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Dynamics for a fractional-order predator-prey model with group defense

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In the present article, a new fractional order predator-prey model with group defense is put up. The dynamical properties such as the existence, uniqueness and boundness of solution, the stability of equilibrium point and the existence of Hopf bifurcation of the involved predator-prey model have been discussed. Firstly, we establish the sufficient conditions that guarantee the existence, uniqueness and boundness of solution by applying Lipschitz condition, inequality technique and fractional order differential equation theory. Secondly, we analyze the existence of various equilibrium points by basic mathematical analysis method and obtain some sufficient criteria which guarantee the locally asymptotically stability of various equilibrium points of the involved predator-prey model with the aid of linearization approach. Thirdly, the existence of Hopf bifurcation of the considered predator-prey model is investigated by using the Hopf bifurcation theory of fractional order differential equations. Finally, simulation results are presented to substantiate the theoretical findings.

It is well known that the interaction of predator species and prey species is an important topic in biology and mathematical ecology. Since the classical works of Lotka and Volterra[1,2], the research on the dynamical behavior of predator-prey models has become a crucial research field for mathematicians and biologists. In the natural eco-system, different biological populations take certain measures such as group defense, refuging, escaping and so on, to maintain their own survivals or search for food. To further grasp the law of interaction of predator species and prey species, many scholars pay much attention to the group defense mechanism of the prey. For example, Falconi et al.[3] investigated the stability and global dynamic of a predator-prey model with group defense, Raw et al.[4] reveal the complex dynamical behavior of prey-predator system with group defense, Xu et al.[5] discussed the global dynamics of a predator-prey system with defense mechanism. In details, one can see[6–9].

In 2013, Venturino and Petrovskii[10] put up the following predator-prey model with group defense:

$$
\begin{cases}
\dot{u}_1(t) = \gamma_1\left[1 - \dfrac{u_1(t)}{\kappa}\right]u_1(t) - \gamma_2 u_1^\sigma(t)u_2(t), \\
\dot{u}_2(t) = -\gamma_3 u_2(t) + \gamma_2\gamma_4 u_1^\sigma(t)u_2(t),
\end{cases}
\tag{1.1}
$$

where u_1 and u_2 stand for the densities of prey and predator population, respectively. γ_1 represents the logistic growth rate, γ_2 stands for the search efficiency of predator for prey, γ_3 stands for the mortality rate of predator species, γ_4 stands for the biomass conversion coefficient, κ is the carrying capacity of the environment, and σ denotes aggregation efficiency. All the parameters $\gamma_i (i = 1, 2, 3, 4)$, κ are positive. Considering that the harvesting play an important role in describing the evolution process of a population, Kumar and Kharbanda[11] introduced the Michaelis-Menten type harvesting into predator-prey model (1.1). Then they established the following predator-prey model with group defense and Michaelis-Menten type harvesting:

$$
\begin{cases}
\dot{u}_1(t) = \gamma_1\left[1 - \dfrac{u_1(t)}{\kappa}\right]u_1(t) - \gamma_2 u_1^\sigma(t)u_2(t) - \dfrac{\rho_1\rho_2 u_1(t)}{\rho_2\rho_3 + \rho_4 u_1(t)}, \\
\dot{u}_2(t) = -\gamma_3 u_2(t) + \gamma_2\gamma_4 u_1^\sigma(t)u_2(t),
\end{cases}
\tag{1.2}
$$

where u_1 and u_2 stand for the densities of prey and predator population, respectively, ρ_1, ρ_2 stand for the catchability parameter, the effort applied to harvest the prey species, respectively, ρ_3 and ρ_4 denote appropriate real constants, $\gamma_i (i = 1, 2, 3, 4)$, κ have the same implication as those in model (1.1). Based on the biological viewpoint, we

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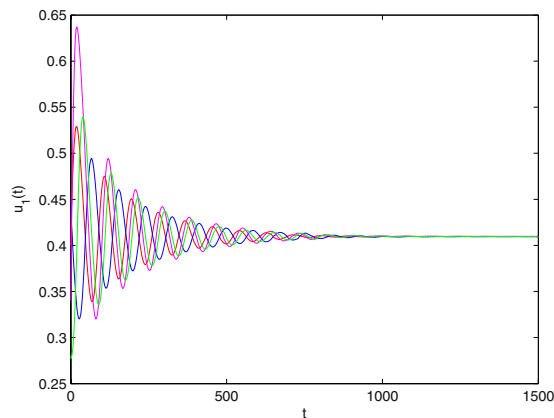
Figure 1. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$, $\theta = 0.78$. The equilibrium point (0.4096, 0.1580) of system (7.1) is asymptotically stable. The relation of t and $u_1(t)$.

assume that all the parameters are positive. Kumar and Kharbanda[11] discussed detailedly the stability and Hopf bifurcation of model (1.2).

As is known to us, fractional calculus is a generalization of classical ordinary differentiation and integration[12–16]. For a long time, the fractional calculus has been maintained a slow development state due to the lack of practical background and technical means. Recently, fractional calculus has been found to be widely applied in numerous areas such as chemical engineering, viscoelasticity, biomedical Science, robotics, physics, mechanics and control science and so on[17–21]. Moreover, fractional-order differential equations can better describe the real objective phenomena than integer-order differential equations since they possess memory and hereditary natures of different materials and processes. Thus we think that the investigation on dynamical behavior of fractional-order differential equations have important theoretical significance and broad potential value.

Model formulation

Considering that fractional calculus is a more suitable tool to describe memory and hereditary properties of numerous processes and materials and fractional-order equations are closely connected with memory for predator-prey systems[22,23]. Based on this viewpoint and inspired by the analysis in Section 1, we modify model (1.2) as the following fractional-order version:

$$
\begin{cases}
D^\theta u_1(t) = \gamma_1 \left[1 - \dfrac{u_1(t)}{\kappa} \right] u_1(t) - \gamma_2 u_1^\sigma(t) u_2(t) - \dfrac{\rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)}, \\
D^\theta u_2(t) = -\gamma_3 u_2(t) + \gamma_2 \gamma_4 u_1^\sigma(t) u_2(t),
\end{cases}
\tag{2.1}
$$

where D^θ denotes the Caputo fractional derivative and $0 < \theta < 1$. We give the initial value of model (1.2) as follows: $u_1(t_0) = u_{10} \geq 0$, $u_2(t_0) = u_{20} \geq 0$. Denote $R_+^2 = \{(u_1, u_2) \in R^2 | u_1 \geq 0, u_2 \geq 0\}$. $\Theta = \{(u_1, u_2) \in R^2 : \max\{|u_1|, |u_2|\} < K\}$.

The key task of this manuscript is to handel two aspects: (1) seek the sufficient conditions to ensure the existence, uniqueness and boundness of solution, the stability of equilibrium point and the existence of Hopf bifurcation for system (2.1); (2) reveal the effect of fractional order on the stability and the existence of Hopf bifurcation of model (2.1).

The bright spots of the manuscript include the following four aspects:

- Based on the earlier works, a new fractional-order predator-prey model with group defense is established.
- Several sufficient criteria to ensure the existence, uniqueness and boundness of solution, the stability of equilibrium point and the existence of Hopf bifurcation of the involved predator-prey model are presented. The investigation reveals that fractional-order is an pivotal parameter in affecting the Hopf bifurcation of the involved fractional-order predator-prey model.
- Skillfully constructing a suitable Lyapunov function to prove the boundness of solution has achieved great success.
- The research approach of this paper will provide useful ideas for future investigation on numerous fractional-order differential systems.

This article is planned as follows. In Sect. 2, a fractional-order predator-prey model is established. In Sect. 3, few necessary definitions and lemmas are prepared. In Sect. 4, the existence, uniqueness and boundness of solution of model (2.1) is analyzed. In Sect. 5, the stability of equilibrium point of system (2.1) is discussed in detail. Hopf bifurcation analysis is carried out in Sect. 6. Some related computer simulations are given to check effectiveness of the main findings in Sect. 7. We ends the article with a simple conclusion in Sect. 8.

Preliminary results
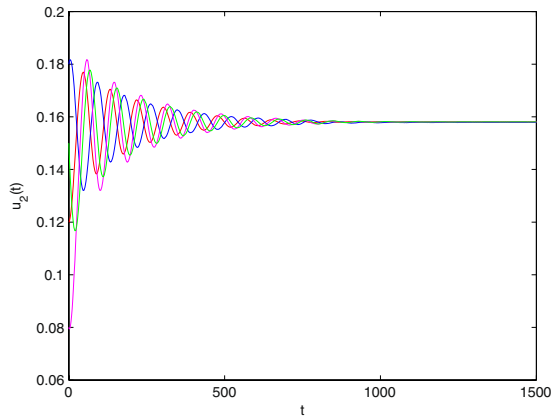
In this section, we give few definitions and lemmas.

Figure 2. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$, $\theta = 0.78$. The equilibrium point (0.4096, 0.1580) of system (7.1) is asymptotically stable. The relation of t and $u_2(t)$.
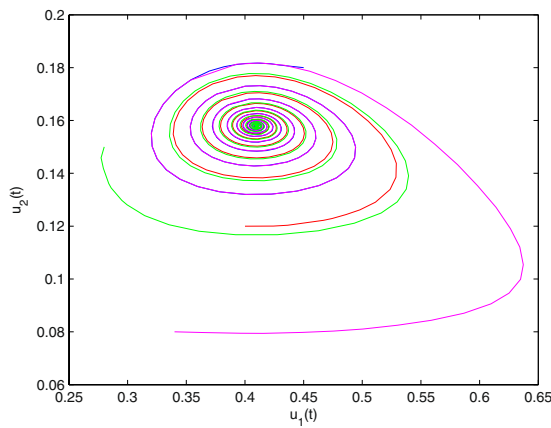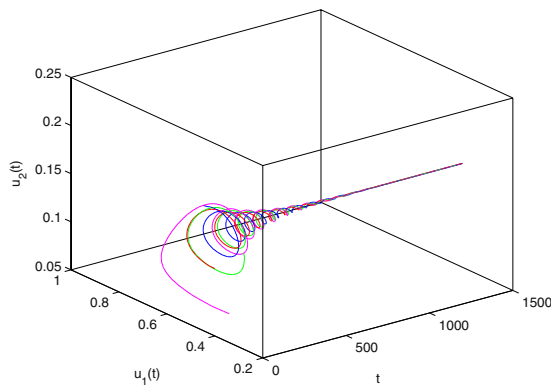

Figure 3. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$, $\theta = 0.78$. The equilibrium point (0.4096, 0.1580) of system (7.1) is asymptotically stable. The relation of $u_1(t)$ and $u_2(t)$.

Figure 4. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$, $\theta = 0.78$. The equilibrium point (0.4096, 0.1580) of system (7.1) is asymptotically stable. The relation of t, $u_1(t)$ and $u_2(t)$.

Definition 3.1[24] *The Caputo fractional derivative of order θ is defined as follows:*

$$D^\theta h(\zeta) = \frac{1}{\Gamma(l - \theta)} \int_{\zeta_0}^{\zeta} \frac{h^{(l)}(\tau)}{(\zeta - \tau)^{\theta - l + 1}} d\tau,$$
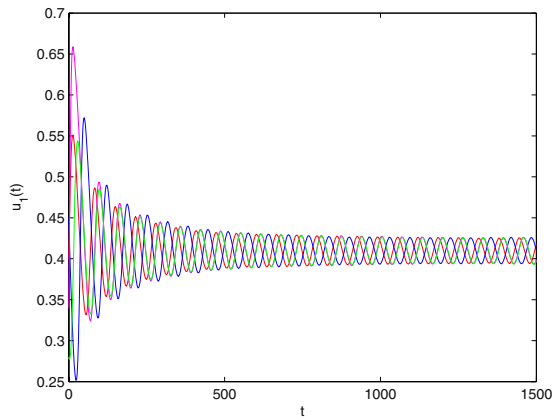
Figure 5. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$ and $\theta = 0.9 > \theta_0 = 0.83$. Hopf bifurcation of system (7.1) appears from the equilibrium point (0.4096, 0.1580). The relation of t and $u_1(t)$.
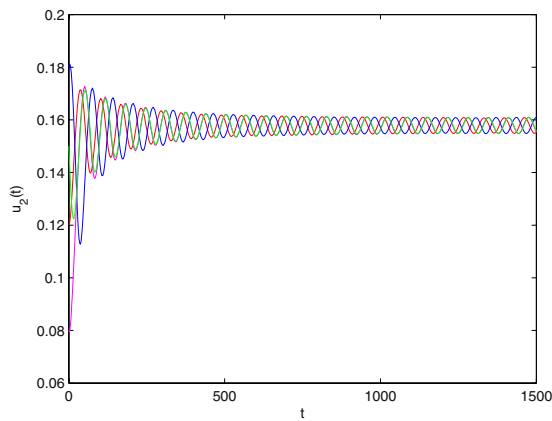

Figure 6. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$ and $\theta = 0.9 > \theta_0 = 0.83$. Hopf bifurcation of system (7.1) appears from the equilibrium point (0.4096, 0.1580). The relation of t and $u_2(t)$.
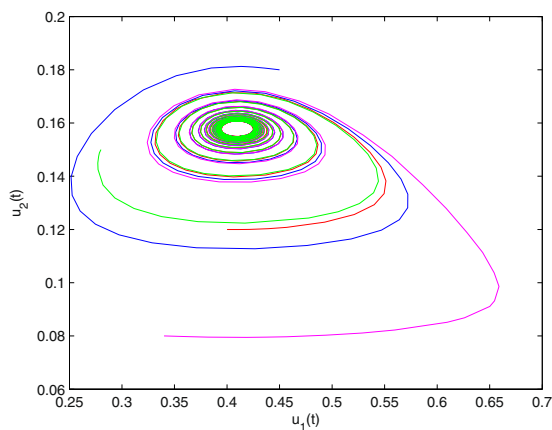

Figure 7. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$ and $\theta = 0.9 > \theta_0 = 0.83$. Hopf bifurcation of system (7.1) appears from the equilibrium point (0.4096, 0.1580). The relation of $u_1(t)$ and $u_2(t)$.

where $h(\zeta) \in ([\zeta_0, \infty), R)$, $\zeta \geq \zeta_0$ and $l \in N$ and satisfies $l - 1 \leq \theta < l$.

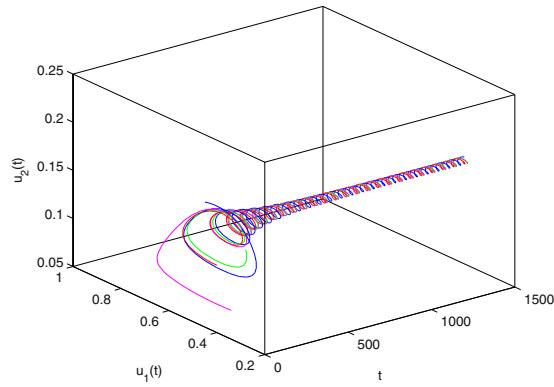Give the following Caputo fractional differential system:

Figure 8. The trajectories of system (7.1) with $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$ and $\theta = 0.9 > \theta_0 = 0.83$. Hopf bifurcation of system (7.1) appears from the equilibrium point $(0.4096, 0.1580)$. The relation of t, $u_1(t)$ and $u_2(t)$.
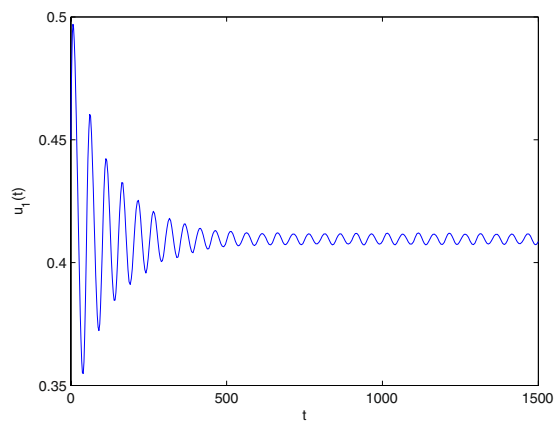

Figure 9. The trajectories of system (7.1) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation appears near the equilibrium point $(0.4168, 0.1465)$. The relation of t and $u_1(t)$.

Figure 10. The trajectories of system (7.1) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation appears near the equilibrium point $(0.4168, 0.1465)$. The relation of t and $u_2(t)$.
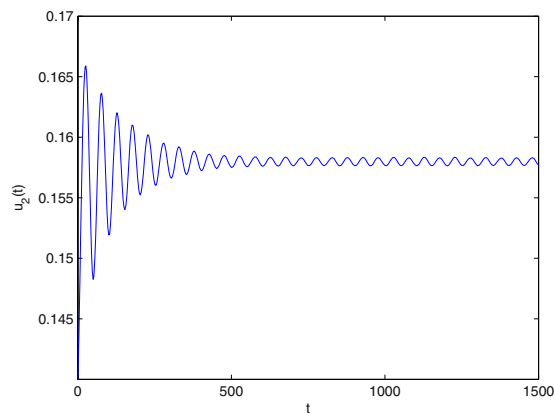
Figure 11. The trajectories of system (7.1) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation appears near the equilibrium point $(0.4168, 0.1465)$. The relation of $u_1(t)$ and $u_2(t)$.
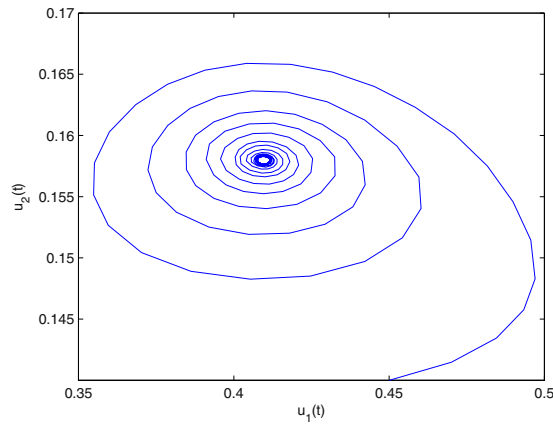

Figure 12. The trajectories of system (7.1) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation appears near the equilibrium point $(0.4168, 0.1465)$. The relation of t, $u_1(t)$ and $u_2(t)$.
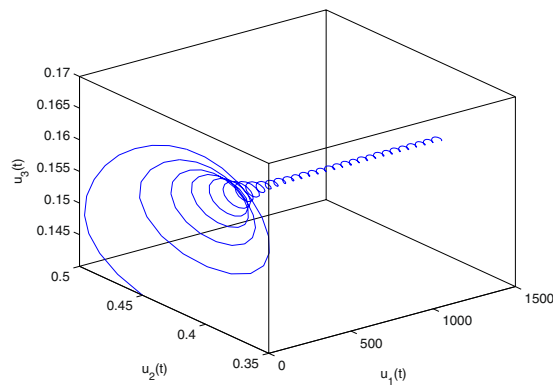
$$D^\theta U(t) = h(t, U(t)), \quad U(t_0) = U_0, \tag{3.1}$$

where $U(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T \in R^n$ and $h: [t_0, \infty) \times \Phi \to R^n$ is piecewise continuous function with respect to t and satisfies locally Lipschitz continuous in U on $[t_0, \infty) \times \Phi$, $\Phi \subset R^n$.

Definition 3.2[25,26] *We say that U^* is an equilibrium point of system (3.1) if and only if $h(t, U^*) = 0$.*
Lemma 3.1[26,27] *Let $J(U^*)$ be the Jacobian matrix of model (3.1) near U^* and $\lambda_i(i = 1, 2, \cdots, n)$ the eigenvalues of $J(U^*)$. If some eigenvalues of λ_i satisfy $|arg(\lambda_i)| > \frac{\theta\pi}{2}$ and some other eigenvalues of λ_i satisfy $|arg(\lambda_i)| < \frac{\theta\pi}{2}$, then the equilibrium point U^* is a saddle point.*

Lemma 3.2[26,27] *Let U^*, $J(U^*)$ be equilibrium point, Jacobian matrix of system (3.1), respectively, and λ_i be the eigenvalues of Jacobian matrix $J(U^*)$ of system (2.1).*

(1) *U^* is locally asymptotically stable $\Leftrightarrow |arg(\lambda_i)| > \frac{\theta\pi}{2}$ for all $\lambda_i(i = 1, 2, \cdots, n)$.*
(2) *U^* is stable $\Leftrightarrow |arg(\lambda_i)| \geq \frac{\theta\pi}{2}$ for all $\lambda_i(i = 1, 2, \cdots, n)$ and the eigenvalues with $\left|arg(\lambda_i)\right| = \frac{\theta\pi}{2}$ have the same geometric multiplicity and algebraic multiplicity.*
(3) *U^* is unstable $\Leftrightarrow \exists \lambda_i$ such that $|arg(\lambda_i)| < \frac{\theta\pi}{2}$.*

Lemma 3.3[26,28] *If the following conditions hold:*

(1) *the Jacobian matrix $J(U^*)$ of system (3.1) near the equilibrium point has a pair of complex conjugate eigenvalues with positive real part;*
(2) *$p(\theta) = \frac{\theta\pi}{2} - \min_{1 \leq i \leq 2}|arg(\lambda_i)| = 0$;*
(3) *$\frac{dp(\theta)}{d\theta}\bigg|_{\theta=\theta_0} \neq 0$,*

Figure 13. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.65$. The equilibrium point $(0.4168, 0.1465)$ of system (7.2) is asymptotically stable. The relation of t and $u_1(t)$.
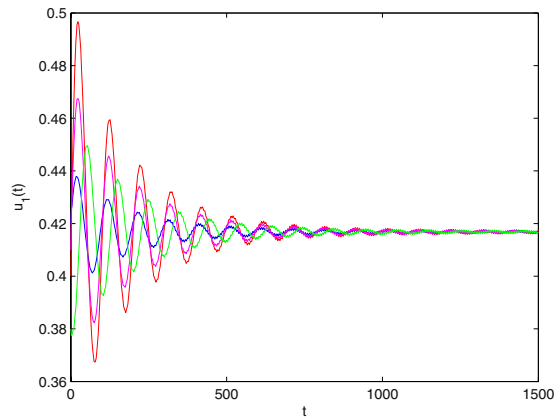

Figure 14. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.65$. The equilibrium point $(0.4168, 0.1465)$ of system (7.2) is asymptotically stable. The relation of t and $u_2(t)$.
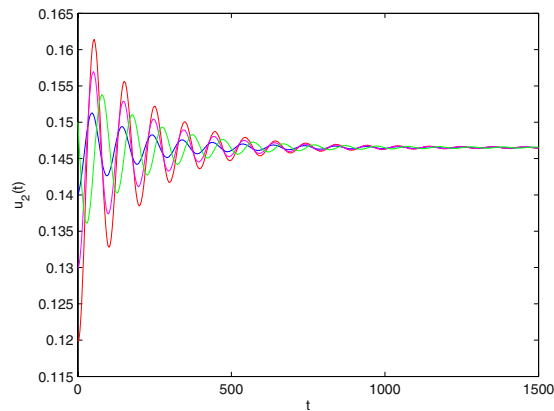

Figure 15. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.65$. The equilibrium point $(0.4168, 0.1465)$ of system (7.2) is asymptotically stable. The relation of $u_1(t)$ and $u_2(t)$.

Figure 16. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.65$. The equilibrium point $(0.4168, 0.1465)$ of system (7.2) is asymptotically stable. The relation of t, $u_1(t)$ and $u_2(t)$.
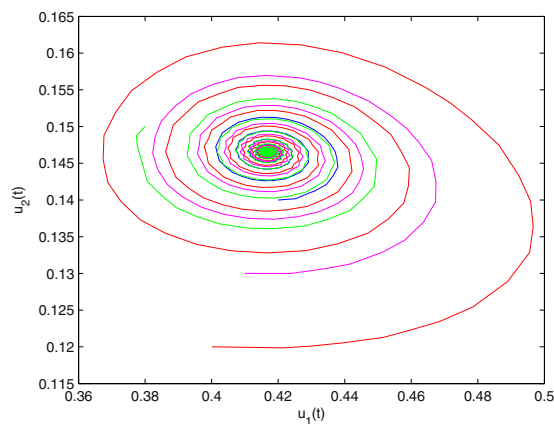

Figure 17. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.823 > \theta_0 = 0.778$. Hopf bifurcation of system (7.2) appears from the equilibrium point $(0.4168, 0.1465)$. The relation of t and $u_1(t)$.
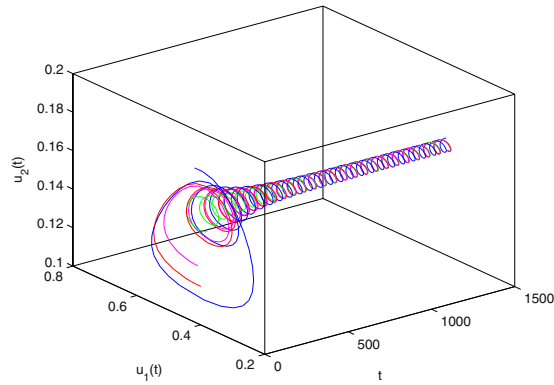

Figure 18. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.823 > \theta_0 = 0.778$. Hopf bifurcation of system (7.2) appears from the equilibrium point $(0.4168, 0.1465)$. The relation of t and $u_2(t)$.
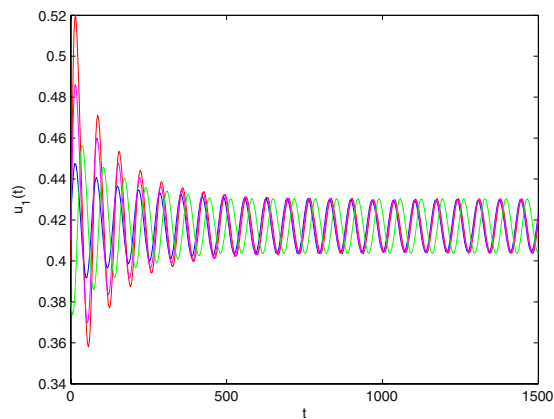
Figure 19. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.823 > \theta_0 = 0.778$. Hopf bifurcation of system (7.2) appears from the equilibrium point (0.4168, 0.1465). The relation of $u_1(t)$ and $u_2(t)$.
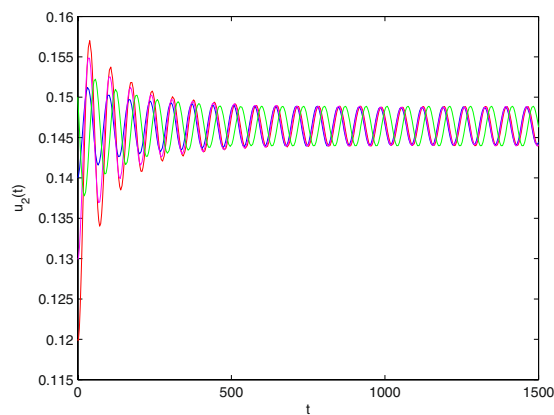

Figure 20. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 0.823 > \theta_0 = 0.778$. Hopf bifurcation of system (7.2) appears from the equilibrium point (0.4168, 0.1465). The relation of t, $u_1(t)$ and $u_2(t)$.

Figure 21. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation of system (7.2) appears neasr the equilibrium point (0.4168, 0.1465). The relation of t and $u_1(t)$.
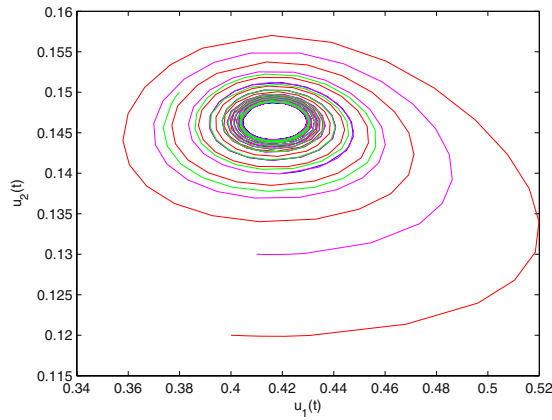
Figure 22. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation of system (7.2) appears neasr the equilibrium point (0.4168, 0.1465). The relation of t and $u_2(t)$.
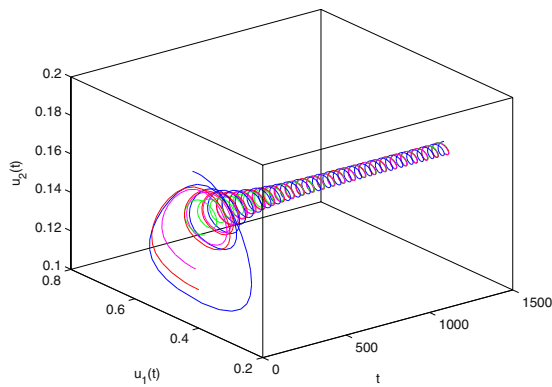

Figure 23. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation of system (7.2) appears neasr the equilibrium point (0.4168, 0.1465). The relation of $u_1(t)$ and $u_2(t)$.
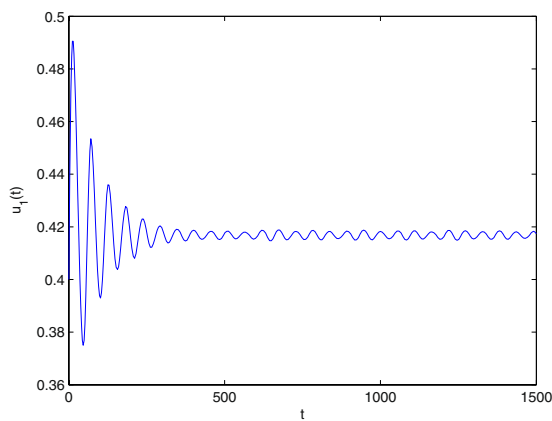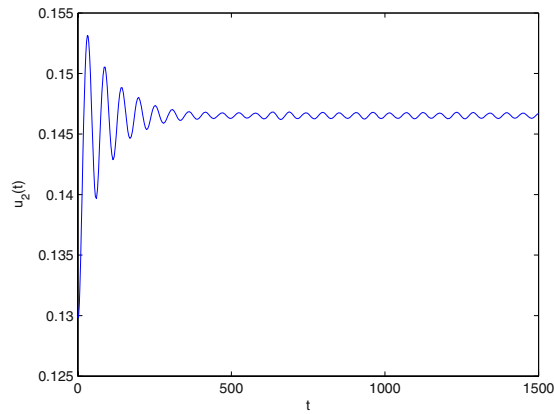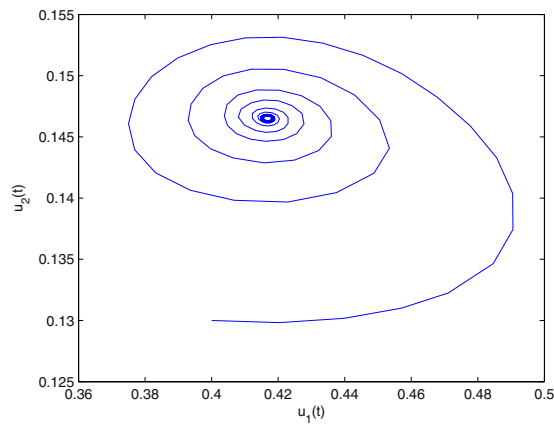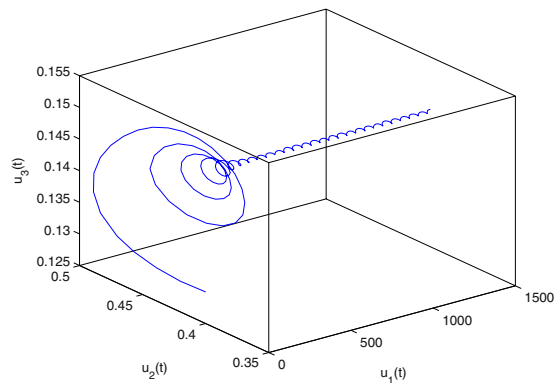

Figure 24. The trajectories of system (7.2) with $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$ and $\theta = 1$. Hopf bifurcation of system (7.2) appears neasr the equilibrium point (0.4168, 0.1465). The relation of t, $u_1(t)$ and $u_2(t)$.

then system (3.1) *undergoes a Hopf bifurcation near the equilibrium point U^* when θ crosses the critical value θ_0.*

Existence, uniqueness and boundness of solution

In this segment, we will discuss the existence, uniqueness and boundness of solution of model (2.1).

Theorem 4.1 *System* (2.1) *with initial value* (u_{10}, u_{20}) *has a unique solution* $(u_1(t), u_2(t)) \in \Theta$ *for all* $t \geq t_0$.
Proof Let

$$
\begin{cases}
F_1(u) = \gamma_1 \left[1 - \dfrac{u_1(t)}{\kappa}\right] u_1(t) - \gamma_2 u_1^{\sigma}(t) u_2(t) - \dfrac{\rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)}, \ F_2(u) \\
\quad = -\gamma_3 u_2(t) + \gamma_2 \gamma_4 u_1^{\sigma}(t) u_2(t),
\end{cases}
\tag{4.1}
$$

where $u = (u_1, u_2)$. Define a mapping as follows:

$$
F(u) = (F_1(u), F_2(u)).
\tag{4.2}
$$

$\forall \ u, \ \bar{u} \in \Theta$, one has

$$
\begin{aligned}
\|F(u) - F(\bar{u})\| &= |F_1(u) - F_1(\bar{u})| + |F_2(u) - F_2(\bar{u})| \\
&= \left| \left\{ \gamma_1 \left[1 - \frac{u_1(t_1)}{\kappa}\right] u_1(t_1) - \gamma_2 u_1^{\sigma}(t_1) u_2(t_1) - \frac{\rho_1 \rho_2 u_1(t_1)}{\rho_2 \rho_3 + \rho_4 u_1(t_1)} \right. \right. \\
&\quad \left. \left. - \left\{ \gamma_1 \left[1 - \frac{u_1(t_2)}{\kappa}\right] u_1(t_2) - \gamma_2 u_1^{\sigma}(t_2) u_2(t) - \frac{\rho_1 \rho_2 u_1(t_2)}{\rho_2 \rho_3 + \rho_4 u_1(t_2)} \right\} \right\} \right| \\
&\quad + \left| -\gamma_3 u_2(t_1) + \gamma_2 \gamma_4 u_1^{\sigma}(t_1) u_2(t_1) - \left[-\gamma_3 u_2(t_2) + \gamma_2 \gamma_4 u_1^{\sigma}(t_2) u_2(t_2) \right] \right| \\
&\leq \left| u_1(t_1) - u_1(t_2) \right| \left| \gamma_1 \left[1 - \frac{u_1(t_1)}{\kappa}\right] - \gamma_2 u_1^{\sigma-1}(t_1) u_2(t_1) - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 u_1(t_1)} \right| \\
&\quad + \max\{|u_1(t_1)|, |u_1(t_2)|\} \left| \left\{ \gamma_1 \left[1 - \frac{u_1(t_1)}{\kappa}\right] - \gamma_2 u_1(t_1)^{\sigma-1} u_2(t_1) - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 u_1(t_1)} \right\} \right. \\
&\quad \left. - \left\{ \gamma_1 \left[1 - \frac{u_1(t_2)}{\kappa}\right] - \gamma_2 u_1^{\sigma-1}(t_2) u_2(t_2) - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 u_1(t_2)} \right\} \right| \\
&\quad + \left| u_2(t_1) - u_2(t_2) \right| \left| -\gamma_3 + \gamma_2 \gamma_4 u_1^{\sigma}(t_1) \right| \\
&\quad + \max\{|u_1(t_1)|, |u_1(t_2)|\} \left| \left(-\gamma_3 + \gamma_2 \gamma_4 u_1^{\sigma}(t_1) \right) - \left(-\gamma_3 + \gamma_2 \gamma_4 u_1(t_2)^{\sigma} \right) \right| \\
&\leq \left| u_1(t_1) - u_1(t_2) \right| \left| \gamma_1 \left[1 - \frac{u_1(t_1)}{\kappa}\right] - \gamma_2 u_1^{\sigma-1}(t_1) u_2(t_1) - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 u_1(t_1)} \right| \\
&\quad + K \max\{|u_1(t_1) - u_1(t_2)|, |u_2(t_1) - u_2(t_2)|\} \left(\frac{\gamma_1}{\kappa} + \gamma_2 K^{\sigma-1} + \frac{\rho_1 \rho_4}{\rho_4} \right) \\
&\quad + \left| u_2(t_1) - u_2(t_2) \right| \left| -\gamma_3 + \gamma_2 \gamma_4 u_1^{\sigma}(t_1) \right| \\
&\quad + K \max\{|u_1(t_1) - u_1(t_2)|, |u_2(t_1) - u_2(t_2)|\} 2\gamma_2 \gamma_4 K^{\sigma} \\
&\leq \left| u_1(t_1) - u_1(t_2) \right| \left| \gamma_1 \left[1 + \frac{K}{\kappa}\right] + \gamma_2 K^{\sigma} - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 K} \right| \\
&\quad + K \max\{|u_1(t_1) - u_1(t_2)|, |u_2(t_1) - u_2(t_2)|\} \left(\frac{\gamma_1}{\kappa} + \gamma_2 K^{\sigma-1} + \frac{\rho_1 \rho_4}{\rho_4} \right) \\
&\quad + \left| u_2(t_1) - u_2(t_2) \right| \left| -\gamma_3 + \gamma_2 \gamma_4 K^{\sigma} \right| \\
&\quad + K \max\{|u_1(t_1) - u_1(t_2)|, |u_2(t_1) - u_2(t_2)|\} 2\gamma_2 \gamma_4 K^{\sigma} \\
&\leq \vartheta |u - \bar{u}|,
\end{aligned}
\tag{4.3}
$$

where

$$
\vartheta = \max \left\{ \left| \gamma_1 \left(1 + \frac{K}{\kappa}\right) + \gamma_2 K^{\sigma} - \frac{\rho_1 \rho_2}{\rho_2 \rho_3 + \rho_4 K} \right| + K \left(\frac{\gamma_1}{\kappa} + \gamma_2 K^{\sigma-1} + \frac{\rho_1 \rho_4}{\rho_4} \right) + 2\gamma_2 \gamma_4 K^{\sigma+1} \right\}.
\tag{4.4}
$$

Then $F(u)$ satisfies the Lipschitz condition with respect to u. Thus we can conclude that system (2.1) with initial value (u_{10}, u_{20}) has a unique solution $(u_1(t), u_2(t)) \in \Theta$ for all $t \geq t_0$. The proof is completed.

Theorem 4.2 *Every solution* $(u_1(t), u_2(t))$ *of system* (2.1) *with the initial condition* $(u_1(t_0), u_2(t_0))$ *is uniformly bounded and non-negative.*
   Proof Let $U(t) = \gamma_4 u_1(t) + u_2(t)$. Then

$$
\begin{aligned}
D^\theta U(t) &= \gamma_4 D^\theta u_1(t) + D^\theta(t) u_2 \\
&= \gamma_4 \left\{ \gamma_1 \left[1 - \frac{u_1(t)}{\kappa} \right] u_1(t) - \gamma_2 u_1^\sigma(t) u_2(t) - \frac{\rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)} \right\} \\
&\quad - \gamma_3 u_2(t) + \gamma_2 \gamma_4 u_1^\sigma(t) u_2(t) \\
&= \gamma_4 \gamma_1 u_1(t) - \frac{\gamma_4 \gamma_1}{\kappa} u_1^2(t) - \frac{\gamma_4 \rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)} - \gamma_3 u_2(t) \\
&= \gamma_4 \gamma_1 u_1(t) - \frac{\gamma_4 \gamma_1}{\kappa} u_1^2(t) - \frac{\gamma_4 \rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)} - \gamma_3 U(t) + \gamma_3 \gamma_4 u_1(t) \\
&\leq (\gamma_1 + \gamma_3) \gamma_4 u_1(t) - \frac{\gamma_4 \gamma_1}{\kappa} u_1^2(t) - \gamma_3 U(t)
\end{aligned}
\tag{4.5}
$$

Then

$$
\begin{aligned}
D^\theta U(t) + \gamma_3 U(t) &\leq (\gamma_1 + \gamma_3) \gamma_4 u_1(t) - \frac{\gamma_4 \gamma_1}{\kappa} u_1^2(t) \\
&= \frac{\gamma_4 \gamma_1}{\kappa} \left[u_1(t) - \left(\frac{\gamma_1 + \gamma_3}{2\gamma_1} \right)^2 \right]^2 + \frac{\gamma_4 (\gamma_1 + \gamma_3)^2}{4\kappa\gamma_1} \\
&\leq \frac{\gamma_4 (\gamma_1 + \gamma_3)^2}{4\kappa\gamma_1}.
\end{aligned}
\tag{4.6}
$$

In view of the results in[29], one has

$$
U(t) \leq U(0) E_\theta(-\gamma_3 t^\theta) + \frac{\gamma_4 (\gamma_1 + \gamma_3)^2}{4\kappa\gamma_1} t^\theta E_{\theta,\theta+1}(-\gamma_3 t^\theta),
\tag{4.7}
$$

where E_θ is the Mittag-Leffler function. In view of Lemma 5 and Corollary 6 of[29], we have

$$
U(t) \leq \frac{\gamma_4 (\gamma_1 + \gamma_3)^2}{4\kappa\gamma_1}, \ t \to \infty.
\tag{4.8}
$$

Thus every solution $(u_1(t), u_2(t))$ of system (2.1) with the initial condition $(u_1(t_0), u_2(t_0))$ is uniformly bounded.

Equilibria and stability

In this section, we consider the equilibria and their stability. In order to obtain the equilibrium point of system (2.1), we can solve the following equations:

$$
\begin{cases}
D^\theta u_1(t) = 0 \\
D^\theta u_2(t) = 0.
\end{cases}
\tag{5.1}
$$

Namely,

$$
\begin{cases}
\gamma_1 \left[1 - \frac{u_1(t)}{\kappa} \right] u_1(t) - \gamma_2 u_1^\sigma(t) u_2(t) - \frac{\rho_1 \rho_2 u_1(t)}{\rho_2 \rho_3 + \rho_4 u_1(t)} = 0, \\
- \gamma_3 u_2(t) + \gamma_2 \gamma_4 u_1^\sigma(t) u_2(t) = 0.
\end{cases}
\tag{5.2}
$$

It is not difficult to obtain the following equilibria of system (2.1): $E_1(0, 0)$, $E_2(u_1^{(1)}, 0)$, $E_3(u_1^{(2)}, 0)$, $E_4(u_{1*}, u_{2*})$, where

$$
\begin{cases}
u_1^{(1)} = \dfrac{\left(1 - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right) + \sqrt{\left(1 - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right)^2 - 4 \left(\frac{\rho_1 \rho_2}{\gamma_1 \rho_4 \kappa} - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right)}}{2}, \\[4mm]
u_1^{(2)} = \dfrac{\left(1 - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right) - \sqrt{\left(1 - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right)^2 - 4 \left(\frac{\rho_1 \rho_2}{\gamma_1 \rho_4 \kappa} - \frac{\rho_2 \rho_3}{\rho_4 \kappa} \right)}}{2}, \\[4mm]
u_{1*} = \left(\dfrac{\gamma_3}{\gamma_2 \gamma_3 \kappa^\sigma} \right)^{\frac{1}{\sigma}}, \\[3mm]
u_{2*} = u_{1*}^{1-\sigma} \left(1 - u_{1*} - \dfrac{\rho_1 \rho_2}{\gamma_1 (\rho_2 \rho_3 + u_{1*} \rho_4 \kappa)} \right).
\end{cases}
\tag{5.3}
$$

Theorem 5.1 *(1) System* (2.1) *always has the zero equilibrium point* $E_1(0, 0)$.

(2) If $\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)^2 < 4\left(\frac{\rho_1\rho_2}{\gamma_1\rho_4\kappa} - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)$, *then system* (2.1) *has no boundary equilibrium points.*

(3) If $\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)^2 = 4\left(\frac{\rho_1\rho_2}{\gamma_1\rho_4\kappa} - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)$ *and* $\frac{\rho_2\rho_3}{\rho_4\kappa} < 1$, *then system* (2.1) *has a unique boundary equilibrium points* $\tilde{E}_1(\tilde{u}_1^{(1)}, 0)$, *where* $\tilde{u}_1^{(1)} = \frac{1 - \frac{\rho_2\rho_3}{\rho_4\kappa}}{2}$.

(4) If $\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)^2 > 4\left(\frac{\rho_1\rho_2}{\gamma_1\rho_4\kappa} - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)$ *and* $\frac{\rho_2\rho_3}{\rho_4\kappa} < 1$, *then system* (2.1) *has two boundary equilibrium points* $\tilde{E}_1^*(\tilde{u}_1^{(1*)}, 0)$ *and* $\tilde{E}_1^{**}(\tilde{u}_1^{(1**)}, 0)$ *where*

$$\tilde{u}_1^{(1*)} = \frac{\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right) + \sqrt{\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)^2 - 4\left(\frac{\rho_1\rho_2}{\gamma_1\rho_4\kappa} - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)}}{2}$$

and

$$\tilde{u}_1^{(1**)} = \frac{\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right) - \sqrt{\left(1 - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)^2 - 4\left(\frac{\rho_1\rho_2}{\gamma_1\rho_4\kappa} - \frac{\rho_2\rho_3}{\rho_4\kappa}\right)}}{2}.$$

(5) If $(1 - u_{1*})\gamma_1(\rho_2\rho_3 + u_{1*}\rho_4\kappa) > \rho_1\rho_2$, *then system* (2.1) *has the interior equilibrium point* $E_4(u_{1*}, u_{2*})$.

Since the proof of Theorem 5.1 is simple in view of (5.3). Here we omit it.

Next we discuss the stability of the equilibrium points. The Jacobian matrix of system (2.1) near the equilibrium point (u_1, u_2) is

$$J(u_1, u_2) = \begin{bmatrix} \gamma_1 - \frac{2\gamma_1 u_1}{\kappa} - \gamma_2\sigma u_1^{\sigma-1}u_2 - \frac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_1)^2} & -\gamma_2 u_1^\sigma \\ \sigma\gamma_2\gamma_4 u_1^{\sigma-1}u_2 & -\gamma_3 + \gamma_2\gamma_4 u_1^\sigma \end{bmatrix}. \quad (5.4)$$

Theorem 5.2 *The equilibrium point* $E_1(0, 0)$ *of system* (2.1) *is locally asymptotically stable if* $\gamma_1 - \frac{\rho_1}{\rho_3} < 0$ *and is a saddle point if* $\gamma_1 - \frac{\rho_1}{\rho_3} > 0$.

Proof In view of (5.4), one can get

$$J(0, 0) = \begin{bmatrix} \gamma_1 - \frac{\rho_1}{\rho_3} & 0 \\ 0 & -\gamma_3 \end{bmatrix} \quad (5.5)$$

It follows that the eigenvalues of $J(0, 0)$ are $\lambda_1 = \gamma_1 - \frac{\rho_1}{\rho_3}$, $\lambda_2 = -\gamma_3$. When $\gamma_1 - \frac{\rho_1}{\rho_3} < 0$, then $\lambda_1 < 0$, $\lambda_2 < 0$ and hence $arg(\lambda_i) = \pi(i = 1, 2)$ and $|arg(\lambda_i)| > \frac{\theta\pi}{2}(i = 1, 2)$. Thus the equilibrium point $E_0(0, 0)$ of system (2.1) is locally asymptotically stable if $\gamma_1 - \frac{\rho_1}{\rho_3} < 0$. When $\gamma_1 - \frac{\rho_1}{\rho_3} > 0$, then $\lambda_1 > 0$, $\lambda_2 < 0$ and hence $arg(\lambda_1) = 0$ and $|arg(\lambda_1)| < \frac{\theta\pi}{2}$. Thus the equilibrium point $E_0(0, 0)$ of system (2.1) is a saddle point if $\gamma_1 - \frac{\rho_1}{\rho_3} > 0$. The proof of Theorem 5.2 is completed.

Theorem 5.3 *(1) The equilibrium point* $E_2(u_1^{(1)}, 0)$ *of system* (2.1) *is locally asymptotically stable if* $\gamma_1 - \frac{2\gamma_1 u_1^{(1)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(1)}\right)^2} < 0$ *and* $-\gamma_3 + \gamma_2\gamma_4(u_1^{(1)})^\sigma > 0$

(2) The equilibrium point $E_2(u_1^{(1)}, 0)$ *of system* (2.1) *is a saddle point if* $\left(\gamma_1 - \frac{2\gamma_1 u_1^{(1)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(1)}\right)^2}\right)$ $(-\gamma_3 + \gamma_2\gamma_4(u_1^{(1)})^\sigma) < 0$.

(3) The equilibrium point $E_2(u_1^{(1)}, 0)$ *of system* (2.1) *is a saddle point if* $\gamma_1 - \frac{2\gamma_1 u_1^{(1)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(1)}\right)^2} > 0$ *and* $-\gamma_3 + \gamma_2\gamma_4(u_1^{(1)})^\sigma > 0$.

Proof In view of (5.4), one can get

$$J(u_1^{(1)}, 0) = \begin{bmatrix} \gamma_1 - \frac{2\gamma_1 u_1^{(1)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_1^{(1)})^2} & -\gamma_2(u_1^{(1)})^\sigma \\ 0 & -\gamma_3 + \gamma_2\gamma_4(u_1^{(1)})^\sigma \end{bmatrix} \quad (5.6)$$

It follows that the eigenvalues of $J(u_1^{(1)}, 0)$ are $\lambda_1 = \gamma_1 - \frac{2\gamma_1 u_1^{(1)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(1)}\right)^2}$, $\lambda_2 = -\gamma_3 + \gamma_2\gamma_4(u_1^{(1)})^\sigma$. Under the assumptions of (1), (2) and (3) of Theorem 5.3, we can easily conclude that the conclusion of Theorem 5.3 holds true. The proof of Theorem 5.3 is completed.

In a same way, we get the following result on the equilibrium point $E_3(u_1^{(2)}, 0)$ of system (2.1).

Theorem 5.4 (1) The equilibrium point $E_3(u_1^{(2)}, 0)$ of system (2.1) is locally asymptotically stable if $\gamma_1 - \frac{2\gamma_1 u_1^{(2)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(2)}\right)^2} < 0$ and $-\gamma_3 + \gamma_2\gamma_4(u_1^{(2)})^\sigma > 0$

(2) The equilibrium point $E_3(u_1^{(2)}, 0)$ of system(2.1)is a saddle point if $\left(\gamma_1 - \frac{2\gamma_1 u_1^{(2)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(2)}\right)^2}\right)$

$(-\gamma_3 + \gamma_2\gamma_4(u_1^{(2)})^\sigma) < 0$.

(3) The equilibrium point $E_3(u_1^{(2)}, 0)$ of system (2.1) is a saddle point if $\gamma_1 - \frac{2\gamma_1 u_1^{(2)}}{\kappa} - \frac{\rho_1\rho_2^2\rho_3}{\left(\rho_2\rho_3 + \rho_4 u_1^{(2)}\right)^2} > 0$ and

$-\gamma_3 + \gamma_2\gamma_4(u_1^{(2)})^\sigma > 0$.

Theorem 5.5 (1) If the one of following inequalities is satisfied: (a) $\alpha \leq 0$ (b) $\alpha > 0$, $\alpha^2 - 4\beta < 0$ and $\frac{\sqrt{4\beta - \alpha^2}}{\alpha} > \tan\frac{\theta\pi}{2}$. Then the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is locally asymptotically stable.
Proof In view of (5.4), we have

$$J(u_{1*}, u_{2*}) = \begin{bmatrix} \gamma_1 - \frac{2\gamma_1 u_{1*}}{\kappa} - \gamma_2\sigma u_{1*}^{\sigma-1}u_{2*} - \frac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_{1*})^2} & -\gamma_2 u_{1*}^\sigma \\ \sigma\gamma_2\gamma_4 u_{1*}^{\sigma-1}u_{2*} & -\gamma_3 + \gamma_2\gamma_4 u_{1*}^\sigma \end{bmatrix}. \tag{5.7}$$

Let

$$\begin{cases} a_{11} = \gamma_1 - \frac{2\gamma_1 u_{1*}}{\kappa} - \gamma_2\sigma u_{1*}^{\sigma-1}u_{2*} - \frac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_{1*})^2}, \\ a_{12} = -\gamma_2 u_{1*}^\sigma, \\ a_{21} = \sigma\gamma_2\gamma_4 u_{1*}^{\sigma-1}u_{2*}, \\ a_{22} = -\gamma_3 + \gamma_2\gamma_4 u_{1*}^\sigma. \end{cases} \tag{5.8}$$

Then (5.7) becomes

$$J(u_{1*}, u_{2*}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{5.9}$$

The eigenvalues of $J(u_{1*}, u_{2*})$ are

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \quad \lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}, \tag{5.10}$$

where

$$\alpha = a_{11} + a_{22}, \quad \beta = a_{11}a_{22} - a_{12}a_{21}. \tag{5.11}$$

If $\alpha \leq 0$, then we consider three cases:

(i) If $\alpha = 0$, then the eigenvalues of $J(u_{1*}, u_{2*})$ are a pair of complex conjugate λ_1 and $\overline{\lambda}_1$. Hence $Re(\lambda_1) = Re(\overline{\lambda}_1) = 0$ and $arg(\lambda_1) = \frac{\pi}{2}$, $arg(\overline{\lambda}_1) = -\frac{\pi}{2}$. Thus $\left|arg(\lambda_1)\right| > \frac{\theta\pi}{2}$ and $|arg(\overline{\lambda}_1)| > \frac{\theta\pi}{2}$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is locally asymptotically stable.

(ii) If $\alpha < 0$, $\alpha^2 - 4\beta \geq 0$, then the both eigenvalues of $J(u_{1*}, u_{2*})$ are $\lambda_1 < 0$ and $\lambda_2 < 0$. Hence $|arg(\lambda_i)| = \frac{\pi}{2} > \frac{\theta\pi}{2} (i = 1, 2)$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is locally asymptotically stable.

(iii) If $\alpha < 0$, $\alpha^2 - 4\beta < 0$, then the eigenvalues of $J(u_{1*}, u_{2*})$ are a pair of complex conjugate λ_1 and $\overline{\lambda}_1$. Hence $Re(\lambda_1) = Re(\overline{\lambda}_1) < 0$ and $arg(\lambda_i) > \frac{\pi}{2}$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is locally asymptotically stable.

If $\alpha > 0$, $\alpha^2 - 4\beta < 0$, $\frac{\sqrt{4\beta - \alpha^2}}{\alpha} > \tan\frac{\theta\pi}{2}$, then the both eigenvalues of $J(u_{1*}, u_{2*})$ are λ_1 and $\overline{\lambda}_1$ satisfy $Re(\lambda_1) = Re(\overline{\lambda}_1) > 0$, $Im(\lambda_1) = -Im(\overline{\lambda}_1) = \frac{\sqrt{4\beta - \alpha^2}}{2} > 0$. Hence

$$\frac{\mathrm{Im}\,(\lambda_1)}{\mathrm{Re}\,(\lambda_1)} > \tan\frac{\theta\pi}{2}, \quad -\frac{\mathrm{Im}\,(\overline{\lambda}_1)}{\mathrm{Re}\,(\overline{\lambda}_1)} > \tan\frac{\theta\pi}{2}.$$

Then $|arg(\lambda_1)| > \frac{\alpha\pi}{2}$, $|arg(\overline{\lambda}_1)| > \frac{\alpha\pi}{2}$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is locally asymptotically stable.

Theorem 5.6 *(1) If the one of following inequalities is satisfied: (a) $\alpha > 0$, $\alpha^2 - 4\beta \geq 0$ (b) $\alpha^2 - 4\beta < 0$, $\alpha > 0$ and $\frac{\sqrt{4\beta - \alpha^2}}{\alpha} < \tan\frac{\theta\pi}{2}$. Then the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is unstable.*
Proof In view of (5.4), we have

$$J(u_{1*}, u_{2*}) = \begin{bmatrix} \gamma_1 - \dfrac{2\gamma_1 u_{1*}}{\kappa} - \gamma_2\sigma u_{1*}^{\sigma-1}u_{2*} - \dfrac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_{1*})^2} & -\gamma_2 u_{1*}^{\sigma} \\[3mm] \sigma\gamma_2\gamma_4 u_{1*}^{\sigma-1}u_{2*} & -\gamma_3 + \gamma_2\gamma_4 u_{1*}^{\sigma} \end{bmatrix}.$$

Let

$$\begin{cases} a_{11} = \gamma_1 - \dfrac{2\gamma_1 u_{1*}}{\kappa} - \gamma_2\sigma u_{1*}^{\sigma-1}u_{2*} - \dfrac{\rho_1\rho_2^2\rho_3}{(\rho_2\rho_3 + \rho_4 u_{1*})^2}, \\[3mm] a_{12} = -\gamma_2 u_{1*}^{\sigma}, \\[2mm] a_{21} = \sigma\gamma_2\gamma_4 u_{1*}^{\sigma-1}u_{2*}, \\[2mm] a_{22} = -\gamma_3 + \gamma_2\gamma_4 u_{1*}^{\sigma}. \end{cases}$$

Then (5.7) becomes

$$J(u_{1*}, u_{2*}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The eigenvalues of $J(u_{1*}, u_{2*})$ are

$$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \quad \lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2},$$

where

$$\alpha = a_{11} + a_{22}, \quad \beta = a_{11}a_{22} - a_{12}a_{21}. \tag{5.12}$$

(i) If $\alpha > 0$, $\alpha^2 - 4\beta \geq 0$, then the both eigenvalues of $J(u_{1*}, u_{2*})$ are $\lambda_1 > 0$ and $\lambda_2 > 0$. Hence $|arg(\lambda_i)| < \frac{\theta\pi}{2}(i = 1, 2)$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is unstable.

(ii) If $\alpha > 0$, $\alpha^2 - 4\beta < 0$ and $\frac{\sqrt{4\beta - \alpha^2}}{\alpha} < \tan\frac{\theta\pi}{2}$, then the eigenvalues of $J(u_{1*}, u_{2*})$ are a pair of complex conjugate λ_1 and $\overline{\lambda}_1$. Hence $\mathrm{Im}\,(\lambda_1) = -\mathrm{Im}\,(\overline{\lambda}_1) > 0$, $\mathrm{Re}\,(\lambda_1) = \mathrm{Re}\,(\overline{\lambda}_1) = \alpha > 0$. Then

$$\frac{\mathrm{Im}\,(\lambda_1)}{\mathrm{Re}\,(\lambda_1)} < \tan\frac{\theta\pi}{2}, \quad -\frac{\mathrm{Im}\,(\overline{\lambda}_1)}{\mathrm{Re}\,(\overline{\lambda}_1)} < \tan\frac{\theta\pi}{2}.$$

Thus $|arg(\lambda_1)| < \frac{\alpha\pi}{2}$, $|arg(\overline{\lambda}_1)| < \frac{\alpha\pi}{2}$. In view of Lemma 3.2, we can conclude that the equilibrium point $E_4(u_{1*}, u_{2*})$ of system (2.1) is unstable.

Bifurcation analysis
In this section, we will establish the sufficient condition that guarantees the existence of Hopf bifurcation of system (2.1).

Theorem 6.1 *If $\alpha^2 - 4\beta > 0$ and $\alpha > 0$, then a Hopf bifurcation of system (2.1) will appear around $E_4(u_{1*}, u_{2*})$ when the fractional order θ crosses the critical value $\theta_0 = \frac{2}{\pi}\arctan\frac{\sqrt{|\alpha^2 - 4\beta|}}{\alpha}$.*

Proof Denote $\vartheta = \frac{\alpha}{2}$ and $\psi = \frac{\sqrt{|\alpha^2 - 4\beta|}}{2}$. In view of the assumption $\alpha > 0$, one get $\vartheta > 0$. By the assumption $\alpha^2 - 4\beta > 0$ and (4.10), the the eigenvalues of $J(u_{1*}, u_{2*})$ of system (2.1) are a pair of complex conjugate $\lambda_{1,2} = \vartheta \pm i\psi$. Next, $p(\theta_0) = \frac{\theta_0\pi}{2} - \min_{1 \leq i \leq 2}|arg(\lambda_i)| = \frac{\theta_0\pi}{2} - arg\left(\frac{\psi}{\vartheta}\right) = arg\left(\frac{\psi}{\vartheta}\right) - arg\left(\frac{\psi}{\vartheta}\right) = 0$. Finally, $\frac{dp(\theta)}{d\theta}\Big|_{\theta=\theta_0} = \frac{\pi}{2} \neq 0$. In view of Lemma 3.3, we can conclude that a Hopf bifurcation of system (2.1) will appear around $E_4(u_{1*}, u_{2*})$ when the fractional order θ crosses the critical value $\theta_0 = \frac{2}{\pi}\arctan\frac{\sqrt{|\alpha^2 - 4\beta|}}{\alpha}$. The proof of Theorem 6.1 is completed.

Remark 6.1 *In[11], the authors studied the stability, Hopf bifurcation and chaotic behavior of integer order predator-pry system. In this article, we mainly focus on the existence, uniqueness and boundness of solution, the stability of equilibrium point and the existence of Hopf bifurcation of fractional order predator-prey model. The research method and theoretical findings are different from those in[11]. According to this viewpoint, the results of this paper complete the works of Kumar and Kharbanda[11].*

Numerical simulation

Example 7.1 We give the fractional-order system as follows:

$$\begin{cases} D^\theta u_1(t) = (1 - u_1(t))u_1(t) - u_1^{0.25}(t)u_2(t) - \dfrac{0.2u_1(t)}{0.3 + u_1(t)}, \\ D^\theta u_2(t) = -0.2u_2(t) + 0.25u_1^{0.25}(t)u_2(t), \end{cases} \tag{7.1}$$

where $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0.2$, $\gamma_4 = 0.25$, $\kappa = 1$, $\sigma = 0.25$, $\rho_1 = 0.2$, $\rho_2 = 1$, $\rho_3 = 0.3$, $\rho_4 = 1$. Obviously, system (7.1) has a unique coexistence equilibrium point $E_3(0.4096, 0.1580)$. By direct computation, one has $\theta_0 = 0.83$. Let $\theta = 0.78$. We can easily check that all the assumptions of Theorem 5.4 and Theorem 6.1 are fulfilled. Thus the equilibrium point $E_3(0.4096, 0.1580)$ of system (7.1) is locally asymptotically stable. This fact is depicted in Figs. 1–4. When the parameter θ crosses the critical value θ_0, then a Hopf bifurcation will appear. This result can be shown in Figs. 5–8 (here let $\theta = 0.9$). If $\theta = 1$ (integer order), a Hopf bifurcation appears near the equilibrium point $E_3(0.4096, 0.1580)$. This result can be illustrated in Figs. 9–12.

Example 7.2 We give the fractional-order system as follows:

$$\begin{cases} D^\theta u_1(t) = (0.999 - 0.998u_1(t))u_1(t) - u_1^{0.255}(t)u_2(t) - \dfrac{0.22u_1(t)}{0.25 + u_1(t)}, \\ D^\theta u_2(t) = -0.2u_2(t) + 0.25u_1^{0.255}(t)u_2(t), \end{cases} \tag{7.2}$$

w h e r e $\gamma_1 = 0.999$, $\gamma_2 = 0.9$, $\gamma_3 = 0.2$, $\gamma_4 = \frac{25}{9}$, $\kappa = \frac{500}{499}$, $\sigma = 0.255$, $\rho_1 = 0.2$, $\rho_2 = 1.1$, $\rho_3 = 0.25$, $\rho_4 = 1$. Obviously, system (7.2) has a unique coexistence equilibrium point $E_3(0.4168, 0.1465)$. By direct computation, one has $\theta_0 = 0.778$. Let $\theta = 0.65$. We can easily check that all the assumptions of Theorem 5.5 and Theorem 6.1 are fulfilled. Thus the equilibrium point $E_3(0.4168, 0.1465)$ of system (7.2) is locally asymptotically stable. This fact is depicted in Figs. 13–16. When the parameter θ crosses the critical value θ_0, then a Hopf bifurcation will appear. This result can be shown in Figs. 10–20 (here let $\theta = 0.823$). If $\theta = 1$ (integer order), a Hopf bifurcation appears near the equilibrium point $E_3(0.4168, 0.1465)$. This result can be illustrated in Figs. 21–24.

Conclusions

In this article, we have discussed a fractional order predator-prey model with group defense. Some sufficient conditions that guarantee the the existence, uniqueness and boundness of solution, the stability of equilibrium point and the existence of Hopf bifurcation of the considered fractional order predator-prey model are established. The study shows that under some suitable parameter conditions, the various equilibrium points are locally asymptotically stable, when the parameter θ cross the critical value, the Hopf bifurcation will appear. The research also reveal that the fractional order has an important effect on the stability and the existence of Hopf bifurcation of the involved fractional order predator-prey model. At last, computer simulations are performed to illustrate the theoretical results. Numerical simulation results show that Hopf bifurcation value of Example 7.1 is $\theta_0 = 0.83$ and Hopf bifurcation value of Example 7.2 is $\theta_0 = 0.778$. The obtained results of this article can be applied to keep the coexistence of biological populations and maintain ecological balance. In addition, we must point out that time delay often exist in predator-prey system. But we still do not consider this case. We will deal with the dynamical behavior of fractional order delayed predator-prey models in the near future.

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Author contributions

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Competing interests

The author declares no competing interests.

Additional information