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Soft ideals of soft ternary semigroups

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ABSTRACT

In this paper, we introduce the notions of certain classes of soft ideals in soft ternary semigroups and study some inter-relations between different types of soft ideals in a soft ternary semigroup. We also characterize completely regular soft ternary semigroups with the help of these soft ideals of soft ternary semigroups.

1. Introduction

Now days in our daily life we have to deal with some situations for which complete information is unavailable. Mathematical models are developed to tackle the situation where uncertainty prevailing. Most of these models are based on an extension of ordinary set theory. Till now perhaps the most appropriate theory to handle the uncertainty related situation is the theory of fuzzy sets. But major difficulty arises in this theory probably due to the inadequacy of parameters. So naturally people are trying to overcome this situation. For this purpose the notion of soft set was introduced by D. Molodtsov [1] by involving enough parameters. The theory of soft set is a generalized mathematical tool for dealing the uncertain phenomenon. Soft sets are used successfully in medical diagnosis, decision-making problem, data analysis etc.

Ternary semigroup is now a widely discussed topic in the area of ternary algebra. Lehmer [2] introduced the concept of ternary semigroup. After the introduction of ternary semigroups, many researcher study the ideal theory of ternary semigroups. A detailed discussion about the ideal theory of ternary semigroups can be found in the paper of Sioson [3]. Good and Hughes [4] introduced the notion of bi-ideal in semigroups. Kar and Maity in [5] and Dutta, Kar and Maity in [6] added some interesting properties of prime ideals and bi-ideals of ternary semigroups and also characterized regular ternary semigroups, completely regular ternary semigroups, intra-regular ternary semigroups by using these classes of ideals. Recently group, semigroup, ring, semiring are studied by using soft sets. In 2010, Ali, Shabir and Shum [7] studied about soft ideals and characterized soft ideals and discussed about soft regular semigroups with the help of soft quasi-ideals, soft bi-ideals of soft semigroups. Latter on Shabir and Ahmad [8] developed soft set in ternary algebra and also studied regularity property, quasi ideal, bi-ideal in soft ternary semigroup. Maji and Roy [9, 10] showed application of soft sets and fuzzy soft set in decision making problem. In 2017 Nasef and EL-Sayed [11] discussed a real life application of soft set theory. In 2016, Garg, Agarwal, Tripathi [12] introduced fuzzy number intuitionistic fuzzy soft sets and studied about different properties operations such as union, intersection, complement, max, min, AND, OR etc. and Mukerjee, Das, Saha [13] studied about fuzzy set, intuitionistic fuzzy set, intuitionistic fuzzy soft set. Kar and Shikari [14] introduced the notion of soft ternary semiring and discussed the properties of ideals, regularity, intra regularity in soft ternary semiring. In 2015 Khan and Sarwar [15] discussed about uni-soft ideals of ternary semigroup. Sezgin and Atagün [16] introduced the concept of soft normalistic groups and discussed some properties of soft groups. Maji, Biswas and Roy [17] studied about super set of a soft set and complement of a soft set and also provide some nice examples. Feng, Ali and Shabir established connection between binary relations and soft set theory in [18].

In this paper, we discuss about soft ternary semigroup. We are motivated by the papers of Ali, Shabir [7], Abbasi, Kahan and Ali [19], Yiarayong [20], Chinram, Panitykul [21] and also try to extend the properties of prime ideal, semiprime ideal, irreducible ideal, prime bi-ideal in view of soft set and characterized completely regular soft

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ternary semigroup by using these soft ideals. Our aim is to introduce a new concept of soft ideals of soft ternary semigroup like soft prime ideal, soft semiprime ideal, soft prime bi-ideal, soft semiprime bi-ideal. Although we have not added any direct application of soft ternary semigroup in our present manuscript but we have developed the theory behind the application. The results discussed in the manuscripts can be applied to solve a real life problem described in soft set theory.

2. Preliminaries and prerequisites

In this section, we recall some definitions which will be used in our later section of this paper.

Definition 2.1. [1] Let U be an universal set and E be the set of parameters. Suppose that $P(U)$ be the power set of U and A be a nonempty subset of E. A pair (F,A) is said to be soft set over U, where $F: A \longrightarrow P(U)$ is a mapping from A to $P(U)$.

Definition 2.2. [17] Let (F,A) and (G,B) be two soft sets over a common universe U. Then (G,B) is said to be soft subset of (F,A) if (i) $B \subseteq A$, (ii) $G(a) \subseteq F(a)$ for all $a \in B \subseteq A$ and it is denoted by $(G,B) \subseteq (F,A)$.

Definition 2.3. [17] Let (F,A) and (G,B) be two soft sets over a common universe U. Then (G,B) is said to be proper subset of (F,A) if $B \subseteq A$ and $G(a) \subset F(a)$ for all $a \in B \subseteq A$ and it is denoted by $(G,B) \subset (F,A)$.

Note 2.4. *[17] Two soft sets (F,A) and (G,B) over a common universe U is said to be equal if $(F,A) \subseteq (G,B)$ and $(G,B) \subseteq (F,A)$.*

Example 2.5. Let $E = \{1,2,3,4,5\}$, $U = \{a,b,c,d,e\}$, $A = \{1,3,4,5\}$, $B = \{1,3,5\}$. Let us define $F(1) = \{a\}$, $F(3) = \{a,b,c\}$, $F(4) = \{b,c\}$, $F(5) = \{a,d,e\}$, $G(1) = \{a\}$, $G(3) = \{b,c\}$, $G(5) = \{a\}$. Then $(G,B) \subseteq (F,A)$ but (G,B) is not a proper soft subset of (F,A). Let $C = \{3,5\}$. Then (G,C) is proper soft subset of (F,A).

Definition 2.6. [8] A soft set (U,E) over U is said to be absolute soft set over U w.r.t. the parameter set E, if $U(e) = U$ for all $e \in E$ and it is denoted by $\tilde{A}_U$.

Definition 2.7. [17] Let (F,A) and (G,B) be two soft sets over a common universe U. Then "$(F,A) AND (G,B)$" is denoted by $(F,A) \bigwedge (G,B)$ and defined by $(H, A \times B)$, where $H(a,b) = F(a) \cap G(b)$ for all $(a,b) \in A \times B$.

Definition 2.8. [17] Let (F,A) and (G,B) be two soft sets over a common universe U. Then "$(F,A) OR (G,B)$" is denoted by $(F,A) \bigvee (G,B)$ and defined by $(K, A \times B)$ where $K(a,b) = F(a) \cup G(b)$ for all $(a,b) \in A \times B$.

Example 2.9. 1. Let $A = \{1,2,3\}$, $B = \{1,2\}$.
Suppose $U = \{-1,-2,-3,-4,-5,-6,-7,-8,-9,-10\}$ be the universal set. Then (F,A) and (G,B) be two soft sets over U defined by $F(1) = \{-1,-5,-7\}$, $F(2) = \{-2,-4,-6,-8,-10\}$, $F(3) = \{-3,-6,-9\}$ and $G(1) = \{-1,-3,-5,-7,-9\}$, $G(2) = \{-2,-4,-6,-8,-10\}$. Thus $C = A \times B = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$. Now $(F,A) \bigwedge (G,B) = (H,C)$. Hence $H(1,1) = F(1) \cap G(1) = \{-1,-5,-7\}$, $H(1,2) = F(1) \cap G(2) = \emptyset$, $H(2,1) = F(2) \cap G(1) = \emptyset$, $H(2,2) = F(2) \cap G(2) = \{-2,-4,-6,-8,-10\}$, $H(3,1) = F(3) \cap G(1) = \{-3,-9\}$, $H(3,2) = F(3) \cap G(2) = \{-6\}$.
2. Again define $(F,A) OR (G,B) = (K,C)$.
Then $K(1,1) = \{-1,-3,-5,-7,-9\}$, $K(1,2) = \{-1,-2,-3,-4,-5,-6,-7,-8,-9,-10\}$, $K(2,1) = \{-1,-2,-3,-4,-5,-6,-7,-8,-9,-10\}$, $K(2,2) = \{-2,-4,-6,-8,-10\}$,
$K(3,1) = \{-1,-3,-5,-6,-8,-9,-10\}$, $K(3,2) = \{-2,-3,-4,-6,-8,-9,-10\}$.

Definition 2.10. [18] Let (F,A) and (G,B) be two soft sets over a common universe U. Then the extended union is denoted by $(F,A) \tilde{\cup}_E (G,B)$ and is defined by $(F,A) \tilde{\cup}_E (G,B) = (H,C)$, where $C = A \cup B$ and

$$\begin{aligned} H(a) &= F(a), \quad a \in A - B \\ &= G(a), \quad a \in B - A \\ &= F(a) \cup G(a), \quad a \in A \cap B. \end{aligned}$$

Definition 2.11. [18] Let (F,A) and (G,B) be two soft sets over a common universe U. Then the restricted union is denoted by $(F,A) \Cup_R (G,B)$ and defined by $(F,A) \Cup_R (G,B) = (H,C)$, where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

Definition 2.12. [18] Let (F,A) and (G,B) be two soft sets over a common universe U. Then the extended intersection is denoted by $(F,A) \tilde{\cap}_E (G,B)$ and defined by $(F,A) \tilde{\cap}_E (G,B) = (H,C)$, where $C = A \cup B$ and

$$\begin{aligned} H(a) &= F(a), \quad a \in A - B \\ &= G(a), \quad a \in B - A \\ &= F(a) \cap G(a), \quad a \in A \cap B. \end{aligned}$$

Definition 2.13. [18] Let (F,A) and (G,B) be two soft sets over a common universe U. Then the restricted intersection is denoted by $(F,A) \Cap_R (G,B)$ and defined by $(F,A) \Cap_R (G,B) = (H,C)$, where $C = A \cap B$, $H(c) = F(c) \cap G(c)$ for all $c \in C$.

Example 2.14. Consider (F,A) and (G,B) as Example 2.9.
1. Let $(F,A) \tilde{\cup}_E (G,B) = (H,C)$. Then $C = A \cup B = \{1,2,3\}$ and $H(1) = F(1) \cup G(1) = \{-1,-3,-5,-7,-9\}$, $H(2) = F(2) \cup G(2) = \{-2,-4,-6,-8,-10\}$, $H(3) = F(3) = \{-3,-6,-9\}$.
2. Let $(F,A) \tilde{\cap}_E (G,B) = (H,C)$. Then $C = A \cup B = \{1,2,3\}$ and $H(1) = F(1) \cap G(1) = \{-1,-5,-7\}$, $H(2) = F(2) \cap G(2) = \{-2,-4,-6,-8,-10\}$, $H(3) = F(3) = \{-3,-6,-9\}$.
3. Let $(F,A) \Cup_R (G,B) = (H,C)$. Then $C = A \cap B$ and $H(1) = F(1) \cup G(1) = \{-1,-3,-5,-7,-9\}$, $H(2) = F(2) \cup G(2) = \{-2,-4,-6,-8,-10\}$.
4. Let $(F,A) \Cap_R (G,B) = (H,C)$. Then $C = A \cap B = \{1,2,3\}$ and $H(1) = F(1) \cap G(1) = \{-1,-5,-7\}$, $H(2) = F(2) \cap G(2) = \{-2,-,-6,-8,-10\}$.

Definition 2.15. [18] Let (F,A), (G,B), (H,C) be three soft sets over a common universe U. Then the restricted ternary product of these three soft sets is denoted by $(F,A) \odot (G,B) \odot (H,C)$ and defined by $(F,A) \odot (G,B) \odot (H,C) = (K,D)$, where $D = A \cap B \cap C$ and $K(a) = F(a)G(a)H(a)$ for all $a \in D$.

Definition 2.16. [18] Let (F,A), (G,B), (H,C) be three soft sets over a common universe U. Then the ternary star product of these three soft sets is denoted by $(F,A) \star (G,B) \star (H,C)$ and defined by $(F,A) \star (G,B) \star (H,C) = (K,D)$, where $D = A \times B \times C$ and $K(a,b,c) = F(a)G(b)H(c)$ for all $(a,b,c) \in D$.

3. Soft prime and soft semiprime ideals

In this section, we study about soft prime and soft semiprime ideals discuss some of their properties.

Definition 3.1. [8] A soft set (F,A) over a ternary semigroup S is said to be a soft ternary semigroup over S if $(F,A) \odot (F,A) \odot (F,A) \subseteq (F,A)$.

Definition 3.2. [8] A non null and non empty soft set (F,A) over a ternary semigroup S is said to be a soft left ideal over S, if $\tilde{A}_S \odot \tilde{A}_S \odot (F,A) \subseteq (F,A)$.

A non null and non empty soft set (F,A) over a ternary semigroup S is said to be a soft right ideal over S, if $(F,A)\odot\tilde{A}_S\odot\tilde{A}_S\subseteq(F,A)$.

A non null and non empty soft set (F,A) over a ternary semigroup S is said to be a soft lateral ideal over S, if $\tilde{A}_S\odot(F,A)\odot\tilde{A}_S\subseteq(F,A)$.

A non null non empty soft set (F,A) over a ternary semigroup S is said to be a soft ideal over S, if (F,A) is soft left, right and lateral ideal over S.

Definition 3.3. A proper soft ideal (F,A) over a ternary semigroup S is said to be a soft prime ideal over S if for any three proper soft ideals (G,B), (H,C), (K,D) over S satisfying $(G,B)\odot(H,C)\odot(K,D)\subseteq(F,A)\Longrightarrow(G,B)\subseteq(F,A)$ or $(H,C)\subseteq(F,A)$ or $(K,D)\subseteq(F,A)$, where $B,C,D\subseteq A$.

Example 3.4. Let $A=\{\alpha\}$ and $S=\mathbb{Z}^-$ be a ternary semigroup w.r.t. ordinary ternary multiplication. Suppose $F:A\longrightarrow P(S)$ is such that $F(\alpha)=<-3>$, where $<-3>$ is the ideal generated by -3. Then (F,A) is a soft prime ideal over the ternary semigroup S.

Definition 3.5. A proper soft ideal (F,A) over a ternary semigroup S is said to be strongly prime ideal over S if for any three proper soft ideals (G,B), (H,C), (K,D) over S satisfying $((G,B)\odot(H,C)\odot(K,D))\Cap_R((H,C)\odot(K,D)\odot(G,B))\Cap_R((K,D)\odot(G,B)\odot(H,C))\subseteq(F,A)$ implies that $(G,B)\subseteq(F,A)$ or $(H,C)\subseteq(F,A)$ or $(K,D)\subseteq(F,A)$, where $B,C,D\subseteq A$.

M. Shabir et al. [8] proved in their paper that a soft set (F,A) over a ternary semigroup S is soft ideal over S if and only if $F(a)\neq\emptyset$ is ideal of S for all $a\in A$. Now we discuss similar results for soft prime and soft strongly prime ideals over ternary semigroup.

Theorem 3.6. *Let (F,A) be a soft prime ideal over a ternary semigroup S. Then $F(\alpha)$ is a prime ideal of S for all $\alpha\in A$ where $F(\alpha)\neq\emptyset$.*

Proof. Let (F,A) be a soft prime ideal over a ternary semigroup S. Let $\alpha\in A$ be such that $F(\alpha)\neq\emptyset$. Let I,J,K be three ideals of S such that $IJK\subseteq F(\alpha)$. Now define $G(\alpha)=I$, $H(\alpha)=J$, $K(\alpha)=K$ and $G(\beta)=H(\beta)=K(\beta)=F(\beta)$ for all $\beta\in A-\{\alpha\}$. Then $(G,A)\odot(H,A)\odot(K,A)\subseteq(F,A)$. This implies that $(G,A)\subseteq(F,A)$ or $(H,A)\subseteq(F,A)$ or $(K,A)\subseteq(F,A)$. Hence $I\subseteq F(\alpha)$ or $J\subseteq F(\alpha)$ or $K\subseteq F(\alpha)$. Since α is arbitrarily chosen element of A, $F(\alpha)$ is a prime ideal of S for all $\alpha\in A$. $\square$

Note 3.7. *The converse of the above result is not true i.e. if $F(\alpha)$ is a prime ideal of a ternary semigroup S for all $\alpha\in A$, where $A\subseteq S$, it may possible that (F,A) is not a soft prime ideal over S.*

In support of our above Note 3.7 we are producing the following example:

Example 3.8. Let $S=\mathbb{Z}^-$, $A=\{-2,-3,-5,-7\}$, $F(-2)=<-2>$, $F(-3)=<-3>$, $F(-5)=<-5>$, $F(-7)=<-7>$. Therefore, $F(\alpha)$ is prime ideal of S for all $\alpha\in A$. We define (G,B), (H,C), (K,D) such that $B=C=D=\{-2,-3,-5\}$, $G(-2)=<-2>$, $G(-3)=<-5>$, $G(-5)=<-7>$, $H(-2)=<-5>$, $H(-3)=<-3>$, $H(-5)=<-2>$, $K(-2)=<-3>$, $K(-3)=<-2>$, $K(-5)=<-5>$. Hence $(G,B)\odot(H,C)\odot(K,D)\subseteq(F,A)$. But $(G,B)\nsubseteq(F,A)$, $(H,C)\nsubseteq(F,A)$, $(K,D)\nsubseteq(F,A)$. Thus (F,A) is not a soft prime ideal over S.

Theorem 3.9. *Let (F,A) be a soft strongly prime ideal over a ternary semigroup S. Then $F(\alpha)$ is a strongly prime ideal of S for all $\alpha\in A$, if $F(\alpha)\neq\emptyset$.*

Proof. Let (F,A) be a soft strongly prime ideal over a ternary semigroup S and $\alpha\in A$ be such that $F(\alpha)\neq\emptyset$. Let I,J,K be three ideals of S such that $IJK\Cap_R JKI\Cap_R KIJ\subseteq F(\alpha)$. Let us define $G(\alpha)=I$, $H(\alpha)=J$, $K(\alpha)=K$ and $G(\beta)=H(\beta)=K(\beta)=F(\beta)$ for all $\beta\in A-\{\alpha\}$. Then we find that $((G,A)\odot(H,A)\odot(K,A))\Cap_R((H,A)\odot(K,A)\odot(G,A))\Cap_R((K,A)\odot(G,A)\odot(H,A))\subseteq(F,A)$. This implies that $(G,A)\subseteq(F,A)$ or $(H,A)\subseteq(F,A)$ or $(K,A)\subseteq(F,A)$. Therefore, $I\subseteq F(\alpha)$ or $J\subseteq F(\alpha)$ or $K\subseteq F(\alpha)$. Hence each $F(\alpha)$ is a strongly prime ideal over S for $F(\alpha)\neq\emptyset$. $\square$

Proposition 3.10. *Every soft strongly prime ideal over a ternary semigroup S is a soft prime ideal over S.*

Proof. Let (F,A) be a soft strongly prime ideal over a ternary semigroup S and (G,B), (H,C), (K,D) be three soft ideals over S such that $(G,B)\odot(H,C)\odot(K,D)\subseteq(F,A)$. Then $((G,B)\odot(H,C)\odot(K,D))\Cap_R((H,C)\odot(K,D)\odot(G,B))\Cap_R((K,D)\odot(G,B)\odot(H,C))\subseteq(G,B)\odot(H,C)\odot(K,D)\subseteq(F,A)$. This implies that $(G,B)\subseteq(F,A)$ or $(H,C)\subseteq(F,A)$ or $(K,D)\subseteq(F,A)$. Hence (F,A) is a soft prime ideal over S. $\square$

Note 3.11. *In a commutative ternary semigroup S, there is no difference between soft strongly prime and soft prime ideal over S.*

Now we define the chain of soft ideals and review some of its properties in view of soft ternary semigroup.

Definition 3.12. Let $\{(F_i,A_i)\}_{i\in I}$ be a collection of soft ideals over a ternary semigroup S. This collection is said to be chain of ideals if $(F_1,A_1)\subseteq(F_2,A_2)\subseteq(F_3,A_3)\subseteq\ldots$.

If all the soft ideals of a chain are soft prime ideals then the chain is called chain of soft prime ideals over a ternary semigroup of S. Following results illustrate some results of chain of soft ideals and chain of soft prime ideals.

Proposition 3.13. *The collection of soft ideals $\{(F_i,A)\}_{i\in I}$ is a chain of soft ideals over a ternary semigroup S if and only if $\{F_i(a)\}_{i\in I}$ is a chain of ideals of S for all $a\in A$.*

Proof. Let S be a ternary semigroup. Then (F_i,A) is soft ideal over S for all $i\in I$ if and only if $F_i(a)$ are ideals of S for all $i\in I$ and $a\in A$. Again $(F_1,A)\subseteq(F_2,A)\subseteq(F_3,A)\subseteq\ldots$ if and only if $F_1(a)\subseteq F_2(a)\subseteq F_3(a)\subseteq\ldots$ for all $a\in A$. Therefore, $\{F_i(a)\}_{i\in I}$ is a chain of ideals for all $a\in A$. $\square$

Proposition 3.14. *Let $\{(F_i,A)\}_{i\in I}$ be a chain of soft prime ideals over a ternary semigroup S. Then $\bigcap_{i\in I}(F_i,A)$ is a soft prime ideal over S.*

Proof. Let $(F,A)=\bigcap_{i\in I}(F_i,A)$. Suppose that (G,B), (H,C) and (K,D) are any three proper soft ideals over S such that $(G,B)\odot(H,C)\odot(K,D)\subseteq(F,A)$. This implies that $(G,B)\odot(H,C)\odot(K,D)\subseteq(F_i,A)$ for all $i\in I$. Therefore, $(G,B)\odot(H,C)\odot(K,D)\subseteq(F_1,A)$. Then by definition of soft prime ideal it is clear that $(G,B)\subseteq(F_1,A)$ or $(H,C)\subseteq(F_1,A)$ or $(K,D)\subseteq(F_1,A)$. Again by definition of chain, we get $(F_1,A)\subseteq(F_i,A)$ for all $i\in I$. Thus it is clear that $(G,B)\subseteq(F,A)$ or $(H,C)\subseteq(F,A)$ or $(K,D)\subseteq(F,A)$. Hence (F,A) is a soft prime ideal over S. $\square$

Proposition 3.15. *Let (F,A) be a soft prime ideal over a ternary semigroup S and (G,B) be any soft ideal over S. Then $(F,A)\Cap_R(G,B)$ is a soft prime ideal of (G,B).*

Proof. Let $(H,C)=(F,A)\Cap_R(G,B)$. Then $C=A\cap B$. Let $(H_1,C_1),(H_2,C_2),(H_3,C_3)$ be three soft ideals of (G,B) such that $(H_1,C_1)\odot(H_2,C_2)\odot(H_3,C_3)\subseteq(F,A)\Cap_R(G,B)$. Therefore, $(H_1,C_1)\odot(H_2,C_2)\odot(H_3,C_3)\subseteq(F,A)$. This implies that $(H_1,C_1)\subseteq(F,A)$ or $(H_2,C_2)\subseteq(F,A)$ or $(H_3,C_3)\subseteq(F,A)$. Since $(H_1,C_1),(H_2,C_2),(H_3,C_3)$ all are soft ideals of (G,B), we have $(H_1,C_1)\subseteq(F,A)\Cap_R(G,B)$ or $(H_2,C_2)\subseteq(F,A)\Cap_R(G,B)$

or $(H_3, C_3) \subseteq (F, A) \Cap_R (G, B)$. Thus $(F, A) \Cap_R (G, B) = (H, C)$ is soft prime ideal of G, B. □

Definition 3.16. [8] A proper soft (left, right, lateral) ideal (F, A) over a ternary semigroup S is said to be a soft (left, right, lateral) semiprime ideal over S if for any proper soft (left, right, lateral) ideal (G, B) over S, $(G, B) \odot (G, B) \odot (G, B) \subseteq (F, A)$ implies that $(G, B) \subseteq (F, A)$.

Lemma 3.17. *A proper soft ideal (F, A) over a ternary semigroup S is soft semiprime ideal over S if and only if $F(a) \neq \emptyset$ is semiprime ideal of S for each $a \in A$.*

Proof. Let (F, A) be a soft semiprime ideal over a ternary semigroup S. Therefore, (F, A) is a soft ideal over S. Let $a \in A$. Then $F(a)$ is an ideal of S. Let I be an ideal of S such that $I^3 \subseteq F(a)$. Let us define a soft ideal (G, B) over S such that $B = \{a\}$, $G(a) = I$. Therefore, $(G, B) \odot (G, B) \odot (G, B) \subseteq (F, A) \Longrightarrow (G, B) \subseteq (F, A) \Longrightarrow G(a) \subseteq F(a) \Longrightarrow I \subseteq F(a)$. Thus $F(a)$ is a semiprime ideal of S for all $a \in A$.

Conversely, suppose that $F(a)$ is a semiprime ideal of S for all $a \in A$. Let (G, B) be any soft ideal over S such that $(G, B) \odot (G, B) \odot (G, B) \subseteq (F, A)$. Therefore, $G(a)G(a)G(a) \subseteq F(a)$ for all $a \in B \subseteq A$. Since $F(a)$ is a semiprime ideal of S, $G(a) \subseteq F(a)$ for all $a \in B \subseteq A \Longrightarrow (G, B) \subseteq (F, A)$. Hence (F, A) is a soft semiprime ideal over the ternary semigroup S. □

Proposition 3.18. *Every soft prime ideal over a ternary semigroup S is a soft semiprime ideal over S.*

Proof. Let (F, A) be a soft prime ideal over S and (G, B) be any proper soft ideal over S such that $(G, B) \odot (G, B) \odot (G, B) \subseteq (F, A)$. Since (F, A) is soft prime ideal over S we have, $(G, B) \subseteq (F, A)$. Hence (F, A) is a soft semiprime ideal over S. □

Corollary 3.19. *Every soft strongly prime ideal over a ternary semigroup S is a soft semiprime ideal over S.*

Definition 3.20. Let (F, A) be a soft ideal over a ternary semigroup S. Then (F, A) is said to be soft irreducible ideal over S if for soft ideals $(G, B), (H, C), (K, D)$ over S satisfying $(G, B) \Cap_R (H, C) \Cap_R (K, D) = (F, A)$ implies that $(G, B) = (F, A)$ or $(H, C) = (F, A)$ or $(K, D) = (F, A)$.

Definition 3.21. Let (F, A) be a soft ideal over a ternary semigroup S. Then (F, A) is said to be soft strongly irreducible ideal over S if for any three soft ideals (G, B), (H, C), (K, D) over S satisfying $(G, B) \Cap_R (H, C) \Cap_R (K, D) \subseteq (F, A)$ implies that $(G, B) \subseteq (F, A)$ or $(H, C) \subseteq (F, A)$ or $(K, D) \subseteq (F, A)$.

The followings are some properties of soft irreducible and soft strongly irreducible ideal over a ternary semigroup S.

Lemma 3.22. *Let (F, A) be a soft irreducible ideal over a ternary semigroup S. Then $F(\alpha)$ is irreducible ideal of S for all $\alpha \in A$, where $F(\alpha) \neq \emptyset$.*

Proof. Let (F, A) be a soft irreducible ideal over a ternary semigroup S. Let $\alpha \in A$. Then $F(\alpha)$ is an ideal of S. Let I, J, K be ideals of S such that $I \cap J \cap K = F(\alpha)$. Now we define $G(\alpha) = I$, $H(\alpha) = J$, $K(\alpha) = K$ and $G(\beta) = H(\beta) = K(\beta) = F(\beta)$ for all $\beta \in A - \{\alpha\}$. Then $(G, A), (H, A), (K, A)$ are soft ideals over S such that $(G, A) \Cap_R (H, A) \Cap_R (K, A) = (F, A)$. Since (F, A) is soft irreducible ideal over S, $(G, A) = (F, A)$ or $(H, A) = (F, A)$ or $(K, A) = (F, A)$. Therefore, $G(\alpha) = F(\alpha)$ or $H(\alpha) = F(\alpha)$ or $K(\alpha) = F(\alpha)$ for all $\alpha \in A$. This implies that $I = F(\alpha)$ or $J = F(\alpha)$ or $K = F(\alpha)$. Thus $F(\alpha)$ is irreducible ideal of S. Since α is arbitrary element of A, $F(\alpha)$ is irreducible ideal of S for all $\alpha \in A$. □

Note 3.23. *The converse of the above Lemma 3.22 is not true, in general.*

Lemma 3.24. *Let (F, A) be a soft strongly irreducible ideal over a ternary semigroup S. Then $F(a)$ is a strongly irreducible ideal of S for all $a \in A$, where $F(a) \neq \emptyset$.*

The Proof is similar to the Proof of Lemma 3.22.

Lemma 3.25. *Every soft strongly irreducible ideal over a ternary semigroup S is a soft irreducible ideal over S.*

In the following proposition we show the inter relation between soft prime ideal and soft strongly irreducible ideal over a ternary semigroup S.

Proposition 3.26. *Every soft prime ideal over a ternary semigroup S is a soft strongly irreducible ideal over S.*

Proof. Let (F, A) be a soft prime ideal over S. Let $(G, B), (H, C), (K, D)$ be three soft ideals over S such that $(G, B) \Cap_R (H, C) \Cap_R (K, D) \subseteq (F, A)$. Now $(G, B) \odot (H, C) \odot (K, D) \subseteq (G, B)$, $(G, B) \odot (H, C) \odot (K, D) \subseteq (H, C)$ and $(G, B) \odot (H, C) \odot (K, D) \subseteq (K, D)$. Therefore, $(G, B) \odot (H, C) \odot (K, D) \subseteq (G, B) \Cap_R (H, C) \Cap_R (K, D) \subseteq (F, A)$. This implies that $(G, B) \subseteq (F, A)$ or $(H, C) \subseteq (F, A)$ or $(K, D) \subseteq (F, A)$. Thus (F, A) is a strongly irreducible ideal over S. □

Note 3.27. *Since every soft prime ideal over a ternary semigroup S is soft strongly irreducible ideal over S and every soft strongly irreducible ideal over a ternary semigroup S is soft irreducible ideal over S, every soft prime ideal over a ternary semigroup S is a soft irreducible ideal over S.*

Theorem 3.28. *A soft semiprime ideal over a ternary semigroup S is soft prime ideal if it is soft irreducible ideal over S.*

Proof. Let (F, A) be a soft semiprime ideal over S. Let $(G, B), (H, C)$, (K, D) be three proper soft ideals over S such that $(G, B) \odot (H, C) \odot (K, D) \subseteq (F, A)$. Suppose (F, A) is soft irreducible ideal over S.

Now $((G, B) \Cap_R (H, C) \Cap_R (K, D)) \odot ((G, B) \Cap_R (H, C) \Cap_R (K, D)) \odot ((G, B) \Cap_R (H, C) \Cap_R (K, D)) \subseteq (G, B) \odot (H, C) \odot (K, D) \subseteq (F, A)$. This implies that $(G, B) \Cap_R (H, C) \Cap_R (K, D) \subseteq (F, A)$. Therefore, $((G, B) \Cap_R (H, C) \Cap_R (K, D)) \cup (F, A) = (F, A)$. This implies that $(G, B) \cup (F, A) = (F, A)$ or $(H, C) \cup (F, A) = (F, A)$ or $(K, D) \cup (F, A) = (F, A)$. Therefore, $(G, B) \subseteq (F, A)$ or $(H, C) \subseteq (F, A)$ or $(K, D) \subseteq (F, A)$. Hence (F, A) is a soft prime ideal over S. □

Let us discuss the following equivalent conditions.

Theorem 3.29. *If (F,A) is a soft ideal over a ternary semigroup S with identity and $a \in S$. Then the following conditions are equivalent:*

(i) (F, A) is soft semiprime ideal over S.

(ii) $F(\alpha) \neq \emptyset$ is a semiprime ideal of S.

(iii) $aSSaSSa \subseteq F(\alpha)$ implies that $a \in F(\alpha)$.

(iv) $(SSa)(SSa)(SSa) \subseteq F(\alpha)$ implies that $a \in F(\alpha)$, $(aSS)(aSS)(aSS) \subseteq F(\alpha)$ implies that $a \in F(\alpha)$, $(SaS)(SaS)(SaS) \subseteq F(\alpha)$ implies that $a \in F(\alpha)$.

Proof. $(i) \Longleftrightarrow (ii)$: follows from the Lemma 3.17.

$(ii) \Longrightarrow (iii)$:

Suppose that $F(\alpha)$ is a semiprime ideal of S for all $\alpha \in A$ and $F(\alpha) \neq \emptyset$. Let $(aSSaSSa) \subseteq F(\alpha)$. Therefore, $SS(aSSaSSa)SS \subseteq SSF(\alpha)SS \subseteq (SSF(\alpha))SS \subseteq F(\alpha)SS \subseteq F(\alpha)$. Now $(SSaSS)(SSaSS)(SSaSS) \subseteq SSaSSaSSaSS \subseteq F(\alpha)$. Therefore, $SSaSS \subseteq F(\alpha)$. Again $SSaSS$ is an ideal containing a. Thus $a \in F(\alpha)$.

$(iii) \Longrightarrow (iv)$:

Let $(SSa)(SSa)(SSa) \subseteq F(\alpha)$. Now $aSSaSSa \subseteq SS(aSSaSSa) \subseteq F(\alpha)$. This implies that $a \in F(\alpha)$. Again $(aSS)(aSS)(aSS) \subseteq F(\alpha)$. Therefore, $(aSSaSSa)SS \subseteq (aSSaSSa)SS \subseteq F(\alpha)$. This implies that $a \in F(\alpha)$. Similarly, $(SaS)(SaS)(SaS) \subseteq F(\alpha)$ implies that $a \in F(\alpha)$.

$(iv) \Longrightarrow (ii)$:

Since (F,A) is a soft ideal over S, $F(\alpha) \neq \emptyset$ is ideal of S for all $\alpha \in A$. Let I be an ideal of S such that $I^3 \subseteq F(\alpha)$. Let $a \in I$. Now $(SSI)(SSI)(SSI) \subseteq I^3 \subseteq F(\alpha)$. Hence $(SSa)(SSa)(SSa) \subseteq (SSI)(SSI)(SSI) \subseteq F(\alpha)$. This implies that $a \in F(\alpha)$. Thus $I \subseteq F(\alpha)$. Therefore, $F(\alpha)$ is a semiprime ideal of S for all $\alpha \in A$ such that $F(\alpha) \neq \emptyset$. □

4. Soft prime bi-ideals and soft semiprime bi-ideals

In this section, we extend the results of soft prime and soft semiprime ideal in terms of soft prime bi-ideal and soft semiprime bi-ideal.

Definition 4.1. A soft bi-ideal (F,A) over a ternary semigroup S is called soft prime bi-ideal if $(G,B) \odot (H,C) \odot (K,D) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$, whenever $B,\ C,\ D \subseteq A$ and (G,B), (H,C), (K,D) are proper soft bi-ideals over S.

Definition 4.2. A soft bi-ideal (F,A) over a ternary semigroup S is said to be strongly prime bi-ideal if $((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$, whenever $B,\ C,\ D \subseteq A$ and (G,B), (H,C), (K,D) are proper soft bi-ideals over S.

Definition 4.3. A soft bi-ideal (F,A) over a ternary semigroup S is said to be soft semiprime bi-ideal over S if $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ for every soft bi-ideal (G,B) over S.

Proposition 4.4. *Let (F,A) be a proper soft bi-ideal over a ternary semigroup S. Then (F,A) is soft semiprime bi-ideal over S if and only if $F(\alpha)$ is semiprime bi-ideal of S for all $\alpha \in A$ whenever $F(\alpha) \neq \emptyset$.*

Proof. Let (F,A) be a proper soft bi-ideal over S. Then (F,A) is a soft ternary semigroup and $(F,A) \odot \tilde{A}_S \odot (F,A) \odot \tilde{A}_S \odot (F,A) \subseteq (F,A)$. Therefore, $F(\alpha)SF(\alpha)SF(\alpha) \subseteq F(\alpha)$ for all $\alpha \in A$. This shows that $F(\alpha)$ is a bi-ideal of S for all $\alpha \in A$. Let I be a bi-ideal of S such that $III \subseteq F(\alpha)$. Let us define $G(\alpha) = I$ and $B = \{\alpha\}$. Then (G,B) is a soft bi-ideal over S and $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$. This implies that $(G,B) \subseteq (F,A)$, since (F,A) is a soft semiprime bi-ideal over the ternary semigroup S. Therefore, $G(\alpha) \subseteq F(\alpha)$. This implies $I \subseteq F(\alpha)$. Hence $F(\alpha)$ is a semiprime bi-ideal of S, whenever $F(\alpha) \neq \emptyset$.

Conversely, let $F(\alpha)$ be a semiprime bi-ideal of S for all $\alpha \in A$, when $F(\alpha) \neq \emptyset$. Let (G,B) be a soft bi-ideal over S, such that $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$. Therefore, $G(\alpha)G(\alpha)G(\alpha) \subseteq F(\alpha)$ for all $\alpha \in B \subseteq A$. If $G(\alpha) = \emptyset$, then clearly $G(\alpha) \subseteq F(\alpha)$. If $G(\alpha) \neq \emptyset$, then $G(\alpha)$ is a bi-ideal and $F(\alpha)$ is a semiprime bi-ideal of S. This implies that $G(\alpha) \subseteq F(\alpha)$. Hence $G(\alpha) \subseteq F(\alpha)$ for all $\alpha \in B \subseteq A$. Thus $(G,B) \subseteq (F,A)$. Therefore, (F,A) is a soft semiprime bi-ideal over S. □

Definition 4.5. A soft bi-ideal (F,A) over a ternary semigroup S is said to be soft irreducible bi-ideal over S if $(G,B) \Cap_R (H,C) \Cap_R (K,D) = (F,A)$ implies that $(G,B) = (F,A)$ or $(H,C) = (F,A)$ or $(K,D) = (F,A)$, whenever $B,\ C,\ D \subseteq A$ and (G,B), (H,C), (K,D) are soft bi-ideals over S.

A soft bi-ideal (F,A) over a ternary semigroup S is said to be strongly irreducible bi-ideal over S if $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$, whenever $B,\ C,\ D \subseteq A$ and (G,B), (H,C), (K,D) are soft bi-ideals over S.

Proposition 4.6. *Let (F,A) be a soft irreducible bi-ideal over a ternary semigroup S. Then for all $\alpha \in A$, $F(\alpha)$ is an irreducible bi-ideal of S, when $F(\alpha) \neq \emptyset$.*

Proof. Same as Lemma 3.22. □

Corollary 4.7. *Let (F,A) be a soft strongly irreducible bi-ideal over a ternary semigroup S. Then for all $\alpha \in A$, $F(\alpha)$ is a strongly irreducible bi-ideal of S when $F(\alpha) \neq \emptyset$.*

Lemma 4.8. *Let (F,A) be a soft bi-ideal over a ternary semigroup S. If (F,A) is both strongly irreducible and semiprime bi-ideal over a ternary semigroup S, then (F,A) is strongly prime bi-ideal over S.*

Proof. Let (F,A) be soft semiprime and strongly irreducible bi-ideal over S. Let (G,B), (H,C), (K,D) be any three soft bi-ideals such that $((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$. Now $((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \subseteq (G,B) \odot (H,C) \odot (K,D)$, $((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \subseteq (H,C) \odot (K,D) \odot (G,B)$ and $((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \subseteq (K,D) \odot (G,B) \odot (H,C)$. Hence $((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \subseteq ((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$. Since (F,A) is a semiprime bi-ideal, $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$. Thus (F,A) is soft strongly prime bi-ideal over S. □

Theorem 4.9. *For a ternary semigroup the following conditions are equivalent:*

(1) *Every soft bi-ideals over S is idempotent.*

(2) $(G,B) \Cap_R (H,C) \Cap_R (K,D) = ((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C))$ *for all soft bi-ideals (G,B), (H,C), (K,D) over S.*

(3) *Every soft bi-ideals over S is soft semiprime bi-ideal over S.*

Proof. $(1) \Longrightarrow (2)$ Let (G,B), (H,C), (K,D) be three soft bi-ideals over S. Then $(G,B) \Cap_R (H,C) \Cap_R (K,D)$ is a soft bi-ideal over S. Since every soft bi-ideal over S is idempotent, it follows that $(G,B) \Cap_R (H,C) \Cap_R (K,D) = ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \odot ((G,B) \Cap_R (H,C) \Cap_R (K,D)) \subseteq (G,B) \odot (H,C) \odot (K,D)$. Similarly, $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq (H,C) \odot (K,D) \odot (G,B)$ and $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq (K,D) \odot (G,B) \odot (H,C)$. Therefore, $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq ((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C))$. Let $((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) = (L,M)$. Now $((G,B) \odot (H,C) \odot (K,D))$, $((H,C) \odot (K,D) \odot (G,B))$, $((K,D) \odot (G,B) \odot (H,C))$ are soft bi-ideals over S. Therefore, (L,M) is a soft bi-ideal over S. Thus $(L,M) = (L,M) \odot (L,M) \odot (L,M) \subseteq ((G,B) \odot (H,C) \odot (K,D)) \odot ((K,D) \odot (G,B) \odot (H,C)) \odot ((H,C) \odot (K,D) \odot (G,B)) \subseteq (G,B) \odot \tilde{A}_S \odot \tilde{A}_S \odot \tilde{A}_S \odot (G,B) \odot \tilde{A}_S \odot \tilde{A}_S \odot \tilde{A}_S \odot (G,B) \subseteq (G,B) \odot \tilde{A}_S \odot (G,B) \odot \tilde{A}_S \odot (G,B) \subseteq (G,B)$. Similarly, $(L,M) \subseteq (H,C)$ and $(L,M) \subseteq (K,D)$. Therefore, $(L,M) \subseteq (G,B) \Cap_R (H,C) \Cap_R (K,D)$. Hence $(G,B) \Cap_R (H,C) \Cap_R (K,D) = ((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C))$ for all soft bi-ideals over S.

$(2) \Longrightarrow (3)$

Let (F,A) be a soft bi-ideal over a ternary semigroup S and (G,B) be a soft bi-ideal over S such that $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$. Now $(G,B) = (G,B) \Cap_R (G,B) \Cap_R (G,B) = ((G,B) \odot (G,B) \odot (G,B)) \Cap_R ((G,B) \odot (G,B) \odot (G,B)) \Cap ((G,B) \odot (G,B) \odot (G,B)) = (G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$. This implies that $(G,B) \subseteq (F,A)$. Hence (F,A) is a soft semiprime bi-ideal over S.

$(3) \Longrightarrow (1)$

Let (F,A) be a soft bi-ideal over S. Then (F,A) is soft semiprime bi-ideal over S. Let $(F,A) \odot (F,A) \odot (F,A) = (G,B)$. Also (G,B) is a soft bi-ideal over S. Therefore, (G,B) is a soft semiprime bi-ideal over S. Hence $(F,A) \odot (F,A) \odot (F,A) \subseteq (G,B)$. This implies that $(F,A) \subseteq (G,B)$. Thus $(F,A) \subseteq (F,A) \odot (F,A) \odot (F,A)$. Again let $(G,B) \odot (G,B) \odot (G,B) = (H,C)$.

Since (G,B) is soft semiprime bi-ideal over S, (H,C) is also soft bi-ideal over S. Thus $(G,B) \subseteq (H,C)$. Hence $(G,B) \subseteq (G,B) \odot (G,B) \odot (G,B) = ((F,A) \odot (F,A) \odot (F,A)) \odot ((F,A) \odot (F,A) \odot (F,A)) \odot ((F,A) \odot (F,A) \odot (F,A)) \subseteq (F,A) \odot \tilde{A}_S \odot (F,A) \odot \tilde{A}_S \odot (F,A) \subseteq (F,A)$. This shows that $(F,A) \subseteq (G,B) \subseteq (F,A)$. Thus $(F,A) \odot (F,A) \odot (F,A) = (F,A)$. Therefore, (F,A) is idempotent. Hence every soft bi-ideal over S is idempotent. □

Theorem 4.10. *Each soft bi-ideal over a ternary semigroup S is strongly prime bi-ideal over S if and only if every soft bi-ideal over S is idempotent and set of all soft bi-ideals over S is totally ordered under set inclusion.*

Proof. Let (F,A) be a strongly prime soft bi-ideal over S. For any three soft bi-ideals (G,B), (H,C), (K,D) satisfying $((G,B)\odot(H,C)\odot(K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$. Now substituting $(H,C) = (K,D) = (G,B)$, we get $(G,B) \odot (G,B) \odot (G,B) = ((G,B) \odot (G,B) \odot (G,B)) \Cap_R ((G,B) \odot (G,B) \odot (G,B)) \Cap_R ((G,B) \odot (G,B) \odot (G,B))$. Therefore, $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$ implies that $((G,B) \odot (G,B) \odot (G,B)) \Cap_R ((G,B) \odot (G,B) \odot (G,B)) \Cap_R ((G,B) \odot (G,B) \odot (G,B)) \subseteq (F,A)$, which implies that $(G,B) \subseteq (F,A)$. Therefore, (F,A) is a soft semiprime bi-ideal over S. Then by Theorem 4.9, we can say that (F,A) is idempotent, i.e. every soft bi-ideal over S is idempotent.

Let (F,A), (G,B), (H,C) be any three soft bi-ideals over S. Since every soft bi-ideal over S is idempotent, by Theorem 4.9, we get $(F,A) \Cap_R (G,B) \Cap_R (H,C) = ((F,A) \odot (G,B) \odot (H,C)) \Cap_R ((G,B) \odot (H,C) \odot (F,A)) \Cap_R ((H,C) \odot (F,A) \odot (G,B))$. Again $(F,A) \Cap_R (G,B) \Cap_R (H,C)$ is a soft bi-ideal over S, it follows that it is strongly prime bi-ideal over S. Now, $((F,A) \odot (G,B) \odot (H,C)) \Cap_R ((G,B) \odot (H,C) \odot (F,A)) \Cap_R ((H,C) \odot (F,A) \odot (G,B)) \subseteq (F,A) \Cap_R (G,B) \Cap_R (H,C)$. This implies that $(F,A) \subseteq (F,A) \Cap_R (G,B) \Cap_R (H,C)$ or $(G,B) \subseteq (F,A) \Cap_R (G,B) \Cap_R (H,C)$ or $(H,C) \subseteq (F,A) \Cap_R (G,B) \Cap_R (H,C)$. Suppose $(F,A) \subseteq (F,A) \Cap_R (G,B) \Cap_R (H,C)$. This implies that $(F,A) \subseteq (G,B) \Cap_R (H,C)$, i.e. $(F,A) \subseteq (G,B)$ and $(F,A) \subseteq (H,C)$. Now substituting (F,A) by (G,B) in the relation $(F,A) \Cap_R (G,B) \Cap_R (H,C) = ((F,A) \odot (G,B) \odot (H,C)) \Cap_R ((G,B) \odot (H,C) \odot (F,A)) \Cap_R ((H,C) \odot (F,A) \odot (G,B))$, we get $(G,B) \Cap_R (G,B) \Cap_R (H,C) = ((G,B) \odot (G,B) \odot (H,C)) \Cap_R ((G,B) \odot (H,C) \odot (G,B)) \Cap_R ((H,C) \odot (G,B) \odot (G,B))$. This implies that $(G,B) \subseteq (G,B) \Cap_R (G,B) \Cap_R (H,C)$ or $(H,C) \subseteq (G,B) \Cap_R (G,B) \Cap_R (H,C)$. Thus $(G,B) \subseteq (H,C)$ or $(H,C) \subseteq (G,B)$. Therefore, the set of all soft bi-ideals are totally ordered under set inclusion.

Conversely, suppose that every soft bi-ideals over S are idempotent and set of all bi-ideals over S are totally ordered under set inclusion. Let (F,A), (G,B), (H,C), (K,D) be soft bi-ideals over S such that $((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$. Since every soft bi-ideals are idempotent by Theorem 4.9, we get $(G,B) \Cap_R (H,C) \Cap_R (K,D) = ((G,B) \odot (H,C) \odot (K,D)) \Cap_R ((H,C) \odot (K,D) \odot (G,B)) \Cap_R ((K,D) \odot (G,B) \odot (H,C)) \subseteq (F,A)$. Again set of all bi-ideals are totally ordered under set inclusion. Therefore, $(G,B) \Cap_R (H,C) \Cap_R (K,D) = (G,B)$ or (H,C) or (K,D). Thus $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$. Hence (F,A) is a strongly prime soft bi-ideal over S. Therefore, every soft bi-ideals over S are strongly prime bi-ideal over S. □

Theorem 4.11. *Let the set of all soft bi-ideals over a ternary semigroup S be totally ordered under set inclusion. Then soft semiprime bi-ideals over S are soft prime bi-ideals over S.*

Proof. Let (F,A) be a soft semiprime bi-ideal over S and (G,B), (H,C), (K,D) be any three soft bi-ideals over S such that $(G,B) \odot (H,C) \odot (K,D) \subseteq (F,A)$. Since set of all soft bi-ideals over S are totally ordered under set inclusion, $(G,B) \subseteq (H,C) \Cap_R (K,D)$ or $(H,C) \subseteq (G,B) \Cap_R (K,D)$ or $(K,D) \subseteq (G,B) \Cap_R (H,C)$. Suppose $(G,B) \subseteq (H,C) \Cap_R (K,D)$. This implies $(G,B) \odot (G,B) \odot (G,B) \subseteq (G,B) \odot (H,C) \odot (K,D) \subseteq (F,A)$. Since (F,A) is soft semiprime bi-ideal over S, $(G,B) \odot (G,B) \odot (G,B) \subseteq (F,A)$ implies that $(G,B) \subseteq (F,A)$. Similarly, $(H,C) \subseteq (G,B) \Cap_R (K,D)$ implies that $(H,C) \subseteq (F,A)$ and $(K,D) \subseteq (G,B) \Cap_R (H,C)$ implies that $(K,D) \subseteq (F,A)$. Hence (F,A) is a soft prime bi-ideal over S. □

Theorem 4.12. *For a ternary semigroup S the following conditions are equivalent:*

(1) *The set of soft bi-ideals over S is totally ordered under set inclusion.*
(2) *Each soft bi-ideal over S is soft strongly irreducible bi-ideal over S.*
(3) *Each soft bi-ideal over S is soft irreducible bi-ideal over S.*

Proof. $(1) \Longrightarrow (2)$

Let the set of soft bi-ideals over S be totally ordered under set inclusion. Let (F,A), (G,B), (H,C), (K,D) be four soft bi-ideals over S such that $(G,B) \Cap_R (H,C) \Cap_R (K,D) \subseteq (F,A)$. Again $(G,B) \Cap_R (H,C) \Cap_R (K,D) = (G,B)$ or $(G,B) \Cap_R (H,C) \Cap_R (K,D) = (H,C)$ or $(G,B) \Cap_R (H,C) \Cap_R (K,D) = (K,D)$. Thus $(G,B) \subseteq (F,A)$ or $(H,C) \subseteq (F,A)$ or $(K,D) \subseteq (F,A)$. Therefore, (F,A) is soft strongly irreducible bi-ideal over S.

$(2) \Longrightarrow (3)$

It is clear that soft strongly irreducible ideal over S is a soft irreducible ideal over S.

$(3) \Longrightarrow (1)$

Let (F,A), (G,B), (H,C) be soft bi-ideals over S. Then $(F,A) \Cap_R (G,B) \Cap_R (H,C)$ is also soft bi-ideal. Therefore, this is soft irreducible bi-ideal over S. Now $(F,A) \Cap_R (G,B) \Cap_R (H,C) = (F,A) \Cap_R (G,B) \Cap_R (H,C)$. This implies that $(F,A) = (F,A) \Cap_R (G,B) \Cap_R (H,C)$ or $(G,B) = (F,A) \Cap_R (G,B) \Cap_R (H,C)$ or $(K,D) = (F,A) \Cap_R (G,B) \Cap_R (H,C)$. Suppose $(F,A) = (F,A) \Cap_R (G,B) \Cap_R (H,C)$. Therefore, $(F,A) \subseteq (G,B) \Cap_R (H,C)$, i.e. $(F,A) \subseteq (G,B)$ and $(F,A) \subseteq (K,D)$. Substituting (F,A) by (G,B) in $(F,A) \Cap_R (G,B) \Cap_R (H,C)$, we get $(G,B) \subseteq (H,C)$ or $(H,C) \subseteq (F,A)$. Hence set of all soft bi-ideals over S are totally ordered under set inclusion. □

5. Soft completely regular ternary semigroup

In this section, we define soft left regular, soft right regular and soft completely regular ternary semigroup. Then we discuss some results on these soft ternary semigroups and also characterize soft completely regular ternary semigroup.

Definition 5.1. [6] An element a of a ternary semigroup S is said to be left (resp. right) regular if there exists an element $x \in S$ such that $xaa = a$ (resp. $aax = a$). If every element of S is left (resp. right) regular then S is said to be left (resp. right) regular ternary semigroup.

Definition 5.2. A proper soft set (F,A) over a ternary semigroup S is said to be soft left (resp. right) regular if $F(\alpha)$ is left (resp. right) regular for all $\alpha \in A$.

Lemma 5.3. *A ternary semigroup S is right (resp. left) regular ternary semigroup if and only if every soft ideal (F,A) over S is right (resp. left) regular.*

Proof. Let S be a right regular ternary semigroup and (F,A) be a soft ideal over S. Let $\alpha \in A$ be such that $F(\alpha) \neq \emptyset$. Then $F(\alpha)$ is an ideal of S. Let $a \in F(\alpha)$. Therefore, $a \in S$. Thus there exists $x \in S$ such that $aax = a$. Let us define $b = xax$. Then $b \in SaS \subseteq SF(\alpha)S \subseteq F(\alpha)$. Now $aab = aa(xax) = (aax)ax = aax = a$. Therefore, for $a \in F(\alpha)$ there exists an element $b \in F(\alpha)$ such that $aab = a$. Hence $F(\alpha)$ is right regular. Since α is arbitrary element of A, $F(\alpha)$ is right regular for all $\alpha \in A$. Hence (F,A) over S is soft right regular.

Similarly, we can show that if S is left regular ternary semigroup then every soft ideal (F,A) over S is soft left regular.

Conversely, suppose that every soft ideal over S is right regular. Let $a \in S$ and I be an ideal of S containing a. Define (F,A) such that $A = \{a\}$ and $F(a) = I$. Then (F,A) is a soft ideal over S. Therefore, (F,A) over S is right regular. This implies that $F(a) = I$ is right regular. Thus $a \in I$ is right regular. Since a is an arbitrary element of S, every element of S is right regular. Thus S is right regular ternary semigroup. Similarly, we can show that S is left regular ternary semigroup if every soft ideal over S is left regular. □

Definition 5.4. [6] A proper ideal I of a ternary semigroup S is called a completely semiprime ideal of S if $x^3 \in I$ implies that $x \in I$ for any element x of S.

Definition 5.5. A proper soft ideal (F, A) over a ternary semigroup S is said to be soft completely semiprime ideal over S if $F(\alpha)$ is completely semiprime ideal for all $\alpha \in A$, where $F(\alpha) \neq \emptyset$.

Definition 5.6. [6] An element a of a ternary semigroup S is said to be completely regular if there exists an element $x \in S$ such that $axa = a$ and $axz = xaz$, $zax = zxa$ for all $z \in S$.

If all the elements of a ternary semigroup S is completely regular then S is said to be completely regular ternary semigroup.

Definition 5.7. A soft set (F, A) over a ternary semigroup S is said to be soft completely regular if $F(\alpha)$ is completely regular ternary semigroup for all $\alpha \in A$, where $F(\alpha) \neq \emptyset$.

Lemma 5.8. *A ternary semigroup S is completely regular if and only if every soft ideal over S is completely regular.*

Proof. Let S be completely regular ternary semigroup and (F, A) be any soft ideal over S. Let $\alpha \in A$ be such that $F(\alpha) \neq \emptyset$. Let $a \in F(\alpha)$. Then $a \in S$. Thus there exists $x \in S$ such that $axa = a$ and $axz = xaz$ for all $z \in S$. Let us define $b = xax$. Then $aba = a(xax)a = (axa)xa = axa = a$ and $b = xax \in SF(\alpha)S \subseteq F(\alpha)$. Therefore, a is regular in $F(\alpha)$. Again let $c \in F(\alpha)$ be any element of $F(\alpha)$. Then $abc = a(xax)c = (axa)xc = axc = xac = x(axa)c = (xax)ac = bac$, $cab = ca(xax) = c(axa)x = cax = cxa = cx(axa) = c(xax)a = cba$. Therefore, $a \in F(\alpha)$ is completely regular in $F(\alpha)$. Then $F(\alpha)$ is completely regular for all $\alpha \in A$. Therefore, (F, A) over S is completely regular. So we can say that every soft ideal over S is completely regular.

Conversely, suppose that every soft ideal over S is completely regular. Define a soft ideal (F, A) over S such that $F(\alpha) = S$ for some $\alpha \in A$. Since (F, A) is completely regular, $F(\alpha)$ is completely regular for all $\alpha \in A$. Therefore, S is completely regular. □

Theorem 5.9. *[6] The following conditions in a ternary semigroup S are equivalent:*

(1) *S is completely regular.*

(2) *S is left and right regular i.e. $a \in a^2S \cap Sa^2$ for all $a \in S$.*

(3) *$a \in a^2Sa^2$ for all $a \in S$.*

Theorem 5.10. *Let S be a ternary semigroup. Then the following conditions are equivalent:*

(1) *S is completely regular ternary semigroup.*

(2) *Every soft ideal over S is soft left and soft right regular.*

(3) *Let (F, A) be a soft ideal over S. Then $a \in a^2F(\alpha)a^2$ for all $a \in F(\alpha)$ and for all $\alpha \in A$*

Proof. (1) $\Longleftrightarrow$ (2)

By Theorem 5.9, S is completely regular ternary semigroup if and only if S is left and right regular. Again by Lemma 5.3, S is left and right regular if and only if every soft ideal over S is left and right regular. Therefore, S is completely regular if and only if every soft ideal over S is soft left and soft right regular.

(2) $\Longrightarrow$ (3)

Let (F, A) be a soft ideal over S. Then (F, A) is soft left and soft right regular over S. Let $\alpha \in A$ be such that $F(\alpha) \neq \emptyset$. Then $F(\alpha)$ is left and right regular ternary semigroup, by definition. Let $a \in F(\alpha)$. Then there exists c, $d \in F(\alpha)$ such that $daa = aac = a$. Now $acx = (daa)cx = d(aac)x = dax$ for all $x \in S$. So $a = daa = aca = aacca = aaccdaa$. Since $F(\alpha)$ is ideal of S and c, $d \in F(\alpha)$, we have $ccd \in F(\alpha)$. Therefore, $a = aa(ccd)aa \in aaF(\alpha)aa$, i.e. $a \in a^2F(\alpha)a^2$.

(3) $\Longrightarrow$ (1)

Let $a \in S$ be an element of S and I be an ideal of S containing a. Define (F, A) such that $A = \{\alpha\}$ and $F(\alpha) = I$. Then $a \in F(\alpha)$ implies that $a \in a^2F(\alpha)a^2 \subseteq a^2Sa^2$. Therefore, for all $a \in S$, $a \in a^2Sa^2$. Then from Theorem 5.9, we get S is completely regular. □

Theorem 5.11. *[6] A ternary semigroup S is completely regular if and only if every bi-ideal of S is completely semiprime.*

Theorem 5.12. *A ternary semigroup S is completely regular if and only if every soft bi-ideal over S is completely semiprime.*

Proof. Let (F, A) be a soft bi-ideal over S and S is completely regular ternary semigroup. Then $F(\alpha)$ is a bi-ideal of S for all $\alpha \in A$, where $F(\alpha) \neq \emptyset$. Then by Theorem 5.11, $F(\alpha)$ is completely semiprime for all $\alpha \in A$, where $F(\alpha) \neq \emptyset$. Therefore, (F, A) is completely semiprime.

Conversely, suppose that every soft bi-ideal over S is completely semiprime. Let B be any bi-ideal of S. Define (F, A) such that $A = \{\alpha\}$ and $F(\alpha) = B$. Then (F, A) is soft bi-ideal over S. Hence (F, A) is completely semiprime and this implies that $F(\alpha) = B$ is completely semiprime. Therefore, every bi-ideal of S is completely semiprime. Thus by Theorem 5.11, S is completely regular. □

Theorem 5.13. *[6] If S is a completely regular ternary semigroup, then every bi-ideal of S is idempotent.*

Theorem 5.14. *If S is a completely regular ternary semigroup, then every soft bi-ideal over S is idempotent.*

Proof. Let S be a completely regular ternary semigroup and (F, A) be a soft bi-ideal over S. If $F(\alpha) = \emptyset$, then $F(\alpha)F(\alpha)F(\alpha) = F(\alpha)$. Again if $F(\alpha) \neq \emptyset$, then $F(\alpha)$ is a bi-ideal of S. Now by Theorem 5.13, $F(\alpha)$ is idempotent, i.e. $F(\alpha)F(\alpha)F(\alpha) = F(\alpha)$. Therefore, $F(\alpha)F(\alpha)F(\alpha) = F(\alpha)$ for all $\alpha \in A$. Hence $(F, A) \odot (F, A) \odot (F, A) = (F, A)$. Therefore, every soft bi-ideal over S is idempotent. □

6. Conclusion

Soft set is a very useful tool in mathematics and its related areas. In this paper, we study about soft prime ideal, semiprime ideal, prime bi-ideal, semiprime bi-ideal and their interrelations. Here we also characterize soft right regular, soft left regular and soft completely regular ternary semigroup and their relations. It can be developed the notions of other soft ideals like soft quasi prime and soft quasi semiprime ideals over a ternary semigroup. In future, soft ternary semigroups can be developed in the light of spherical sets and also can be applied to solve decision-making problem as in the paper [22], spherical fuzzy sets are discussed. Since soft set theory has a large number of application in different area of real life problem, development of the theory of soft ternary semigroup will be helpful to enlarge the area of application. As we have developed various area of soft ideals of soft ternary semigroup this is a motivational work for future. We may generalize soft rough semigroups of [23] and soft filters of ordered semigroup of [24] in soft ternary semigroup and soft ordered ternary semigroup also.

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Author contribution statement

S. Kar, I. Dutta: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

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