winding angles θ with respect to the origin of the curve in a lattice of size L. We determined the winding angle at each point z_N of the perimeter according to the definition in Eq. (4) and then calculated the variance of the resulting distribution. In Fig. 3 we present our numerical results for different lattice sizes L and different values of H. We show that for all
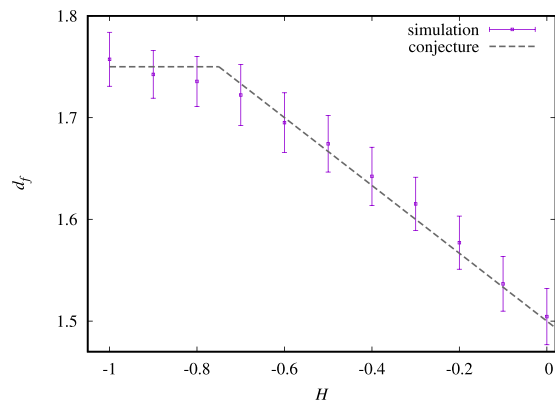
Figure 2. Fractal dimension of the full perimeter as a function of H, calculated using the yardstick method[8]. The dashed gray line represents the conjecture proposed by Schrenk *et al.*[4]. All values are averages over 10^4 samples and the error bars are defined by the variance of the distribution of the fractal dimension values.
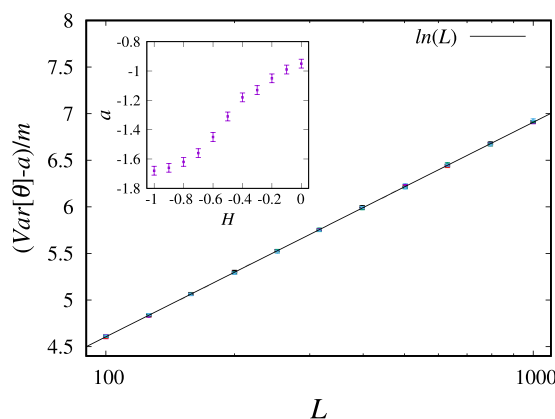

Figure 3. Rescaled variance of the winding angle distribution as a function of system size for different values of the Hurst exponent. The points (error bars are only slightly larger than the symbols) correspond to values of H in the interval $[-1, 0]$ with increments of 0.1. The results confirm the relationship in Eq. 5 and therefore support the presence of conformal invariance in the complete perimeter of percolating clusters considered in this study. Insert: Values obtained numerically for the constant a in Eq. (5) as a function of H.

values of $H \in [-1, 0]$ considered, the variance does indeed grow logarithmically with system size. The expression in Eq. (5) fits all our data, which is one condition for the curves to be conformally invariant.

Given that the winding angle test alone is not sufficient to determine whether a curve is *SLE*, we focus on the direct approach and study the properties of the driving function of the complete perimeter. In the case where the random curve is *SLE* in the scaling limit, the resulting driving function is a Brownian motion with mean square displacement that scales with time as

$$\langle \zeta_t^2 \rangle \sim \kappa t. \tag{11}$$

We therefore investigate this dependence for critical site percolation interfaces of random landscapes with H values in the interval $[-1, 0]$. As shown in Fig. 4, we obtain a good linear dependence of the variance on time. The different slopes (κ values) are due the different Hurst exponents of the random surfaces from which the curves were extracted. The mean square displacement error $\Delta \langle \zeta_t^2 \rangle$ was computed as follows:

$$\Delta \langle \zeta_t^2 \rangle = \sqrt{\frac{1}{N}\left[\langle \zeta_t^4 \rangle - \langle \zeta_t^2 \rangle^2\right]}, \qquad \langle \zeta_t^4 \rangle = \frac{1}{N}\sum_{k=1}^{N}\zeta_t^4, \tag{12}$$

where N is the total number of samples of driving functions.

In order to determine the value of κ we used the function,

$$f(t) = \kappa_{lim}t + \frac{t^d}{ct^e + b}, \qquad e > d > 0, \quad b > c > 0, \quad b, c, d, e \in \mathbb{R} \quad . \tag{13}$$
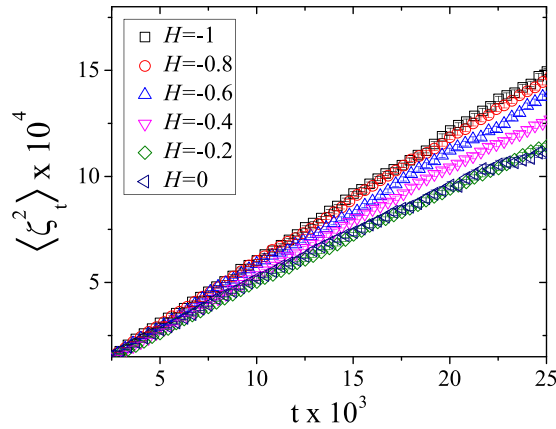
Figure 4. The linear time dependence of the mean square displacement of the driving function for different values of Hurst exponent. Without loss of information, in this plot we did not show all the data points used for the full calculation of the κ. All values are average over 10^4 samples and the error bars (inside the symbols) are defined by the variance of the mean square displacement distribution of the driving function.
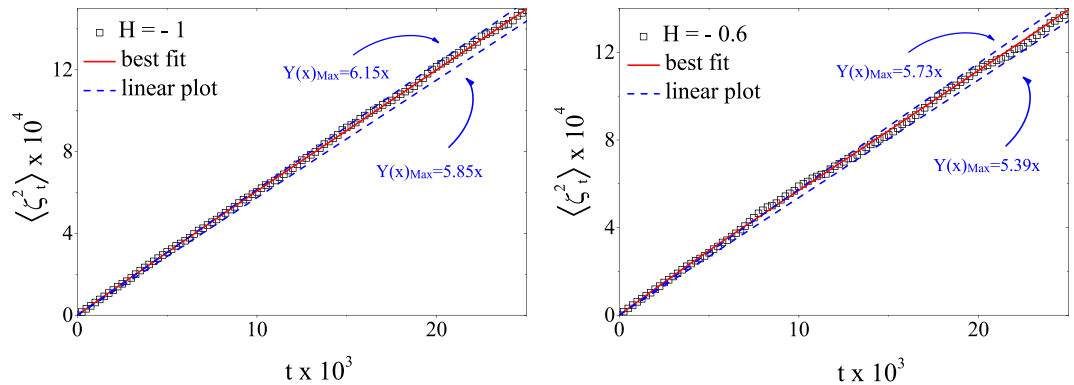

Figure 5. Linear time dependence of the mean square displacement of the driving function for two different Hurst exponents. The red line corresponds to the best fit using the model in Eq. (13). The dashed blue lines $Y_{Max}(t)$ and $Y_{Min}(t)$ are linear functions with maximum and minimum slopes (κ values) which delimit the time evolution of $\langle \zeta_t^2 \rangle$. These limits were used to define the error bars of the numerical estimate of κ, as shown in the Fig. 6.

as our model to fit the data of the time evolution of $\langle \zeta_t^2 \rangle$ (Fig. 5). The proposed form is justified by the fact that, in the small time regime, the curve has a non-linear behavior that, if not properly taken into account, could compromise the evaluation of the linear coefficient κ, which describes the behavior of $\langle \zeta_t^2 \rangle$ in the limit where $t \to \infty$. When $t \to \infty$, the linear term prevails due to the condition $d > e$, being the non linear term only relevant in the low t range. Indeed, the second term in $f(t)$ is sufficient to describe this localized effect for small values of t, and that it does not affect the asymptotic region. We then considered two straight lines $Y(t)_{Max}$ and $Y(t)_{Min}$, that bound the evolution of $\langle \zeta_t^2 \rangle$, estimating then the maximum and minimum values of κ, respectively (see Fig. 5). Finally, we calculated κ and its corresponding error with the following expression:

$$\kappa = \kappa_{lim} \pm \left(\frac{\kappa_{max} - \kappa_{min}}{2} \right).$$

(14)

Following Eq. (14), we calculated κ for a family of curves associated with different values of H. In Fig. 6 we compare the values of κ calculated numerically with the conjecture mentioned earlier in the text. Furthermore, we used the numerical results obtained for the slope m of the variance of the winding angle defined in Eq. (5) and the expression in Eq. (6) to determine indirectly the values of κ for different values of H. Similarly, Eq. (10) and the data for the fractal dimension d_f were also used to determine indirectly the values of κ. The plot on the right of Fig. 6 shows that within error bars our results are compatible with Eqs (6) and (10).

In order to confirm that a random curve is *SLE* it is not sufficient that the evolution of the mean square displacement of the corresponding driving function is linear in time as shown in Fig. 5. It is also necessary that the driving function is uncorrelated in time. We therefore tested for the Markov property of the driving function by computing its time correlation function $c(t,\tau)$, defined by:
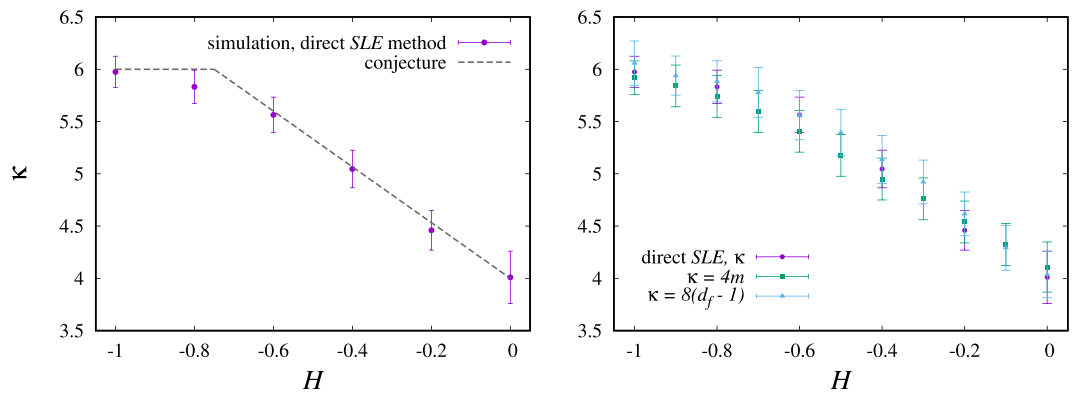
Figure 6. Left: Dependence of the diffusion coefficient (κ) on the Hurst exponent (H) estimated by Eq. (14). The values of the points were determined via the direct SLE test. The dashed gray line was derived by combining Eq. (10) with the conjecture put forward by Schrenk *et al.*[4], $\left(d_f = \frac{3}{2} - \frac{H}{3}\right)$. Right: Comparison of the values of κ obtained from the direct *SLE* method with κ determined indirectly using the data for m and the expression $m = 4\kappa$ in Eq. (6) and using the measured fractal dimension d_f with the expression $d_f = 1 + \kappa/8$ in Eq. (10).
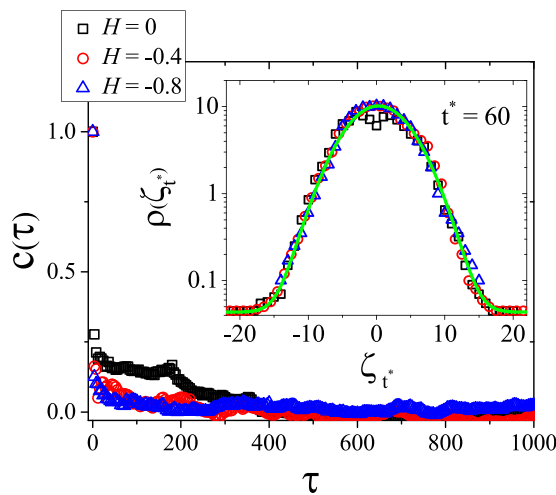
Figure 7. Correlation time of the driving function for three different values of Hurst, $H = -0.8, -0.4, 0$. The inset shows the probability density distribution $\rho(\zeta)$ for a specific Loewner time $t^* = 60$ for the same Hurst values described above. The solid green line is a guide to the eye $\rho(\zeta) = \frac{1}{\sqrt{2\pi\kappa t^*}}\exp\left(\frac{-\zeta_{t^*}^2}{2\kappa t^*}\right)$.

$$c(t, \tau) = \frac{\langle \zeta_{t+\tau}\zeta_t \rangle - \langle \zeta_{t+\tau} \rangle \langle \zeta_t \rangle}{\sqrt{\left(\langle \zeta_{t+\tau}^2 \rangle - \langle \zeta_{t+\tau} \rangle^2\right)\left(\langle \zeta_t^2 \rangle - \langle \zeta_t \rangle^2\right)}}.$$

(15)

As shown in Fig. 7 the correlation $c(t, \tau)$ goes to zero after a few times steps, as expected for Brownian motion. The short time correlation is associated to the discretization of the curve, i.e. due to the finite grid size. To complete the investigation of the Markov property we also calculated the distribution of the driving function for a specific time (t^*), which is shown to follow a Gaussian distribution (see inset of Fig. 7). It should be noted that the length of the perimeters used in our calculations is larger than 2.5×10^5 points. Therefore, the results displayed in Fig. 7 were obtained after only zippering a small fraction (0.5 to 5%) of the perimeter. The same is valid for the distribution $\rho(\zeta)$ evaluated at $t^* = 60$. Hence, justifying our claim regarding the lack of correlation.

Conclusion

Given that many systems can be viewed as long-range correlated landscapes, properties of the iso-height lines extracted from them become relevant. Our results suggest that the complete perimeter of the percolating cluster of long-range correlated landscapes ($-1 \leq H \leq 0$) is statistically equivalent to *SLE* curves. We found consistent agreements between the diffusion constant κ calculated by the zipper algorithm and the value obtained via the fractal dimension of the *SLE* curves[7]. We also proposed a new conjecture for the dependence between κ and H, in the assumed interval, on correlated random surfaces. In addition, we also showed that, in the scaling limit, the curves are Markovian in nature, in the sense that their driving functions are uncorrelated in time and Gaussian

distributed at specific points in time. A practical consequence of having established that the curves under study are *SLE* is that we can extend the established results from *SLE* theory to iso-height lines that correspond to the completer perimeter of percolating clusters. Indeed, it is possible to generate an ensemble of such curves just by solving a stochastic differential equation, without the need to generate the entire landscape.

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Author Contributions

C.P. de Castro, M. Luković, R.F.S. Andrade and H.J. Herrmann conceived the research, C.P. de Castro conducted the numerical simulations. G. Pompanin contributed to the simulations and results relative to the winding angle. All authors contributed to the writing of the manuscript.

Additional Information