Heliyon 10 (2024) e27712

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Multi-attribute group decision making method for sponge iron factory location selection problem using multi-polar fuzzy EDAS approach

Chiranjibe Jana [a,*], Ibrahim M. Hezam [b]

[a] Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences (SIMATS), Chennai 602105, Tamil Nadu, India
[b] Department of Statistics & Operations Research, College of Sciences, King Saud University, Riyadh, Saudi Arabia

ARTICLE INFO

Keywords:
Multi-polar fuzzy numbers
Einstein operators
EDAS method
Multi-attribute group decision-making

ABSTRACT

This paper presents new averaging operators, such as mpF Einstein weighted averaging (mpFEWA), mpF Einstein ordered weighted averaging (mpFEOWA), mpF Einstein hybrid weighted averaging (mpFEHWA), mpF Einstein weighted geometric (mpFEWG), and mpF Einstein hybrid weighted geometric (mpFEHWG), as well as new Einstein operations (mpFNs) for handling multi-polar fuzzy numbers. We evaluate these operators for idempotency, boundedness, monotonicity, and commutativity, and we design them to deal with multi-polar fuzzy numbers (mpFNs). Furthermore, the study investigates the use of these operators in MAGDM settings, namely mpFEWA and mpFEWG operators, to expand on this. Additionally, it proposes a procedure for determining the best site for a sponge iron production plant by use of the created MAGDM method. The EDAS method, which stands for “Evaluation based on Distance from Average Solution,” verifies that the solutions are effective. Finally, the suggested model highlights the benefits and possible improvements provided by these creative strategies by comparing the new approach to conventional methods and evaluating its efficiency and practicality.

1. Introduction

In light of significant complexity, decision science suggests that real attribute values hold little relevance. In 1965, Zadeh [1] introduced the Theory of Fuzzy Sets (FS), offering a novel mathematical logic that effectively addresses problems in multi-attribute decision making (MADM) and multi-attribute group decision making (MADGM). Despite its reliability, FS lacked a robust mathematical foundation. Atnassov [2] sought to rectify this by introducing intuitionistic fuzzy sets (IFS) in 2012, capable of managing intricate fuzzy information. IFS delineates elements in the universe and articulates both membership and non-membership functions. This sparked scholarly interest in the information aggregation process, leading to notable advancements within the realm of IFS

Despite the high level of complexity, decision science shows that exact attribute values don’t matter much. As a solution to problems with MADM and MADGM, Zadeh [1] presented the Theory of Fuzzy Sets (FS). There was not a solid mathematical basis for FS at first, but it was nonetheless beneficial. Intuitionistic fuzzy sets (IFS), introduced by Atnassov [2], can handle complicated fuzzy information and fill this need. IFS defines membership and non-membership functions and classifies items in the universe. As a consequence, there was a surge of academic interest in information aggregation methods, which led to major advancements in IFS

* Corresponding author.
E-mail addresses: jana.chiranjibe7@gmail.com (C. Jana), ialmishnanah@ksu.edu.sa (I.M. Hezam).

https://doi.org/10.1016/j.heliyon.2024.e27712
Received 16 June 2023; Received in revised form 29 February 2024; Accepted 5 March 2024

Available online 12 March 2024

(See [3–10]). According to researchers [11–15], among others, a number of aggregation processes are used to aggregate a set of real values. Many researchers have come up with more complex and brilliant methods of decision-making in recent years [16–19].

An alternate to IFS and IVIFS that Zhang [20,21] suggested was bipolar fuzzy sets (BFS), which would handle practical ambiguity better than the prior two. Membership degrees are assigned by BFS on a scale from −1 to 1, with 0 representing negative membership and 1 representing positive membership. Multiple attribute decision-making (MADM) issues based on Dombi norms within BF structures, such the ones given in Jana's publications [22,23], which use aggregation operators, emerged as a result of this idea. A dual hesitant bipolar Hamacher aggregation MADM model was created by Gao et al. [25], expanding on the work of Wei et al. [24] about BF Hamacher operator-based MADM issues. Using a bipolar soft aggregation operator, Jana et al. [27] supported Xu and Wei's [26] dual hesitant bipolar multiple criteria decision-making (MCDM) method. The authors Akram and Waseem [28] looked at decision-making issues in a bipolar fuzzy graph framework, and they remarked that projects like diesel power plants and fuel pumps require multipolar data. In response, they brought forward M-polar (mF) fuzzy sets, which were first suggested by Chen et al. [29] to expand upon BFS [30–32]. Afterwards, m-polar fuzzy sets were used in certain areas including graph theory, group theory, Lie algebras, and BCK/BCI algebras [33–35]. In 2018, Khameneh and Kilicman [36] created operators for m-polar fuzzy soft weighted aggregation. The versatile uses of m-polar fuzzy sets in different settings are asserted by Akram et al. [37,38]. Recent research has demonstrated an increasing interest in investigating various aggregation strategies to address MADM issues in fuzzy settings; nonetheless, there is currently a lack of research on Multi-Attribute Decision Making (MADM) procedures that are specifically designed for m-polar fuzzy structures. To aggregate mFNs data, Waseem et al. [39] used Hamacher operators; to solve Multiple Attribute Group Decision Making (MAGDM) issues, Akram et al. [40] presented an m-polar hesitant fuzzy TOPSIS method. In addition, using an m-polar fuzzy framework, Akram et al. [41] suggested a method for group decision-making that combines the PROMETHEE Approach with the Analytic Hierarchy Process (AHP).

Some other notable techniques include those based on MADMs that use Einstein operators. In interval-valued intuitionistic fuzzy structures, Liu et al. [42] studied generalised Einstein aggregation operations, and in an Intuitionistic Fuzzy Set (IFS) framework, Xia and Wei [43] used Einstein hybrid operators. Fuzzy Pythagorean operator [45,46], complex q-rung Orthopair fuzzy operator [47], and fuzzy picture Einstein operator [44] are just a few examples of the Einstein operators that have been investigated by multiple authors in conjunction with fuzzy concepts.

Despite some progress, there is still a lot of unanswered questions about a specific Multiple Attribute Decision Making (MADM) method that uses Einstein operators for m-polar fuzzy structures. This might be a great opportunity for researchers to go more into this area in the future. Uncertain data seen in real-world scenarios poses a difficulty that has not been adequately investigated: how well mFS can manage this type of data. There is a strong justification for doing this study due to the well-established difficulties in classical literature [13–15] and Multi-Attribute Group Decision Making (MAGDM) problems using Einstein norms [42–45,47], as well as practical applications such as hospital location [48] selection in different fuzzy frameworks. In order to address difficulties in sponge iron placement selection, we aim to include mpFNs into the mFS domain using Einstein norms and an MCDM model. The use of these operators in the analysis of the sponge iron mill location selection model provided the basis for this method. Using a hybrid Pythagorean fuzzy framework, our goal is to build a MAGDM model that draws on the TOPSIS approach used to ATM site selection challenges. The main objectives of this paper are:

- In relation to some m-polar fuzzy operators, a novel technique is taken into consideration.
- employ the MAGDM technique suggested strategy.
- A case study with a numerical example is given to illustrate the method.
- The effectiveness of the method is quantified.

The next section delves into the fundamental ideas that underpin both Einstein on mpFNs and mpFS, which stand for multi-polar fuzzy sets. Within the framework of m-polar fuzzy logic, Section 3 defines the operators ordered weighted, hybrid weighted, and Einstein weighted averaging in detail. Proposals for Einstein weighted geometric, ordered weighted, and hybrid weighted operators in the m-polar fuzzy context are made in Section 4. An technique called Multiple Attribute Group Decision Making (MAGDM) was developed using these operators; the specifics of which are presented in Section 5. Section 6 is a hypothetical situation that serves to illustrate the process of choosing an appropriate location for a sponge iron manufacturing. Conclusion and last thoughts on the study are included in Section 7.

2. Preliminaries

Definition 1. [39] Let a multi-polar fuzzy set be $mpFS$ defined over the universe U is represented by a function $\zeta : U \to [0,1]^m$. This function assigns membership grades to each object in U, where the membership grade for an object u is expressed as $\zeta(u) = (p_1 * \zeta(u), p_2 * \zeta(u), \dots, p_m * \zeta(u))$. Here, $p_l * \zeta : [0,1]^m \to [0,1]$ represents the l-th projection mapping, and $\zeta = (p_1 * \zeta, \dots, p_m * \zeta)$ denotes the multi-polar fuzzy numbers (mpFNs), with each $p_l * \zeta$ falling within the interval $[0,1]$ for $l = 1, 2, \dots, m$.

Definition 2. [39] Let $MPSF$ be a score function of an $mpFNs$ $\zeta = (p_1 * \zeta, \dots, p_m * \zeta)$ is defined in Equ. (1) below:

$$MPSF(\zeta) = \frac{1}{m}\left(\sum_{l=1}^{m}(p_l * \zeta)\right), \Phi(\zeta) \in [0,1]. \tag{1}$$

Definition 3. [39] Let $MPAF$ be a accuracy function of an $mpFN$ $\zeta = (p_1 * \zeta, \dots, p_m * \zeta)$ is defined as follows:

$$MPAF(\zeta) = \frac{1}{m}\left(\sum_{l=1}^{m} (-1)^l (p_l * \zeta - 1) \right), \Phi(\zeta) \in [-1, 1].$$

Using the definitions of the score and accuracy functions, we have created a prioritised connection between two mpFNs.

Definition 4. Let $\zeta_1 = (p_1 * \zeta_1, \dots, p_m * \zeta_1)$ and $\zeta_2 = (p_1 * \zeta_2, \dots, p_m * \zeta_2)$ be two mpFNs. Then

(i) If $MPSF(\zeta_1) < MPSF(\zeta_2)$, implies $\zeta_1 \prec \zeta_2$
(ii) If $MPSF(\zeta_1) > MPSF(\zeta_2)$, implies $\zeta_1 \succ \zeta_2$
(iii) If $MPSF(\zeta_1) = MPSF(\zeta_2)$, then
 (1) If $MPAF(\zeta_1) < MPAF(\zeta_2)$, implies $\zeta_1 \prec \zeta_2$.
 (2) If $MPAF(\zeta_1) > MPAF(\zeta_2)$, implies $\zeta_1 \succ \zeta_2$.
 (3) If $MPAF(\zeta_1) = MPAF(\zeta_2)$, implies $\zeta_1 \sim \zeta_2$.

Here, some basic operations on mpFNs are defined below:

Definition 5. [39] Let $\zeta_1 = (p_1 * \zeta_1, \dots, p_m * \zeta_1)$ and $\zeta_2 = (p_1 * \zeta_2, \dots, p_m * \zeta_2)$ be two mpFNs, and $\tau > 0$. Then

(1) $\zeta_1 \oplus \zeta_2 = \left(p_1 * \zeta_1 + p_2 * \zeta_2 - p_1 * \zeta_1 p_2 * \zeta_2, \dots, p_m * \zeta_1 + p_m * \zeta_2 - p_m * \zeta_1 p_m * \zeta_2 \right)$

(2) $\zeta_1 \otimes \zeta_2 = \left(p_1 * \zeta_1 p_1 * \zeta_2, \dots, p_m * \zeta_1 p_m * \zeta_2 \right)$

(3) $\tau\zeta_1 = \left(1 - (1 - p_1 * \zeta_1)^\tau, \dots, 1 - (1 - p_m * \zeta_1)^\tau \right)$

(4) $(\zeta_1)^\tau = \left((p_1 * \zeta_1)^\tau, \dots, (p_m * \zeta_1)^\tau \right)$

(5) $(\zeta_1)^c = \left((p_1 * \zeta_1)^c, \dots, (p_m * \zeta_1)^c \right)$

(6) $\zeta_1 \subseteq \zeta_2$ if and only if $p_1 * \zeta_1 \leq p_1 * \zeta_2, \dots, p_m * \zeta_1 \leq p_m * \zeta_2$

(7) $\zeta_1 \bigcup \zeta_2 = \left\{ \max\left(p_1 * \zeta_1, p_1 * \zeta_2 \right), \dots, \max\left(p_m * \zeta_1, p_m * \zeta_2 \right) \right\}$

(8) $\zeta_1 \bigcap \zeta_2 = \left\{ \min\left(p_1 * \zeta_1, p_1 * \zeta_2 \right), \dots, \min\left(p_m * \zeta_1, p_m * \zeta_2 \right) \right\}$.

2.1. Einstein operations on mpFNs

Two prime examples of t-norms and t-conorms, the Einstein operations, are the Einstein product and the Einstein sum. The following are the definitions of each of them:

Definition 6. [43] The following are the definitions of the Einstein product $\bigotimes$ and the Einstein sum $\bigoplus$ between two real numbers u and v.

$$u \oplus_E v = \frac{u + v}{1 + u \cdot v}$$
$$u \otimes_E v = \frac{u \cdot v}{1 + (1 - u) \cdot (1 - v)}$$

where, for all $(u, v) \in [0, 1]^2$.

We described Einstein operations with respect to mpFNs in light of Einstein-norm and Einstein-conorms.

Definition 7. Let $\zeta_1 = (p_1 * \zeta_1, \dots, p_m * \zeta_1)$ and $\zeta_2 = (p_1 * \zeta_2, \dots, p_m * \zeta_2)$ be two mpFNs, and $\tau > 0$. Now, we defined below Einstein operations on mpFNs.

(1) $\zeta_1 \oplus_E \zeta_2 = \left\langle \frac{p_1 * \zeta_1 + p_1 * \zeta_2}{1 + p_1 * \zeta_1 \cdot p_1 * \zeta_2}, \dots, \frac{p_m * \zeta_1 + p_m * \zeta_2}{1 + p_m * \zeta_1 \ p_m * \zeta_2} \right\rangle$

(2) $\zeta_1 \otimes_E \zeta_2 = \left\langle \frac{p_1*\zeta_1 . p_1*\zeta_2}{1+(1-p_1*\zeta_1)(1-p_1*\zeta_2)}, \ldots, \frac{p_m*\zeta_1 \; p_m*\zeta_2}{1+(1-p_m*\zeta_1)(1-p_m*\zeta_2)} \right\rangle$

(3) $\tau . \zeta_1 = \left\langle \frac{(1+p_1*\zeta_1)^\tau-(1+p_1*\zeta_2)^\tau}{(1+p_1*\zeta_1)^\tau+(1+p_1*\zeta_2)^\tau}, \ldots, \frac{(1+p_m*\zeta_1)^\tau-(1+p_m*\zeta_2)^\tau}{(1+p_m*\zeta_1)^\tau+(1+p_m*\zeta_2)^\tau} \right\rangle$

(4) $(\zeta_1)_1^\tau = \left\langle \frac{2(p_1*\zeta_1)^\tau}{(2-p_1*\zeta_1)^\tau+(p_1*\zeta)^\tau}, \ldots, \frac{2(p_m*\zeta_1)^\tau}{(2-p_m*\zeta_1)^\tau+(p_m*\zeta)^\tau} \right\rangle.$

3. mpF Einstein operators

Introduced the terms mpFEWA, mpFEOWA, and mpFEHWA operators in this section, as well as established their features.

Definition 8. Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. Then an mF Einstein weighted averaging (mpFEWA) operator is a function $mpFEWA : \zeta^s \to \zeta$ such that:

$$mpFEWA_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) = \bigoplus_{r=1}^{s} \left(\delta_r \zeta_r\right)$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_s)^T$ a set of weight vector on ζ_r such that $\delta_r > 0$ and $\sum_{r=1}^{s} \delta_r = 1$.

Theorem 1. *Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. Then accumulated value of all mpFNs based on mpFEWA operator is also an mFN, which follows as*

$$\begin{aligned} & mpFEWA_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) \\ &= \bigoplus_{r=1}^{s} \left(\delta_r \zeta_r\right) \\ &= \left(\begin{array}{c} \dfrac{\prod_{r=1}^{s}(1 + p_1 * \zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1 - p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1 + p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1 - p_1 * \zeta_r)^{\delta_r}}, \\ \ldots, \dfrac{\prod_{r=1}^{s}(1 + p_m * \zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1 - p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1 + p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1 - p_m * \zeta_r)^{\delta_r}} \end{array} \right) \end{aligned}$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_s)$ be weight vector of ζ_r $(r = 1, 2, \ldots, s)$ such that $\delta_r > 0$, and $\sum_{r=1}^{s} \delta_r = 1$.

Now we can prove this theorem via induction in mathematics. **Proof:** If we take, $r = 1$, then equation (6) becomes

$$\begin{aligned} & mpFEWA_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) = \delta_1 \zeta_1 = \zeta_1 \quad (since\ \delta_1 = 1) \\ &= \left\langle \frac{(1 + p_1 * \zeta_1) - (1 - p_1 * \zeta_1)}{(1 + p_1 * \zeta_1) + (1 - p_1 * \zeta_1)}, \ldots, \frac{(1 + p_m * \zeta_1) - (1 - p_m * \zeta_1)}{(1 + p_m * \zeta_1) + (1 - p_m * \zeta_1)} \right\rangle. \end{aligned}$$

Thus, above equation (6) is true for $r = 1$.
Assume equation (6) is accurate for $r \geq \theta$, where $\theta \in \mathbb{N}$, which gives

$$\begin{aligned} & mpFEWA_\delta(\zeta_1, \zeta_2, \ldots, \zeta_\theta) \\ &= \bigoplus_{r=1}^{\theta} \left(\delta_r \zeta_r\right) \end{aligned}$$

$$=\left(\begin{array}{c}\dfrac{\prod_{r=1}^{\theta}(1+p_1*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta}(1-p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta}(1+p_1*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta}(1-p_1*\zeta_r)^{\delta_r}},\\ \ldots,\dfrac{\prod_{r=1}^{\theta}(1+p_m*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta}(1-p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta}(1+p_m*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta}(1-p_m*\zeta_r)^{\delta_r}}\end{array}\right).$$

Now for $r=\theta+1$, then

$$mpFEWA_{\delta}(\zeta_1,\zeta_2,\zeta_\theta\ldots,\zeta_{\theta+1})$$

$$=\bigoplus_{r=1}^{\theta}(\delta_r\zeta_r)\oplus\delta_{\theta+1}\zeta_{\theta+1}$$

$$=\left(\begin{array}{c}\dfrac{\prod_{r=1}^{\theta}(1+p_1*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta}(1-p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta}(1+p_1*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta}(1-p_1*\zeta_r)^{\delta_r}},\ldots,\\ \dfrac{\prod_{r=1}^{\theta}(1+p_m*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta}(1-p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta}(1+p_m*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta}(1-p_m*\zeta_r)^{\delta_r}}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\dfrac{(1+p_1*\zeta_{\theta+1})^{\delta_{\theta+1}}-(1-p_1*\zeta_{\theta+1})^{\delta_{\theta+1}}}{(1+p_1*\zeta_{\theta+1})^{\delta_{\theta+1}}+(1-p_1*\zeta_{\theta+1})^{\delta_{\theta+1}}},\\ \ldots,\dfrac{(1+p_m*\zeta_{\theta+1})^{\delta_{\theta+1}}-(1-p_m*\zeta_{\theta+1})^{\delta_{\theta+1}}}{(1+p_m*\zeta_{\theta+1})^{\delta_{\theta+1}}+(1-p_m*\zeta_{\theta+1})^{\delta_{\theta+1}}}\end{array}\right)$$

$$=mpFEWA_{\delta}(\zeta_1,\zeta_2,\ldots,\zeta_{\theta+1})$$

$$=\bigoplus_{r=1}^{\theta+1}\left(\delta_r\zeta_r\right)$$

$$=\left(\begin{array}{c}\dfrac{\prod_{r=1}^{\theta+1}(1+p_1*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta+1}(1-p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta+1}(1+p_1*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta+1}(1-p_1*\zeta_r)^{\delta_r}},\\ \ldots,\dfrac{\prod_{r=1}^{\theta+1}(1+p_m*\zeta_r)^{\delta_r}-\prod_{r=1}^{\theta+1}(1-p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{\theta+1}(1+p_m*\zeta_r)^{\delta_r}+\prod_{r=1}^{\theta+1}(1-p_m*\zeta_r)^{\delta_r}}\end{array}\right).$$

Hence, the Theorem 1 is true for all natural numbers.

Example 1. Let $\zeta_1=(0.4,0.6,0.7,0.5)$, $\zeta_2=(0.3,0.4,0.5,0.6)$, $\zeta_3=(0.5,0.7,0.4,0.2)$ be 4pFNs with a weight vector $\delta=(0.3,0.3,0.4)$ for 4pFNs. Then by Theorem 1 gives

$$mpFEWA_{\delta}(\zeta_1,\zeta_2,\ldots,\zeta_s)$$

$$=\bigoplus_{r=1}^{s}\left(\delta_r\zeta_r\right)$$

$$= \left(\begin{array}{c} \dfrac{\prod_{r=1}^{s}(1+p_1*\zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1-p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1+p_1*\zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1-p_1*\zeta_r)^{\delta_r}}, \\ \dots, \dfrac{\prod_{r=1}^{s}(1+p_m*\zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1-p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1+p_m*\zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1-p_m*\zeta_r)^{\delta_r}} \end{array} \right)$$

$$= \left(\begin{array}{c} \dfrac{(1+0.4)^{0.3}(1+0.3)^{0.3}(1+0.5)^{0.4} - (1-0.4)^{0.3}(1-0.3)^{0.3}(1-0.5)^{0.4}}{(1+0.4)^{0.3}(1+0.3)^{0.3}(1+0.5)^{0.3} + (1-0.4)^{0.3}(1-0.3)^{0.3}(1-0.5)^{0.4}}, \\ \dfrac{(1+0.6)^{0.3}(1+0.4)^{0.3}(1+0.7)^{0.4} - (1-0.6)^{0.3}(1-0.4)^{0.3}(1-0.7)^{0.4}}{(1+0.6)^{0.3}(1+0.4)^{0.3}(1+0.7)^{0.4} + (1-0.6)^{0.3}(1-0.4)^{0.3}(1-0.7)^{0.4}}, \\ \dfrac{(1+0.7)^{0.3}(1+0.5)^{0.3}(1+0.4)^{0.4} - (1-0.7)^{0.3}(1-0.5)^{0.3}(1-0.4)^{0.4}}{(1+0.7)^{0.3}(1+0.5)^{0.3}(1+0.4)^{0.4} + (1-0.7)^{0.3}(1-0.5)^{0.3}(1-0.4)^{0.4}}, \\ \dfrac{(1+0.5)^{0.3}(1+0.6)^{0.3}(1+0.2)^{0.4} - (1-0.5)^{0.3}(1-0.6)^{0.3}(1-0.2)^{0.4}}{(1+0.5)^{0.3}(1+0.6)^{0.3}(1+0.2)^{0.4} + (1-0.5)^{0.3}(1-0.6)^{0.3}(1-0.2)^{0.4}} \end{array} \right)$$

$$= \left\langle 0.4134, 0.5928, 0.5331, 0.4251 \right\rangle.$$

The following characteristics are observed by mpFEWA operators.

Theorem 2. *(Idempotency property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. If these all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all $(r = 1, 2, \dots, s)$. Then*

$$mpFEWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) = \zeta.$$

Proof. Let $\zeta_r = (p_1 * \zeta_b, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a s mpFNs. If these all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all $(r = 1, 2, \dots, s)$. Then, from equation (6), we have

$$\begin{aligned} & mpFEWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) \\ &= \bigoplus_{r=1}^{s} \left(\delta_r \zeta_r \right) \\ &= \left(\begin{array}{c} \dfrac{\prod_{r=1}^{s}(1+p_1*\zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1-p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1+p_1*\zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1-p_1*\zeta_r)^{\delta_r}}, \\ \dots, \dfrac{\prod_{r=1}^{s}(1+p_m*\zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1-p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1+p_m*\zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1-p_m*\zeta_r)^{\delta_r}} \end{array} \right) \\ &= \left(\begin{array}{c} \dfrac{(1+p_1*\zeta) - (1-p_1*\zeta)}{(1+p_1*\zeta) + (1-p_1*\zeta)}, \\ \dots, \dfrac{(1+p_m*\zeta) - (1-p_m*\zeta)}{(1+p_m*\zeta) + (1-p_m*\zeta)} \end{array} \right) \\ &= \left\langle p_1 * \zeta, \dots, p_m * \zeta \right\rangle = \zeta. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3. *(Boundedness property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. If $\zeta^- = \bigcap_{r=1}^{s} \zeta_r$ and $\zeta^+ = \bigcup_{r=1}^{s} \zeta_r$. Then,*

$$\zeta^- \leq mpFEWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_t) \leq \zeta^+.$$

Theorem 4. *(Monocity property) Let $\zeta_r = (p_1 * \zeta_b, \dots, p_m * \zeta_r)$ and $\zeta'_r = (p'_1 * \zeta'_r, \dots, p'_m * \zeta'_r)$ $(r = 1, 2, \dots, s)$ be two sets of 's' mpFNs such that $\zeta_r \leq \zeta'_r$ for all r, then*

$$mpFEWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) \leq mpFEWA_\delta(\zeta'_1, \zeta'_2, \dots, \zeta'_s).$$

Next, we introduced the definition of mFEOWA operator.

Definition 9. Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. Then an mpFNs Einstein ordered weighted averaging (mpFEOWA) operator is a function $mpFEOWA : \zeta^s \to \zeta$ is defined below:

$$mpFEOWA_\gamma(\zeta_1, \zeta_2, \dots, \zeta_s) = \bigoplus_{r=1}^{s} \left(\gamma_r \zeta_{\sigma(r)}\right)$$

where a collection of weight vectors $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_s)^T$ on ζ_r such that $\gamma_r > 0$ and $\sum_{r=1}^{s} \gamma_r = 1$. Also, $\sigma(1), \sigma(2), \dots, \sigma(s)$ is the permutation of $(1, 2, \dots, s)$ for which $\zeta_{\sigma(s-1)} \geq \zeta_{\sigma(s)}$ for all $r = 1, 2, \dots, s$.

Theorem 5. *Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. Then accumulated value of all mpFNs based on mpFEOWA operator is also an mpFN, which follows as*

$$\begin{aligned}
&mpFEOWA_\gamma(\zeta_1, \zeta_2, \dots, \zeta_s) \\
&= \bigoplus_{r=1}^{s} \left(\gamma_\sigma(r)\zeta_r\right) \\
&= \left(\begin{array}{c} \dfrac{\prod_{r=1}^{s}(1 + p_1 * \zeta_\sigma(r))^{\gamma_r} - \prod_{r=1}^{s}(1 - p_1 * \zeta_\sigma(r))^{\gamma_r}}{\prod_{r=1}^{s}(1 + p_1 * \zeta_\sigma(r))^{\gamma_r} + \prod_{r=1}^{s}(1 - p_1 * \zeta_\sigma(r))^{\gamma_r}}, \\ \dots, \dfrac{\prod_{r=1}^{s}(1 + p_m * \zeta_\sigma(r))^{\gamma_r} - \prod_{r=1}^{s}(1 - p_m * \zeta_\sigma(r))^{\gamma_r}}{\prod_{r=1}^{s}(1 + p_m * \zeta_\sigma(r))^{\gamma_r} + \prod_{r=1}^{s}(1 - p_m * \zeta_\sigma(r))^{\gamma_r}} \end{array} \right)
\end{aligned}$$

where a collection of weight vectors be $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_s)^T$ on ζ_r. Each γ_r must be larger than zero, and the total of all γ_r must be one. If $\zeta_{\sigma(s-1)}$ is larger than or equal to $\zeta_{\sigma(s)}$ for every r from 1 to s, then $\sigma(1), \sigma(2), \dots, \sigma(s)$ are permutations of $(1, 2, \dots, s)$.

It is simple to demonstrate the following mpFEOWA characteristics.

Theorem 6. *(Idempotency property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. If these all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all r $(r = 1, 2, \dots, s)$. Then*

$$mpFEOWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) = \zeta.$$

Theorem 7. *(Boundedness property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. If $\zeta^- = \bigcap_{r=1}^{s} \zeta_r$ and $\zeta^+ = \bigcup_{r=1}^{s} \zeta_r$. Then,*

$$\zeta^- \leq mpFEOWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) \leq \zeta^+.$$

Theorem 8. *(Monocity property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ and $\zeta'_r = (p'_1 * \zeta'_r, \dots, p'_m * \zeta'_r)$ $(r = 1, 2, \dots, s)$ be two sets of 's' mpFNs such that $\zeta_r \leq \zeta'_r$ for all r, then*

$$mpFEOWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) \leq mpFEOWA_\delta(\zeta'_1, \zeta'_2, \dots, \zeta'_s).$$

Theorem 9. *(Commutative property) Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ and $\zeta'_r = (p'_1 * \zeta'_r, \dots, p'_m * \zeta'_r)$ $(r = 1, 2, \dots, s)$ be two sets of 's' mpFNs such that $\zeta_r = \zeta'_r$ for all r, then*

$$mpFEOWA_\delta(\zeta_1, \zeta_2, \dots, \zeta_s) = mpFEOWA_\delta(\zeta'_1, \zeta'_2, \dots, \zeta'_s),$$

where ζ'_r is arbitrary permutation of ζ_r for all $(r = 1, 2, \dots, s)$.

The mpFEWA operator used the mpFv weights in Definitions 8 and 9. In contrast, the mpFEOWA operator's weight does not represent the weights themselves but rather the ordered position of the mpFv. Therefore, we provide a new operator, the mpF Einstein hybrid averaging (mpFEHWA) operator, which is a qualitative combination of the mpFEWA and mpFEOWA operators.

Definition 10. Let $\zeta_r = (p_1 * \zeta_r, \dots, p_m * \zeta_r)$ $(r = 1, 2, \dots, s)$ be a 's' mpFNs. Then an mpFNs Einstein hybrid averaging (mpFEHWA) operator is a function $mpFEHWA : \zeta^s \to \zeta$ is defined below:

$$mpFEHWA_{\delta,\gamma}(\zeta_1,\zeta_2,\ldots,\zeta_s)=\bigoplus_{b=1}^{s}\left(\gamma_r\dot{\zeta}_{\sigma(r)}\right)$$

Also, $\sigma(1),\sigma(2),\ldots,\sigma(s)$ be a permutation of $(1,2,\ldots,s)$ and $\zeta_{\sigma(s-1)}\geq\zeta_{\sigma(s)}$ for all $r=1,2,\ldots,s$ for mpFNs ζ_r and $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_s)^T$ is the vector associated weighted of the mpFNs $(\zeta_1,\zeta_2,\ldots,\zeta_s)$ such that $\gamma_r>0$ and $\sum_{r=1}^{s}\gamma_r=1$. $\dot{\zeta}_r$ is biggest mpFNs, where, $\dot{\zeta}_r=(s\delta)\zeta_s$, $(r=1,2,\ldots,s)$ for which $\delta=(\delta_1,\delta_2,\ldots,\delta_s)^T$ is the vector weight such that $\delta_r>0$ and $\sum_{r=1}^{s}\delta_r=1$.

Theorem 10. *Let $\zeta_r=(p_1*\zeta_r,\ldots,p_m*\zeta_r)$ $(r=1,2,\ldots,s)$ be a 's' mpFNs. Then accumulated values of mpFNs ζ_r using mpFEHWA operator is also a mFN. Further, we get*

$$\begin{aligned}
&mpFEHWA_{\gamma}(\zeta_1,\zeta_2,\ldots,\zeta_s)\\
&=\bigoplus_{r=1}^{s}\left(\gamma_r\dot{\zeta}_{\sigma}(r)\right)\\
&=\left(\begin{array}{c}\dfrac{\prod_{r=1}^{s}(1+p_1*\dot{\zeta}_{\sigma}(r))^{\gamma_r}-\prod_{r=1}^{s}(1-p_1*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}{\prod_{r=1}^{s}(1+p_1*dot\zeta_{\sigma}(r))^{\gamma_r}+\prod_{r=1}^{s}(1-p_1*\dot{\zeta}_{\sigma}(r))^{\gamma_r}},\\ \ldots,\dfrac{\prod_{r=1}^{s}(1+p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}-\prod_{r=1}^{s}(1-p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}{\prod_{r=1}^{s}(1+p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}+\prod_{r=1}^{s}(1-p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}\end{array}\right).
\end{aligned}$$

Proof. It can be proved by mathematical induction. □

4. mmF Einstein geometric operators

Here, the definitions and properties of the mpFEWG, mpFEOWG, and mFEHWG operators were introduced.

Definition 11. Let $\zeta_r=(p_1*\zeta_r,\ldots,p_m*\zeta_r)$ $(r=1,2,\ldots,s)$ be a 's' mpFNs. Then mpFNs Einstein weighted geometric (mpFEWG) operator is a function $mpFEWG:\zeta^s\rightarrow\zeta$ is defined below:

$$mpFEWG_{\delta}(\zeta_1,\zeta_2,\ldots,\zeta_s)=\bigotimes_{r=1}^{s}\left(\zeta_r\right)^{\delta_r}$$

where $\delta=(\delta_1,\delta_2,\ldots,\delta_s)^T$ be a weight vector of ζ_r, and $\delta_r>0$ and $\sum_{r=1}^{s}\delta_r=1$.

Theorem 11. *Let $\zeta_r=(p_1*\zeta_r,\ldots,p_m*\zeta_r)$ $(r=1,2,\ldots,s)$ be a 's' mpFNs. Then accumulated value of all mpFNs based on mpFEWG operator is also a mpFN, which follows as:*

$$\begin{aligned}
&mpFEWG_{\delta}(\zeta_1,\zeta_2,\ldots,\zeta_s)\\
&=\bigotimes_{r=1}^{s}\left(\zeta_r\right)^{\delta_r}\\
&=\left(\begin{array}{c}\dfrac{2\prod_{r=1}^{s}(p_1*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2-p_1*\zeta_r)^{\delta_r}+\prod_{r=1}^{s}(p_1*\zeta_r)^{\delta}},\\ \ldots,\dfrac{2\prod_{r=1}^{s}(p_m*\zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2-p_m*\zeta_r)^{\delta_r}+\prod_{r=1}^{s}(p_m*\zeta_r)^{\delta_r}}\end{array}\right)
\end{aligned}$$

where $\delta=(\delta_1,\delta_2,\ldots,\delta_s)$ be a weight vector of ζ_r $(r=1,2,\ldots,s)$, and $\delta_r>0$, and $\sum_{r=1}^{s}\delta_r=1$.

Example 2. Let $\zeta_1 = (0.4, 0.6, 0.7, 0.5)$, $\zeta_2 = (0.3, 0.4, 0.5, 0.6)$, $\zeta_3 = (0.5, 0.7, 0.4, 0.2)$ be 4pFNs with a weight vector $\delta = (0.3, 0.3, 0.4)$ for 4pFNs. Then by Theorem 11 gives

$$
\begin{aligned}
& mpFEWG_\delta(\zeta_1, \zeta_2, \zeta_3) \\
&= \bigotimes_{r=1}^{3} \left(\zeta_r\right)^{\delta_r} \\
&= \left(\begin{array}{c} \dfrac{2\prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2 - p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta}}, \\ \ldots, \dfrac{2\prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2 - p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}} \end{array} \right) \\
&= \left(\begin{array}{c} \dfrac{2 \times (0.4)^{0.3} \times (0.3)^{0.3} \times (0.5)^{0.4}}{(2-0.4)^{0.3} \times (2-0.3)^{0.3} \times (2-0.5)^{0.4} + (0.4)^{0.3} \times (0.3)^{0.3} \times (0.5)^{0.4}}, \\ \dfrac{2 \times (0.6)^{0.3} \times (0.4)^{0.3} \times (0.7)^{0.4}}{(2-0.6)^{0.3} \times (2-0.4)^{0.3} \times (2-0.7)^{0.4} + (0.6)^{0.3} \times (0.4)^{0.3} \times (0.7)^{0.4}}, \\ \dfrac{2 \times (0.7)^{0.3} \times (0.5)^{0.3} \times (0.4)^{0.3}}{(2-0.7)^{0.3} \times (2-0.5)^{0.3} \times (2-0.4)^{0.4} + (0.7)^{0.3} \times (0.5)^{0.3} \times (0.4)^{0.4}}, \\ \dfrac{2 \times (0.5)^{0.3} \times (0.6)^{0.3} \times (0.2)^{0.4}}{(2-0.5)^{0.3} \times (2-0.6)^{0.3} \times (2-0.2)^{0.4} + (0.5)^{0.3} \times (0.6)^{0.3} \times (0.2)^{0.4}}, \end{array} \right) \\
&= \left\langle 0.4034, 0.5708, 0.5109, 0.3761 \right\rangle.
\end{aligned}
$$

It is possible to prove Theorem 11 via mathematical induction.

Proof. If $r = 1$ and $\delta = 1$, then left side of the Theorem 11 becomes $mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) = \bigotimes_{r=1}^{s} \left(\zeta_r\right)^{\delta_r} = (\zeta_1)^{\delta_1}$ at that point, the right-hand side of the Theorem 11 changes into

$$
\begin{aligned}
& mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) \\
&= \bigotimes_{r=1}^{s} \left(\zeta_r\right)^{\delta_r} \\
&= \left(\zeta_1\right)^{\delta_1} = \zeta_1 \\
&= \left(\begin{array}{c} \dfrac{2(p_1 * \zeta_1)}{(2 - p_1 * \zeta_1) + (p_1 * \zeta_1)}, \\ \ldots, \dfrac{2(p_m * \zeta_1)}{(2 - p_m * \zeta_1) + (p_m * \zeta_1)} \end{array} \right)
\end{aligned}
$$

Thus, above technique is true for $r = 1$. □

Let us follows from Theorem 11 holds for $r \geq q$, where $q \in \mathbb{N}$, which gives

$$
\begin{aligned}
& mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_q) \\
&= \bigotimes_{r=1}^{q} \left(\zeta_r\right)^{\delta_r}
\end{aligned}
$$

$$= \left(\begin{array}{c} \dfrac{2\prod_{r=1}^{q}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q}(2 - p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{q}(p_1 * \zeta_r)^{\delta_r}}, \\ \ldots, \dfrac{2\prod_{r=1}^{q}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q}(2 - p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{q}(p_m * \zeta_r)^{\delta_r}} \end{array} \right)$$

Now for $r = q + 1$, then

$$mpFEWG_\delta(\zeta_1, \zeta_2, \zeta_q \ldots, \zeta_{q+1})$$

$$= \bigotimes_{r=1}^{q}(\zeta_r)^{\delta_r} \otimes (\zeta_{q+1})^{\delta_{q+1}}$$

$$= \left(\begin{array}{c} \dfrac{2\prod_{r=1}^{q}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q}(2 - p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{q}(p_1 * \zeta_r)^{\delta}}, \\ \ldots, \dfrac{2\prod_{r=1}^{q}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q}(2 - p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{q}(p_m * \zeta_r)^{\delta_r}} \end{array} \right)$$

$$\otimes \left(\begin{array}{c} \dfrac{2(p_1 * \zeta_{q+1})^{\delta_{q+1}}}{(2 - p_1 * \zeta_{q+1})^{\delta_{q+1}} + (p_1 * \zeta_{q+1})^{\delta_{q+1}}}, \\ \ldots, \dfrac{2(p_m * \zeta_{q+1})^{\delta_{q+1}}}{(2 - p_m * \zeta_{q+1})^{\delta_{q+1}} + (p_m * \zeta_{q+1})^{\delta_{q+1}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \dfrac{2\prod_{r=1}^{q+1}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q+1}(2 - p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{q+1}(p_1 * \zeta_r)^{\delta}}, \\ \ldots, \dfrac{2\prod_{r=1}^{q+1}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{q+1}(2 - p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{q+1}(p_m * \zeta_r)^{\delta_r}} \end{array} \right).$$

As a result, for all natural numbers satisfy the Theorem 11.

The mpFEWG operators abide by the subsequent characteristics.

Theorem 12. *(Idempotency property) Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. If there all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all r $(r = 1, 2, \ldots, s)$. Then*

$$mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) = \zeta.$$

Proof. Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. If there all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all r $(r = 1, 2, \ldots, s)$. Then, from equation (6), we have

$$mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s)$$

$$= \bigotimes_{r=1}^{s}\left(\zeta_r\right)^{\delta_r}$$

$$
= \left(\frac{2\prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2-p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta}}, \ldots, \frac{2\prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2-p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}} \right)
$$

$$
= \left\langle \frac{2(p_1 * \zeta)}{(2-p_1 * \zeta)+(p_1 * \zeta)}, \ldots, \frac{2(p_m * \zeta)}{(2-p_m * \zeta)+(p_m * \zeta)} \right\rangle
$$

$$
= \left\langle p_1 * \zeta, \ldots, p_m * \zeta \right\rangle = \zeta.
$$

Therefore, the theorem's proof is finished. □

Theorem 13. *(Boundedness property) Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, t)$ be a 's' mpFNs. If $\zeta^- = \bigcap_{r=1}^{s} \zeta_r$ and $\zeta^+ = \bigcup_{r=1}^{s} \zeta_r$. Then,*

$$
\zeta^- \le mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) \le \zeta^+.
$$

Theorem 14. *(Monocity property) Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ and $\zeta'_r = (p'_1 * \zeta'_r, \ldots, p'_m * \zeta'_r)$ $(r = 1, 2, \ldots, s)$ be two sets of 's' mpFNs such that $\zeta_r \le \zeta'_r$ for all r, then*

$$
mpFEWG_\delta(\zeta_1, \zeta_2, \ldots, \zeta_s) \le mpFEWG_\delta(\zeta'_1, \zeta'_2, \ldots, \zeta'_s).
$$

The mpFEOWG operator will now be defined.

Definition 12. Let mpFNs Einstein ordered weighted geometric (mpFEOWG) operator be a function $mpFEOWG : \zeta^s \to \zeta$, where $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ for $r = 1, 2, \ldots, s$, such that

$$
mpFEOWG_\delta(\zeta_\sigma(1), \zeta_\sigma(2), \ldots, \zeta_\sigma(s)) = \bigotimes_{r=1}^{s} \left(\zeta_{\sigma(r)}\right)^{\gamma_r}
$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)^T$ a set of weight vector on ζ_r such that $\gamma_r > 0$ and $\sum_{r=1}^{s} \gamma_r = 1$. Also, $\sigma(1), \sigma(2), \ldots, \sigma(s)$ is the permutation of $(1, 2, \ldots, s)$ for which $\zeta_{\sigma(s-1)} \ge \zeta_{\sigma(s)}$ for all $r = 1, 2, \ldots, s$.

Theorem 15. *Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. Then accumulated value of all mpFNs based on mpFEOWG operator is also a mpFN, which further follows as:*

$$
mpFEOWG_\delta(\zeta_\sigma(1), \zeta_\sigma(2), \ldots, \zeta_\sigma(s))
$$

$$
= \bigotimes_{r=1}^{s} \left(\zeta_\sigma(r)\right)^{\gamma_r}
$$

$$
= \left(\frac{2\prod_{r=1}^{s}(p_1 * \zeta_\sigma(r))^{\gamma_r}}{\prod_{r=1}^{s}(2-p_1 * \zeta_\sigma(r))^{\gamma_r} + \prod_{r=1}^{s}(p_1 * \zeta_\sigma(r))^{\gamma}}, \ldots, \frac{2\prod_{r=1}^{s}(p_m * \zeta_\sigma(r))^{\gamma_r}}{\prod_{r=1}^{s}(2-p_m * \zeta_\sigma(r))^{\gamma_r} + \prod_{r=1}^{s}(p_m * \zeta_\sigma(r))^{\gamma_r}} \right)
$$

where a collection of weight vectors be $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)^T$ on ζ_r. Each γ_r must be larger than zero, and the total of all γ_r must be one. If $\zeta_{\sigma(s-1)}$ is larger than or equal to $\zeta_{\sigma(s)}$ for every r from 1 to s, then $\sigma(1), \sigma(2), \ldots, \sigma(s)$ are permutations of $(1, 2, \ldots, s)$.

The following properties of mpFEOWG can be proved easily.

Theorem 16. *(Idempotency property) Let $\zeta_r = (p_1 * \zeta_r, \ldots, p_m * \zeta_r)$ $(r = 1, 2, \ldots, s)$ be a 's' mpFNs. If these all mpFNs are equal, i.e., $\zeta_r = \zeta$ for all r $(r = 1, 2, \ldots, s)$. Then*

$$mpFEOWG_{\delta}(\zeta_1,\zeta_2,\dots,\zeta_s)=\zeta.$$

Theorem 17. *(Boundedness property) Let* $\zeta_r=(p_1*\zeta_r,\dots,p_m*\zeta_r)$ $(r=1,2,\dots,s)$ *be a 's' mpFNs. If* $\zeta^-=\bigcap_{r=1}^{s}\zeta_r$ *and* $\zeta^+=\bigcup_{r=1}^{s}\zeta_r$. *Then,*

$$\zeta^-\le mpFEOWG_{\delta}(\zeta_1,\zeta_2,\dots,\zeta_s)\le\zeta^+.$$

Theorem 18. *(Monocity property) Let* $\zeta_r=(p_1*\zeta_r,\dots,p_m*\zeta_r)$ *and* $\zeta_r'=(p_1'*\zeta_r',\dots,p_m'*\zeta_r')$ $(r=1,2,\dots,s)$ *be two sets of 's' mpFNs such that* $\zeta_r\le\zeta_r'$ *for all r, then*

$$mpFEOWG_{\delta}(\zeta_1,\zeta_2,\dots,\zeta_s)\le mpFEOWG_{\delta}(\zeta_1',\zeta_2',\dots,\zeta_s'),$$

where ζ_r' *is any permutation of* ζ_r $(r=1,2,\dots,s)$.

Theorem 19. *(Commutative property) Let* $\zeta_r=(p_1*\zeta_r,\dots,p_m*\zeta_r)$ *and* $\zeta_r'=(p_1'*\zeta_r',\dots,p_m'*\zeta_r')$ $(r=1,2,\dots,s)$ *be two sets of mpFNs such that* $\zeta_r=\zeta_r'$ *for all r, then*

$$mpFEOWG_{\delta}(\zeta_1,\zeta_2,\dots,\zeta_s)=mpFEOWG_{\delta}(\zeta_1',\zeta_2',\dots,\zeta_s'),$$

where ζ_r' *is arbitrary permutation of* ζ_r *for all* $(r=1,2,\dots,s)$.

In Definitions 11 and 12, the mpFEOWG operator's weight now shows the ordered position of the mpFv rather than the weights themselves, whereas in the past the mpFEWG operator used the weights from the mFv. The mpF Einstein hybrid geometric (mpFE-HWG) operator is a new operator that takes use of the mpFEWG and mFEOWG operations' qualitative features.

Definition 13. Let $\zeta_r=(p_1*\zeta_r,\dots,p_m*\zeta_r)$ $(r=1,2,\dots,s)$ be a 's' mpFNs. Then mpF Einstein hybrid weighted geometric (mpFEHWG) operator is a function $mpFEHWG:\zeta^s\to\zeta$ is defined below:

$$mpFEHWG_{\delta,\gamma}(\zeta_1,\zeta_2,\dots,\zeta_s)=\bigotimes_{b=1}^{s}\left(\dot{\zeta}_{\sigma(r)}\right)^{\gamma_r}$$

Also, $\sigma(1),\sigma(2),\dots,\sigma(s)$ is the permutation of $(1,2,\dots,s)$ for which $\zeta_{\sigma(s-1)}\ge\zeta_{\sigma(s)}$ for all $r=1,2,\dots,s$ for mpFNs ζ_r and $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_s)^T$ is the associated weighted vector of the mpFNs $(\zeta_1,\zeta_2,\dots,\zeta_s)$ such that $\gamma_r>0$ and $\sum_{r=1}^{s}\gamma_r=1$. $\dot{\zeta}_r$ is biggest mpFNs, where, $\dot{\zeta}_r=(s\delta)\zeta_s$, $(r=1,2,\dots,s)$ for which $\delta=(\delta_1,\delta_2,\dots,\delta_s)^T$ is the weight vector such that $\delta_r>0$ and $\sum_{r=1}^{s}\delta_r=1$.

Theorem 20. *Let* $\zeta_r=(p_1*\zeta_r,\dots,p_m*\zeta_r)$ $(r=1,2,\dots,s)$ *be a 's' mpFNs. Then accumulated values of mpFNs* ζ_r *using mpFEHWG operator is also a mpFN. Further, we get*

$$\begin{aligned}
&mpFEHWG_{\delta}(\zeta_{\sigma}(1),\zeta_{\sigma}(2),\dots,\zeta_{\sigma}(s))\\
&=\bigotimes_{r=1}^{s}\left(\dot{\zeta}_{\sigma}(r)\right)^{\gamma_r}\\
&=\left(\begin{array}{c}\dfrac{2\prod_{r=1}^{s}(p_1*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}{\prod_{r=1}^{s}(2-p_1*\dot{\zeta}_{\sigma}(r))^{\gamma_r}+\prod_{r=1}^{s}(p_1*\dot{\zeta}_{\sigma}(r))^{\gamma}},\\ \dots,\dfrac{2\prod_{r=1}^{s}(p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}{\prod_{r=1}^{s}(2-p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}+\prod_{r=1}^{s}(p_m*\dot{\zeta}_{\sigma}(r))^{\gamma_r}}\end{array}\right)
\end{aligned}$$

Proof. By induction method it can be proved. □

5. MAGDM method based on multi-polar fuzzy sets

In this part, we build a MAGDM method with real attribute weights and mpFN values utilising mF Einstein aggregation procedures. Let $A=\{A_1,A_2,\dots,A_s\}$ be a set of alternatives and a set of attributes be $G=\{G_1,G_2,\dots,G_r\}$, and $E=\{e_1,e_2,\dots,e_s\}$ be a group of s experts. Let $\delta=(\delta_1,\delta_2,\dots,\delta_r)$ be a group of weighting vector of G_q $(q=1,2,\dots,\upsilon)$ are assigned by DMs such that $\delta_q>0$ and

$\sum_{q=1}^{v} \delta_q = 1$. Let R represent an m-polar fuzzy decision matrix denoted by $\gamma^k vs$, where $\gamma^k vs$ is a multi-polar fuzzy number specified by the experts $e^k \in E$. In this context, $\gamma^k vs$ is composed of elements $(p_1 * \zeta^k qr, \dots, p_m * \zeta^k qr)v \times s$, where $p_j * A_{qr}$ $(j = 1, 2, \dots, s) \in [0, 1]$ indicates the membership degree for the alternatives A_q satisfying the attribute G_r.

In the accompanying algorithm, we suggest employing the mpFEWA and mpFEWG operators to solve the MAGDM issue using the mpFNs data.

Step 1. Make a decision matrix-based arrangement of the mpFN data for each option $R^{(k)}$ as:

$$R^{(k)} = \begin{matrix} \\ G_1 \\ G_2 \\ \vdots \\ G_r \end{matrix} \begin{matrix} A_1 & A_2 & \cdots & A_s \\ \left[\begin{matrix} \gamma_{11}^{(k)} & \gamma_{12}^{(k)} & \cdots & \gamma_{1r}^{(k)} \\ \gamma_{21}^{(k)} & \gamma_{22}^{(k)} & \cdots & \gamma_{2r}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{s1}^{(k)} & \gamma_{s2}^{(k)} & \cdots & \gamma_{sr}^{(k)} \end{matrix}\right] \end{matrix}$$

Step 2: Normalize decision matrices, if required, cost type attribute change to benefit type by the following equations

$$\tilde{\gamma} = \begin{cases} (p_1 * \zeta_1), \dots, (p_m * \zeta_1), & \text{If } C_q \ are\ benefit\ type \\ (p_1 * \zeta_1)^c, \dots, (p_m * \zeta_1)^c, & \text{If } C_q \ are\ cost\ type. \end{cases}$$

Step 2. Using operator mpFEWA to calculate expert information using expert weights $\Upsilon_q^k = mpFEWA_\omega(\zeta_{q1}^k, \zeta_{q2}^k, \dots, \zeta_{qs}^k) = \bigoplus_{r=1}^{s} \left(\omega_r^k \zeta_r^k\right)$

$$= \left(\frac{\prod_{r=1}^{s}(1 + p_1 * \zeta_r^k)^{\omega_r^k} - \prod_{r=1}^{s}(1 - p_1 * \zeta_r^k)^{\omega_r^k}}{\prod_{r=1}^{s}(1 + p_1 * \zeta_r^k)^{\omega_r^k} + \prod_{r=1}^{s}(1 - p_1 * \zeta_r^k)^{\omega_r^k}}, \dots, \frac{\prod_{r=1}^{s}(1 + p_m * \zeta_r^k)^{\omega_r^k} - \prod_{r=1}^{s}(1 - p_m * \zeta_r^k)^{\omega_r^k}}{\prod_{r=1}^{s}(1 + p_m * \zeta_r^k)^{\omega_r^k} + \prod_{r=1}^{s}(1 - p_m * \zeta_r^k)^{\omega_r^k}} \right)$$

or $\Upsilon_q^k = mpFEWG_\omega(\zeta_{q1}^k, \zeta_{q2}^k, \dots, \zeta_{qs}^k) = \bigotimes_{r=1}^{s} \left(\zeta_r^k\right)^{\omega_r^k}$

$$= \left(\frac{2\prod_{r=1}^{s}(p_1 * \zeta_r^k)^{\omega_r^k}}{\prod_{r=1}^{s}(2 - p_1 * \zeta_r^k)^{\omega_r^k} + \prod_{r=1}^{s}(p_1 * \zeta_r^k)^{\omega^k}}, \dots, \frac{2\prod_{r=1}^{s}(p_m * \zeta_r^k)^{\omega_r^k}}{\prod_{r=1}^{s}(2 - p_m * \zeta_r^k)^{\omega_r^k} + \prod_{r=1}^{s}(p_m * \zeta_r^k)^{\omega_r^k}} \right)$$

to compute accumulated values Ψ_q $(q = 1, 2, \dots, v)$ of the alternatives A_v.

Step 3. Now employ the criteria weighted decision data stated in matrix R, and the operator mpFEWA $\Upsilon_q = mpFEWA_\delta(\zeta_{q1}, \zeta_{q2}, \dots, \zeta_{qs}) = \bigoplus_{r=1}^{s} \left(\delta_r \zeta_r\right)$

$$= \left(\frac{\prod_{r=1}^{s}(1 + p_1 * \zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1 - p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1 + p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1 - p_1 * \zeta_r)^{\delta_r}}, \dots, \frac{\prod_{r=1}^{s}(1 + p_m * \zeta_r)^{\delta_r} - \prod_{r=1}^{s}(1 - p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(1 + p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(1 - p_m * \zeta_r)^{\delta_r}} \right)$$

or $\Upsilon_q = mpFEWG_\delta(\zeta_{q1}, \zeta_{q2}, \dots, \zeta_{qs}) = \bigotimes_{r=1}^{s} \left(\zeta_r\right)^{\delta_r}$

Table 1
3-polar fuzzy decision matrix.

	A_1	A_2	A_3	A_4	A_5
G_1	(.5,.7,.5)	(.6,.7,.8)	(.4,.5,.6)	(.5,.7,.4)	(.8,.7,.9)
G_2	(.6,.8,.4)	(.4,.3,.6)	(.7,.9,.3)	(.3,.6,.4)	(.7,.6,.4)
G_3	(.5,.6,.7)	(.7,.4,.5)	(.6,.3,.5)	(.5,.7,.8)	(.8,.5,.3)
G_4	(.8,.5,.5)	(.7,.8,.9)	(.6,.5,.5)	(.4,.6,.8)	(.6,.5,.8)

$$= \left(\frac{2\prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2 - p_1 * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_1 * \zeta_r)^{\delta}}, \ldots, \frac{2\prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}}{\prod_{r=1}^{s}(2 - p_m * \zeta_r)^{\delta_r} + \prod_{r=1}^{s}(p_m * \zeta_r)^{\delta_r}} \right)$$

to compute accumulated values Ψ_q $(q = 1, 2, \ldots, v)$ of the alternatives A_v.

Step 3. Use the aggregated information from the mpFNs data Υ_q to get the scoring function, $MPSF(\Upsilon_q)$, for each A_q alternative $(q = 1, 2, \ldots, v)$. In order to find the best option, this scoring method will help rate all of the possibilities. In situations when there is no difference between the score functions $\Phi(\Upsilon_q)$ and $MPSF(\Upsilon_r)$, the evaluation of the accuracy degrees of $MPAF(\Upsilon_q)$ and $MPAF(\Upsilon_r)$ is carried out. For the purpose of determining the degrees of accuracy of the scoring functions $MPSF(\Upsilon_q)$ and $MPSF(\Upsilon_r)$, this assessment makes use of all accessible mpFNs data. Then, the degrees of accuracy of their respective scoring functions, $MPSF(\Upsilon_q)$, are used to rank the alternatives A_q.

Step 4. To choose the best option(s), rank all of the alternatives A_q $(q = 1, 2, \ldots, v)$ according to $\Phi(\Upsilon_q)$ $(q = 1, 2, \ldots, v)$.

Step 5. Stop.

6. Numerical example

We employ the suggested method to address an MAGDM issue.

6.1. *Location selection for sponge iron factory*

Because of the serious risks they pose to human health and the environment, sponge iron plants are considered "red category" enterprises. During production, these plants release silica, smoke, extremely high temperatures, and unburned carbon particles. Although electrostatic precipitators (ESPS) are required by the majority of states to reduce emissions, the extent to which they effectively decrease dust is still up for debate. Careful management is also required for the disposal of contaminants such as carbon dust, iron dust, and coal char that are collected from ESPS. A team of three specialists is appointed by a company to investigate five possible sites for a sponge iron factory. To choose the best site among alternatives A_1, A_2, A_3, A_4, and A_5, the expert team takes into account the intermediate evaluations given by decision makers (DMs) and uses a weight vector of $(0.5, 0.23, 0.27)^T$ to aggregate their judgements.

G_1 : Infrastructures
G_2 : Environmental conditions
G_3 : Social impact
G_4 : Governmental policies.

Each criterion was then broken down into three sub-criteria in order to construct a 3-polar fuzzy set. The accessibility of transit, water, and coal are crucial to the infrastructure. Temperature, humidity, and wind speed are three environmental factors. The influence on society is dictated by institutions like hospitals, schools, and clinics. Among the many facets of government policy are licencing, subsidies, and institutional financing. The decision-makers need to weed out five possible locations because none of these matters more than the others. Locations are represented as A_q $(q = 1, 2, \ldots, 4, 5)$ in the decision matrices that are given in Tables 1, 2, and 3. Those in charge of making decisions give each characteristic a weight of $(0.4, 0.3, 0.2, 0.1)^T$, respectively.

Step 1: In this case, experts have rated five different places, A_1, A_2, A_3, A_4, and A_5, using four different criteria, G_1, G_2, G_3, and G_4. Here, we welcome three decision-makers, represented by $e = (e_1, e_2, e_3)$, whose tastes are encapsulated in the weight vector $w = (0.5, 0.23, 0.27)^T$. Their mission is to find the best spot to set up a sponge iron manufacturing. Tables 1, 2, and 3 present the decision-makers' rankings of the sites, correspondingly.

Step 2: Since each characteristic represents a sort of benefit, standardisation is unnecessary.

Step 3: Utilizing the mpFEWA operator, we combine the insights provided by three experts, as outlined in Table 4, employing a weighting vector of $(.5, .23, .27)^T$.

Table 2
3-polar fuzzy decision matrix.

	A_1	A_2	A_3	A_4	A_5
G_1	(.6,.8,.4)	(.5,.6,.5)	(.3,.3,.2)	(.4,.3,.2)	(.7,.6,.8)
G_2	(.3,.5,.5)	(.6,.4,.7)	(.4,.6,.5)	(.5,.6,.6)	(.6,.5,.5)
G_3	(.4,.5,.6)	(.8,.5,.2)	(.5,.6,.7)	(.4,.7,.3)	(.7,.6,.4)
G_4	(.6,.4,.7)	(.2,.4,.6)	(.7,.8,.9)	(.6,.4,.9)	(.5,.6,.7)

Table 3
3-polar fuzzy decision matrix.

	A_1	A_2	A_3	A_4	A_5
G_1	(.5,.7,.5)	(.4,.5,.6)	(.6,.2,.6)	(.5,.4,.4)	(.4,.5,.6)
G_2	(.4,.6,.4)	(.6,.5,.8)	(.3,.5,.9)	(.3,.5,.6)	(.7,.6,.8)
G_3	(.5,.4,.5)	(.7,.6,.4)	(.6,.4,.5)	(.5,.6,.7)	(.6,.5,.5)
G_4	(.7,.6,.7)	(.4,.3,.5)	(.8,.6,.7)	(.8,.4,.3)	(.4,.3,.6)

Table 4
Aggregated values for the decision-makers.

	A_1	A_2	A_3	A_4	A_5
G_1	(.5244,.7261,.4780)	(.5280,.6300,.6977)	(.4383,.3804,.5229)	(.4780,.5506,.3565)	(.6979,.6300,.8262)
G_2	(.4870,.6977,.4240)	(.5068,.3084,.6872)	(.5474,.7794,.5906)	(.3493,.5746,.5068)	(.6790,.5784,.5613)
G_3	(.4780,.5280,.6300)	(.7261,.4818,.4100)	(.5784,.4041,.5528)	(.4780,.6752,.6935)	(.7333,.5244,.3804)
G_4	(.7366,.5074,.6097)	(.5322,.6233,.7794)	(.6872,.6135,.6897)	(.5836,.5068,.7494)	(.5280,.4759,.7333)

Table 5
Aggregated values for the decision-makers.

	A_1	A_2	A_3	A_4	A_5
G_1	(.5220,.7225,.4756)	(.5182,.6196,.6707)	(.4213,.3526,.4781)	(.4756,.5060,.3436)	(.6550,.6196,.7919)
G_2	(.4641,.6707,.4216)	(.4932,.3700,.6745)	(.5007,.7124,.4722)	(.3392,.5719,.4932)	(.6763,.5760,.5164)
G_3	(.4756,.5182,.6196)	(.7255,.4720,.3865)	(.5760,.3842,.5425)	(.4756,.6722,.6335)	(.7207,.5220,.3700)
G_4	(.7249,.5006,.5952)	(.4688,.5419,.7124)	(.6749,.5900,.6362)	(.5389,.4932,.6560)	(.5182,.4585,.7207)

Employing the mpFEWA operator to ascertain aggregated data, we may select an appropriate place A_m $(m = 1, 2, \dots, u)$:

- **Step 4.** Here, using the mpFEWA operator to accumulated preference values Ω_m of A_m $(m = 1, 2, \dots, 5)$ are $\Omega_1 = (.5306, .6643, .5102)$, $\Omega_2 = (.5689, .5156, .6581)$, $\Omega_3 = (.5290, .5584, .5682)$, $\Omega_4 = (.4530, .5809, .5261)$, $\Omega_5 = (.6855, .5801, .6829)$
- **Step 5.** Score values of $\Phi(\Omega_m)$ $(m = 1, 2, \dots, 5)$ using Equation (1) are as follows: $\Phi(\Omega_1) = .5684$, $\Phi(\Omega_2) = .5807$, $\Phi(\Omega_3) = .5519$, $\Phi(\Omega_4) = .5200$, $\Phi(\Omega_5) = .6495$
- **Step 6.** Ranking results of A_m $(m = 1, 2, \dots, 5)$ in accordance with the score values of $\Phi(\Omega_m)$ $(m = 1, 2, \dots, 5)$ of the overall mpFNs as $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$.
- **Step 6.** A_5 is suggested as most favourable location.

If the operator mpFEWG is employed instead, the issue can be resolved in a similar fashion.

Step 1: As previous from Step 1-2.

Step 3: To apply the mpFEWG operator with the weighting vector $(.5, .23, .27)^T$ for aggregating information from the three experts listed in Table 5.

- **Step 4.** Using mpFEWG operator to obtain the overall accumulated values of Q_a $(a = 1, 2, \dots, 5)\Omega_1 = (.5182, .6403, .4965)$, $\Omega_2 = (.5433, .4998, .6110)$, $\Omega_3 = (.4973, .4742, .5035)$, $\Omega_4 = (.4368, .5556, .4666)$, $\Omega_5 = (.6596, .5694, .6033)$
- **Step 5.** Compute score values $\Phi(\Omega_m)$ $(m = 1, 2, \dots, 5)$ of Ω_m $(m = 1, 2, \dots, 5)$ as: $\Phi(\Omega_1) = .5517$, $\Phi(\Omega_2) = .5514$, $\Phi(\Omega_3) = .4917$, $\Phi(\Omega_4) = .4863$, $\Phi(\Omega_5) = .6108$
- **Step 6.** Ranking all A_m $(m = 1, 2, \dots, 5)$ in the value of score functions $\Phi(\Omega_a)$ $(a = 1, 2, \dots, 5)$ of the overall mpFNs as $A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$.
- **Step 4.** Return A_5 is selected as suitable location.

According to the previous study, location A_5 is often favoured when the mpFEWA and mpFEWG operators are used.

What follows is a comparison of the suggested study's efficiency and efficacy to other current approaches.

Table 6
The values AV_q of alternatives for mpFEWA operator.

Alternatives	G_1	G_2	G_3	G_4
A_1	.5762	.5362	.5453	.6179
A_2	.6186	.5008	.5393	.6449
A_3	.4472	.6391	.5118	.6635
A_4	.4617	.4769	.6156	.6133
A_5	.7180	.6062	.5460	.5791
AV	.5807	.5638	.5618	.6288

Table 7
The values of PDA_q under mpFEWA.

Alternatives	G_1	G_2	G_3	G_4
A_1	.0000	.0000	.0000	.0000
A_2	.0653	.0000	.0000	.0256
A_3	.0000	.1336	.0000	.0552
A_4	.0000	.0000	.0958	.0000
A_5	.2364	.0755	.0000	.0000

Table 8
The values of NDA_q under mpFEWA.

Alternatives	G_1	G_2	G_3	G_4
A_1	.0078	.0490	.0294	.0173
A_2	.000	.1117	.0400	.0000
A_3	.2299	.0000	.0889	.0000
A_4	.2049	.1541	.0000	.0623
A_5	.0000	.0000	.0281	.0790

7. Verification by EDAS method

We use an EDAS approach to verify the outcomes of the suggested mpFEWA or mpFEWG procedure. Tables 4 and 5 display the combined assessments of all decision makers using the m-polar fuzzy weighted Einstein geometric operator or the m-polar fuzzy weighted Einstein averaging operator. Here, we use the criterion weight vector $\omega = (0.4, 0.3, 0.2, 0.1)^T$. The data from Table 4 is used to use the EDAS approach. The steps listed below are what we use to resolve this issue.
Step 1-3: It is same as the previous Algorithm.
Step 4: Table 6 lists the score values for the options, and this is where we compute the average solution (AV).

$$AV = \left[AV_r\right] = \left[\frac{\sum_{q=1}^{v} A_{qr}}{v}\right]_{1\times s} = \left(\begin{array}{c} \dfrac{\prod_{q=1}^{v}(1+p_1*\zeta_r)^{1/v} - \prod_{q=1}^{v}(1-p_1*\zeta_r)^{1/v}}{\prod_{q=1}^{v}(1+p_1*\zeta_r)^{1/v} + \prod_{q=1}^{v}(1-p_1*\zeta_r)^{1/v}}, \\ \dots, \dfrac{\prod_{q=1}^{v}(1+p_m*\zeta_r)^{1/v} - \prod_{q=1}^{v}(1-p_m*\zeta_r)^{1/v}}{\prod_{q=1}^{v}(1+p_m*\zeta_r)^{1/v} + \prod_{q=1}^{v}(1-p_m*\zeta_r)^{1/v}} \end{array} \right)$$

or

=

For convenience, we get the modified form of PDA and NDA based on score function for the operator mpFEWA are given in Table 7 and Table 8 by the formulas Equations (2) and (3) as follows:

$$PDA = \left[P_{qr}\right]_{v\times s} = \frac{\max\left(0, \left(\Phi(A_{qr}) - \Phi(AV_q)\right)\right)}{\Phi(AV_q)} \tag{2}$$

$$NDA = \left[N_{qr}\right]_{v\times s} = \frac{\max\left(0, \left(\Phi(AV_q) - \Phi(A_{qr})\right)\right)}{\Phi(AV_r)} \tag{3}$$

Step 5: The weighted sums of PDA and NDA using attribute's weight denoted as SP_r and NP_r are obtained in the following formulas given in equation (4):

Table 9
Aggregated values using mFHWA (mFHWG) operators on 3FNs.

Alternative	$mFHWA$	$mFHWG$
A_1	(.5699, .7014, .5192)	(.5562, .6866, .5036)
A_2	(.5814, .5611, .7185)	(.5616, .5048, .6832)
A_3	(.5633, .6517, .4891)	(.5401, .5552, .4669)
A_4	(.4340, .6627, .5554)	(.4222, .6590, .5025)
A_5	(.7568, .6167, .7103)	(.7486, .6067, .5849)

Table 10
Score of the alternatives using mFHWA and mFHWG operators.

Methods	A_1	A_2	A_3	A_4	A_5
$mFHWA$	.5968	.6203	.5680	.5507	.6946
$mFHWG$	.5821	.5832	.5207	.5279	.6467

Table 11
Comparison of Methods.

Operators	Ranking orders
Waseem et al. [39] mpFHWA	$A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$
Waseem et al. [39] mpFHWG	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$
Proposed mpFEWA	$A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$
Proposed mpFEWG	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$.

$$SP_q = \sum_{r=1}^{s} \delta_r PDA_{qr}, \quad NP_q = \sum_{r=1}^{s} \delta_r NDA_{qr}. \tag{4}$$

The obtained results are: $SP_1 = .0000$, $SP_2 = .0287$, $SP_3 = .0456$, $SP_4 = .0192$, $SP_5 = .1172 NP_1 = .0254$, $NP_2 = .0415$, $NP_3 = .1097$, $NP_4 = .1344$, $SP_5 = .0135$

Step 6: The normalized results of equation (5) can be obtained based on equation (4):

$$NSP_q = \frac{SP_q}{\max_q \left(SP_q\right)}, \quad NSN_q = 1 - \frac{NP_q}{\max_q \left(NP_q\right)}. \tag{5}$$

Here, the results are as follows: $NSP_1 = .0000$, $NSP_2 = .2449$, $NSP_3 = .3891$, $SP_4 = .1638$, $SP_5 = 1.0000 NSN_1 = .8110$, $NSN_2 = .6912$, $NSN_3 = .1838$, $NSN_4 = .0000$, $NSN_5 = .8996$.

Step 7: The appraisal score (AS) based on each alternatives values NSP_q and NSN_q as per equation (6) below:

$$AS_p = \frac{1}{2}\left(NSP_q + NSN_q\right). \tag{6}$$

are $AS_1 = .4055$, $AS_2 = .4681$, $AS_3 = .2865$, $AS_4 = .0819$ and $AS_5 = .9498$.

Step 8: The order of alternatives as per AS_p values is $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$. Hence, the favourable location is A_5.

7.1. Comparative results

In an effort to provide the groundwork for comparing their study to current challenges, Waseem and colleagues [39] were pioneers in introducing the m-polar fuzzy environment. To conduct their study, they utilised two operators: the m-polar fuzzy Hamacher weighted geometric (mFHWG) operator and the m-polar fuzzy Hamacher weighted average (mFHWA). Table 9 displays the assessment matrix that was used for this comparative analysis. It was created from Tables 4 and 5. Afterwards, Table 10 displays the computed score values, which are defined in Equation 2.

Table 11 shows the state-of-the-art approach using m-polar fuzzy Hamacher aggregation operators compared to the suggested model [39]. The table shows that the ranking order $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ is reached when the present mFHWA operator is used, which is the same order as when the proposed mpFEWA operator is used. Likewise, the ranking order $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ is maintained when the mFHWG operator is used, matching the ranking that was achieved with the suggested mpFEWG operator. The recommended choice is A_5, as highlighted by all operator selections. Therefore, the proposed method is trustworthy, providing decision-makers with a new and flexible way to tackle m-polar fuzzy MADM problems.

8. Conclusion

In that article, the weighted geometric operator, the order weighted geometric operator, and the hybrid weighted operator under Einstein triangular norms are among the novel operators introduced in this study that are designed for m-polar fuzzy environments.

It delves into the basic characteristics of these operators, such as their idempotency, monotonicity, boundedness, and commutativity. The study proposes using m-polar Einstein weighted geometric and Einstein weighted averaging operations to address Multiple Attribute Group Decision Making (MAGDM). The success of this technique is illustrated by a practical case and comparative analysis. To provide decision-makers with more alternatives for optimising parameters, parameter sensitivity analysis is also performed. So, the suggested model is able to reliably process complicated data and produce accurate computational results, complicated bipolar fuzzy sets, m-polar hesitant fuzzy systems, and other types of complicated fuzzy sets are all under its purview. It also works in settings with operators that may go in both directions, such as the Bonferroni Mean, Heronian Mean, Interactive, and Einstein operators. In addition, group decision-making using TOPSIS is useful for ATM site selection [49], developing the Weighted Spearman's correlation coefficient [50] or WS rank similarity coefficient [51], Enhancing Industry 4.0 Adoption [52], Energy-Product Systems Modelling [53], distributive risk assessment [54], sustainable strategies [55], and city supply chain management [56]. Economic models, intelligent diagnostics, business and management, and three-way decision-making all feature potentially unpredictable contexts.

CRediT authorship contribution statement

Chiranjibe Jana: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Conceptualization. **Ibrahim M. Hezam:** Writing – review & editing, Supervision, Methodology, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

We have no restriction to use the data described in the article.

Acknowledgements

This research is supported by the Researchers Supporting Project number (RSP2024R389), King Saud University, Riyadh, Saudi Arabia. Grant number: RSP2024R389.

References

[1] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.
[2] K.T. Atanassov, On Intuitionistic Fuzzy Sets Theory, Studies in Fuzziness and Soft Computing, vol. 283, Springer-Verlag, Berlin Heidelberg, 1999.
[3] H. Garg, Generalised Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making, J. Exp. Theor. Artif. Intell. 30 (2018) 763–794.
[4] H. Garg, Nancy, Multiple attribute decision making based on immediate probabilities aggregation operators for single-valued and interval neutrosophic sets, J. Appl. Math. Comput. 63 (2020) 619–653.
[5] C. Jana, T. Senapati, M. Pal, R.R. Yager, Picture fuzzy Dombi aggregation operators: application to MADM process, Appl. Soft Comput. 74 (1) (2019) 99–109.
[6] C. Jana, M. Pal, F. Karaaslan, J.Q. Wang, Trapezoidal neutrosophic aggregation operators and its application in multiple attribute decision-making process, Sci. Iran. Trans. Ind. Eng. E 27 (3) (2020) 1655–1673.
[7] C. Jana, G. Muhiuddin, M. Pal, Some Dombi aggregation of Q-rung orthopair fuzzy numbers in multiple-attribute decision making, Int. J. Intell. Syst. 34 (12) (2019) 3220–3240.
[8] C. Jana, G. Muhiuddin, M. Pal, Multi-criteria decision making approach based on SVTrN Dombi aggregation functions, Artif. Intell. Rev. 54 (2021) 3685–3723.
[9] C. Jana, M. Pal, A dynamical hybrid method to design decision making process based on GRA approach for multiple attributes problem, Eng. Appl. Artif. Intell. 100 (2021) 104203.
[10] M.M. Xia, Z.S. Xu, B. Zhu, Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm, Knowl.-Based Syst. 31 (1) (2012) 78–88.
[11] A.A. Khan, L. Wang, Generalized and group-generalized parameter based Fermatean fuzzy aggregation operators with application to decision-making, Int. J. Knowl. Innov. Stud. 1 (1) (2023) 10–29, https://doi.org/10.56578/ijkis010102.
[12] M. Saqlain, Sustainable hydrogen production: a decision-making approach using VIKOR and intuitionistic hypersoft sets, J. Intell. Manag. Decis. 2 (3) (2023) 130–138, https://doi.org/10.56578/jimd020303.
[13] Z.S. Xu, Intuitionistic fuzzy aggregation operators, IEEE Trans. Fuzzy Syst. 15 (6) (2007) 1179–1187.
[14] Z.S. Xu, R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen. Syst. 35 (4) (2006) 417–433.
[15] R.R. Yager, J. Kacprzyk, The Ordered Weighted Averaging Operators: Theory and Applications, M.A., Kluwer, Boston, 1997.
[16] A. Sarkar, T. Senapati, L.S. Jin, R. Mesiar, A. Biswas, R.R. Yager, Sugeno–Weber triangular norm-based aggregation operators under T-spherical fuzzy hypersoft context, Inf. Sci. 645 (2023) 119305, https://doi.org/10.1016/j.ins.2023.119305.
[17] A. Puska, I. Stojanović, Fuzzy multi-criteria analyses on green supplier selection in an agri-food company, J. Intell. Manag. Decis. 1 (1) (2022) 2–16, https://doi.org/10.56578/jimd010102.
[18] J. Chakraborty, S. Mukherjee, L. Sahoo, Intuitionistic fuzzy multi-index multi-criteria decision-making for smart phone selection using similarity measures in a fuzzy environment, J. Ind. Intell. 1 (1) (2023) 1–7, https://doi.org/10.56578/jii010101.
[19] T. Senapati, G. Chen, R. Mesiar, R.R. Yager, Intuitionistic fuzzy geometric aggregation operators in the framework of Aczel-Alsina triangular norms and their application to multiple attribute decision making, Expert Syst. Appl. 212 (2023) 118832, https://doi.org/10.1016/j.eswa.2022.118832.
[20] W.R. Zhang, Bipolar fuzzy sets and relations: a computational frame work for cognitive modelling and multiagent decision analysis, in: Proceedings of IEEE Conference, 1994, pp. 305–309.
[21] W.R. Zhang, Bipolar fuzzy sets, in: Proceedings of FUZZYIEEE, 1998, pp. 835–840.

[22] C. Jana, M. Pal, J.Q. Wang, Bipolar fuzzy Dombi aggregation operators and its application in multiple attribute decision making process, J. Ambient Intell. Humaniz. Comput. 10 (9) (2019) 3533–3549.
[23] C. Jana, M. Pal, J.Q. Wang, Bipolar fuzzy Dombi prioritized aggregation operators in multiple attribute decision making, Soft Comput. 24 (5) (2020) 3631–3646.
[24] G.W. Wei, F.E. Alsaadi, H. Tasawar, A. Alsaedi, Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making, Int. J. Fuzzy Syst. 20 (1) (2018) 1–12.
[25] H. Gao, G.W. Wei, Y.H. Huang, Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making, IEEE Access 6 (1) (2018) 11508–11522.
[26] X.R. Xu, G.W. Wei, Dual hesitant bipolar fuzzy aggregation operators in multiple attribute decision making, Int. J. Knowl. Based Intell. Eng. Syst. 21 (2017) 155–164.
[27] C. Jana, M. Pal, J.Q. Wang, A robust aggregation operator for multi-criteria decision-making method with bipolar fuzzy soft environment, Iran. J. Fuzzy Syst. 16 (6) (2019) 1–16.
[28] M. Akram, N. Waseem, Novel applications of bipolar fuzzy graphs to decision making problems, J. Appl. Math. Comput. 56 (2018) 73–91.
[29] J. Chen, S. Li, S. Ma, X. Wang, m-polar fuzzy sets: an extension of bipolar fuzzy sets, Sci. World J. 2014 (2014) 416530.
[30] T. Mahmood, U. ur Rehman, Z. Ali, Analysis and application of Aczel-Alsina aggregation operators based on bipolar complex fuzzy information in multiple attribute decision making, Inf. Sci. 619 (2023) 817–833.
[31] U. ur Rehman, T. Mahmood, M. Naeem, Dombi Bonferroni mean operators under bipolar complex fuzzy environment and their applications in Internet world, IEEE Access 11 (2023) 22727–22755.
[32] U. ur Rehman, T. Mahmood, The generalized Dice similarity measures for bipolar complex fuzzy set and its applications to pattern recognition and medical diagnosis, Comput. Appl. Math. 41 (2022) 265.
[33] A. Al-Masarwah, A.G. Ahmad, m-Polar fuzzy ideals of BCK/BCI-algebras, J. King Saud Univ., Sci. 31 (4) (2019) 1220–1226.
[34] M. Akram, A. Farooq, K.P. Shum, On m-polar fuzzy Lie subalgebras, Ital. J. Pure Appl. Math. 36 (2016) 445–454.
[35] G. Ghorai, M. Pal, Some properties of m-polar fuzzy graphs, Pac. Sci. Rev. A, Nat. Sci. Eng. 18 (1) (2016) 38–46.
[36] A.Z. Khameneh, A. Kilicman, m-polar fuzzy soft weighted aggregation operators and their applications in group decision-making, Symmetry 10 (2018) 636.
[37] M. Akram, N. Waseem, Novel approach in decision-making with m-polar fuzzy ELECTRE-I, Int. J. Fuzzy Syst. 21 (2019) 1117–1129.
[38] M. Akram, m-polar fuzzy graphs, in: Studies in Fuzziness and Soft Computing, Springer, Cham, Switzerland, 2019, p. 371.
[39] N. Waseem, M. Akram, J.C.R. Alcantud, Multi-attribute decision-making based on m-polar fuzzy Hamacher aggregation operators, Symmetry 11 (2019) 1498.
[40] M. Akram, A.R. Adeel, J.C. Alcantud, Multi-criteria group decision-making using an m-polar hesitant fuzzy TOPSIS approach, Symmetry 11 (6) (2019) 795.
[41] M. Akram Shumaiza, J.C.R. Alcantud, An m-polar fuzzy PROMETHEE approach for AHP-assisted group decision-making, Math. Comput. Appl. 25 (2) (2020) 26.
[42] P. Liu, Y.H. Li, Y.B. Chen, Some generalized Einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making, Sci. Iran. E 22 (6) (2015) 2684–2701.
[43] X. Zhao, G.W. Wei, Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making, Knowl.-Based Syst. 37 (2013) 472–479.
[44] S. Khan, S. Abdullah, S. Asraf, Picture fuzzy aggregation information based on Einstein operations and their application in decision making, Math. Sci. 13 (2019) 213–229.
[45] K. Rahman, S. Abdullah, R. Ahmad, M. Ullah, Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making, J. Intell. Fuzzy Syst. 33 (1) (2017) 1–13.
[46] T.K. Paul, C. Jana, M. Pal, Enhancing multi-attribute decision making with Pythagorean fuzzy Hamacher aggregation operators, J. Ind. Intell. 1 (1) (2023) 30–54, https://doi.org/10.56578/jii010103.
[47] P. Liu, Z. Ali, T. Mahmood, Generalized complex q-rung orthopair fuzzy Einstein averaging aggregation operators and their application in multi-attribute decision making, Complex Intell. Syst. 7 (2021) 511–538, https://doi.org/10.1007/s40747-020-00197-6.
[48] F. Eldemir, I. Onden, Geographical information systems and multicriteria decision integration approach for hospital location selection, Int. J. Inf. Technol. Decis. Mak. 15 (5) (2016) 975–999.
[49] A. Yildiz, E. Ayyildiz, A.T. Gumus, C. Ozkan, A modified balanced scorecard based hybrid Pythagorean fuzzy ahp-topsis methodology for atm site selection problem, Int. J. Inf. Technol. Decis. Mak. 19 (2) (2020) 365–384.
[50] H. Li, Y. Cao, L. Su, Pythagorean fuzzy multi-criteria decision-making approach based on Spearman rank correlation coefficient, Soft Comput. 26 (2022) 3001–3012.
[51] A. Shekhovtsov, W. Sałabun, A comparative case study of the VIKOR and TOPSIS rankings similarity, Proc. Comput. Sci. 176 (2020) 3730–3740.
[52] Y.J. Qiu, M.B. Bouraima, C.K. Kiptum, E. Ayyildiz, Z. Stevic, I. Badi, K.M. Ndiema, Strategies for enhancing industry 4.0 adoption in East Africa: an integrated spherical fuzzy SWARA-WASPAS approach, J. IndIntell. 1 (2) (2023) 87–100, https://doi.org/10.56578/jii010202.
[53] V. Selicati, N. Cardinale, Sustainability assessment techniques and potential sustainability accreditation tools for energy-product systems modelling, J. Sustain. Energy 2 (1) (2023) 1–18, https://doi.org/10.56578/jse020101.
[54] V. Pajić, M. Andrejić, M. Sternad, FMEA-QFD approach for effective risk assessment in distribution processes, J. Intell. Manag. Decis. 2 (2) (2023) 46–56, https://doi.org/10.56578/jimd020201.
[55] C. Jana, M. Dobrodolac, V. Simic, M. Pal, B. Sarkar, Ž. Stević, Evaluation of sustainable strategies for urban parcel delivery: linguistic q-rung orthopair fuzzy Choquet integral approach, Eng. Appl. Artif. Intell. 126 (2023) 106811.
[56] M. Riaz, H.M.A. Farid, C. Jana, M. Pal, B. Sarkar, Efficient city supply chain management through spherical fuzzy dynamic multistage decision analysis, Eng. Appl. Artif. Intell. 126 (2023) 106712.