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# Distribution of quantum Fisher information in asymmetric cloning machines

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An unknown quantum state cannot be copied and broadcast freely due to the no-cloning theorem. Approximate cloning schemes have been proposed to achieve the optimal cloning characterized by the maximal fidelity between the original and its copies. Here, from the perspective of quantum Fisher information (QFI), we investigate the distribution of QFI in asymmetric cloning machines which produce two nonidentical copies. As one might expect, improving the QFI of one copy results in decreasing the QFI of the other copy. It is perhaps also unsurprising that asymmetric phase-covariant cloning outperforms universal cloning in distributing QFI since a priori information of the input state has been utilized. However, interesting results appear when we compare the distributabilities of fidelity (which quantifies the full information of quantum states), and QFI (which only captures the information of relevant parameters) in asymmetric cloning machines. Unlike the results of fidelity, where the distributability of symmetric cloning is always optimal for any  $d$ -dimensional cloning, we find that any asymmetric cloning outperforms symmetric cloning on the distribution of QFI for  $d \leq 18$ , whereas some but not all asymmetric cloning strategies could be worse than symmetric ones when  $d > 18$ .

Classical information can be replicated perfectly and broadcast without fundamental limitations. However, information encoded in quantum states is subject to several intrinsic restrictions of quantum mechanics, such as Heisenberg's uncertainty relations<sup>1</sup> and quantum no-cloning theorem<sup>2</sup>. The no-cloning theorem tells us that an unknown quantum state cannot be perfectly replicated because of the linearity of the time evolution in quantum physics, which is the essential prerequisite for the absolute security of quantum cryptography<sup>3</sup>. Nevertheless, it is still possible to clone a quantum state approximately, or instead, clone it perfectly with certain probability<sup>4,5</sup>. Therefore, various types of quantum cloning machines have been designed for different quantum information tasks, including universal quantum cloning machine (UQCM)<sup>6,7</sup>, state-dependent cloning machines<sup>8</sup> and phase-covariant quantum cloning machine (PQCM)<sup>9–12</sup>.

So far, the optimality of the approximate cloning machine is judged generally by whether the obtained fidelity between the cloning output state and the ideal state achieves its optimal bound. Although the fidelity may have qualified the complete information of the quantum states, in most scenarios, only the information of certain parameters which are physically encoded in quantum states is our practical concern. For example, the relative phase estimation is an extremely important issue in the field of quantum metrology<sup>13,14</sup>. Thus it is not necessary to gain complete information of the whole quantum states themselves, but rather the relevant parameter information. QFI is a natural candidate to quantify the physical information about the involved parameters<sup>15</sup>. In a recent work<sup>16</sup>, the authors pointed that the QFI of relevant parameter encoded in quantum states also cannot be cloned perfectly, while it might be broadcast even in some non-commuting quantum states. Furthermore, from the perspective of QFI, Song *et al.* showed that Wootters-Zurek cloning performs better than universal cloning for the symmetric cloning cases<sup>17</sup>. In our recent work, the multiple phase estimation problem was investigated in the framework of symmetric quantum cloning machines<sup>18</sup>.

On the other hand, we note that quantum cloning machines not only provide a good platform for investigating distribution of quantum information, but also have been proved to be very efficient eavesdropping attacks on the quantum key distribution (QKD) protocols<sup>19–22</sup>. In this context, asymmetric quantum cloning machines would be



of particular interest since the eavesdropper can adjust the trade-off between the information gained from a quantum communication channel and the error rate of information transmitted to the authorized receiver. Motivated by these considerations, we investigate the problem of distributing QFI in asymmetric quantum cloning machines for any dimensionality. We focus on the following four questions: (i) Is it possible to improve the QFI of one copy by decreasing that of the other copy? If YES, what's the trade-off relation between them? (ii) Does asymmetric PQCM always perform better than asymmetric UQCM on the capability of distributing QFI? (iii) Does asymmetric cloning always outperform symmetric cloning in distributing QFI for any dimensionality? (iv) What's the difference between fidelity and QFI on the characterization of distributability in asymmetric cloning? Except for the fourth question need to be clarified in detail, we can briefly answer the first two questions in the affirmative but the third in the negative. Our results shed an alternative light on quantum cloning and may be exploited for quantum phase estimation.

## Results

**Quantum Fisher information.** We start with a brief introduction of QFI and give a useful form of QFI for a special kind of mixed qudit states, which usually represents the output states of qudit cloning. Recall that QFI of parameter  $\theta$  encoded in  $d$ -dimensional quantum state  $\rho_\theta$  is generally defined as<sup>15,23,24</sup>

$$\mathcal{F}_\theta = \text{Tr}(\rho_\theta \mathcal{L}_\theta^2), \quad (1)$$

where  $\mathcal{L}_\theta$  is the so-called symmetric logarithmic derivative, which is defined by  $\partial_\theta \rho_\theta = (\mathcal{L}_\theta \rho_\theta + \rho_\theta \mathcal{L}_\theta)/2$  with  $\partial_\theta = \partial/\partial\theta$ . By diagonalizing the matrix as  $\rho_\theta = \sum_n \lambda_n |\psi_n\rangle \langle \psi_n|$ , one can rewrite the QFI as<sup>25,26</sup>

$$\mathcal{F}_\theta = \sum_n \frac{(\partial_\theta \lambda_n)^2}{\lambda_n} + \sum_n \lambda_n \mathcal{F}_{\theta,n} - \sum_{n \neq m} \frac{8\lambda_n \lambda_m}{\lambda_n + \lambda_m} |\langle \psi_n | \partial_\theta \psi_m \rangle|^2, \quad (2)$$

where  $\mathcal{F}_{\theta,n}$  is the QFI for pure state  $|\psi_n\rangle$  with the form

$$\mathcal{F}_{\theta,n} = 4 \left[ \langle \partial_\theta \psi_n | \partial_\theta \psi_n \rangle - |\langle \psi_n | \partial_\theta \psi_n \rangle|^2 \right]. \quad (3)$$

Note that equation (2) suggests the QFI of a non-full-rank state is only determined by the subset of  $\{|\psi_i\rangle\}$  with nonzero eigenvalues. Physically, the QFI can be divided into three parts<sup>26,27</sup>. The first term is just the classical Fisher information determined by the probability distribution; The second term is a weighted average over the QFI for all the nonzero eigenstates; The last term stemming from the mixture of pure states reduces the QFI and hence the estimation precision below the pure-state case. Though the equation (2) is powerful, there is no explicit expression for an arbitrary  $d$ -dimensional mixed state. However, it is worth noting that the output reduced states of UQCM and PQCM all have a form as

$$\rho_{out} = \eta |\psi\rangle \langle \psi| + \frac{1-\eta}{d} \mathbf{I}_d, \quad (4)$$

which is completely characterized by a parameter independent shrinking factor  $\eta$  and the dimensionality  $d$ . Here,  $|\psi\rangle$  and  $\mathbf{I}_d$  are the input state and  $d$ -dimensional identity matrix, respectively. Although  $\rho_{out}$  has such a simple form, an analytical expression of QFI is still difficult to achieve<sup>28</sup>. Fortunately, if we restrict our discussions to the special case of input states in the form

$$|\psi\rangle_{in} = \frac{1}{\sqrt{d}} \sum_k e^{i\theta_k} |k\rangle, \quad (5)$$

which are covariant with respect to rotations of the phases, a general form of QFI for any parameter  $\theta_k$  could be given by (see methods)

$$\mathcal{F}_{\theta_k} = \frac{4(d-1)\eta^2}{2d+d(d-2)\eta}. \quad (6)$$

The above equation is a key mathematical tool for our analysis of this paper. Although this expression only holds for the combination of equations (4) and (5), it is powerful since the scaling form of  $\rho_{out}$  is usually satisfied in quantum cloning machines or in the case of a pure state under white noises. On the other hand, the equatorial states are widely employed in the physical implementations of quantum communication ideas (such as BB84 protocol<sup>29</sup>) as well as in the demonstration of fundamental questions in quantum information processing. After a simple calculation, we find that  $\mathcal{F}_{\theta_k}$  is a monotonically increasing function of the shrinking factor  $\eta$ . This is to be expected because the larger  $\eta$  indicates more information the reduced output state  $\rho_{out}$  contains about the relevant parameter.

**Distributability.** Before moving to the discussion of distribution, a proper measure that quantify the distributability of cloning machines should be well defined. Note that one can define the distributability in different ways which depends on what is distributed in the procedure. For example, it can be defined from the perspectives of fidelity which quantifies the total information of the state, and QFI which quantifies the information of particular parameters in the state.

It is well known that, unlike the symmetric cloning in which all the copies are the same, the outputs of asymmetric cloning are nonidentical. Hence, the optimality of asymmetric cloning can be judged by maximizing the sum of all copies' fidelities, as discussed in references<sup>30,31</sup>. Intuitively, when we consider the distribution of quantum states, the distributability of asymmetric cloning can be defined as

$$F = \sum_i F_i, \quad (7)$$

where  $F_i$  is the fidelity between the original and the  $i$ th copy. Namely, the larger  $F$  indicates the better capability of distribution on the quantum states.

In a seminal work<sup>24</sup>, the authors pointed that both fidelity and QFI are highly related to the distinguishability of the states, which is measured by Bures distance<sup>32</sup>. Therefore, from the perspective of QFI, the measure

$$\mathcal{F} = \sum_i \mathcal{F}_{i,\theta}, \quad (8)$$

also qualifies the capability of distributing information of relevant parameter encoded in the input state, where  $\mathcal{F}_{i,\theta}$  denotes the QFI of parameter  $\theta$  in the  $i$ th copy. In the following discussions, we will use the definitions (7) and (8) to quantify the distributabilities of fidelity and QFI respectively in asymmetric cloning machines.

**QFI distribution for 2-dimensional cloning.** As we mentioned above, one is particularly interested in the asymmetric cloning machines which produce two copies with different qualities within the framework of quantum cryptography. Two typical asymmetric cloning machines are asymmetric UQCM<sup>30</sup>, which clones all input states equally well, and asymmetric PQCM<sup>9</sup>, which works equally well only for equatorial input states with the form of equation (5). We first discuss the 2-dimensional cloning by obtaining the analytical results, and then generalize it to  $d$ -dimensional cloning by the assistance of numerical simulations.

**Asymmetric 2-dimensional UQCM.** The  $1 \rightarrow 1 + 1$  optimal asymmetric UQCM was independently proposed by Niu and Griffiths<sup>33</sup>, Bužek *et al.*<sup>34</sup> and Cerf<sup>35</sup>. Though their formalisms are slightly different, the results are exactly the same. For the sake of convenience, we adopt the quantum circuit approach developed by Bužek. The transformation of asymmetric UQCM can be written in the following form



$$\begin{aligned} & |\psi\rangle_A (a|\Phi^+\rangle_{BR} + b|0\rangle_B |+\rangle_R) \\ & \rightarrow a|\psi\rangle_A |\Phi^+\rangle_{BR} + b|\psi\rangle_B |\Phi^+\rangle_{AR}, \end{aligned} \quad (9)$$

where  $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$  and  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . The parameters  $a$  and  $b$  are real, and satisfy the normalization condition  $a^2 + b^2 + ab = 1$ . With the transformation (9) in mind, it is now an easy exercise to verify that the two different copies of the original state  $|\psi\rangle = (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$  are

$$\rho_A = (1 - b^2)|\psi\rangle\langle\psi| + \frac{b^2}{2}I_2, \quad (10)$$

$$\rho_B = (1 - a^2)|\psi\rangle\langle\psi| + \frac{a^2}{2}I_2. \quad (11)$$

From the point of geometry, the asymmetric UQCM shrinks the original Bloch vector by two different shrinking factors ( $\eta_A = 1 - b^2$ ,  $\eta_B = 1 - a^2$ ) regardless of its direction. As special cases, we can see that if  $\eta_A = 1$ , then no information has been transferred from the original system, while, if  $\eta_B = 1$ , then all of the information in system A has been transferred to system B. In addition, if  $\eta_A = \eta_B = 2/3$  (i.e.,  $a = b = 1/\sqrt{3}$ ), then it reduces to the symmetric UQCM case. Obviously, the two shrinking factors  $\eta_A$  and  $\eta_B$  are related, and should satisfy the no-cloning inequality<sup>30,34</sup>

$$\eta_A^2 + \eta_B^2 + (1 - \eta_A)(1 - \eta_B) \leq 1, \quad (12)$$

which is an ellipse in the  $(\eta_A, \eta_B)$  space. An optimal asymmetric UQCM is characterized by a point  $(\eta_A, \eta_B)$  which lies on the boundary of this ellipse.

According to equation (6), when  $d = 2$ , the QFI of parameter  $\theta$  is proportional to  $\eta^2$ . Immediately, the QFIs are

$$\mathcal{F}_A = (1 - b^2)^2, \mathcal{F}_B = (1 - a^2)^2. \quad (13)$$

Here and henceforth we omit the subscript  $\theta_k$  for brevity since we restrict our discussions to the single-parameter scenario. Remarkably, a trade-off relation exists for the two QFIs: if one QFI is large, correspondingly another QFI will become small. Combining equations (12) and (13), the trade-off relation of QFI is expressed as

$$\mathcal{F}_A + \mathcal{F}_B + (\sqrt{\mathcal{F}_A} - 1)(\sqrt{\mathcal{F}_B} - 1) \leq 1. \quad (14)$$

This trade-off tells us that even we only concern cloning the information of a particular parameter encoded in quantum states, two close-

to-perfect copies cannot be achieved simultaneously, imposed by quantum mechanics. An intuitive presentation of this trade-off relation is shown in Fig. 2(b) (dashed line).

In the following, we consider the distribution of QFI in the asymmetric UQCM. As we defined in equation (8), the distributability of QFI is measured by

$$\mathcal{F}^{UQCM} = \mathcal{F}_A + \mathcal{F}_B. \quad (15)$$

Therefore, the larger  $\mathcal{F}$  is, the more QFI of the relevant parameter has been distributed to the two copies. We find that the asymmetric UQCM always performs better than symmetric UQCM in distributing QFI, which means

$$\mathcal{F}^{UQCM}(\eta_A \neq \eta_B) > \mathcal{F}^{UQCM}(\eta_A = \eta_B). \quad (16)$$

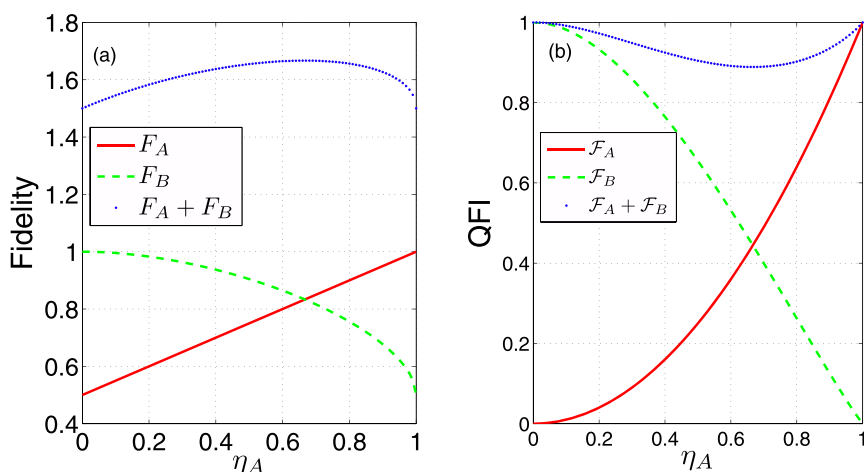
This can be proved by the method of Lagrange multiplier. One will find three extreme points:  $(\eta_A = 0, \eta_B = 1)$ ,  $(\eta_A = \eta_B = 2/3)$  and  $(\eta_A = 1, \eta_B = 0)$ . It is easy to verify that  $\mathcal{F}(\eta_A = \eta_B) = 8/9$  is the minimum value.

In order to show the difference of distributability between QFI and fidelity, we also write the corresponding fidelities defined as  $F_{A(B)} = \langle \psi | \rho_{A(B)} | \psi \rangle$

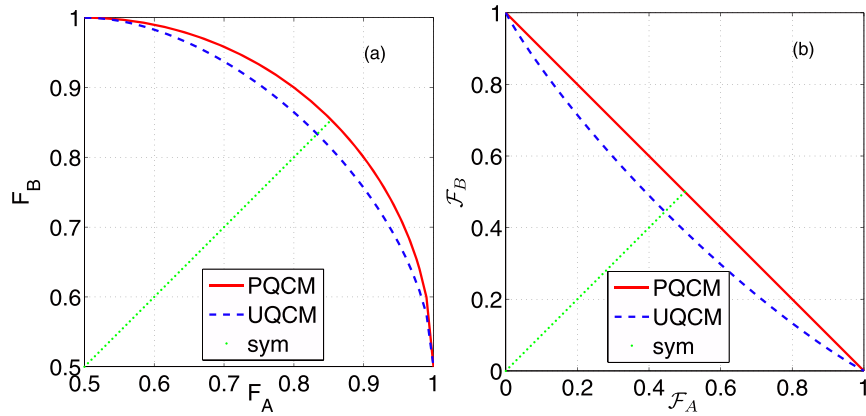
$$F_A = 1 - \frac{b^2}{2}, F_B = 1 - \frac{a^2}{2}. \quad (17)$$

According to equation (7), we adopt  $F = F_A + F_B$  to qualify the capability of asymmetric UQCM in distributing the entire quantum state. Figure 1 shows the results of asymmetric 2-dimensional UQCM. It is remarkable that, from the perspective of QFI, the asymmetric UQCM is always works better than symmetric UQCM, which is a sharp contrast to the result of fidelity<sup>31</sup>, where the symmetric UQCM is always optimal.

**Asymmetric 2-dimensional PQCM.** In the context of quantum cryptography<sup>3</sup>, the UQCM studied in the previous subsection might be optimal if the detail setup of QKD protocol is not specified. But it may not be optimal for the quantum states involved in a special QKD protocol. Practically, it is possible that we already know a priori information of the input states. Thus, a state-dependent cloning machine would perform better than UQCM. The best-known example of state-dependent cloning machine is the so-called PQCM. The symmetric PQCM was firstly proposed by Bruß *et al* for the equatorial qubit state<sup>10</sup> and then an asymmetric version was demonstrated by Niu and Griffiths<sup>9</sup>. Recently, the asymmetric PQCM has been experimentally realized using NMR<sup>36</sup> and fiber optics<sup>37</sup>.



**Figure 1** | Results of asymmetric 2-dimensional UQCM. (a) Fidelities as a function of shrinking factor  $\eta_A$  and (b) QFIs as a function of  $\eta_A$ .



**Figure 2** | Trade-off relations for 2-dimensional asymmetric UQCM and PQCM. (a) Fidelity (b) QFI. The dotted lines denote the symmetric cases.

Previous studies suggest that the equatorial qubit state PQCM can be realized by both economic<sup>9</sup> and non-economic<sup>10,38</sup> transformations. One can check that both economic and non-economic methods achieve the same distributability of QFI. In the text we discuss the non-economic case as it can be directly generalized to  $d$ -dimensional PQCM. The transformation of asymmetric PQCM can be written in the following form

$$|i\rangle_A |0\rangle_B |\Sigma\rangle_R \rightarrow c|i\rangle_A |i\rangle_B |\Sigma_i\rangle_R + (a|i\rangle_A |j\rangle_B + b|j\rangle_A |i\rangle_B) |\Sigma_j\rangle_R, \quad (18)$$

with  $i, j = 0, 1$  and  $i \neq j$ .  $a^2 + b^2 + c^2 = 1$  is the normalization condition.  $|\Sigma_i\rangle_R$  is a set of orthogonal normalized ancillary state. For the equatorial qubit-state  $|\psi\rangle = (|0\rangle + e^{i\theta}|1\rangle) / \sqrt{2}$ , the output states have the form as

$$\rho_A = 2ac|\psi\rangle\langle\psi| + \frac{1-2ac}{2} I_2, \quad (19)$$

$$\rho_B = 2bc|\psi\rangle\langle\psi| + \frac{1-2bc}{2} I_2. \quad (20)$$

Then, the shrinking factors are  $\eta_A = 2ac$ ,  $\eta_B = 2bc$ . Using the normalization condition, the shrinking factors can be simplified as

$$\eta_A = 2a\sqrt{1-a^2-b^2}, \quad (21)$$

$$\eta_B = 2b\sqrt{1-a^2-b^2}. \quad (22)$$

As is seen, in the scenario of asymmetric PQCM, there are two free parameters to be optimized. Therefore, an optimal asymmetric PQCM is defined as the following: if we fix the quality of one copy, then the other copy is optimal with the highest quality. From the equations (21) and (22), one can eliminate  $b$  and obtain the trade-off relation between  $\eta_A$  and  $\eta_B$

$$\eta_B = \frac{\eta_A}{a} \sqrt{1 - \frac{\eta_A^2}{4a^2} - a^2}. \quad (23)$$

Assuming  $\eta_A$  is constant, the optimal value of  $a_{\text{optimal}} = \eta_A / \sqrt{2}$  can be found, and the optimal trade-off relation reduces to

$$\eta_A^2 + \eta_B^2 = 1. \quad (24)$$

The corresponding QFIs are

$$\mathcal{F}_A = \eta_A^2, \quad \mathcal{F}_B = \eta_B^2 \quad (25)$$

In the symmetric case, we have  $\eta_A = \eta_B = 1/\sqrt{2}$ . Remarkably, according to equation (14), one can immediately find the inequality

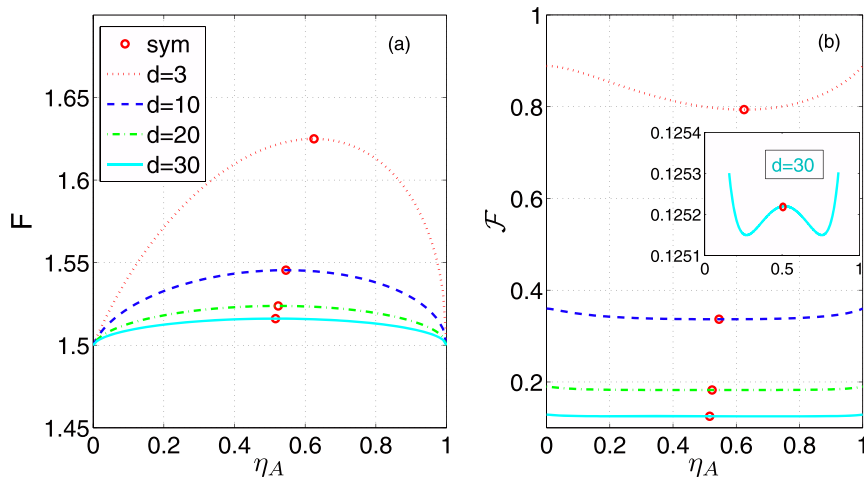
$$\mathcal{F}^{\text{PQCM}} = \mathcal{F}_A + \mathcal{F}_B = 1 \geq \mathcal{F}^{\text{UQCM}}. \quad (26)$$

The meanings of above equation are twofold. On one hand, it indicates that asymmetric PQCM performs better than asymmetric UQCM in distributing QFI by virtue of the known information. This result is essentially in agreement with that of fidelity. To be clear, we plot the trade-off relations between A's fidelity (QFI) and B's fidelity (QFI) for both asymmetric UQCM and PQCM in Fig. 2. It is evident that the lines of asymmetric PQCM are always above those of asymmetric UQCM, except for the start points and end points. On the other hand, it should be noted that, for the 2-dimensional PQCM, the asymmetric case is as good as the symmetric case on the capability of distributing QFI, while the later always performs better than the former with the measure of fidelity. The reason is that the fidelity  $F_B$  as a function of  $F_A$  is strictly concave, but the QFI  $\mathcal{F}_B$  as a function of  $\mathcal{F}_A$  is convex. This results in the sum of two fidelities and QFIs achieving its maximal and minimal value, respectively, in the symmetric case.

**QFI distribution for  $d$ -dimensional cloning.** Until now, we have restricted our discussions to the 2-dimensional cloning. Although all quantum information tasks can be performed by using only two-level systems, it has been recently recognized that higher-dimensional quantum states (i.e., qudits) can offer significant advantages for improving the security of quantum cryptographic protocols<sup>39</sup>, achieving higher information-density coding<sup>40,41</sup> and reducing the required resources for quantum computation and simulation<sup>42,43</sup>.

Based on these considerations, it would be essential to extend above discussions to  $d$ -dimensional cloning. One may think the results will be trivial and analogous conclusions will be obtained as well as the 2-dimensional cloning. However, as we will show below, there are some similarities between them, but more importantly, significant differences will appear with increasing dimensionality  $d$ .

**Asymmetric  $d$ -dimensional UQCM.** The optimal asymmetric  $d$ -dimensional UQCM was proposed by Cerf<sup>30</sup> and Braunstein *et al.*<sup>44</sup>. For a  $d$ -dimensional quantum system, the corresponding asymmetric UQCM can be generalized directly from the transformation (9) with  $|\Phi^+\rangle$  and  $|+\rangle_R$  instead defined in higher-dimension,  $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle$ , and  $|+\rangle_R = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_R$  respectively. Hence, the normalization condition now reads  $a^2 + b^2 + 2ab/d = 1$ . The output reduced density matrices are written in the form of equation (4)



**Figure 3** | Results of asymmetric  $d$ -dimensional UQCM. (a) Fidelity and (b) QFI as a function of  $\eta_A$  for  $d = 3, 10, 20, 30$ , respectively from top to down. The red circles denote the corresponding results of symmetric UQCM, and the insertion is magnified plot of  $d = 30$ .

$$\rho_A = (1 - b^2)|\psi\rangle\langle\psi| + \frac{b^2}{d}I_d, \quad (27)$$

$$\rho_B = (1 - a^2)|\psi\rangle\langle\psi| + \frac{a^2}{d}I_d, \quad (28)$$

with shrinking factors  $\eta_A = 1 - b^2$  and  $\eta_B = 1 - a^2$ . In particular, if  $a^2 = b^2 = d/(2d + 2)$ , we recover the results of symmetric UQCM. Similarly, we can obtain a trade-off relation between two shrinking factors  $\eta_A$  and  $\eta_B$ .

$$\eta_A^2 + \eta_B^2 + \frac{2d^2 - 4}{d^2}(1 - \eta_A)(1 - \eta_B) \leq 1, \quad (29)$$

which corresponds to a set of ellipses in the space of shrinking factors that their eccentricities vary with dimensionality. It should be noted that in the infinite dimensional case, the corresponding ellipse shrinks to the line  $\eta_A + \eta_B = 1$ .

Now we turn to the calculation of QFI. Assuming the input state is a  $d$ -dimensional equatorial state, then the QFIs of output states (27) and (28) are obtained directly by (6).

$$\mathcal{F}_A = \frac{4(d-1)(1-b^2)^2}{2d + d(d-2)(1-b^2)}, \quad (30)$$

$$\mathcal{F}_B = \frac{4(d-1)(1-a^2)^2}{2d + d(d-2)(1-a^2)}. \quad (31)$$

The tradeoff relation between  $\mathcal{F}_A$  and  $\mathcal{F}_B$  can be derived by substituting

$$\eta_X = \frac{(d^2 - 2d)\mathcal{F}_X + \sqrt{(d^2 - 2d)^2\mathcal{F}_X^2 + 32(d^2 - d)\mathcal{F}_X}}{8(d-1)} \quad (32)$$

into (29) with  $X = A, B$ , but it is too complicated to present in the text. Nevertheless, there is no doubt that one cannot gain, at the same time, two copies whose QFIs are above values allowed by the trade-off relation.

We are concerned with whether the  $d$ -dimensional asymmetric UQCM still performs better than symmetric UQCM in distributing QFI. Against all expectations, the results become subtle with increasing  $d$ . Unlike the 2-dimensional case where asymmetric UQCM is always better than symmetric UQCM, the  $d$ -dimensional asymmetric UQCM may be worse than symmetric UQCM under certain

conditions. As shown in Fig. 3(a), we find the sum of two fidelities still reaches its largest value at the point of  $\eta_A = \eta_B = (d + 2)/(2d + 2)$ , which means, by the measure of fidelity, the symmetric UQCM will optimally copy the state regardless of the dimension<sup>31</sup>. However, Figure 3(b) shows that the distributability of QFI achieves its smallest value at the point of  $\eta_A = \eta_B$  when  $d = 3, 10$ , while it is interesting to note that the point of  $\eta_A = \eta_B$  becomes a local maximum point when  $d = 30$ . Namely, the asymmetric UQCM is no longer always better than symmetric UQCM with increasing  $d$ . The reason is that even though QFI is a monotonically function of the shrinking factor, it is not a linear function of it. This sophisticated relation between  $\mathcal{F}$  and  $\eta$  reveals above interesting results.

Naturally, we start wondering when the asymmetric UQCM may become worse than symmetric UQCM. The numerical simulation shows that when  $d \leq 18$ , the  $\eta_A$  of global minimal  $\mathcal{F}$  is equal to the symmetric case. While a bifurcation appears at  $d = 18$ , which means  $\eta_A = (d + 2)/(2d + 2)$  is no longer the global point of minimal  $\mathcal{F}$ , as shown in Fig. 4. Mathematically, we can understand the bifurcation as follows:  $\mathcal{F}$  as a function of  $\eta_A$  has three extreme points which are physically allowed when  $d \leq 18$ , and  $\eta_A = \eta_B$  is the point of global minima. When  $d > 18$ , it has five extreme points, as seen from the insertion in Fig. 3b. Moreover,  $\eta_A = \eta_B$  is no longer the global minima but a local maximum point. Thus, the symmetric UQCM may outperform the asymmetric case. However, it is hard to understand why the critical point is  $d = 18$  in physical, we conjecture this critical point is related to the Hilbert space structure of qudits. We leave this as an open question and the further study is underway.

**Asymmetric  $d$ -dimensional PQCM.** The generalization of asymmetric PQCM to  $d$ -dimensional is much more difficult, and in particular, it is too complicated to present an analytical trade-off relation between two copies. However, with the aid of numerical simulations, we can confirm two main results about the distribution of QFI in asymmetric  $d$ -dimensional PQCM: (i) PQCM gains an advantage over UQCM by utilizing the priori information, and (ii) a sudden change of the point of minimum also exists in asymmetric PQCM with increasing  $d$ .

The cloning transformation of asymmetric  $d$ -dimensional PQCM can be introduced<sup>45</sup>

$$|i\rangle_A |0\rangle_B |\Sigma\rangle_R \rightarrow c|i\rangle_A |i\rangle_B |\Sigma_i\rangle_R + \left( a \sum_{i \neq j} |i\rangle_A |j\rangle_B + b \sum_{i \neq j} |j\rangle_A |i\rangle_B \right) |\Sigma_j\rangle_R, \quad (33)$$

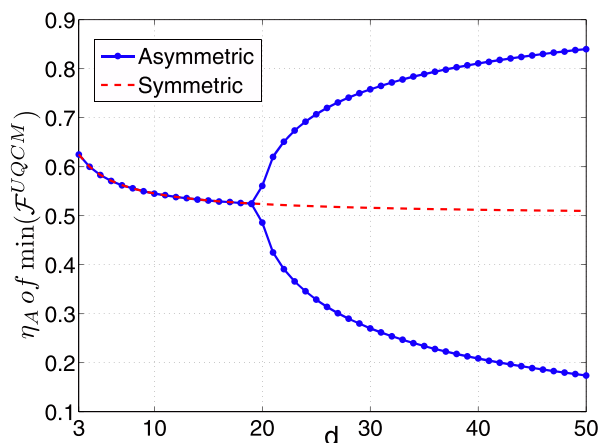


Figure 4 |  $\eta_A$  of minimal  $\mathcal{F}^{UQCM}$  as a function of dimensionality  $d$ .

with the normalization condition  $(d - 1)(a^2 + b^2) + c^2 = 1$ . Given the input state in the form of (5), then, the shrinking factors of the output copies read as

$$\eta_A = (d - 2)a^2 + 2a\sqrt{1 - (d - 1)(a^2 + b^2)}, \quad (34)$$

$$\eta_B = (d - 2)b^2 + 2b\sqrt{1 - (d - 1)(a^2 + b^2)}, \quad (35)$$

where we have use the normalization condition to eliminate the parameter  $c$ . Similar to the 2-dimensional case, here we again need to optimize two free tuning parameters. Therefore, in the same way, an optimal asymmetric PQCM is defined by optimizing  $\eta_B$  as large as possible when  $\eta_A$  is fixed, and vice versa. By eliminating the parameter  $b$ , we can obtain the trade-off relation (not the optimal one)

$$\eta_B = \frac{d - 2}{d - 1} \left[ 1 - (d - 1)a^2 - \left( \frac{\eta_A - (d - 2)a^2}{2a} \right)^2 \right] + \frac{\eta_A - (d - 2)a^2}{a} \sqrt{\frac{1 - \left( \frac{\eta_A - (d - 2)a^2}{2a} \right)^2 - a^2}{d - 1}}. \quad (36)$$

The optimal trade-off relation need to be further optimized by choosing a proper value of  $a_{optimal}$  to make the largest  $\eta_B$ . Unfortunately, there is not a closed analytical form of  $a_{optimal}$  for any dimensionality  $d$ . However, by simple numerical simulations, we find that asymmetric PQCM indeed always performs better than asymmetric UQCM in distributing QFI as shown in

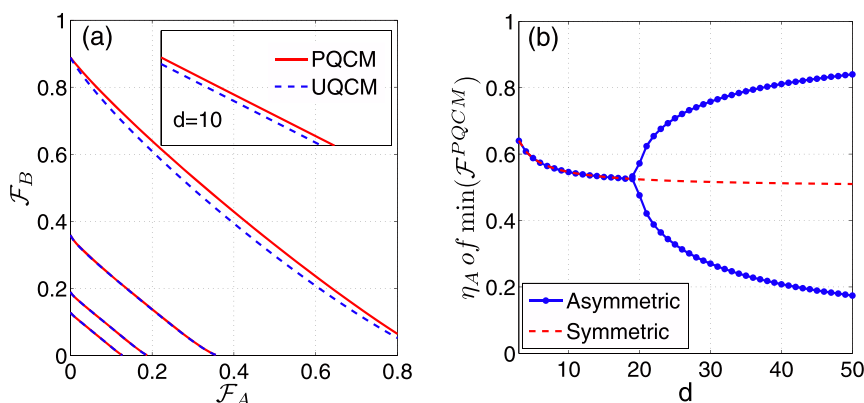


Figure 5 | Results of asymmetric  $d$ -dimensional PQCM. (a) Trade-off relations between  $\mathcal{F}_A$  and  $\mathcal{F}_B$  for  $d$ -dimensional asymmetric UQCM (dashed lines) and PQCM (solid lines) with  $d = 3, 10, 20, 30$ , respectively from top to bottom. The insertion is magnified plot of  $d = 10$ . Note that when  $d = 20$  and  $30$ , the two lines overlap so greatly that cannot be resolved. (b)  $\eta_A$  of minimal  $\mathcal{F}^{PQCM}$  as a function of dimensionality  $d$ .

Fig. 5(a). When the dimensionality  $d$  is large (e.g.,  $d = 20$ ), it should be noted that the advantage of PQCM over UQCM almost disappears. Moreover, similar to the case of asymmetric  $d$ -dimensional UQCM, the asymmetric  $d$ -dimensional PQCM is not always better than the symmetric case for any dimensionality  $d$ . Figure 5(b) shows that a bifurcation of the point of global minimum also occurs at  $d = 18$ . This phenomena stresses that the critical point appearing at  $d = 18$  is not in any sense accidental. The physical reason behind this is worth further study.

### Discussions

In this paper, we have investigated the distribution of QFI in asymmetric cloning machines which produce two nonidentical copies. In particular, we have elucidated four questions as we mentioned before. Here, we summarize our results by replying these questions. (i) The answer is YES. It is definite that improving the QFI in one copy results in decreasing the QFI of the other copy, and the trade-off relation can be obtained analytically except for the asymmetric  $d$ -dimensional PQCM. (ii) The answer is also YES. Thanks to a priori knowledge of the input states, PQCM always performs better than UQCM in distributing QFI. (iii) The answer is not so straightforward. It should be divided into two categories: for 2-dimensional cloning, asymmetric cloning always outperforms symmetric cloning on the distribution of QFI; While for the  $d$ -dimensional cloning case, the above conclusion only holds when  $d \leq 18$  and becomes invalid when  $d > 18$ , i.e., the asymmetric cloning is not always better than symmetric cloning for any dimensionality. (iv) The most significant difference between fidelity and QFI is that fidelity is a linear function of the shrinking factor while QFI is nonlinear. This leads to the counterintuitive result that symmetric cloning is always optimal from the perspective of fidelity, but asymmetric cloning usually works better than symmetric cloning on the distribution of QFI, except for some particular situations (e.g., when  $d = 30$  and  $\eta_A = 0.25$ , see the insertion in Fig. 3b).

In view of these findings, we note that there are some problems in need of further clarifications. The first important issue is to understand why does the critical point appear at  $d = 18$ , not other numbers. Secondly, we should realize that we have confined our discussion to the distributability of single parameter in asymmetric cloning machines. However, from both theoretical and practical points of view, it seems to be interesting to examine the problem of multi-parameter distribution in asymmetric quantum cloning. Intuitively, there would be a trade-off relation of the quantum Fisher information matrices between two nonidentical copies. These would be very intriguing topics that need further studies.



## Methods

Here, we give the details of the derivation of equation (6) from equations (4) and (5). To be clear, we can rewrite the equation (4) as

$$\rho = \frac{(d-1)\eta+1}{d} |\psi\rangle\langle\psi| + \frac{1-\eta}{d} (I_d - |\psi\rangle\langle\psi|). \quad (37)$$

Note that the eigenvalues of  $\rho$  consists of only two categories:  $\lambda_0 = [(d-1)\eta+1]/d$ , and  $\lambda_n = (1-\eta)/d$  with  $1 \leq n \leq d-1$ . Obviously,  $|\psi_0\rangle = |\psi\rangle$  is an eigenstate of  $\rho$ . Thus the problem is converted to construct a complete orthogonal set (containing  $d-1$  bases) of the operator  $\hat{\Pi} = I - |\psi\rangle\langle\psi|$  which is also orthogonal to  $|\psi\rangle$  at the same time. The procedure can be divided into three steps: (i) finding  $d-1$  bases of  $\hat{\Pi}$  which are orthogonal to  $|\psi\rangle$ ; (ii) using the Gram-Schmidt procedure to orthogonalize them and (iii) the normalization.

*Step (i)* Intuitively, the  $d-1$  bases which are orthogonal to  $|\psi\rangle$  can be written as

$$|\varphi_n\rangle = \frac{1}{\sqrt{2}} \left[ -e^{-i(\theta_n - \theta_0)}, 0, \dots, \underbrace{1}_{\text{nth}}, \dots, 0 \right] \quad (38)$$

where  $1 \leq n \leq d-1$ .

*Step (ii)* According to the procedure of Gram-Schmidt orthonormalization, one can construct a set of orthogonal but un-normalized bases:

$$|\tilde{\psi}_n\rangle = |\varphi_n\rangle - \frac{1}{n} \sum_{m=1}^{n-1} e^{i(\theta_m - \theta_n)} |\varphi_j\rangle. \quad (39)$$

*Step (iii)* Notice that  $\langle\tilde{\psi}_n|\tilde{\psi}_n\rangle = (n+1)/2n$ , we finally obtain the  $d-1$  orthogonal and normalized bases of the operator  $\hat{\Pi}$

$$|\psi_n\rangle = \sqrt{\frac{2n}{n+1}} \left[ |\varphi_n\rangle - \frac{1}{n} \sum_{m=1}^{n-1} e^{i(\theta_m - \theta_n)} |\varphi_j\rangle \right], \quad (40)$$

which are also orthogonal to  $|\psi\rangle$ .

By this time, we have diagonalized the state (4) in the bases  $\{|\psi\rangle, |\psi_1\rangle, \dots, |\psi_n\rangle\}$ , and then the QFI can be calculated by equation (2). Note that all the parameters  $\theta_k$  is equally weighted due to the symmetry of  $|\psi\rangle$ , thus the QFI of any parameter is the same. Furthermore, we observe that the part of classical Fisher information vanishes since the probability distribution is independent of parameters  $\theta_k$ , if measured in this bases. The remaining work is to determine the last two terms in equation (2). After lots of complicated but straightforward calculations, we obtain the QFI of any parameter  $\theta_k$

$$\begin{aligned} \mathcal{F}_{\theta_k} &= \sum_n \lambda_n \mathcal{F}_{\theta_n} - \sum_{n \neq m} \frac{8\lambda_n \lambda_m}{\lambda_n + \lambda_m} |\langle\psi_n|\partial_{\theta_k}\psi_m\rangle|^2, \\ &= \frac{4(d-1)\eta^2}{2d+d(d-2)\eta}. \end{aligned} \quad (41)$$

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## Author contributions

X.X. and Y.Y. proposed the idea and carried out the calculations under the guidance of X.G.W. L.M.Z. made some numerical simulations and plotted the figures. X.X. wrote the paper. All authors discussed the results and commented on the manuscript.



## Additional information

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