

Research



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Handlebody decompositions of three-manifolds and polycontinuous patterns

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We introduce the concept of a handlebody decomposition of a three-manifold, a generalization of a Heegaard splitting, or a trisection. We show that two handlebody decompositions of a closed orientable three-manifold are stably equivalent. As an application to materials science, we consider a mathematical model of polycontinuous patterns and discuss a topological study of microphase separation of a block copolymer melt.

1. Introduction

A *Heegaard splitting* is a decomposition of a closed orientable three-manifold into two handlebodies of the same genus. It is well known that every closed orientable three-manifold admits a Heegaard splitting. By the Reidemeister–Singer theorem [1,2], two Heegaard splittings of a given three-manifold are stably equivalent, i.e. isotopic after a finite number of stabilizations.

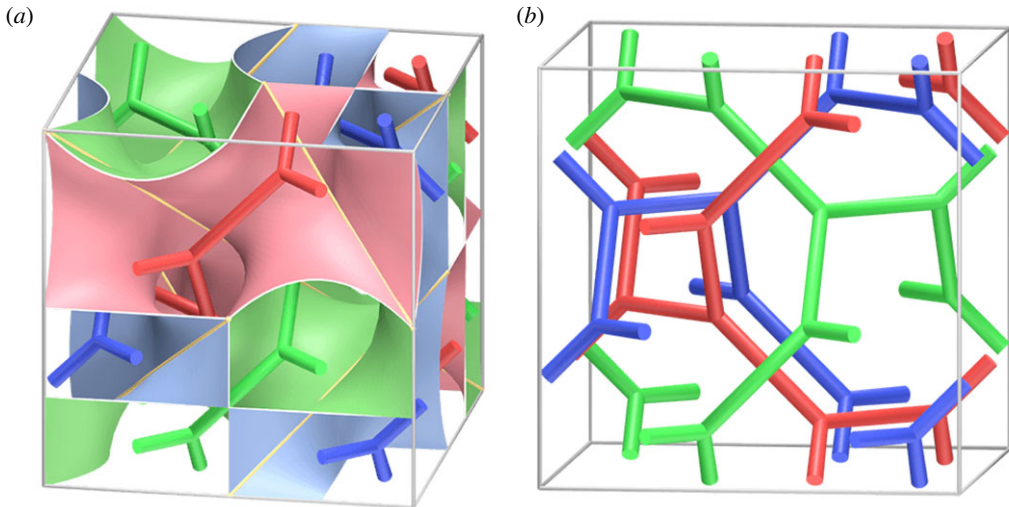


Figure 1. A tricontinuous pattern and an entangled network. (Online version in colour.)

Many generalizations of Heegaard splittings have been investigated. Gómez-Larrañaga [3] studied orientable three-manifolds decomposed into three solid tori. Coffey and Rubinstein analysed orientable three-manifolds built from three π_1 -injective handlebodies [4]. In [5], Koenig considered a *trisection* of a closed orientable three-manifold, which is an embedded branched surface decomposing the manifold into three handlebodies with connected pairwise intersections. Koenig introduced the notion of stabilization for a trisection and showed an analogue of the Reidemeister–Singer theorem for trisections of three-manifolds.

In this paper, we consider a generalization of all of the above. We define a *handlebody decomposition* to be a decomposition of a closed orientable three-manifold into a finite number of handlebodies (see definition 2.1 for the detailed definition). We will also introduce *stabilizations* for handlebody decompositions and show an analogue of the Reidemeister–Singer theorem for handlebody decompositions (see theorem 3.5).

The primary motivation of this study comes from materials science. We are interested in the characterization of *bicontinuous patterns*, *tricontinuous patterns* and *polycontinuous patterns* of microphase separation of a block copolymer melt (see §6b). See [6,7] for related research. A mathematical model of a bicontinuous (resp. tricontinuous or polycontinuous) pattern is a triply periodic non-compact surface (resp. tribranched surface or polyhedron) embedded in \mathbb{R}^3 that divides it into two (resp. three or a finite number of) possibly disconnected submanifolds as shown in figure 1 (see definition 5.8 for more details). We are particularly interested in the case where the submanifolds are the open neighbourhood of networks.

If a bicontinuous pattern is triply periodic, then by considering the quotient of the action, the pattern induces a Heegaard splitting of the three-dimensional torus T^3 (see remark 7.1). If a polycontinuous pattern is triply periodic and satisfies suitable conditions, then it corresponds to a handlebody decomposition of T^3 (corollary 5.12). Hence a characterization of handlebody decompositions of T^3 gives that of triply periodic polycontinuous patterns. The Reidemeister–Singer-type theorem of polycontinuous patterns (corollary 6.3) follows from that of handlebody decompositions of T^3 . This point of view allows us to explain how two polycontinuous patterns are related, which will be discussed in §6b.

This paper is organized as follows. In §2, we define a handlebody decomposition of a three-manifold. In §3, we introduce several types of stabilization operations of handlebody decompositions and prove an analogue of the Reidemeister–Singer theorem for them. In §4, we particularly focus on decompositions of three-manifolds into three handlebodies. In §5, we study

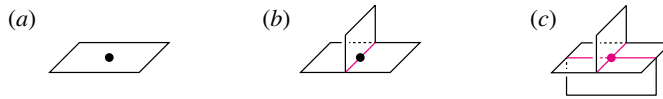


Figure 2. A neighbourhood of each point of a simple polyhedron. (a) A non-singular point, (b) a triple point and (c) a vertex. (Online version in colour.)

a mathematical model of polycontinuous patterns. We define polycontinuous patterns and, more generally, net-like patterns. The correspondence between triply periodic net-like patterns and handlebody decompositions of T^3 is given. In §6, we discuss stabilizations of net-like patterns. We also present how this research relates to the subject of materials science. In §7, we give characterizations of net-like patterns.

2. Handlebody decompositions of three-manifolds

We work in the piecewise linear category throughout this paper.

By a *two-dimensional polyhedron* P , we mean the underlying space of a non-collapsible locally finite two-dimensional complex such that the link of each vertex contains no isolated vertices. A connected component of the set of points of P having neighbourhoods homeomorphic to discs is called a *sector*. The set of all points not contained in the sectors is called its *singular graph*. A two-dimensional polyhedron P is said to be *simple* if, after giving a structure of a complex in a suitable way, the link of each point in P is homeomorphic to one of the three models shown in figure 2. A point whose link is homeomorphic to the model in figure 2c is called a *vertex* of its singular graph. See Matveev [8] for more details.

Definition 2.1 (Handlebody decomposition). Let M be a closed, connected, orientable three-manifold and P a connected compact two-dimensional polyhedron embedded in M . We call $(H_1, H_2, \dots, H_n; P)$ a *type- (g_1, g_2, \dots, g_n) handlebody decomposition* of M if $M \setminus P = \bigsqcup_{i=1}^n H_i$, where H_i is the interior of a handlebody of genus g_i . The polyhedron P is called a *partition* for the decomposition. A handlebody decomposition is said to be *proper* if there is no simple closed curve in $M \setminus B$ that intersects a sector of P transversely once, where B is the singular graph of P . A handlebody decomposition is said to be *simple* if its partition is a simple polyhedron.

Remark 2.2. (1) Let $(H_1, H_2, \dots, H_n; P)$ be a type- (g_1, g_2, \dots, g_n) handlebody decomposition of M , and W_i a handlebody of genus g_i for $i = 1, 2, \dots, n$. Then there exists a continuous map $\iota_i: W_i \rightarrow M$ such that the restriction of ι_i to the interior of W_i is a homeomorphism to H_i . Then we have $\iota_i(W_i) \cap P = \iota_i(\partial W_i) \cap P$. Suppose that the handlebody decomposition is proper. Then for the closure F of each sector, there exists a pair of handlebodies (W_i, W_j) ($i \neq j$) such that $F \subset \iota_i(\partial W_i) \cap \iota_j(\partial W_j)$. We denote the union of all such surfaces F by F_{ij} . (Note that $F_{ij} = F_{ji}$.)

(2) In general, ι_i may not be injective on the singular graph of P . If the decomposition is simple and proper, then ι_i is a homeomorphism.

The notion of handlebody decompositions generalizes both Heegaard splittings [9] and trisections [5] of closed orientable three-manifolds. In fact, a simple proper handlebody decomposition with $n = 2$ is nothing but a Heegaard splitting, while that with $n = 3$, where each F_{ij} is connected, is a trisection. By [10], any closed, connected, three-manifold M admits a simple (non-proper) type-(0) handlebody decomposition. Therefore, it is easily seen that for any sequence (g_1, \dots, g_n) of non-negative integers, there exists a simple (possibly non-proper) type- (g_1, \dots, g_n) handlebody decomposition of M .

3. Stable equivalence

This section discusses the stable equivalence of simple proper handlebody decompositions of a three-manifold. We assume that a handlebody decomposition is simple and proper

throughout this section. By remark 2.2, for a handlebody decomposition $(H_1, \dots, H_n; P)$, there exist handlebodies W_1, \dots, W_n and continuous maps ι_1, \dots, ι_n such that the restriction of each ι_i to the interior of W_i is an embedding $\text{int}(W_i) \rightarrow H_i$. For simplicity, we regard H_i as $\iota_i(W_i)$ and ∂H_i as $\iota_i(\partial W_i)$. Then the intersection of H_i and H_j is a possibly disconnected surface with boundary. We denote it by F_{ij} .

(a) Stabilizations and destabilizations of handlebody decompositions

The following operations for handlebody decompositions are a generalization of the ‘stabilization’ for Heegaard splittings.

Definition 3.1. Let $(H_1, \dots, H_n; P)$ be a simple proper type- (g_1, \dots, g_n) handlebody decomposition of a closed, connected, orientable three-manifold M .

- (0) Take a properly embedded arc α in H_i , and an arc β in ∂H_i such that the endpoints of α lie in the interior of F_{ij} , and α is parallel to β in H_i relative to the endpoints, i.e. the endpoints of α are equal to that of β , and $\alpha \cup \beta$ bounds a disc in H_i . Then we get a type- (g'_1, \dots, g'_n) handlebody decomposition (H'_1, \dots, H'_n) of M with

$$g'_l = \begin{cases} g_i + 1 & (l = i) \\ g_j + 1 & (l = j) \\ g_l & (l \neq i, j) \end{cases} \quad H'_l = \begin{cases} H_i \setminus \text{int}(N(\alpha)) & (l = i) \\ H_j \cup N(\alpha) & (l = j) \\ H_l & (l \neq i, j), \end{cases}$$

where $N(\alpha)$ and $\text{int}(N(\alpha))$ are a regular neighbourhood of α and its interior in H_i , respectively. We call this operation a *type-0 stabilization* (along α). Conversely, we assume that there exist properly embedded discs D_j of H_j and E in H_i such that the boundary of D_j is in F_{ij} , and the boundary of D_j intersects that of E transversely exactly one point. Then we can perform the inverse operation of a type-0 stabilization. We call this operation a *type-0 destabilization* (along D_j). See figure 3a.

- (1) Take a properly embedded arc α on F_{jk} such that the endpoints of α lie in the boundary of H_i for $i \neq j, k$. Then we get a type- (g'_1, \dots, g'_n) handlebody decomposition (H'_1, \dots, H'_n) of M with

$$g'_l = \begin{cases} g_i + 1 & (l = i) \\ g_l & (l \neq i) \end{cases} \quad H'_l = \begin{cases} H_i \cup N(\alpha) & (l = i) \\ H_l \setminus \text{int}(N(\alpha)) & (l = j, k) \\ H_l & (l \neq i, j, k). \end{cases}$$

We call this operation a *type-1 stabilization* (along α). Conversely, if there exists a non-separating disc D_i of H_i such that the boundary of D_i intersects the singular graph of the partition P transversely exactly two points, then we can perform the inverse operation of a type-1 stabilization. We call this operation a *type-1 destabilization* (along D_i). See figure 3b.

- (2) Take two points on the interior of F_{ij} and that of F_{ik} for $j \neq k$, and we connect the points by a properly embedded arc α in H_i . Let β be an arc in ∂H_i such that α is parallel to β . Then we get a type- (g'_1, \dots, g'_n) handlebody decomposition (H'_1, \dots, H'_n) of M with

$$g'_l = \begin{cases} g_i + 1 & (l = i) \\ g_l & (l \neq i) \end{cases} \quad H'_l = \begin{cases} H_i \setminus \text{int}(N(\alpha)) & (l = i) \\ H_j \cup N(\alpha) & (l = j) \\ H_l & (l \neq i, j). \end{cases}$$

We call this operation a *type-2 stabilization* (along α). Conversely, if there exists a disc component D_{jk} of F_{jk} whose boundary intersects a properly embedded non-separating disc in H_i transversely once, then we can perform the inverse operation of a type-2 stabilization. We call this operation a *type-2 destabilization* (along D_{jk}). See figure 3c.

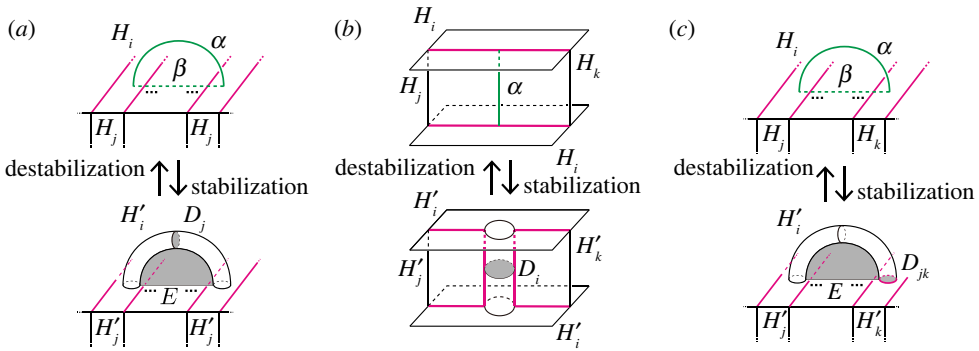


Figure 3. Stabilizations and destabilizations. Red curves represent the singular graphs. Both ends of the arc α of a type-0 stabilization are contained in F_{ij} , whereas one end of the arc α of a type-2 stabilization is contained in F_{ij} and the other is in F_{ik} with $j \neq k$. A type-0 stabilization connects two parts of H_j by the 1-handle $N(\alpha)$. On the other hand, a new branch locus and a new component D_{jk} of F_{jk} appear after a type-2 stabilization. (a) Type-0, (b) type-1 and (c) type-2. (Online version in colour.)

Remark 3.2. Consider a type- (g_1, g_2, \dots, g_n) handlebody decomposition of a closed, connected, orientable three-manifold M with $3 \leq n$. For every $g_i \leq g'_i$, we can obtain a type- $(g'_1, g'_2, \dots, g'_n)$ handlebody decomposition of M by performing type-1 stabilizations repeatedly in a suitable way.

Definition 3.3. A handlebody decomposition is said to be *stabilized* if it is obtained from another handlebody decomposition by a stabilization.

When $n = 2$, a type-0 stabilization is nothing but a stabilization of Heegaard splittings. In electronic supplementary material, we discuss the independence of these stabilizations.

(b) Stable equivalence theorem

This subsection will generalize Koenig's argument [5] on the stable equivalence of decompositions. We first recall the following operations on simple polyhedra embedded in a closed orientable three-manifold introduced by Matveev [8] and Piergallini [11] under our setting.

Definition 3.4. Let P be the partition of a handlebody decomposition of M .

- (1) Let α be a properly embedded arc in F_{jk} . A modification of P in a neighbourhood of α , as in figure 4a, is called a *0-2 move along α* . By this operation, the number of vertices of P increases by two, and a new disc component appears in F_{ij} . Conversely, we can perform the inverse operation of 0-2 move along a disc component, D , of F_{ij} . We call the operation a *2-0 move along D* . By this operation, the number of vertices of P decreases by two, and the disc component is removed from F_{ij} .
- (2) Let α be an edge of the singular graph of P . A modification of P in a neighbourhood of α , as in figure 4b, is called a *2-3 move along α* . By this operation, the number of vertices of P increases by one, and a new disc component appears in F_{im} . Conversely, we can perform the inverse operation of 2-3 move along a disc component, D , of F_{im} . We call the operation a *3-2 move along D* . By this operation, the number of vertices of P decreases by one, and the disc component is removed from F_{im} .

We note that the above moves do not change the topological type of each handlebody of a decomposition.

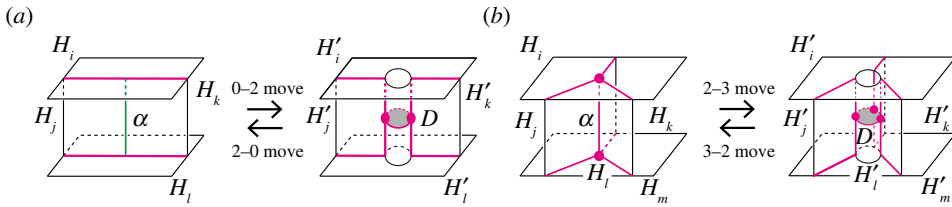


Figure 4. Moves on a handlebody decomposition. As the arc α of a 0-2 (resp. 2-3) move connects H_i and H_l , the 0-2 (resp. 2-3) move produces a new component D of F_{ij} . The boundary ∂D is contained in the singular graph and has two (resp. three) vertices. A 0-2 move increases the number of vertices by two and a 2-3 move increases by one. (a) 0-2 move and 2-0 move and (b) 2-3 move and 3-2 move. (Online version in colour.)

We say that two handlebody decompositions $(H_1, H_2, \dots, H_n; P)$ and $(H'_1, H'_2, \dots, H'_n; P')$ of a closed orientable three-manifold M are *equivalent* if there exists an ambient isotopy of M that moves P to P' and each H_i to H'_i ($i = 1, \dots, n$) simultaneously.

Theorem 3.5. *Let $\mathcal{H} = (H_1, H_2, \dots, H_n; P)$ and $\mathcal{H}' = (H'_1, H'_2, \dots, H'_n; P')$ be simple proper handlebody decompositions of a closed orientable three-manifold M . Then \mathcal{H} and \mathcal{H}' are equivalent after applying 0-2, 2-0, 2-3 moves and types-0 and -1 stabilizations finitely many times.*

Proof. Set $F_{ij} = H_i \cap H_j$ and $F'_{ij} = H'_i \cap H'_j$ as in remark 2.2. We will prove the theorem in the following steps.

- Step 0. In the case of $n \geq 4$, we perform 0-2, 2-0, 2-3 moves and type-1 stabilization appropriately until it holds $F_{ij} = \emptyset$ for any $3 \leq i < j \leq n$. Then P becomes a simple polyhedron without vertices.
- Step 1. For each $j \in \{3, \dots, n\}$, we deform F_{2j} into a disc by type-1 stabilizations. Then, $(H_1, (H_2 \cup \dots \cup H_n))$ is a Heegaard splitting. By applying the same process for \mathcal{H}' , $(H'_1, (H'_2 \cup \dots \cup H'_n))$ becomes a Heegaard splitting, so by the Reidemeister–Singer theorem, we have $H_1 = H'_1$ after applying type-0 stabilizations.
- Step 2. For each $j \in \{3, \dots, n\}$, we deform F_{1j} into a disc by type-1 stabilizations. We denote by S_1 the surface F_{12} at this stage, and keep it throughout the steps hereafter.
- Step 3. We cover H_1 along S_1 with H_3 by type-1 stabilizations. Then it holds that $H_3 = H'_3$ after handle slides.
- Step i . ($4 \leq i \leq n$) We cover H_{i-1} along S_1 with H_i by 0-2, 2-0 moves and type-1 stabilizations. Then it holds that $H_i = H'_i$ after handle slides.

If $n = 3$, after performing the operations described in the first half of Step 0, the decompositions \mathcal{H} and \mathcal{H}' become trisections. Then, they are equivalent by using Koenig’s theorem. Hence, in this proof, we assume that $n \geq 4$.

Step 0. Put $J = \{(i, j) \mid 3 \leq i < j \leq n, F_{ij} \neq \emptyset\}$. Let (i, j) be the minimum element of J in the lexicographical order. First, we change F_{ij} to be connected if it is disconnected as follows. Take an arc properly embedded in the closure of $\partial H_i \setminus F_{ij}$ that connects different components of F_{ij} . If the arc is contained in some F_{ik} , a type-1 stabilization along the arc decreases the number of components by one. Otherwise, we can perform a type-1 stabilization after 0-2 moves along the arc to decrease the number of components. Hence, by repeatedly applying this process finitely many times, we may assume that F_{ij} is connected.

Next, take mutually disjoint arcs properly embedded in F_{ij} so that they cut open F_{ij} into a disc. We perform either a type-1 stabilization or a 0-2 move along each of the arcs according to whether both ends of the arc lie in ∂H_k for $k \neq i, j$ or not. Then F_{ij} becomes a disc. Since P gives the simple proper handlebody decomposition \mathcal{H} , the boundary ∂F_{ij} has either at least two vertices or no vertex of P .

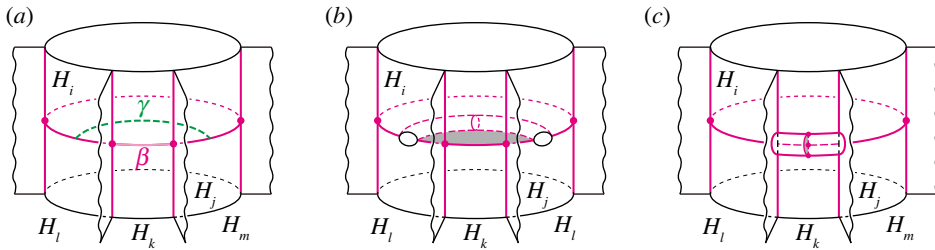


Figure 5. (a) The disc sector F_{ij} . (b) Performing a type-1 stabilization along γ . We can perform a 2-0 move along the greyed region. (c) Performing 2-3 move along β . (Online version in colour.)

Suppose F_{ij} is a disc and ∂F_{ij} has more than two vertices. Let β be a sub-arc of ∂F_{ij} cut off by the vertices, γ a properly embedded arc in F_{ij} parallel to β , and H_k ($k = 1, 2$, or $j < k$) the handlebody with $\beta \subset \partial H_k$ (figure 5a). We perform either a 2-0 move along β after a type-1 stabilization along γ or a 2-3 move along β according to whether there exists a different handlebody H_l ($l = 1, 2$ or $j < l$) from H_k with $\partial\beta \subset H_k \cap H_l$ or not (figure 5b,c). Each operation reduces the number of vertices in ∂F_{ij} by two or one. By continuing this process, the number of vertices in ∂F_{ij} can be reduced to two. Then we have $F_{ij} = \emptyset$ after performing a 2-0 move on F_{ij} .

Suppose F_{ij} is a disc and ∂F_{ij} has no vertices. There is a handlebody H_k ($k = 1, 2$, or $j < k$) with $\partial F_{ij} \subset H_k$. In other words, F_{ik} and F_{jk} share their boundary components in ∂F_{ij} . Since P is connected, at least one of F_{ik} and F_{jk} has another boundary component. If F_{ik} shares another boundary component with F_{il} (and F_{kl}), we take an arc in F_{ik} which connects ∂F_{ij} and ∂F_{il} . Then we can remove F_{ij} by a 2-0 move after applying a 0-2 move along the arc. A disc component of F_{jl} arises in this operation. It follows that $l > j$ or $l = 1, 2$, as (i, l) is greater than (i, j) in the lexicographical order by the minimality of (i, j) . If F_{jk} shares another boundary component with F_{jl} (and F_{kl}), similarly, we can remove F_{ij} by a 2-0 move after applying a 0-2 move. In this case, there is a possibility that F_{il} changes to a disc from the empty set by this operation with $3 \leq l < j$. This implies that the minimal element of J varies from (i, j) to (i, l) . In such a case, we take an oriented arc in P from a point in ∂F_{ij} to a point in ∂H_1 or ∂H_2 . Then we can remove F_{ij} , and the minimal element of J increases after successively applying 0-2 and 2-0 moves along the arc from the start to the end.

By repeating the same process, we have $J = \emptyset$. Namely, $F_{ij} = \emptyset$ for $3 \leq i < j \leq n$. Since each vertex of P is contained in four different handlebodies, this condition implies that P has no vertex.

Step 1. For each $j \geq 3$, we will deform F_{2j} into a disc by applying similar operations in Step 0. Since $\partial H_j = F_{1j} \cup F_{2j}$, we may assume that F_{2j} is connected, if necessary, by performing type-1 stabilizations along arcs in F_{1j} . Take a maximal set of non-separating arcs properly embedded in F_{2j} . By performing type-1 stabilizations along the arcs, F_{2j} becomes a disc. Then $H_2 \cup \dots \cup H_n$ is a handlebody. By applying the same process for \mathcal{H}' , $H'_2 \cup \dots \cup H'_n$ becomes a handlebody. Hence $(H_1, (H_2 \cup \dots \cup H_n))$ and $(H'_1, (H'_2 \cup \dots \cup H'_n))$ are Heegaard splittings of M . By the Reidemeister–Singer theorem, these two Heegaard splittings become equivalent after performing a finite sequence of type-0 stabilizations. In particular, we can assume $H_1 = H'_1$.

Step 2. Similarly to Step 1, for each $j \geq 3$, we can deform F_{1j} into a disc by performing type-1 stabilizations along suitable arcs properly embedded in F_{1j} .

Claim 3.6. For $i \in \{3, \dots, n\}$, let D_{i1}, \dots, D_{ig_i} be a complete meridian disc system of H_i such that $\partial D_{ij} \subset F_{2i}$ for $j \in \{1, \dots, g_i\}$. Then there exist disjoint meridian discs E_{ij} ($i \in \{3, \dots, n\}$, $j \in \{1, \dots, g_i\}$) of H_2 such that $\partial E_{ij} \subset F_{12} \cup F_{2i}$, $E_{ij} \cap D = E_{ij} \cap D_{ij}$, and ∂E_{ij} intersects ∂D_{ij} transversely in a single point, where D denotes the union $\cup_{i,j} D_{ij}$ of all D_{ij} .

Proof of claim 3.6. According to the deformation of H_2 at this step, there exist mutually disjoint separating discs E_3, \dots, E_n in H_2 such that each E_i cuts off a handlebody W_i from H_2 so that (W_i, F_{2i}) is homeomorphic to $(F_{2i} \times [0, 1], F_{2i} \times \{0\})$. (The union $H_i \cup W_i$ can be regarded as the

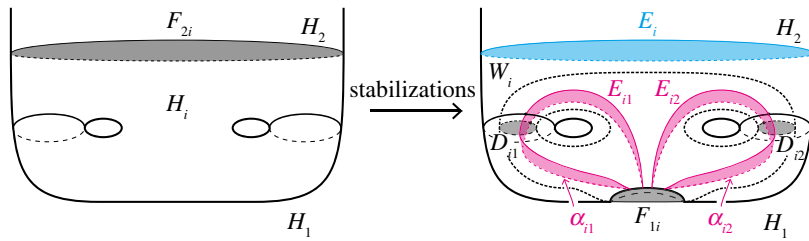


Figure 6. A deformation at Step 2. (Online version in colour.)

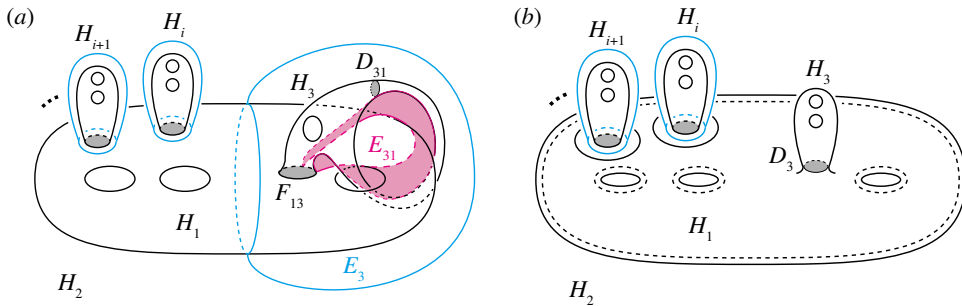


Figure 7. (a) Before performing the operation in Step 3. (b) After performing the operation. The handlebody H_3 is a local unknotted handlebody. (Online version in colour.)

handlebody H_i at the end of Step 1.) We can take mutually disjoint arcs $\alpha_{i1}, \dots, \alpha_{ig_i}$ properly embedded in F_{2i} so that $\alpha_{ij} \cap D = \alpha_{ij} \cap D_{ij}$, and α_{ij} intersects D_{ij} transversely in a single point. See figure 6. Let E_{ij} be a disc corresponding to $\alpha_{ij} \times [0, 1]$ such that $E_{ij} \cap E_i = \emptyset$ for each $j \in \{1, \dots, g_i\}$. Then the assertion holds since $(\partial W_i \setminus E_i) \subset F_{12} \cup F_{2i}$. ■

Let S_1 denote the surfaces F_{12} at this stage. Claim 3.6 implies that any one-handle of each handlebody H_i ($i \geq 3$) can be a local one-handle after a handle slide on S_1 .

Step 3. For handle slide of H_3 , we will cover H_1 along S_1 with H_3 by type-1 stabilizations. Take a maximal set of mutually non-parallel, non-boundary parallel arcs properly embedded in S_1 whose endpoints lie in ∂H_3 . We perform type-1 stabilizations along those arcs. The surface $F_{12} = S_1 \setminus F_{13}$ becomes the union of $n - 3$ annuli A_{14}, \dots, A_{1n} such that $A_{1j} \cap H_j = \partial F_{1j} = \partial F_{2j}$ for each $j \in \{4, \dots, n\}$. Since all spines of S_1 are covered by H_3 , H_3 becomes a local unknotted handlebody after performing handle slides by claim 3.6 (figure 7). Applying the same process for \mathcal{H}' and arranging genera of H_3 and H'_3 by performing type-0 stabilizations if necessary, we can assume that $H_3 = H'_3$.

According to the deformation of H_3 at this step, there exists a separating disc D_3 in H_3 that cuts off a handlebody V_3 from H_3 so that (V_3, F_{13}) is homeomorphic to $(F_{13} \times [0, 1], F_{13} \times \{0\})$. ($H_3 \setminus V_3$ can be regarded as the previous H_3 at the end of Step 2.) Let S_3 be the surface $\partial V_3 \setminus (D_3 \cup F_{13})$, which is a subsurface of F_{23} and homeomorphic to S_1 .

Step i ($4 \leq i \leq n$). At the beginning of Step i , we may have S_3, \dots, S_{i-1} as subsurfaces of $\partial H_3, \dots, \partial H_{n-1}$, respectively, that are homeomorphic to $S_1 \subset \partial H_1$, and $(i - 3)$ annuli $A_{1i}, A_{3i}, \dots, A_{(i-2)i} \subset \partial H_2$ between ∂F_{2i} and a component of ∂S_{i-1} , where $A_{ji} \subset S_j \setminus F_{j(j+1)}$ for each $j \in \{3, \dots, i - 2\}$, $A_{3i} \cap S_1 = A_{3i} \cap A_{1i} = \partial A_{3i} \cap \partial A_{1i}$, and $A_{ji} \cap S_{j-1} = A_{ji} \cap A_{(j-1)i} = \partial A_{ji} \cap \partial A_{(j-1)i}$ for each $j \in \{4, \dots, i - 2\}$.

Similar to Step 3, we will cover H_{i-1} along S_1 with H_i by performing handle slides of H_i . By a 0-2 move and a 2-0 move on A_{1i} , F_{1i} extends to A_{1i} , and an annulus F_{3i} arises. Continuing the same

operation on $A_{3i}, \dots, A_{(i-2)i}, F_{ji}$ ($j \in \{1, 3, \dots, i-2\}$) becomes the annulus A_{ji} . By the same operation as Step 3 on $S_{i-1}, S_{i-1} \setminus F_{(i-1)i}$ becomes the union of $n-4$ annuli including $A_{(i-1)(i+1)}, \dots, A_{(i-1)n}$. (In the case of $n=4$, F_{34} includes S_3 at Step 4.) By the same argument at Step 3, then, H_i can be a local unknotted handlebody after handle slide, and we can assume that $H_i = H'_i$.

When we finish Step n , we have $H_1 = H'_1, H_3 = H'_3, \dots, H_n = H'_n$. Since this automatically implies that $H_2 = H'_2$, the proof is completed. ■

4. Handlebody decomposition consisting of three handlebodies

In this section, we provide several results of handlebody decompositions consisting of three handlebodies. We keep assuming that all handlebody decompositions are simple and proper unless otherwise specified.

Section 4a will consider stabilizability on handlebody decompositions containing a three-ball. In [12], Waldhausen showed that any genus- g Heegaard splitting of S^3 is stabilized for $g \geq 1$. On the other hand, Koenig found an infinite family of unstabilized type-(1,2,2) handlebody decompositions of S^3 (see §6 in [5]). We will show that a closed connected orientable three-manifold not containing a non-separating sphere admits an unstabilized type-(0,0, g) handlebody decomposition, where g is the Heegaard genus of the manifold (proposition 4.2). Furthermore, we will see that almost all lens spaces admit a type-(0,1,2) handlebody decomposition (proposition 4.9). In §4b, we will study handlebody decompositions of the three-dimensional torus T^3 . These decompositions play an important role in *polycontinuous patterns* (§5).

(a) Handlebody decompositions containing a three-ball

We first introduce the result of Gómez-Larrañaga [3]. That result gave a complete classification of all closed connected three-manifolds that admit handlebody decompositions with small genera.

Theorem 4.1 ([3, propositions 1–3, theorem 1]). *Let (H_1, H_2, H_3) be a type- (g_1, g_2, g_3) handlebody decomposition of a closed connected orientable three-manifold M with $g_1 \leq g_2 \leq g_3$. We denote by \mathcal{B} the connected sum of some copies of $S^2 \times S^1$, and denote by \mathcal{L} or \mathcal{L}_i a lens space with non-trivial finite fundamental group. Then the following hold:*

- (1) *If all g_i are equal to 0, then M is homeomorphic to S^3 or \mathcal{B} . Conversely, S^3 and \mathcal{B} admit such a handlebody decomposition.*
- (2) *If $g_1 = g_2 = 0$ and $g_3 = 1$, then M is homeomorphic to S^3 , \mathcal{B} , \mathcal{L} or $\mathcal{B} \# \mathcal{L}$. Conversely, these manifolds admit such a handlebody decomposition.*
- (3) *If $g_1 = 0$ and $g_2 = g_3 = 1$, then M is homeomorphic to S^3 , \mathcal{B} , \mathcal{L} , $\mathcal{B} \# \mathcal{L}$, $\mathcal{L}_1 \# \mathcal{L}_2$ or $\mathcal{L}_1 \# \mathcal{L}_2 \# \mathcal{B}$. Conversely, these manifolds admit such a handlebody decomposition.*
- (4) *If all g_i are equal to 1, then M is homeomorphic to S^3 , \mathcal{B} , \mathcal{L} , $\mathcal{B} \# \mathcal{L}$, $\mathcal{L}_1 \# \mathcal{L}_2$, $\mathcal{L}_1 \# \mathcal{L}_2 \# \mathcal{B}$, $\mathcal{L}_1 \# \mathcal{L}_2 \# \mathcal{L}_3$, $\mathcal{L}_1 \# \mathcal{L}_2 \# \mathcal{L}_3 \# \mathcal{B}$, $S(3)$ or $S(3) \# \mathcal{B}$, where $S(3)$ denotes a Seifert fibre space with at most three exceptional fibres. Conversely, these manifolds admit such a handlebody decomposition.*

Let M be a closed orientable three-manifold with a Heegaard splitting (W_1, W_2) of genus l . Then, we can take $l+1$ non-separating discs in W_1 so that they separate W_1 into two three-balls. Hence, M admits a type-(0,0, l) handlebody decomposition (see [13, example 1.2]). The following proposition classifies such a decomposition.

Proposition 4.2. *Let M be a closed, connected, orientable three-manifold of Heegaard genus g . Suppose that M does not contain a non-separating sphere. Then M admits a type-(0,0, l) handlebody decomposition if and only if we have $g \leq l$. In particular, a type-(0,0, g) handlebody decomposition of M is unstabilized.*

To prove the above proposition, we first show the following lemma.

Lemma 4.3. *Let $(H_1, H_2, H_3; P)$ be a type-(0,0, l) handlebody decomposition of a closed, connected, orientable three-manifold M . Suppose that M does not contain a non-separating sphere. Then $(H_1 \cup H_2, H_3)$ is a genus- l Heegaard splitting of M .*

Proof. Let F_{ij} denote a surface as in remark 2.2. We show that the surface F_{12} consists of discs. Assume that F_{12} contains a non-disc component S . Then there exists an essential simple loop C in S such that each complementary region of C in $\partial H_1 \cong S^2$ contains a connected component of F_{13} . Since H_1 and H_2 are three-balls, the simple loop C bounds a disc in each of H_1 and H_2 . Then the union of the two discs is a non-separating disc, which is a contradiction.

Thus, F_{12} consists of only discs. It follows that the union of H_1 and H_2 is a handlebody, which implies the assertion. ■

Proof of proposition 4.2. Let g be the Heegaard genus of a closed, connected, orientable three-manifold M . Then, as explained above, M admits a type- $(0,0,g)$ handlebody decomposition. Then, by remark 3.2, we can obtain a type- $(0,0,l)$ handlebody decomposition of M for each $g \leq l$. Conversely, assume that M admits a type- $(0,0,l)$ handlebody decomposition. By lemma 4.3, this handlebody decomposition induces a Heegaard splitting of genus l . Thus, we have $g \leq l$. This particularly implies that a type- $(0,0,g)$ handlebody decomposition of M is unstabilized. ■

Example 4.4. An unstabilized type- $(0,0,2)$ handlebody decomposition of S^3 is constructed as follows. Let (W_1, W_2) be a genus-2 Heegaard splitting of S^3 . By using [14, §5 and fig. 4], we can take a non-primitive disc triple of W_1 , which separates W_1 into two three-balls. We denote the three-balls by H_1 and H_2 , and put $H_3 = W_2$. Then (H_1, H_2, H_3) forms a type- $(0,0,2)$ handlebody decomposition of S^3 . Because each component of F_{12} is a non-primitive disc in W_1 , we can see that the boundary of any properly embedded disc in H_3 transversely intersects the singular graph of the partition in at least six points. Hence we cannot perform a destabilization along any properly embedded discs in H_3 . Therefore, the decomposition is unstabilized.

Next, we will consider the stabilizability of type- $(0,1,l)$ handlebody decompositions.

Proposition 4.5. *Let M be a closed, connected, orientable, irreducible three-manifold. Suppose that M is not a lens space with non-trivial finite fundamental group. Then, for each $1 \leq l$, any type- $(0,1,l)$ handlebody decomposition of M is stabilized. In fact, such a decomposition is obtained from a type- $(0,0,l)$ handlebody decomposition by performing a type-1 stabilization.*

Proof. We first assume that F_{12} consists of discs. Then there exists a meridian disc of H_2 whose boundary intersects ∂F_{12} transversely exactly twice. Hence we can perform a type-1 destabilization along the meridian disc.

In the remainder, we assume that F_{12} has a non-disc component.

Claim 4.6. We have $\chi(S) \geq 0$ for each component S of F_{12} .

Proof of claim 4.6. Suppose that $\chi(S) < 0$. Since $S \subset \partial H_1 \cong S^2$, the boundary ∂S has at least three components. Then there exists a component c of ∂S such that it is an inessential loop in $\partial H_2 \cong T^2$. Hence c bounds a properly embedded disc in H_2 . The closed curve c also bounds a properly embedded disc in H_1 since H_1 is a three-ball. Because each complementary region of c in $\partial H_1 \cong S^2$ contains a component of F_{13} , the two properly embedded discs in H_1 and H_2 form a non-separating sphere. This contradicts the irreducibility of M . ■

Claim 4.7. A core curve of each annulus component of F_{12} is essential in $\partial H_2 \cong T^2$.

Proof of claim 4.7. Assume that a core curve C of an annulus component of F_{12} is inessential in ∂H_2 . Then C bounds a properly embedded disc in H_2 , and each complementary region of C intersects F_{23} . Since H_1 is a three-ball, C also bounds a properly embedded disc in H_1 . Hence the two discs form a non-separating sphere. This is a contradiction. ■

Claim 4.8. The surface F_{12} contains precisely one annulus component.

Proof of claim 4.8. We assume that F_{12} contains two annulus components. Then, by claim 4.7, their core curves, C_1 and C_2 , are parallel essential loops in $\partial H_2 \cong T^2$. Thus, $C_1 \cup C_2$ cobounds a properly embedded annulus in H_2 . Since H_1 is a three-ball, each of C_1 and C_2 bounds a disc in H_1 . Because each complementary region of $C_1 \cup C_2$ in ∂H_2 intersects F_{23} , the union of the annulus and the discs is a non-separating sphere. This is a contradiction. ■

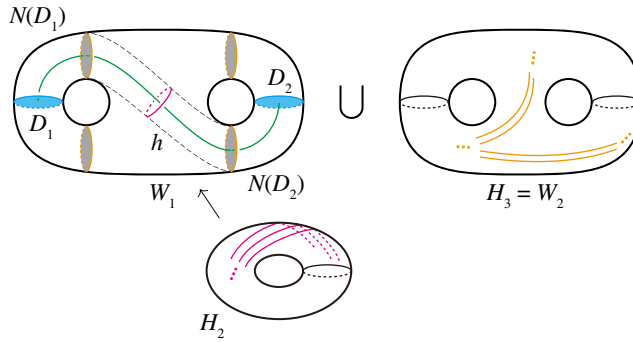


Figure 8. An unstabilized type-(0, 1, 2) handlebody decomposition of a lens space with non-trivial finite fundamental group. The decomposition consists of three handlebodies $H_1 = h \cup N(D_1) \cup N(D_2)$, H_2 and $H_3 = W_2$.

Let C be a core curve of the annulus component A of F_{12} . Then we have $[C] = a\mu + b\lambda \in H_1(\partial H_2)$, where $a, b \in \mathbb{Z}$, and μ and λ denote homology classes of a meridian loop and a longitude loop, respectively. Since H_1 is a three-ball, each component of ∂A bounds a properly embedded disc in H_1 . Then the union S of $\partial H_2 \setminus \text{int}(A)$ and the two discs is a separating sphere in M . Thus, if $b \neq 0$ and ± 1 , then M has a lens space as a connected summand. However, M is irreducible and not a lens space. Hence we have $b = 0$ or ± 1 . If $b = 0$, then C bounds a properly embedded disc in H_2 . The curve C also bounds a properly embedded disc in H_1 since H_1 is a three-ball. So, the discs form a non-separating sphere, which is a contradiction. Hence we have $b = \pm 1$. Thus, we can take a meridian disc of H_2 that intersects the boundary of F_{12} transversely exactly two points. Therefore, we can perform a type-1 destabilization along the meridian disc. ■

The following proposition implies that the assumption that M is not a lens space in proposition 4.5 is essential.

Proposition 4.9. *Any lens space with non-trivial finite fundamental group admits an unstabilized type-(0, 1, 2) handlebody decomposition.*

Proof. Let (W_1, W_2) be a genus-2 Heegaard splitting of S^3 . Then there exists a pair of non-primitive discs D_1 and D_2 in W_1 as in example 4.4. Note that $B := N(D_1; W_1) \cup N(D_2; W_1) \cup W_2$ is a three-ball, where $N(D_1; W_1)$ and $N(D_2; W_1)$ are regular neighbourhoods of D_1 and D_2 , respectively. We take an unknotted arc δ in the interior of W_1 that joins D_1 and D_2 and intersects them at only its endpoints. Let h be a one-handle attached to sides of each $N(D_1; W_1)$ and $N(D_2; W_1)$ along δ . Then, $h \cup B$ is a solid torus (figure 8). Hence, for any lens space M , there exists a homeomorphism ψ from $\partial(h \cup B)$ to the boundary of a solid torus H_2 such that M is homeomorphic to the manifold pasted by $h \cup B$ and H_2 along ψ . We put $H_1 = h \cup N(D_1; W_1) \cup N(D_2; W_1)$, $H_3 = W_2$. Thus, (H_1, H_2, H_3) is a type-(0, 1, 2) handlebody decomposition of M . By the construction, each meridian disc of H_2 and H_3 intersects the singular graph at least four and six times, respectively. Hence, the handlebody decomposition is unstabilized. ■

(b) Examples: the three-dimensional torus

We will show some examples of handlebody decompositions of the three-dimensional torus T^3 .

First, we consider handlebody decompositions consisting of one ball and two handlebodies. By proposition 4.2, T^3 admits a unstabilized type-(0, 0, 3) handlebody decomposition (figure 9a). Thus, for $k \geq 0$ and $l \geq 3$, T^3 admits a type-(0, k , l) handlebody decomposition by remark 3.2. Figure 9b illustrates a type-(0, 2, 2) handlebody decomposition of T^3 . On the other hand, by theorem 4.1, T^3 admits neither type-(0, 0, 0), type-(0, 0, 1), nor type-(0, 1, 1) handlebody decompositions. In addition, by propositions 4.2 and 4.5, there is no type-(0, 1, 2) handlebody

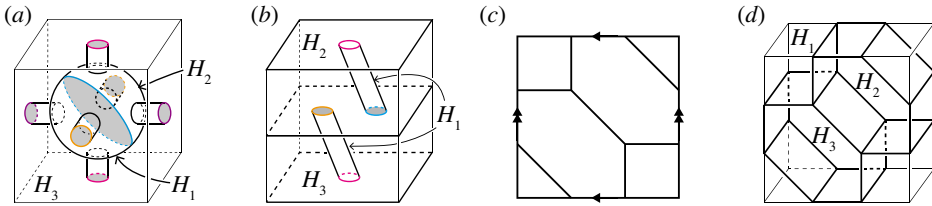


Figure 9. (a) A type-(0, 0, 3) handlebody decomposition of T^3 . Coloured edges illustrate the singular graph, and greyed discs are all components of F_{12} as in remark 2.2. (b) A type-(0, 2, 2) handlebody decomposition of T^3 . (c) A decomposition of T^2 into three hexagons. (d) The hexagonal honeycomb decomposition of T^3 . (Online version in colour.)

decomposition of T^3 . Hence, any type-(0, 2, 2) handlebody decomposition of T^3 is unstabilized. In summary, we have the following proposition.

Proposition 4.10. *Let (k, l) be a pair of non-negative integers with $k \leq l$. The 3-dimensional torus T^3 admits a type-(0, k, l) handlebody decomposition if and only if the pair (k, l) is not in $\{0, 1\} \times \{0, 1, 2\}$.*

Theorem 4.1 guarantees that T^3 admits a type-(1, 1, 1) handlebody decomposition. Figure 9c shows a decomposition of T^2 into three hexagons. By taking the product with S^1 , we have a decomposition of T^3 into three solid tori. We call this handlebody decomposition the *hexagonal honeycomb decomposition* (figure 9d). In general, a three-manifold admits a lot of handlebody decompositions of the same type. The next proposition asserts that the hexagonal honeycomb decomposition is the unique type-(1, 1, 1) handlebody decomposition of T^3 up to self-homeomorphism of T^3 .

Proposition 4.11. *For a simple and proper type-(1, 1, 1) handlebody decomposition of T^3 , there exists a self-homeomorphism of T^3 that maps the partition of the decomposition to that of the hexagonal honeycomb decomposition.*

Proof. Let (H_1, H_2, H_3) be a simple and proper type-(1, 1, 1) handlebody decomposition of T^3 . Let F_{ij} denote a surface as in remark 2.2.

Claim 4.12. For any $1 \leq i < j \leq 3$, there is no disc component in $F_{ij} = H_i \cap H_j$.

Proof of claim 4.12. Suppose there is a disc component D in some F_{ij} . Without loss of generality, we can assume that $D \subset F_{23}$. If ∂D is essential in ∂H_1 , $H'_1 = H_1 \cup N(D)$ is a punctured lens space. Since T^3 is prime, H'_1 is a three-ball. Thus, a triple $(H'_1, \text{cl}(H_2 \setminus N(D)), \text{cl}(H_3 \setminus N(D)))$ gives a simple and proper type-(0, 1, 1) handlebody decomposition of T^3 . However, from proposition 4.10, T^3 does not admit such a decomposition, which is a contradiction.

Suppose ∂D is inessential in ∂H_1 . Then we can take a disc D' in ∂H_1 such that $\partial D' = \partial D$. Put $S = D \cup D'$. Since T^3 does not contain non-separating spheres, S is separating. Thus, F_{23} consists of only the disc D . Hence, $H_2 \cup H_3$ is a genus-2 handlebody. This is impossible because $\partial(H_2 \cup H_3) = \partial H_1$ is a torus. ■

By claim 4.12 and [3, lemma 1], each component of F_{ij} is an annulus. Suppose the core of an annulus of F_{ij} is meridional in H_i . Then, C is longitudinal in H_j ; otherwise, we can find a punctured lens space or a punctured $S^2 \times S^1$ in T^3 , which is a contradiction. Then, by removing the neighbourhood of a meridian disc in H_i and attaching it to H_j , we have a decomposition of T^3 with two three-balls and a solid torus, i.e. a type-(0, 0, 1) decomposition, which contradicts proposition 4.10. Therefore, H_1 , H_2 and H_3 are fibre tori of a Seifert fibration of T^3 . As the Seifert fibre structure of T^3 is unique up to self-homeomorphism of T^3 , we can assume that $H_i = D_i \times S^1$ ($i = 1, 2, 3$), where D_1 , D_2 and D_3 are discs in T^2 satisfying $D_1 \cup D_2 \cup D_3 = T^2$. By the Euler characteristic, the intersection of two of the discs consists of precisely three arcs. Hence, this structure corresponds to the hexagonal honeycomb decomposition in T^3 .

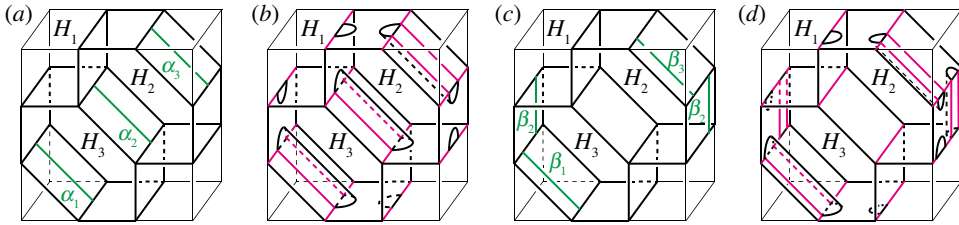


Figure 10. (b,d) A pair of different handlebody decompositions of T^3 . The decomposition (b) (resp. (d)) is of type-(2,2,2) obtained from the hexagonal honeycomb decomposition by type-1 stabilizations along α_1, α_2 and α_3 in (a) (resp. β_1, β_2 and β_3 in (c)). (Online version in colour.)

When it comes to the case of type-(2,2,2) handlebody decompositions, the uniqueness no longer holds, as we see in the following example.

Example 4.13. By performing type-1 stabilizations to the honeycomb decomposition three times, we have the two type-(2,2,2) handlebody decompositions shown in figure 10. In figure 10a, each $F_{ij} = H_i \cap H_j$ is homeomorphic to the disjoint union of a two-holed torus and a disc. On the other hand, in figure 10b, the surface F_{23} is homeomorphic to the disjoint union of a one-holed torus and an annulus, and each F_{1j} is homeomorphic to a three-holed sphere. Hence, they are different decompositions. ■

5. Topological study of polycontinuous patterns

In this section, we will consider ‘polycontinuous patterns’, which are roughly three-periodic structures assembled by polymers. See, for example, [6,15] for studies on polycontinuous patterns. We will suggest a mathematical model of polycontinuous patterns (definition 5.8).

(a) Polycontinuous patterns and net-like patterns

First, we define ‘net-like patterns’ that satisfy the essential properties of polycontinuous patterns.

Definition 5.1. We denote by T^d the d -dimensional torus. Let \tilde{X} be a graph embedded in \mathbb{R}^d such that each component of \tilde{X} is unbounded. If there exists a covering map $\pi : \mathbb{R}^d \rightarrow T^d$ such that all covering transformations of π preserve \tilde{X} , then \tilde{X} is called a *net*.

In this paper, we mainly discuss the case where $d = 3$.

Remark 5.2. In crystal chemistry (e.g. [16]), the term ‘net’ means a periodic, connected, simple, abstract graph. In this paper, we allow a net to be disconnected. Furthermore, all nets are embedded in Euclidean space.

Definition 5.3. Let \tilde{P} be a non-compact connected two-dimensional polyhedron embedded in \mathbb{R}^3 . The polyhedron \tilde{P} is called a *net-like pattern* if there exist a covering map $\pi : \mathbb{R}^3 \rightarrow T^3$ and a net \tilde{X} such that the following conditions hold:

- (1) All covering transformations of π preserve both \tilde{P} and \tilde{X} .
- (2) The polyhedron \tilde{P} divides \mathbb{R}^3 into unbounded open components V_i ($i \in I$), where I is a finite or countable set.
- (3) There exists a strong deformation retraction of $\mathbb{R}^3 \setminus \tilde{P}$ onto \tilde{X} .

We call the pair (\tilde{P}, π) a *framed net-like pattern*, and π its *frame*. We say that a connected component of $\mathbb{R}^3 \setminus \tilde{P}$ (resp. \tilde{X}) is a *labyrinthine domain* (resp. *labyrinthine net*) of \tilde{P} .

A net-like pattern \tilde{P} is said to be *proper* if there is no simple closed curve in \mathbb{R}^3 that does not cross the singular graph and intersects a sector of \tilde{P} transversely once. A net-like pattern \tilde{P} is said to be *simple* if \tilde{P} is a simple polyhedron.

More generally, we can define net-like patterns for any closed prime three-manifold with a (possibly non-Euclidean) crystallographic group and its covering space, but this paper will not deal with it.

Remark 5.4. Consider two net-like patterns that satisfy the following conditions:

- (1) They have the same labyrinthine net.
- (2) They do not have a disc sector.
- (3) The singular graphs of them have no vertices.

Then, they can be transformed to each other by a (possibly infinite) sequence of *IX-moves*, *XI-moves* and isotopies (see [17, theorem 3.1]).

By using [11,18], if two net-like patterns with the same labyrinthine net are simple, then the patterns can be transformed to each other by a (possibly infinite) sequence of 0-2 moves, 2-0 moves, 2-3 moves, 3-2 moves and isotopies.

The following two propositions state a relationship between (framed) net-like patterns and handlebody decompositions of T^3 .

Proposition 5.5. *Let $(H_1, H_2, \dots, H_n; P)$ be a handlebody decomposition of T^3 , and \tilde{P} the preimage of P under the universal covering map π of T^3 . Suppose that, for each i , the induced homomorphism $(\iota_i)_* : \pi_1(H_i) \rightarrow \pi_1(T^3)$ is not trivial, where ι_i is the inclusion map. Then the pair (\tilde{P}, π) is a framed net-like pattern. Furthermore, if P is simple (resp. proper), then the net-like pattern is also simple (resp. proper).*

Proof. Let $\{V_j^i\}$ be the connected components of the preimage of H_i under π . Since the homomorphism $(\iota_i)_*$ is not trivial, each open component V_j^i is unbounded. Each open handlebody H_i contains a simple finite graph X_i that is a strong deformation retract of H_i . Then the preimage, \tilde{X}_i , of X_i under π is a net. Furthermore, each connected component of \tilde{X}_i is a strong deformation retract of some V_j^i . Hence, (\tilde{P}, π) is a framed net-like pattern.

Since π is a local homeomorphism, if P is simple, then the preimage \tilde{P} is also simple. Next, assume that the handlebody decomposition $(H_1, H_2, \dots, H_n; P)$ of T^3 is proper, whereas the net-like pattern \tilde{P} is *not* proper. Then, there exists a simple loop \tilde{c} in \mathbb{R}^3 that transversely intersects P at a single point only in a sector. Thus, there exists a simple loop in T^3 isotopic to $\pi(\tilde{c})$ that intersects a sector of P transversely once. Hence, the handlebody decomposition is not proper, which is a contradiction. Therefore, the net-like pattern \tilde{P} is proper. ■

Proposition 5.6. *Let (\tilde{P}, π) be a framed net-like pattern.*

- (1) *The image $\pi(\tilde{P})$ gives a handlebody decomposition of T^3 . If \tilde{P} is simple, then the handlebody decomposition is also simple.*
- (2) *Let $\{V_i\}_{i \in I}$ be the set of labyrinthine domains of \tilde{P} , where I is a finite or countable set. Suppose that \tilde{P} is proper. Suppose further that for any V_i, V_j with $V_i \neq V_j$, where V_i is the image of V_j under some covering transformation, V_i and V_j are not adjacent to the same sector. Then the handlebody decomposition given by $\pi(\tilde{P})$ is proper.*

Proof. Let Γ be the covering transformation group of π . Set $P = \pi(\tilde{P})$.

(1) Since \tilde{P} is a connected two-dimensional polyhedron, its projection image P is also a connected two-dimensional polyhedron. Furthermore, if \tilde{P} is simple, then P is also simple.

The complement $T^3 \setminus P$ consists of finite open components $\{H_j\}$ because $T^3 = \mathbb{R}^3/\Gamma$ is compact and P is the underlying space of a locally finite complex. We show that each open component H_j is an open handlebody. There exists a labyrinthine domain V_i such that $H_j = \pi(V_i)$. Furthermore,

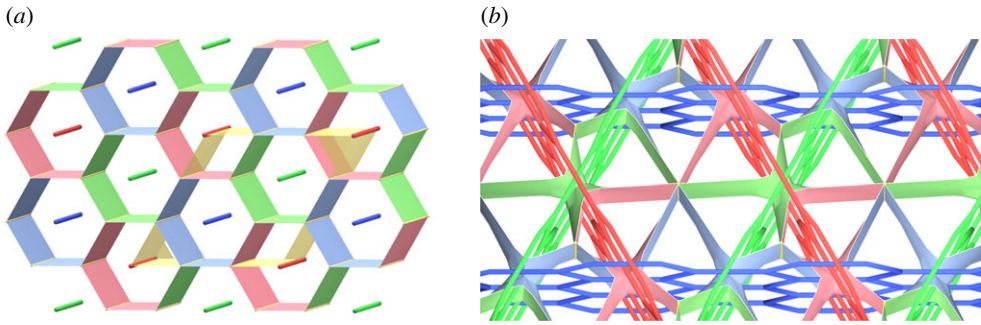


Figure 11. (a) The hexagonal honeycomb pattern and (b) a non-simple net-like pattern. (Online version in colour.)

since \tilde{P} is a net-like pattern, there exists a labyrinthine net \tilde{X}_i such that \tilde{X}_i is a strong deformation retract of V_i . Put $G = \pi(\tilde{X}_i)$. Then, G is an embedding of a graph in T^3 . So, the fundamental group $\pi_1(G)$ is free. Since $\pi|_{V_i}$ is a covering map, and \tilde{X}_i is a strong deformation retract of V_i , the inclusion map $G \rightarrow H_j$ induces an isomorphism from $\pi_1(G)$ to $\pi_1(H_j)$. Hence, H_j is the interior of a handlebody because $\pi_1(H_j)$ is free. Therefore, P gives a handlebody decomposition of T^3 .

(2) We suppose that \tilde{P} is proper and P is not proper. Then there exists a simple loop $c : [0, 1] \rightarrow T^3$ such that it transversely intersects P at a single point only in a sector. Let \tilde{c} be a lift of c . Since \tilde{P} is proper, \tilde{c} is an arc (not a loop) whose initial point v and terminal point w are contained in different labyrinthine domains V_i and V_j , respectively, of $\mathbb{R}^3 \setminus \tilde{P}$. Note that V_i and V_j are adjacent. This is impossible because there exists a covering transformation that takes v to w , so V_i to V_j . ■

Example 5.7. Figure 11a illustrates a simple proper net-like pattern that comes from the hexagonal honeycomb tessellation of \mathbb{R}^2 . A yellow polygon illustrates a fundamental domain of its frame. We call the pattern the *hexagonal honeycomb pattern*. By proposition 5.6, the hexagonal honeycomb pattern with the frame induces the hexagonal honeycomb decomposition (see figure 9d). Note that a tessellation of \mathbb{R}^2 induces a net-like pattern in general. The meanings of colours except yellow in figure 11a will be explained in definition 5.10.

We now suggest a strict mathematical definition of polycontinuous patterns.

Definition 5.8. We say that a net-like pattern \tilde{P} is an n -continuous pattern (or a *polycontinuous pattern*) if the following conditions hold:

- (i) \tilde{P} has precisely n labyrinthine domains.
- (ii) \tilde{P} is proper.
- (iii) Any sector of \tilde{P} is not a disc.

Note that for any positive integer n , there exists an n -continuous pattern. In the remainder, we call a two-continuous (resp. three-continuous) pattern a *bicontinuous* (resp. *tricontinuous*) pattern, according to the conventions of soft materials [6,15].

The following corollary is a polycontinuous pattern version of proposition 5.5.

Corollary 5.9. Let $(H_1, \dots, H_n; P)$ be a proper handlebody decomposition of T^3 , and \tilde{P} the preimage of P under the universal covering map of T^3 . Suppose that the following two conditions hold:

- (1) For each i , we have $(\iota_i)_*(\pi_1(H_i)) \cong Z \oplus Z \oplus Z$.
- (2) Any sector of P is not a disc.

Then, \tilde{P} is a polycontinuous pattern. In particular, if $(\iota_i)_*(\pi_1(H_i)) = \pi_1(T^3)$ for each i , then \tilde{P} is an n -continuous pattern. Furthermore, if P is simple (resp. proper), then \tilde{P} is also simple (resp. proper).

(b) Colourings of patterns

In this subsection, we will define colourings of net-like patterns. Each labyrinthine domain of a net-like pattern is a mathematical model of polymers assembled in one kind of block. In materials science, one kind of block may form many domains of a net-like pattern in general. To describe such a situation, we introduce ‘colours’ of net-like patterns, of which each colour corresponds to one kind of block of polymers.

Definition 5.10. Let \tilde{P} be a net-like pattern. Set $X_n = \{1, 2, \dots, n\}$. Let $\tilde{\mathcal{V}}$ denote the set of all labyrinthine domains of \tilde{P} . A surjection $\varphi: \tilde{\mathcal{V}} \rightarrow X_n$ is called an n -colouring of \tilde{P} if there exists a frame π of \tilde{P} such that the following conditions hold:

- (1) For each covering transformation t of π and for any $V \in \tilde{\mathcal{V}}$, we have $\varphi(V) = \varphi \circ t(V)$.
- (2) Two sides of a local part of each sector have different colours. Namely, for each point p in a sector C , the two labyrinthine domains, V and V' , that have non-trivial intersection with $N(p)$ satisfy $\varphi(V) \neq \varphi(V')$.

The image $\varphi(V)$ is called *the colour of V* . The frame π of \tilde{P} is said to be *compatible with the colouring φ* . A net-like pattern together with a fixed (n -) colouring is called an (n -)coloured net-like pattern (figure 11*b*). We say that two coloured net-like patterns, \tilde{P} and \tilde{Q} , are *equivalent* if there exists an ambient isotopy of \mathbb{R}^3 that moves \tilde{P} to \tilde{Q} , and each pair of corresponding labyrinthine domains has the same colour after permuting the colours. If a surjection φ satisfies only the condition (1), then we call φ a *non-effective n -colouring*, and \tilde{P} is said to be *non-effectively n -coloured*.

Let \tilde{P} be a (possibly non-effectively) n -coloured net-like pattern with a colouring $\varphi: \tilde{\mathcal{V}} \rightarrow X_n$, where $X_n = \{1, 2, \dots, n\}$, and π a frame of \tilde{P} compatible with φ . By proposition 5.6, the image $\pi(\tilde{P})$ gives a handlebody decomposition of T^3 . Denote by \mathcal{V} the set of all handlebodies of the decomposition. Then we say that (\tilde{P}, π) is of *type $(\mathfrak{g}_1, \dots, \mathfrak{g}_n)$* , where $\mathfrak{g}_i = [g_{i_1}, \dots, g_{i_{k_i}}]$ is a sequence of the genera of the handlebodies in \mathcal{V} coloured by $i \in X_n$. (For simplicity, if the length of \mathfrak{g}_i is equal to 1, then we put $\mathfrak{g}_i = g_{i_1}$.)

Note that, as opposed to colourings of graphs on surfaces, for any integers m, n with $n \geq 2$ and $m < n$, there is an n -coloured framed net-like pattern that does not admit m -colouring.

Remark 5.11. If a net-like pattern admits a colouring, then it is necessarily proper.

In fact, a coloured net-like pattern \tilde{P} with its frame π compatible with the colouring satisfies the condition that $\pi(\tilde{P})$ induces a proper handlebody decomposition of T^3 since any two labyrinthine domains sharing a sector have different colours (see proposition 5.6). Hence, the following holds.

Corollary 5.12. Let (\tilde{P}, π) be a framed net-like pattern and let $\mathfrak{g}_i = [g_1^{(i)}, \dots, g_{k_i}^{(i)}]$ be a sequence of positive integers for $i \in \{1, \dots, n\}$. Suppose that (\tilde{P}, π) admits an n -colouring and is of type $(\mathfrak{g}_1, \dots, \mathfrak{g}_n)$. Then $\pi(\tilde{P})$ gives a proper type- $(g_1^{(1)}, \dots, g_{k_1}^{(1)}, \dots, g_1^{(n)}, \dots, g_{k_n}^{(n)})$ handlebody decomposition $(H_1^{(1)}, \dots, H_{k_1}^{(1)}, \dots, H_1^{(n)}, \dots, H_{k_n}^{(n)})$ such that $H_{j_1}^{(i)} \cap H_{j_2}^{(i)} = \emptyset$ for $j_1, j_2 \in \{1, \dots, k_i\}$.

The converse of the above corollary is clear by proposition 5.5.

Corollary 5.13. Let $(H_1^{(1)}, \dots, H_{k_1}^{(1)}, \dots, H_1^{(n)}, \dots, H_{k_n}^{(n)}; P)$ be a proper type- $(g_1^{(1)}, \dots, g_{k_1}^{(1)}, \dots, g_1^{(n)}, \dots, g_{k_n}^{(n)})$ handlebody decomposition of T^3 and let π be the universal covering map $\mathbb{R}^3 \rightarrow T^3$. We assume that $H_{j_1}^{(i)} \cap H_{j_2}^{(i)} = \emptyset$ for $j_1, j_2 \in \{1, \dots, k_i\}$. We further assume that, for each handlebody $H_j^{(i)}$, the induced homomorphism $(\iota_j^{(i)})_*: \pi_1(H_j^{(i)}) \rightarrow \pi_1(T^3)$ is not trivial, where $\iota_j^{(i)}$ is the inclusion map. Then $(\pi^{-1}(P), \pi)$ is a coloured net-like pattern of type $(\mathfrak{g}_1, \dots, \mathfrak{g}_n)$, where $\mathfrak{g}_i = [g_1^{(i)}, \dots, g_{k_i}^{(i)}]$.

(c) A sufficient condition for the equivalence of patterns

Corollaries 5.12 and 5.13 say there is a nice relationship between coloured net-like patterns and proper handlebody decompositions. This subsection gives a sufficient condition for two coloured net-like patterns to be equivalent.

To this end, we first consider adjusting a framed net-like pattern to another frame. Let (\tilde{P}, π) be a framed net-like pattern, and let ρ be a covering map $\mathbb{R}^3 \rightarrow T^3$. Since the two covering maps are equivalent, there exists a self-homeomorphism f of \mathbb{R}^3 such that $\pi = \rho \circ f$. If f is orientation-preserving, we say that π and ρ have the same orientation. Otherwise, we say that π and ρ have different orientations.

If π and ρ have the same orientation, then \tilde{P} is isotopic to $f(\tilde{P})$, and $(f(\tilde{P}), \rho)$ is a framed net-like pattern. Consider the case that π and ρ have different orientations. Let r be an orientation-reversing self-homeomorphism of T^3 . Then, $\pi' := r \circ \pi$ is also a covering map $\mathbb{R}^3 \rightarrow T^3$. Hence, there exists an orientation-preserving homeomorphism $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\pi' = \rho \circ g$. So, \tilde{P} is isotopic to $g(\tilde{P})$, and we have $r(\pi(\tilde{P})) = \rho(g(\tilde{P}))$. Furthermore, $(g(\tilde{P}), \rho)$ is a framed net-like pattern because each covering transformation of π' is also that of π .

To summarize, we obtain the following lemma.

Lemma 5.14. *Let (\tilde{P}, π) be a framed net-like pattern, and let ρ be a covering map $\mathbb{R}^3 \rightarrow T^3$. Then, there exists a net-like pattern \tilde{Q} such that the following three conditions hold:*

- (1) *The covering map ρ is a frame of \tilde{Q} .*
- (2) *The pattern \tilde{P} is isotopic to \tilde{Q} .*
- (3) *Either $\pi(\tilde{P}) = \rho(\tilde{Q})$ or there exists an orientation-reversing self-homeomorphism r of T^3 with $r(\pi(\tilde{P})) = \rho(\tilde{Q})$.*

In particular, if \tilde{P} is (non-effectively) n -coloured, then so is \tilde{Q} . Furthermore, \tilde{P} and \tilde{Q} have the same type.

By lemma 5.14, we can assume that any two net-like patterns have the same frame. Proposition 5.6 says the two framed net-like patterns induce two handlebody decompositions of T^3 , respectively. If the two decompositions are isotopic, then the two patterns are also isotopic. In fact, we can say more as follows.

Lemma 5.15. *Let (\tilde{P}, π) and (\tilde{Q}, ρ) be framed net-like patterns. Suppose that there exists an orientation-preserving self-homeomorphism f of T^3 that maps $\pi(\tilde{P})$ to $\rho(\tilde{Q})$. Then, \tilde{P} is isotopic to \tilde{Q} .*

Proof. By the assumption, we have an orientation-preserving self-homeomorphism f of T^3 that maps $\pi(\tilde{P})$ to $\rho(\tilde{Q})$. Let \tilde{f} be the unique lift of $f \circ \pi$. Then \tilde{f} is a homeomorphism of \mathbb{R}^3 , and we have $\tilde{f}(\tilde{P}) = \tilde{Q}$. Therefore, \tilde{P} and \tilde{Q} are isotopic. ■

By the above lemmas, we have the following proposition.

Proposition 5.16. *Let (\tilde{P}, π) and (\tilde{Q}, ρ) be n -coloured framed net-like patterns. We assume that $\pi(\tilde{P})$ and $\rho(\tilde{Q})$ are homeomorphic under an orientation-preserving or orientation-reversing self-homeomorphism f of T^3 according to whether the covering maps π and ρ have the same orientation or different orientations. Suppose that any two corresponding handlebodies under f are the images of labyrinthine domains with the same colour (after permuting the colours). Then (\tilde{P}, π) and (\tilde{Q}, ρ) are equivalent.*

6. Stabilizations on net-like patterns

In this section, we will discuss (de)stabilizations of net-like patterns and introduce some examples.

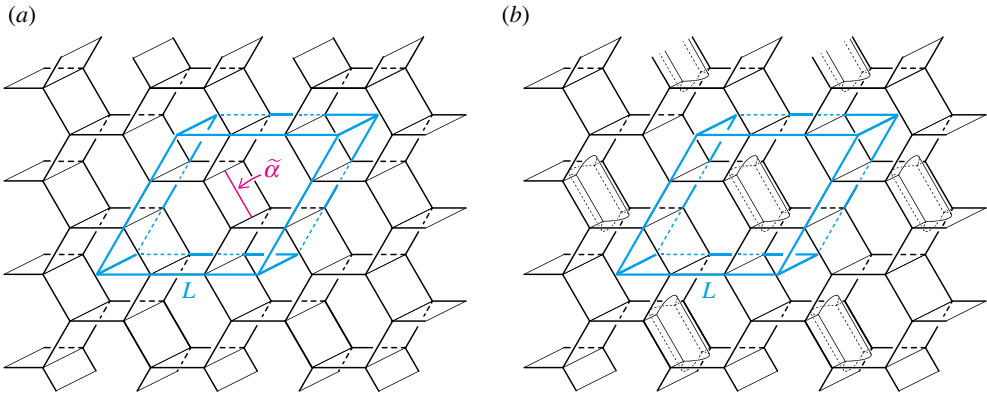


Figure 12. (a) The hexagonal honeycomb pattern introduced in example 5.7. The parallelepiped L illustrates a fundamental domain of the pattern. (b) The net-like pattern obtained by performing a type-1 stabilization on the honeycomb pattern along the properly embedded lifted arc $\tilde{\alpha}$. (Online version in colour.)

(a) Stabilization theorem for net-like patterns

Section 3 showed an analogue of the Reidemeister–Singer theorem for handlebody decompositions (theorem 3.5). This subsection shows a net-like pattern version of the theorem. To do so, we define (de)stabilizations of net-like patterns. First, we will use an example to explain how to define it.

In example 5.7, we introduced the hexagonal honeycomb pattern that is a three-coloured framed net-like pattern of type $(1, 1, 1)$. We consider a type-1 stabilization on the pattern. Let $\tilde{\alpha}$ be a properly embedded arc in a sector of the pattern (figure 12a). We assume that $\tilde{\alpha}$ is *lifted* by π , i.e. the restriction of π to $\tilde{\alpha}$ is injective, where π is the frame of the pattern. By corollary 5.12, $\pi(\tilde{P})$ gives a simple proper handlebody decomposition. As noted in example 5.7, $P := \pi(\tilde{P})$ is a simple proper type- $(1, 1, 1)$ handlebody decomposition (figure 9d). Since $\tilde{\alpha}$ is lifted and contained in a sector, the image $\alpha := \pi(\tilde{\alpha})$ is a properly embedded arc in a sector of P . Furthermore, $\tilde{\alpha}$ connects two labyrinthine domains mapped to the same handlebody by π . So, the arc α connects the same handlebody. Thus, we can perform a type-1 stabilization along α . Hence, a type- $(1, 1, 2)$ handlebody decomposition P' is obtained by performing a type-1 stabilization along α . By corollary 5.13, the preimage of P' under π gives a three-coloured framed net-like pattern \tilde{P}' of type $(1, 1, 2)$ (figure 12b). Hence, we obtain the new net-like pattern \tilde{P}' from \tilde{P} . We will call such an operation a *type-1 stabilization for net-like patterns*.

Based on the above example, we give a strict definition of stabilizations for patterns as follows.

Definition 6.1. Let (\tilde{P}, π) be a simple, n -coloured, framed net-like pattern of type (g_1, \dots, g_n) , where g_i is a sequence of positive integers $[g_1^{(i)}, \dots, g_{m_i}^{(i)}]$ for $1 \leq i \leq n$. Put $P = \pi(\tilde{P})$. By corollary 5.12, P gives a simple proper handlebody decomposition $\mathcal{H} = (H_1^{(1)}, \dots, H_{m_1}^{(1)}, H_1^{(2)}, \dots, H_{m_2}^{(2)}, \dots, H_1^{(n)}, \dots, H_{m_n}^{(n)}; P)$ of T^3 such that $H_j^{(i)}$ is a genus- $g_j^{(i)}$ handlebody coloured by i . Let U, U', V, V' and W be labyrinthine domains. We assume that, for each pair of the labyrinthine domains except for (U, U') , (V, V') and (U, W) , the two domains are different and share a sector. (There is a possibility that $U = U'$ or $V = V'$.) We further assume that $\pi(U) = \pi(U') = H_{j_U}^{(i_U)}$, $\pi(V) = \pi(V') = H_{j_V}^{(i_V)}$ and $\pi(W) = H_{j_W}^{(i_W)}$. Here, $H_{j_U}^{(i_U)}$, $H_{j_V}^{(i_V)}$ and $H_{j_W}^{(i_W)}$ are distinct handlebodies, and $i_U \neq i_V$, $i_V \neq i_W$ and $i_W \neq i_U$.

Depending on the type of stabilization, we take an arc $\tilde{\alpha}$ as follows.

- (type-0) The arc $\tilde{\alpha}$ is a properly embedded lifted arc in V that connects U and U' . We assume that a lifted disc in V contains $\tilde{\alpha}$ as a part of its boundary, and the other part is contained in ∂V .

(type-1) The arc $\tilde{\alpha}$ is a properly embedded lifted arc in a sector of \tilde{P} that connects V and V' .

(type-2) The arc $\tilde{\alpha}$ is a properly embedded lifted arc in V that connects U and W . We assume that a lifted disc in V contains $\tilde{\alpha}$ as a part of its boundary, and the other part is contained in ∂V .

Then, we can obtain a new handlebody decomposition \mathcal{H}' performed by a suitable stabilization on \mathcal{H} along $\pi(\tilde{\alpha})$. We can see by corollary 5.13 that the preimage of the partition of \mathcal{H}' is a simple, coloured, framed net-like pattern of type (g'_1, \dots, g'_n) . Here, each g'_i is equal to g_i except for the following sequences:

$$\begin{aligned} \text{(type-0)} \quad g'_{i_U} &= [g_1^{(i_U)}, \dots, g_{j_U}^{(i_U)} + 1, \dots, g_{m_{i_U}}^{(i_U)}], \quad g'_{i_{V'}} = [g_1^{(i_{V'})}, \dots, g_{j_{V'}}^{(i_{V'})} + 1, \dots, g_{m_{i_{V'}}}^{(i_{V'})}]. \\ \text{(type-1 and type-2)} \quad g'_{i_{V'}} &= [g_1^{(i_{V'})}, \dots, g_{j_{V'}}^{(i_{V'})} + 1, \dots, g_{m_{i_{V'}}}^{(i_{V'})}]. \end{aligned}$$

We call this operation a *type- k stabilization (along $\tilde{\alpha}$ with respect to π)* and its inverse operation a *type- k destabilization* for each k . In electronic supplementary material, we discuss sufficient conditions for performing a destabilization.

Note that the result of a (de)stabilization of a polycontinuous pattern is not necessarily a polycontinuous pattern. Further note that in a type-2 stabilization along an arc for net-like patterns, even if the arc connects different labyrinthine domains, we cannot perform the operation if they are the same colour.

Definition 3.4 introduced some operations for handlebody decompositions called *moves*. We next consider a net-like pattern version of them. Of course, we can perform the original operations on simple coloured net-like patterns, but they generally lose periodicity after performing them. Thus, we give adapted ‘moves’ to net-like patterns in a similar way to the stabilizations.

Definition 6.2. Let (\tilde{P}, π) be a simple, n -coloured, framed net-like pattern. Take a properly embedded lifted arc $\tilde{\alpha}$ in a sector (resp. an edge $\tilde{\alpha}$ of the singular graph of \tilde{P}) so that it connects labyrinthine domains V and V' of different colours. By corollary 5.12, $P := \pi(\tilde{P})$ gives a simple proper handlebody decomposition \mathcal{H} . Then, we can obtain a new handlebody decomposition \mathcal{H}' performed by a 0-2 (resp. 2-3) move on P along $\pi(\tilde{\alpha})$. By corollary 5.13 the preimage of the partition of \mathcal{H}' is a simple, coloured, framed net-like pattern. We call such an operation a 0-2 (resp. 2-3) *move (along $\tilde{\alpha}$ with respect to π)* and its inverse operation a 2-0 (resp. 3-2) *move*.

Note that, similar to type-2 stabilizations of net-like patterns, even if we can perform a move on a handlebody decomposition corresponding to a pattern, it does not necessarily mean that we can perform the corresponding move on the pattern.

An analogue of the Reidemeister–Singer theorem for net-like patterns is as follows.

Corollary 6.3. Let (\tilde{P}, π) and (\tilde{Q}, ρ) be simple, n -coloured, framed net-like patterns of type (g_1, \dots, g_n) and (g'_1, \dots, g'_n) , respectively, where g_i and g'_i are positive integers. Then \tilde{P} and \tilde{Q} are equivalent after applying 0-2, 2-0 and 2-3 moves, and type-0 and type-1 stabilizations finitely many times.

Proof. We assume that π and ρ have the same orientation. The proof of the other case is similar. By corollary 5.12, the images $P := \pi(\tilde{P})$ and $Q := \rho(\tilde{Q})$ give type- (g_1, \dots, g_n) and type- (g'_1, \dots, g'_n) simple proper handlebody decompositions of T^3 , respectively. Hence, by theorem 3.5, there exists a simple proper handlebody decomposition such that $\pi(\tilde{P})$ and $\rho(\tilde{Q})$ are isotopic to the partition R of the decomposition after applying 0-2, 2-0 and 2-3 moves, and type-0 and type-1 stabilizations to them finitely many times. By corollary 5.13, $\tilde{R} := \pi^{-1}(R)$ is a simple n -coloured net-like pattern. Therefore, by proposition 5.16, each of \tilde{P} and \tilde{Q} is equivalent to \tilde{R} after applying 0-2, 2-0 and 2-3 moves, and type-0 and type-1 stabilizations finitely many times. ■

In the above corollary, we assume that, for each colour, all labyrinthine domains coloured by it are mapped to the same handlebody because moves performed in the proof of theorem 3.5 generally do not preserve the colouring. On the concept of colourings, we can regard single-coloured domains as composed of the same kind of blocks, so connecting these parts is a natural idea.

Definition 6.4. Let (\tilde{P}, π) be a simple, n -coloured, framed net-like pattern. Take a properly embedded lifted arc $\tilde{\alpha}$ in a sector so that it connects labyrinthine domains, V and V' , of the same colour. We assume that $H := \pi(V)$ and $H' := \pi(V')$ are different handlebodies of the simple proper handlebody decomposition induced by $P := \pi(\tilde{P})$. By performing a 0-2 move on P along $\pi(\tilde{\alpha})$, the modified H and H' are intersected, and by corollary 5.12, their intersection consists of only the disc created by the operation. So, $H'' := H \cup H'$ is also a handlebody. Hence, we have a new handlebody decomposition by replacing H and H' with H'' . By corollary 5.13, the decomposition induces a new simple, coloured, framed net-like pattern (\tilde{P}', π) . We call such an operation a *domain-connection (along $\tilde{\alpha}$ with respect to π)* and its inverse operation a *domain-disconnection*.

Remark 6.5. We can obtain the type of (\tilde{P}', π) in definition 6.4 as follows. Let \mathfrak{g}_i be the sequence of positive integers in the type of (\tilde{P}, π) corresponding to the colour of the labyrinthine domains V and V' . We remove the integers corresponding to V and V' from \mathfrak{g}_i and append their sum. Then, we denote a new sequence by \mathfrak{g}'_i . By replacing \mathfrak{g}_i with \mathfrak{g}'_i , we obtain the type of (\tilde{P}', π) .

By applying the following to a coloured net-like pattern, it satisfies the assumption of corollary 6.3.

Lemma 6.6. Let (\tilde{P}, π) be a simple, n -coloured, framed net-like pattern of type $(\mathfrak{g}_1, \dots, \mathfrak{g}_n)$, where \mathfrak{g}_i is a sequence of positive integers $[\mathfrak{g}_1^{(i)}, \dots, \mathfrak{g}_{m_i}^{(i)}]$ for $1 \leq i \leq n$. Set $\hat{\mathfrak{g}}_i = \sum_{k=1}^{m_i} \mathfrak{g}_k^{(i)}$. Then, we have a simple, n -coloured, framed net-like pattern of type $(\hat{\mathfrak{g}}_1, \dots, \hat{\mathfrak{g}}_n)$ by applying 0-2 moves and domain-connections with respect to π finitely many times to \tilde{P} .

Proof. Since \tilde{P} is connected, for each colour i , there exist labyrinthine domains, V and V' , with colour i and an embedded lifted arc $\tilde{\delta}$ joining V and V' in \tilde{P} such that the following hold:

- (1) The images $\pi(V)$ and $\pi(V')$ are distinct handlebodies.
- (2) The arc $\tilde{\delta}$ does not cross any labyrinthine domains with colour i .
- (3) The arc $\tilde{\delta}$ intersects the singular graph of \tilde{P} transversely.

Then, by cutting $\tilde{\delta}$ at its intersection with the singular graph, we obtain the sequence of sub-arcs $\tilde{\delta}_1, \dots, \tilde{\delta}_k$. Thus, we can perform 0-2 moves along $\tilde{\delta}_1, \dots, \tilde{\delta}_{k-1}$, and we can finally apply domain-connection along $\tilde{\delta}_k$. By repeating the above process, all labyrinthine domains with colour i are joined. Then, the type corresponding to colour i is $\hat{\mathfrak{g}}_i$ by remark 6.5. Therefore, we have a net-like pattern of type $(\hat{\mathfrak{g}}_1, \dots, \hat{\mathfrak{g}}_n)$. ■

(b) Microphase separation of a block copolymer melt

One motivation for this research comes from materials science. We are interested in the characterization of *polycontinuous patterns* that appear as microphase separation of a block copolymer melt [6,7].

In this subsection, we discuss block copolymers and phase separation of a block copolymer melt. One reference of this subject is [19]. A *polymer* is a molecule of high molecular weight created by chemically coupling large numbers of small reactive molecules, called *monomers*. If a polymer is made of one type of monomer, it is called a *homopolymer*. A polymer containing two or more chemically distinct monomers is referred to as a *copolymer*. A *block copolymer* is an important type of copolymer, in which monomers of a given type form polymerized sequences called *blocks*. If a block copolymer contains two (respectively three) blocks, it is called a *diblock* (resp. *triblock*) copolymer. If a linear diblock copolymer is made of blocks of monomers A and B, it is called an *AB diblock copolymer*. An *ABA triblock copolymer* is a linear triblock copolymer consisting of a sequence of a block of monomer A, a block of monomer B, and a block of monomer A. See figure 13a. SBS (styrene–butadiene–styrene) triblock and SIS (styrene–isoprene–styrene) triblock copolymers are examples of linear triblock copolymers. Polymers with more complex architecture have been synthesized. For example, a *star polymer* has one branched point connecting several linear polymers. An *ABC triblock-arm star-shaped molecule (3-star polymer)* as in figure 13a is an

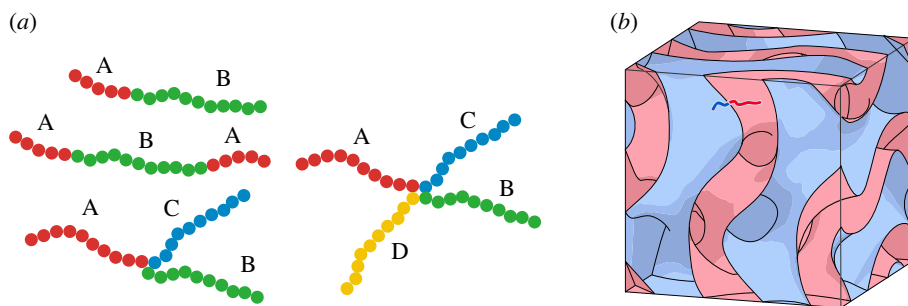


Figure 13. (a) A red dot indicates the monomer A, green monomer B, blue monomer C and yellow monomer D. Left: AB diblock copolymer, ABA triblock copolymer and ABC star-shaped block copolymer. Right: ABCD star-shaped block copolymer. (b) A double gyroid that is a famous bicontinuous pattern. (Online version in colour.)

example of triblock copolymer with a star architecture, where A, B and C blocks are mutually immiscible.

A *block copolymer melt* is a solvent-free viscoelastic liquid composed of block copolymers. Due to the chemical distinction of monomers, we can observe phase separation in a block copolymer melt. A *domain* of phase separation consists of monomers of one type. *Microphase separation* of a block copolymer melt is phase separation with domains of the mesoscopic size scale. Sphere, cylinder, bicontinuous and lamellar structures appear as microphase separation of AB diblock or ABA triblock copolymers [19,20].

An example of *bicontinuous patterns* is the Gyroid surface. In materials science, in the bicontinuous pattern of an AB diblock copolymer melt, the domain of the A monomer is the neighbourhood of the partition surface, and that of the B monomer forms two labyrinths (figure 13b). A *tricontinuous pattern* is a mathematical model of microphase separation of an ABC star-shaped block copolymer melt. The branch line of a tricontinuous pattern consists of the connection points of the A, B and C blocks in the block copolymers [6]. See [21–23] for studies on geometric phases of star polymer melts. Note that a sector of a tricontinuous pattern is the interface of two domains.

Next, we discuss a mathematical model of microphase separation with four phases. Let A, B, C and D be four chemically distinct monomers. We consider the polycontinuous pattern of melts of four types of three-star block copolymers of ABC, ABD, ACD and BCD. In this case, four different branched lines appear. The interface of domains and these branched lines form a simple polyhedron. The vertex of the simple polyhedron of the polycontinuous pattern corresponds to the point where four domains A, B, C and D meet. The four edges corresponding to the connecting points of the ABC, ABD, ACD and BCD triblock star polymers are placed around a vertex. Also, ABCD four-star polymers are synthesized [24,25], and their morphologies have been discussed in [26,27]. The joining point of four blocks of the block copolymer corresponds to a vertex of the simple polycontinuous pattern.

We want to analyse the property of materials with this structure via a topological study of these polycontinuous patterns. We hope the characterization and the classification of polycontinuous patterns will lead to the design of polymeric materials with the desired properties.

As an application of corollary 6.3, we can discuss the relation between two microphase-separated structures of the same type. Here, we discuss the polymer science implications of stabilization and destabilization operation of patterns.

Observation 6.7. The type-0 destabilization for a bicontinuous pattern can be considered as the model of the cancelling of an unstable local one-handle structure of the pattern of the microphase separation. The type-1 destabilization (resp. stabilization) for a polycontinuous pattern can be considered as the model of the separation (resp. amalgamation) of the domains during the uniaxial elongation of polymeric materials.

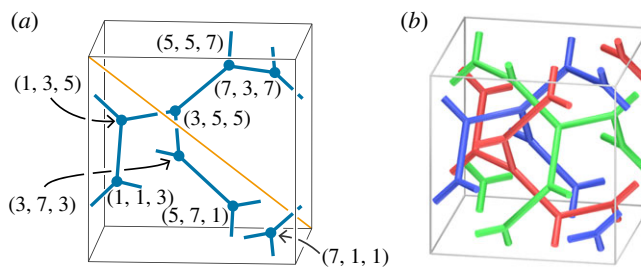


Figure 14. (a) An *srs* net. The orange line l passes through the points $(0, 0, 8)$ and $(8, 8, 0)$. Note that this net is topologically the same as the *srs-b* net (see [30]). (b) A *3srs* net. The $2\pi/3$ rotation around l preserves the net. (Online version in colour.)

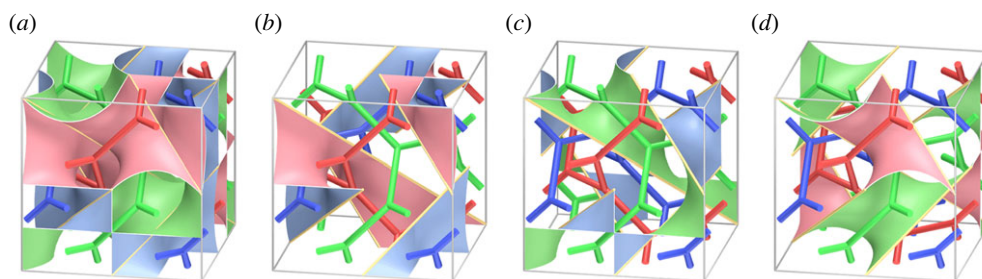


Figure 15. (a) The tricontinuous pattern corresponding to the *3srs* net with a cubical fundamental domain. (b–d) Surfaces with boundary, each of which is shared by exactly two labyrinthine domains. (Online version in colour.)

(c) Example: a *3srs* pattern

A *3srs* pattern is an example of a tricontinuous pattern. In this subsection, we will show the pattern can be destabilized to the hexagonal honeycomb pattern.

First, we introduce a *3srs* net. An *srs* net is a 3-periodic ‘minimal’ net in \mathbb{R}^3 (see [28] and figure 14a). Figure 14a illustrates an *srs* net with a cubical fundamental domain, of which the length of each edge is 8. The net is an infinite trivalent graph, and the *space group* of it is $I4_132$ (see [15,29]). Note that a $2\pi/3$ rotation around the cube diagonal (shown in figure 14a) generates an action of order 3 and preserves the cube. A *3srs* net is the union of the images of the *srs* net under the action (figure 14b).

Figure 15a illustrates a branched surface in \mathbb{R}^3 with a cubical fundamental domain. The branched surface is the union of precisely three surfaces with the boundary (figure 15b–d). It is clear that the branched surface is a simple three-coloured tricontinuous pattern, and each component of the *3srs* net is a labyrinthine net of the pattern. We call the tricontinuous pattern the *3srs pattern*. The *3srs* pattern is of type $(5,5,5)$ as illustrated in figures 14 and 15.

Theorem 6.8. *The 3srs pattern can be destabilized to the hexagonal honeycomb pattern, i.e. the 3srs pattern can be obtained from the hexagonal honeycomb pattern by a finite sequence of type-1 stabilizations.*

Proof. Let \tilde{P} be the *3srs* pattern, and π its frame obtained from a cubical fundamental domain as shown in figure 15. Put $P = \pi(\tilde{P})$. Figure 16 shows a simple proper type- $(5,5,5)$ handlebody decomposition $(H_1, H_2, H_3; P)$ of T^3 induced by \tilde{P} . We denote by F_{12} , F_{13} and F_{23} surfaces with boundary as in remark 2.2. By definition 6.1, if we destabilize the decomposition to the hexagonal honeycomb decomposition by performing a finite sequence of type-1 destabilizations, then we can also destabilize \tilde{P} to the hexagonal honeycomb pattern by corresponding destabilizations.

First, for each i , we take three meridian discs D_{i1} , D_{i2} and D_{i3} of the handlebody H_i as shown in figure 16a–c. Each disc intersects the singular graph of P transversely exactly two points.

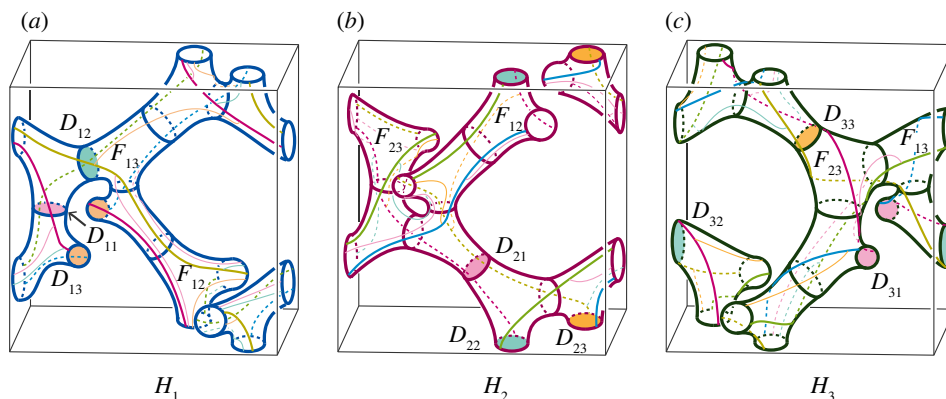


Figure 16. A handlebody decomposition of T^3 induced by the 3srs pattern. The ‘cores’ of handlebodies are the quotient of the 3srs net. The bold curves on the boundaries of handlebodies make up the singular graph. Each F_{ij} denotes a surface defined in remark 2.2.

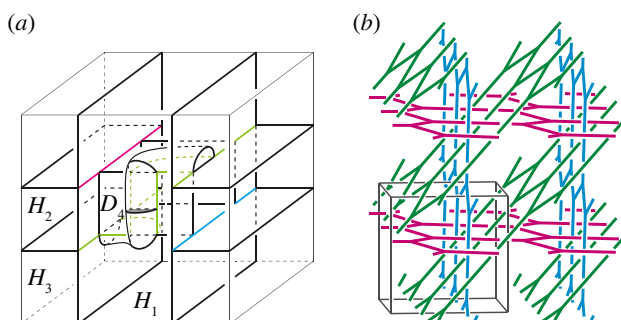


Figure 17. (a) A type-(2, 2, 2) handlebody decomposition of T^3 . (b) A 3hcb net that is the preimage by the universal covering map of the core of the handlebodies H_1 , H_2 and H_3 .

Furthermore, any two different discs are disjoint. Hence, we can perform type-1 destabilizations along them. By this operation, we obtain a type-(2, 2, 2) handlebody decomposition of T^3 (see figure 17a). For simplicity, we denote each handlebody and the partition of the destabilized handlebody decomposition by the same symbol H_1 , H_2 , H_3 and P , respectively. Note that the preimage of the union of spines of H_1 , H_2 and H_3 is isotopic to a 3hcb net as shown in figure 17b. See [31] for examples of materials with this chemical framework. See also [32]. The destabilized net-like pattern is also a simple coloured tricontinuous pattern.

For the type-(2, 2, 2) handlebody decomposition, we can perform a type-1 destabilization along a meridian disc D_4 of H_3 (figures 17a and 18a). The type of resulting decomposition is (2, 2, 1). Figure 18a–e illustrates a destabilization to the type-(2, 2, 1) handlebody decomposition, which produces a type-(1, 1, 1) handlebody decomposition. The type-(1, 1, 1) handlebody decomposition illustrated in figure 18f is the hexagonal honeycomb decomposition (figure 9d).

7. Characterization of patterns

In this section, we will prove that bicontinuous patterns are unique. We will also show that simple, coloured, framed net-like patterns of type (1, 1, 1) are unique. On the other hand, we will provide two different simple coloured net-like patterns of type (1, 1, 1, 1).

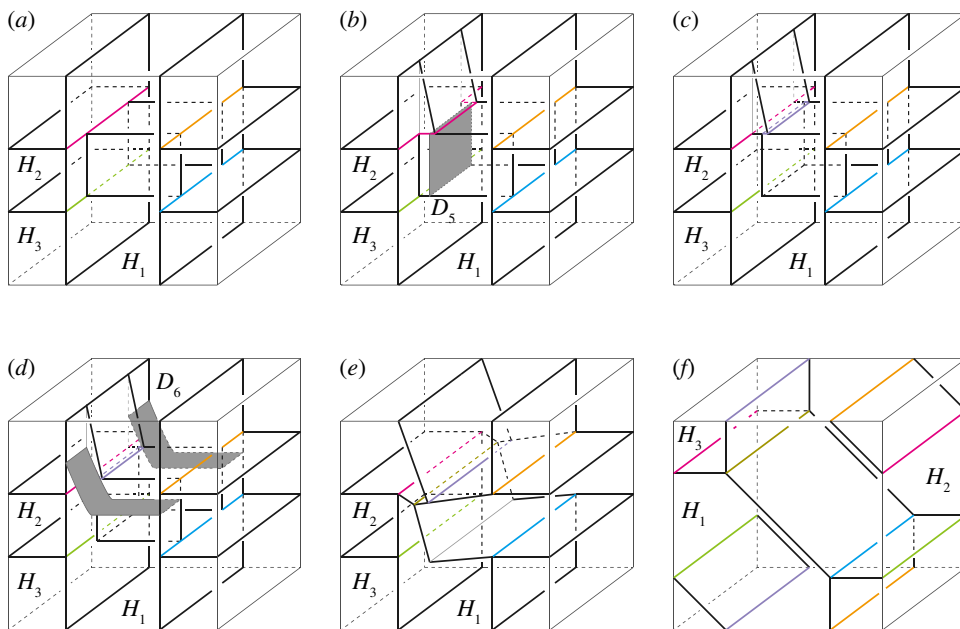


Figure 18. A sequence of type-1 destabilizations from the type-(2, 2, 1) handlebody decomposition to the type-(1, 1, 1) handlebody decomposition. (a) Type-(2, 2, 1), (b) a meridian disc D_5 of H_2 , (c) type-(2, 1, 1), (d) a meridian disc D_6 of H_1 , (e) type-(1, 1, 1), (f) the hexagonal honeycomb decomposition. (Online version in colour.)

(a) Bicontinuous patterns and Heegaard splittings of T^3

By definition, an n -continuous pattern consists of precisely n labyrinthine domains, and it is proper. Hence, by assigning a different colour to each domain, the pattern admits an n -colouring. In general, a frame of the pattern is not compatible with the colouring. However, by expanding the fundamental domain, we can obtain a frame compatible with the colouring. Then, by corollary 5.12, the pattern with the frame gives a proper type- (g_1, \dots, g_n) handlebody decomposition of T^3 . Hence, the pattern is a framed net-like pattern of type (g_1, \dots, g_n) . In particular, we note the following for each simple bicontinuous pattern and such a frame.

Remark 7.1. Any simple bicontinuous pattern and its frame compatible with a colouring induce a Heegaard splitting of T^3 .

By [33,34], Heegaard splittings of T^3 are determined by their Heegaard genera. Hence, we can prove the uniqueness of bicontinuous patterns.

Theorem 7.2. Any two simple bicontinuous patterns are equivalent.

Proof. Let (\tilde{P}, π) and (\tilde{P}', π') be bicontinuous patterns of types (g, g) and (g', g') , respectively. For the frame π , there exists a basis $\langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$ of \mathbb{R}^3 such that the translations t_i defined by the vectors \mathbf{a}_i generate the covering transformation group. We denote by T a group generated by translations t_1^{g-1}, t_2 and t_3 . Hence we have a covering map $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}^3/T \cong T^3$. Since the Euler characteristic of \tilde{P}/ρ is $(g-1)$ -times that of \tilde{P}/π , the surface \tilde{P}/ρ gives a Heegaard splitting of T^3 of genus $(g-1)(g-1) + 1$. Similarly, we can take a covering map $\rho': \mathbb{R}^3 \rightarrow T^3$ so that \tilde{P}'/ρ' also gives a Heegaard splitting of genus $(g-1)(g-1) + 1$. Therefore, by using [34, Théorème] and proposition 5.16, the two simple bicontinuous patterns are equivalent. ■

The Gyroid, the Schwartz D surface and the Schwartz P surface are famous triply periodic minimal surfaces that decompose \mathbb{R}^3 into precisely two open components (see [35]), i.e. the surfaces are simple bicontinuous patterns. In [36, appendix], Squires *et al.* gave an isotopy from

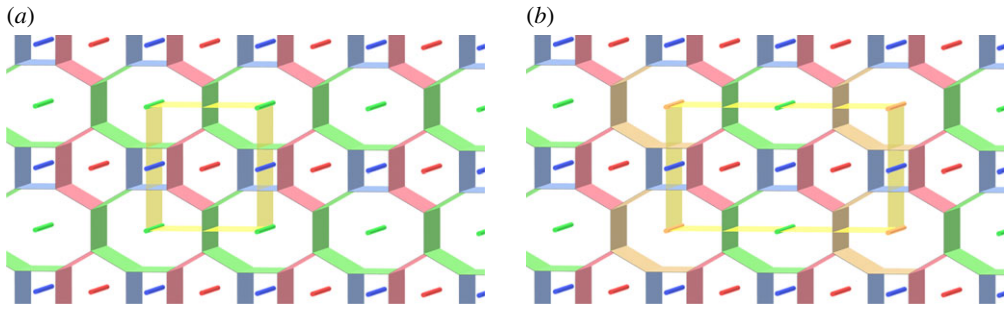


Figure 19. (a) A simple non-effectively coloured net-like pattern of type $(1, 1, 1)$. (b) A simple coloured net-like pattern of type $([1, 1], [1, 1], 1, 1)$. The handlebody decomposition induced by the pattern contains two blue solid tori and two red solid tori. (Online version in colour.)

the Gyroid to the Schwartz D surface and the Schwartz D surface to the Schwartz P surface by an explicit formula. Note that theorem 7.2 is a generalization of the result but does not give a formula for transformation between patterns.

(b) The uniqueness of framed patterns of type $(1, 1, 1)$

We consider the hexagonal honeycomb pattern introduced in example 5.7. Recall that its pattern admits a colouring and a frame compatible with it, as in figure 11a. The pattern induces the hexagonal honeycomb decomposition of T^3 . Hence, the hexagonal honeycomb pattern with the frame is of type $(1, 1, 1)$. By propositions 4.11 and 5.16, the hexagonal honeycomb pattern is a canonical model of simple coloured net-like patterns of type $(1, 1, 1)$. Therefore, we have the following.

Theorem 7.3. *Any simple, coloured, framed net-like pattern of type $(1, 1, 1)$ is equivalent to the hexagonal honeycomb pattern.*

Note that a simple three-coloured net-like pattern whose labyrinthine nets consist of lines is not necessarily equivalent to the hexagonal honeycomb pattern in general (see example 7.4). Also, there are distinct simple coloured net-like patterns of type $(1, 1, 1)$ (see example 7.5).

Example 7.4. We consider a tessellation of the plane \mathbb{R}^2 by three kinds of tiles: square, hexagon and eight-sided polygon. Figure 19 shows a net-like pattern induced by the tessellation. The left side (figure 19a) illustrates a framed net-like pattern of type $(1, 1, 1)$ that is not coloured since eight-sided components are assigned to the same colour, and they are adjacent. On the other hand, the pattern admits a four-colouring (figure 19b). However, it is no longer type $(1, 1, 1)$. This pattern is called $[8, 6, 4; 8, 8, 6]$ in [23, fig. 8(k)] and the colouring given there corresponds to a coloured net-like pattern of type $([1, 1], [1, 1], [1, 1])$.

Example 7.5. Figure 20 illustrates two simple coloured net-like patterns, (\tilde{P}_a, ρ_a) and (\tilde{P}_b, ρ_b) , of type $(1, 1, 1)$ with a cubical fundamental domain, where ρ_a and ρ_b denote their frames compatible with the colourings, respectively. We can see the two patterns are not equivalent as follows. Let \tilde{X}_a and \tilde{X}_b be nets associated with \tilde{P}_a and \tilde{P}_b . We consider the image $(\iota_a)_*(\pi_1(\rho_a(\tilde{X}_a)))$ and $(\iota_b)_*(\pi_1(\rho_b(\tilde{X}_b)))$, where ι_a and ι_b are the inclusion maps, respectively. By figure 20a $(\iota_a)_*(\pi_1(\rho_a(\tilde{X}_a)))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. On the other hand, $(\iota_b)_*(\pi_1(\rho_b(\tilde{X}_b)))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by figure 20b. Hence, \tilde{P}_a is not equivalent to \tilde{P}_b .

By theorem 7.3, any two simple coloured framed net-like patterns of type $(1, 1, 1)$ are equivalent. However, simple coloured net-like patterns of type $(1, 1, 1)$ are not unique.

The labyrinthine nets of these types of patterns are called cubic rod (cylinder) packings [37] or weavings [38]. Many of those structures do not correspond to simple coloured net-like patterns.

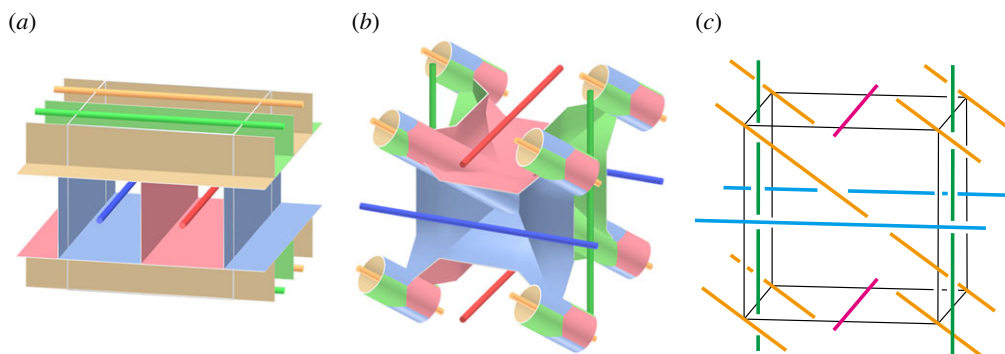


Figure 20. (a,b) Two framed simple coloured net-like patterns of type $(1, 1, 1, 1)$. (c) The labyrinthine nets of (b). (Online version in colour.)

Data accessibility. This article has no additional data.

Authors' contributions. N.S.: conceptualization, formal analysis, investigation, methodology, validation, visualization, writing—original draft, writing—review and editing; R.M.: investigation, methodology, writing—original draft; M.O.: investigation, methodology, writing—original draft, writing—review and editing; K.I.: conceptualization, formal analysis, investigation, methodology, validation, visualization, writing—original draft, writing—review and editing; Y.K.: conceptualization, formal analysis, investigation, methodology, validation, writing—original draft, writing—review and editing; M.O.: conceptualization, formal analysis, investigation, methodology, validation, writing—original draft, writing—review and editing; K.S.: conceptualization, formal analysis, funding acquisition, investigation, methodology, project administration, supervision, validation, writing—original draft, writing—review and editing.

All authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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