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Research article

A formal analysis of inconsistent decisions in intertemporal choice through subjective time perception

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A R T I C L E I N F O

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A B S T R A C T

The **framework** of this paper is subjective time perception in the context of intertemporal choice, that is to say, the process of making decisions on dated outcomes (monetary or not) by an individual or a group of individuals. In this setting, the Discounted Utility model and, more specifically, the exponential discounting have been the paradigmatic **methodology** used to measure the preferences on delayed outcomes. However, this model can only be applied to consistent choices in which individuals do not change their preferences when the involved rewards are delayed the same time interval. Unfortunately, this is not the case of several decision scenarios where time is viewed as a subjective variable. The **objective** of this paper is to formally analyze the consistency of intertemporal choices governed by a discount function, derived from the exponential, where time has been distorted according to certain psychological traits of the subjects involved in the decision-making. More specifically, the different types of decreasing impatience will be characterized by focusing on the distortion derived from the subjective perspective of time. The **findings** of this research are very relevant in order to explain the time-related behavior of decision-makers in some noteworthy fields such as finance, psychology, marketing or sociology.

1. Introduction

Intertemporal choice refers to the process whereby an individual or a group of individuals (e.g., an organization) have to select one among a set of dated outcomes. The simplest case is when the subject has to choose one of two dated rewards, and the most complex situation is when the individual must decide about his/her preferred sequence of rewards [1,2]. Typically, a problem in intertemporal choice considers an immediate or short-term advantage and a long-term disadvantage in a specific choice option, or vice versa. Because the consequences of a decision are not only rewards (in the traditional sense), this choice does not necessarily have to involve money or some material or economic consequence, but it may refer to non-monetary decisions such as health-related outcomes, substance abuse and others [3,4].

From an economic point of view, intertemporal choice may be analyzed by using either preferences or a discount function able to value the different offered options. The bridge between both perspectives can be found in [5]: If order, monotonicity, continuity,

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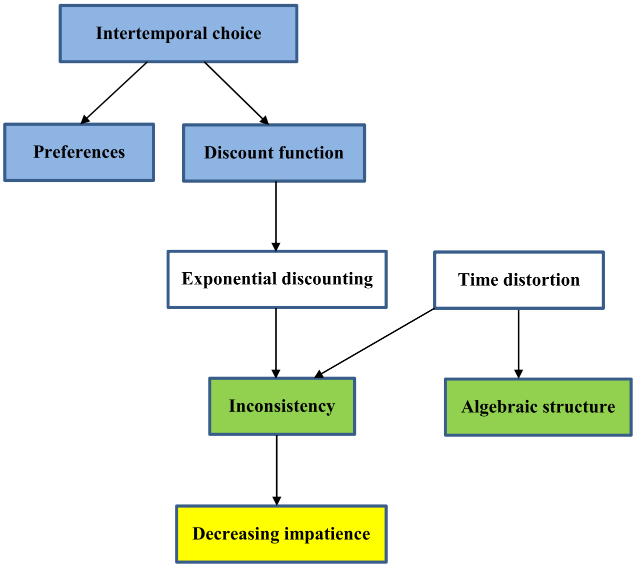

Fig. 1. Structure of the paper.

impatience,[1] and separability[2] hold, and the set of rewards X is an interval, then there are continuous real-valued functions u on X and F on the time interval T such that

$$(x,s) \succeq (y,t) \text{ if, and only if, } u(x)F(s) \geq u(y)F(t). \tag{1}$$

Additionally, $u(0)=0$ and u is increasing, whilst F is decreasing and positive. Focusing on function F, Samuelson [7] proposed the exponential discounting ($F(t)=\exp\{-kt\}$, $k>0$) as the criterion to value dated outcomes in his well-known Discounted Utility (DU) model. This discount function is characterized by a constant discount rate:

$$\delta(t)=k, \tag{2}$$

which leads to consistency in the decisions governed by this financial pattern. However, this model (stationary and consistent) is not able to explain several paradoxes or anomalies shown in intertemporal choice (see, e.g., [6]: delay effect, interval effect, magnitude effect, direction effect, sign effect, sequence effect, date-delay effect and frame effect) and, consequently, this justifies the need to take into account other decision models.

In this paper, we will use these two perspectives when relating consistent (resp. inconsistent) preferences with additive (resp. non-additive) discount functions. However, we will focus on inconsistent preferences [8] and, more specifically, on those decisions guided by a criterion of strongly or moderately decreasing impatience. Thus, the main objective of this paper is to characterize the aforementioned types of decreasing impatience by using discount functions which result from distorting time in the exponential model.

This paper is structured as follows (see Fig. 1). Section 2 introduces the mathematical concept of time distortion and the implications of distorting time in the exponential discount function. The algebraic structure of the family distortions is presented in Section 3. Section 4 analyzes the different possibilities to model both consistent and inconsistent preferences as conditions equivalent, among others, to additivity and non-additivity, respectively, of the underlying discount function. Focusing on stationary discount functions, Section 5 deals with the concept of decreasing impatience and its main modalities: the so-called strongly and moderately decreasing impatience. In this way, stationary discount functions are presented as the exponential discounting where time has been previously distorted. Finally, Section 6 discusses the results obtained in this research, whilst Section 7 summarizes and concludes.

2. Distorting time in intertemporal choice

In the context of intertemporal choice, the *impatience* is given by the instantaneous discount rate of the discount function which describes the preferences of individuals:

[1] Impatience or pure time preference means that "a given amount of utility is preferred the earlier it arrives". More specifically, that "someone who currently expects to experience equal utility at two future times, will want to increase the earlier utility by one unit, in exchange for a decrease in the later utility of more than one unit" [6]. Mathematically, $(x,s) \succ (x,t)$, provided that $s<t$ and $x>0$.

[2] Separability means that the preference of a dated outcome, $F(x,t)$, can be represented as a separable discount utility, $F(x,t)=u(x)F(t)$, where money is evaluated with the increasing function u (utility function) and time is projected with the decreasing function F (unitary discount function).

$$\delta(t) := -\frac{\mathrm{d}\ln F(t)}{\mathrm{d}t}, \tag{3}$$

that is to say, by the derivative of the natural logarithm of the discount function. On the other hand, *inconsistency*, that is to say, varying impatience, can be explained starting from the different perception of calendar time by people. The relative perception of time could be mathematically described by the concept of distortion. Therefore, the objective of this paper is to explain certain variations of impatience (inconsistency) by distorting the variable "time" in the expression of the well-known exponential discounting because, in the same way as [9], "we assume that the "true" internal discounting process is exponential". At this point, we have to clarify that this is a model assumption rather than a firm theory about choice or human behavior.

The manuscript places itself in the Perceived-time-based tradition. In effect, taking into account the particular case of a separable discount function $F(x,t) := u(x)F(t)$ (as derived in Section 1), consumers' decision could be biased either by considering the objective time interval (t) and assigning different weights to outcomes (*Perceived-Value–Based Accounts*), or by substituting the objective time by subjective time and keeping invariable the utility function (*Perceived-Time–Based Accounts*) [9–11]. Observe that this last approach is equivalent to assume that people consider different discount rates. For example, when analyzing the psychological determinants of hyperbolic discounting:

$$F(t) = \frac{1}{1+kt},\ k > 0, \tag{4}$$

the Perceived-Value–Based Accounts focus on "why individuals discount the value of outcomes *per se* at a different rate". Observe that this approach is consistent with the so-called *magnitude effect*, i.e., the discount rate is higher for smaller amounts.

However, this paper will be centered on the Perceived-Time–Based Accounts, that is to say, the effect of anticipatory time (derived from experienced time) in the specific expression of $F(t)$. According to Block [12], psychological time can be considered either as succession, as duration or as temporal perspective. In this paper, we will consider time as duration. In this way, Fraisse [13,14], cited by [12], proposed that "direct time judgments [are] founded immediately on the changes we experience and later on the changes we remember" (p. 234). Observe that this approach gives rise to the so-called *inconsistency* in intertemporal choice and, in particular, to the well-known *delay effect*, i.e., the discount rate is larger the longer the delay.[3]

Cajueiro [15] and Takahashi [16] distorted time in the hyperbolic discount function, giving rise to the so-called q-exponential discounting. Several analyses in the field of econophysics have shown the relationship between the roles of psychophysical effects of time perception and the anomalies in intertemporal choice. On the other hand, Jin [17] introduced a discussion of the current influence of time perception on intertemporal choice by exploring different representations. In particular, Lu and Li [18] studied the psychophysics presented in the consumer's preferences. Recent studies by [19] used Tsallis' statistics-based econophysics to show that the q-exponential discount function may continuously parameterize a subject's consistency in intertemporal choice. This result was generalized by [20] to any discount function, based on the deformed algebra developed in the Tsallis' nonextensive thermostatistics. Later, Cruz Rambaud and Ventre [21] and Cruz Rambaud et al. [22] distorted time by means of the Stevens' "power" law in a subadditive discount function in order to obtain inverse-S discounting curves. In this context, Webb [23] provided a novel model to study the inverse-S discounting behavior. Indeed, the analysis of time with delay functions will help us to better understand those mechanisms of intertemporal choice centered on time such as distorting time or several types of decreasing impatience.

The general principle of mapping a physical dimension (of which duration would be an instance) onto an internal scale through a law which can be expressed as an equation, particularly non-linear ones, is studied since the beginnings of psychophysics, notably in the field of psychophysical scaling. In order to generalize this approach, the following definition provides a general description of the concept of time distortion (for a summary of this psychological concept, see [24] who show behavioral data and their modeling thereof). In this way, the term "distortion" will be used in a very general sense, also encompassing a perfectly linear or even proportional mapping of the objective into the subjective realm as a special case.

Definition 1. A time distortion is a continuous real-valued function, $g(t)$, defined in an interval $[0, t_0)$ (t_0 can be $+\infty$), satisfying the following conditions:

1. $g(0) = 0$.
2. $g(t)$ is strictly increasing.

An alternative choice might have been to present the definition of what should happen at 0 as a limit:

$$\lim_{t\to 0} g(t) = 0$$

and keep t strictly positive rather than including 0 in the domain. The main time distortions in the existing literature on the topic of intertemporal choice are the following functions:

- The so-called Weber-Fechner law [25], defined as $g(t) = \alpha \ln(1 + \beta t)$, where $\alpha > 0$ and $\beta > 0$. In this case, $g'(t) = \frac{\alpha\beta}{1+\beta t}$, from which $g'(0) = \alpha\beta$. The idea of showing the tangent at 0 is to indicate whether the function is below or above the $g_0(t) = t$ reference

[3] In any case, we can analyze the joint perceived value of individuals by considering a general, stationary discount function $F(x,t)$ o, even a dynamic discounting model $F(x,d,t)$. See Section 3.

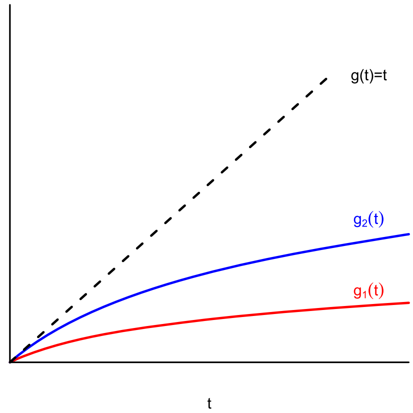

Fig. 2. Weber-Fechner law ($\alpha\beta \le 1$).

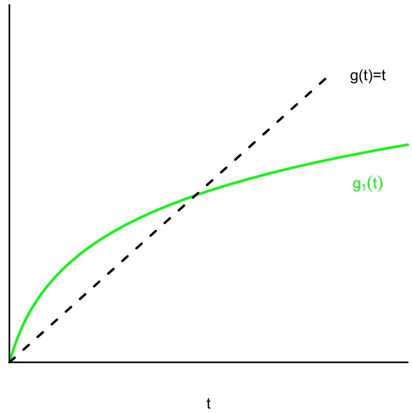

Fig. 3. Weber-Fechner law ($\alpha\beta > 1$).

in order to examine whether there is a range $0 < t < t_0$ in which $g(t) < t$) and another in $t > t_0$ in which $g(t) > t$, and determine the value of t_0. Therefore, this time distortion can describe either a situation in which the perceived time is always below the calendar time and the gap between them is increasing more than proportionally (Fig. 2[4]), or the same former situation but with an initial interval where the perceived time is higher (see Fig. 3).

- The well-known Stevens' "power" law [26], defined as $g(t) = \alpha t^{\beta}$, where $\alpha > 0$ and $\beta > 0$. In this case, $g'(t) = \alpha\beta t^{\beta-1}$, from which

$$g'(0) = \begin{cases} 0, & \text{if} \quad \beta > 1 \\ \alpha, & \text{if} \quad \beta = 1 \\ \infty, & \text{if} \quad 0 < \beta < 1 \end{cases} \tag{5}$$

The internal representation of duration, being mental, has no real unit. Only when this internal representation is mapped back to the physical world, in timing tasks, numerical estimation, reproduction, comparison, etc., of durations, can a measurement be made. Therefore, the comparison between the graphs of function $g(t)$ versus a $g_0(t) = t$ diagonal line does not make much sense as $g(t)$ and $g_0(t)$ are not commensurable. What does make sense is the comparison of the behavior of $g(t)$ versus a linear or, more accurately, a proportional model, in which time passed on the internal magnitude scale is proportional to duration in seconds, hours, months, or whatever unit is being used. However and for the sake of simplicity, when comparing $g(t)$ and αt, for a certain $\alpha > 0$, we assume that a previous, suitable change of variable allows comparing $g(t)$ and t.
Therefore, the above-defined time distortion can describe either a situation in which the perceived time is always below (resp. under) the calendar time and the gap between them is proportionally increasing (Fig. 5, resp. Fig. 6) or the same former situation but with an initial interval where the perceived time is lower (resp. higher) and the gap existing between both times is increasing more than proportionally (Fig. 4, resp. Fig. 7).

- Other time distortions such as $g(t) = \alpha t^2 + \beta t$, where $\alpha > 0$ and $\beta > 0$. In this case, $g'(t) = 2\alpha t + \beta$, from which $g'(0) = \beta$. Therefore, this time distortion can describe either a situation in which the perceived time is always under the calendar time and the gap

[4] In what follows, the ordinate of each graph will represent the values of the functions displayed in such graphs. For the sake of simplicity, these labels have been omitted.

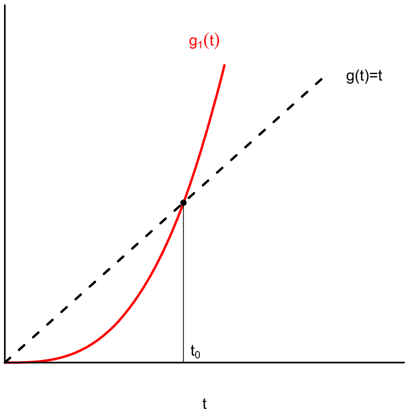

Fig. 4. Stevens' power law ($\beta > 1$).

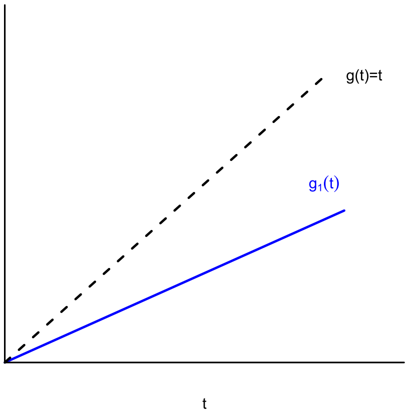

Fig. 5. Stevens' power law ($\beta = 1$, $0 < \alpha < 1$).

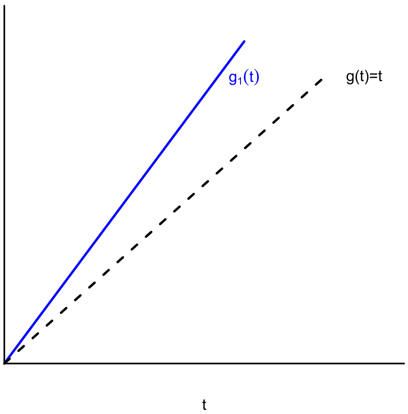

Fig. 6. Stevens' power law ($\beta = 1$, $\alpha > 1$).

between them is increasing more than proportionally (Fig. 8), or the same former situation but with an initial interval where the perceived time is lower (Fig. 9).

As indicated, in this paper, starting from the exponential discount function, defined by $F(t) = \exp\{-kt\}$, with $k > 0$, we are going to distort time by means of a given function $g(t)$ such that the intertemporal choice is now described by $G(t) := \exp\{-kg(t)\}$, with $k > 0$ [39]. Easy calculations show that the instantaneous discount rate of G and its derivative are given by (see Equation (2)):

$$\delta_G(t) = kg'(t) \tag{6}$$

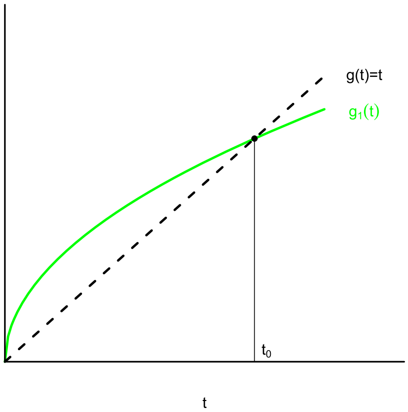

Fig. 7. Stevens' power law ($0 < \beta < 1$).

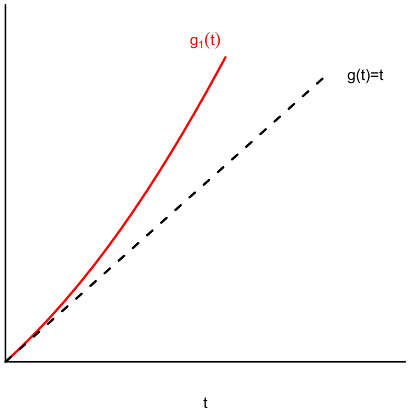

Fig. 8. $g(t) = \alpha t^2 + \beta t$ ($\beta \geq 1$).

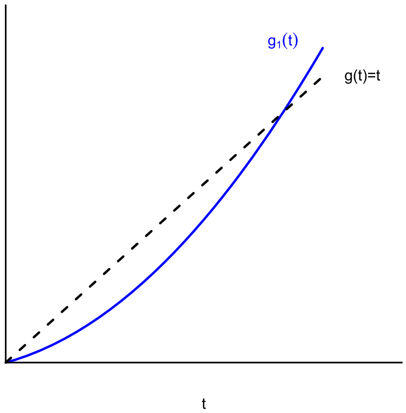

Fig. 9. $g(t) = \alpha t^2 + \beta t$ ($0 < \beta < 1$).

and

$$\delta'_G(t) = kg''(t). \tag{7}$$

With respect to the new discount function $G(t)$, the decision-maker perceives time *above the calendar time* (resp. *under the calendar time*) if $g(x) > x$ (resp. $g(x) < x$). Additionally, if $g(x)$ is convex (resp. concave), then the intertemporal choice is *inconsistent* and shows *increasing* (resp. *decreasing*) impatience. Specifically,

- In Fig. 4, the intertemporal choice exhibits increasing impatience, and time is perceived under (resp. above) the calendar time from 0 to t_0 (resp. from t_0 to ∞).
- In Fig. 5, the intertemporal choice exhibits constant impatience, and time is always perceived under the calendar time.
- In Fig. 6, the intertemporal choice exhibits constant impatience, and time is always perceived above the calendar time.
- In Fig. 7, the intertemporal exhibits decreasing impatience, and time is perceived above (resp. under) the calendar time from 0 to t_0 (resp. from t_0 to ∞).

3. Algebraic structure of the set of distortions

The set of all possible time distortions, denoted by $\mathcal{D}$, with respect to the composition of functions ($\circ$), that is to say, the couple $(\mathcal{D}, \circ)$, is a monoid since, given two arbitrary time distortions g_1 and g_2, the composition $g_1 \circ g_2$ is a new time distortion, and the identity function (no distortion of calendar time) is the null element. The algebraic structure of monoid is very easy and intuitive since it can be viewed as the process of building words by the single juxtaposition of letters being the empty word the null element.

Based on this basic idea, the composition of two time distortions has the following psychological interpretation. The result of distorting time by an individual may be the final output of distorting time due to several reasons (personal traits, addictions, etc.) [27] which, depending on the specific moment, arise in a certain order like the juxtaposition of letters which compose a word. For example, in case of a poly-addiction (multiple addictions at once), if the individual is smoker (addiction #1) and drugs addict (addiction #2), when smoking is the primary need, he/she will distort time according to certain functions g_1 and g_2 (applied in this order). However, if the primary necessity is drugs consumption, he/she will distort time according to g_2 and g_1 (in this order). This will depend on what drug is available, what is affordable or what is easier to get for the person who is interested in.

Moreover, the degree of time distortion could be modulated by taking suitable multiples of the distortions describing subjective time, giving rise to the family of distortions $g_\alpha := \alpha(g_1 \circ g_2) + (1-\alpha)(g_2 \circ g_1)$, where $0 \leq \alpha \leq 1$. In general, given n traits or addictions $\#1, \#2, \ldots, \#n$, the potential distortions may be described by the following alpha-weighted summation over S_n:

$$g_\alpha := \sum_{(i_1,i_2,\ldots,i_n)\in S_n} \alpha_{(i_1,i_2,\ldots,i_n)} (g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_n}), \tag{8}$$

where $(i_1, i_2, \ldots, i_n)$ denotes any permutation of $(1, 2, \ldots, n)$, S_n is the group of permutation of the n first positive integers, and

$$\sum_{(i_1,i_2,\ldots,i_n)\in S_n} \alpha_{(i_1,i_2,\ldots,i_n)} = 1.$$

The following paragraphs will provide a mathematical interpretation of this construction. To do this, we first need the concept of derivation relative to a function [28]. Let $v(t, h)$ be a real function of two variables t and h such that, for every s:

$$\lim_{h\to 0} v(s, h) = s.$$

Let $f(t)$ be a real function differentiable at $t = s$. The *derivative of f relative to v, at $t = s$*, denoted by $D_v(f)(s)$, is defined as the following limit:

$$D_v(f)(s) := \lim_{h\to 0} \frac{f[v(s,h)] - f(s)}{h}. \tag{9}$$

In the particular case in which $s = g(t)$ and $v(s, h) = g(t) + g'(t)h$, where $g(t)$ is a differentiable real function, one has:

$$D_g(f)(s) := \lim_{h\to 0} \frac{f[g(t) + g'(t)h] - f[g(t)]}{h} = f'[g(t)]g'(t). \tag{10}$$

Taking into account that $v(s, h) \approx g(t + h)$, the former derivative is obviously $(f \circ g)'(t)$. Summarizing,

$$D_g(f) = (f \circ g)'. \tag{11}$$

This view of the derivative of function composition is very important because $D_g(f)$ means the variation of f in the direction provided by g, that is to say, g'. Thus, going back to the object of this paper, the dominant trait or addiction (#1) distorts time in the direction pointed by the secondary trait or addiction (#2), that is to say, $(g_1 \circ g_2)'$. Obviously, this is a local approach because, at a given instant, the addiction #1 could be dominant in a beginning but this situation could change in favor of #2, in which case we will be interested in the composition $(g_2 \circ g_1)'$ (observe here that the function composition is not commutative). In other words, the poly-addict will jump back-and-forth from one drug to the next.

According to Monterosso and Ainslie [29], the reason whereby an addiction occurs in the first place is still unclear. On the other hand, addicts show an undeniable inability to escape from frequent alternation of contradictory preferences about their addictive activity. Moreover, Roelofsma and Read [30] claim that "decision makers appear to have as many discount rates as choice situations into which they can be placed". In this context, the relative importance of each cause will determine the specific deformations and the order of their composition (recall again that the composition of functions is not commutative). Summarizing, if an individual is male and drugs addict, he will deform time according to $g_1 \circ g_2$ or $g_2 \circ g_1$, depending on the relative importance of causes 1 and 2 for this individual. Rachlin [31] and Xu et al. [32] used a parameter s to represent the subjective time in the following discounting expression (see also [33]):

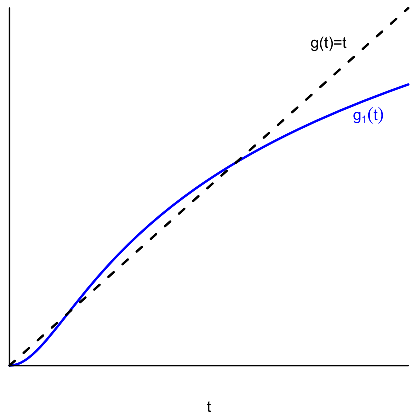

Fig. 10. Composition Stevens with Weber-Fechner.

$$V = \frac{R}{1 + kt^s},\ k > 0, \tag{12}$$

where V is the subjective value of a reward R, available at a delay t. There are other generalizations of the hyperbolic discounting function. In effect, the one proposed by Loewenstein and Prelec [34], which exponentiates the entire denominator, has many properties in common with the expression in (12), especially when considering the limiting cases. One important difference between Loewenstein and Prelec's and Rachlin's discount functions is that the first one is the result of distorting time in the exponential model by means of the Weber-Fechner law, whilst the second one results from plugging a "power" law into hyperbolic discounting. In this case, at $s = 1$, then $t^s = t$ and subjective time coincides with calendar time; for $0 < s < 1$, t^s is concave and the subjective time marginally decreases as t increases; finally, for $s > 1$, t^s is convex and subjective time positively accelerates as t increases.

Observe that, moreover, this discount function is the result of considering the composition of a Weber-Fechner and a Stephens' "power" law, as subjective times, in the exponential discounting.

In general, the composition of two time distortions g_1 and g_2 can give rise to a new distortion with more "alternances" in its convexity and its relative position with respect to calendar time, specifically when one of the composed distortions is convex and the other is concave (or reciprocally). This is not the case when the two distortions are convex (or concave) because the composition of two increasing, convex (or concave) function is convex (resp. concave).[5]

Obviously, if the composition changes its convexity, this is because there exists at least a t_0 (inflection point) such that $(g_1 \circ g_2)''(t_0) = 0$.

Example 1. The composition of the Weber-Fechner law, $g_1(t) = \alpha \ln(1 + \beta t)$, and the Stevens' "power" law, $g_2(t) = \gamma t^\delta$, results in:

- $(g_1 \circ g_2)''(t) = \alpha\beta\gamma\delta t^{\delta-2} \frac{\delta-1-\beta\delta t^\delta}{(1+\beta\gamma t^\delta)^2}$. Therefore, a possible value t_0 must satisfy $\delta - 1 - \beta\gamma t^\delta = 0$, from which:
  - If $0 < \delta < 1$, then the composite function is strictly concave and the equation $\delta - 1 = \beta\gamma t^\delta$ which has no solution.
  - If $\delta > 1$, then $t_0 = \left(\frac{\delta-1}{\beta\gamma}\right)^{1/\delta}$.
- $(g_2 \circ g_1)''(t) = \alpha\beta^2\gamma\delta \ln^{\delta-2}(1 + \beta t) \frac{\delta-1-\ln(1+\beta t)}{(1+\beta t)^2}$. Therefore, a possible value t_0 must satisfy $\delta - 1 - \ln(1 + \beta t) = 0$, from which:
  - If $0 < \delta < 1$, then $\delta - 1 = \ln(1 + \beta t)$ which has no solution.
  - If $\delta > 1$, then $t_0 = \frac{\exp\{\delta-1\}-1}{\beta}$.

More specifically, the composition of the Weber-Fechner law, where $\alpha = 2$ and $\beta = 0.4$, with the Stevens' "power" law, with $\gamma = 1$ and $\delta = 2$, in the two orders, gives the time distortions shown in Figs. 10 ($t_0 = 1.5811$) and 11 ($t_0 = 4.2957$).

In general, given the relevance of points at which the second derivative vanishes, we can enunciate Theorem 1.

Theorem 1. *Let $P(t) = -\frac{(\ln F)''(t)}{(\ln F)'(t)}$ be the Prelec's index corresponding to F [35] and $F_i := \exp\{-kg_i(t)\}$, $k > 0$, where g_i $(i = 1, 2)$ is a time distortion. The following conditions are equivalent:*

(i) $(g_1 \circ g_2)''(t_0) = 0$.
(ii) $P_1(g_2(t_0))g_2'(t_0) = -P_2(t_0)$.

---

[5] In effect, given two deformations of time, g_1 and g_2, one has:

- $(g_1 \circ g_2)'(t) = g_1'[g_2(t)]g_2'(t)$.
- $(g_1 \circ g_2)''(t) = g_1''[g_2(t)][g_2'(t)]^2 + g_1'[g_2(t)]g_2''(t)$.

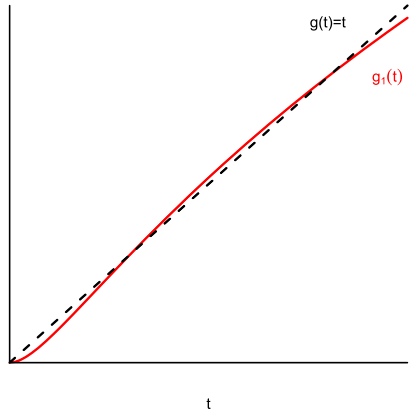

Fig. 11. Composition Weber-Fechner with Stevens.

(iii) $D(D_{g_2} g_1)(t_0) = 0$*, where* D *is the usual derivative and* $D_{g_2} g_1$ *is the derivative of* g_1 *according to* g_2*.*

Proof. (i) ⇔ (ii). In effect, by the formula of the second derivative of a composition of two function, one has:

$$(g_1 \circ g_2)''(t) = g_1''[g_2(t)][g_2'(t)]^2 + g_1'[g_2(t)]g_2''(t).$$

Therefore, $(g_1 \circ g_2)''(t_0) = 0$ is equivalent to:

$$-\frac{g_1''[g_2(t_0)]}{g_1'[g_2(t_0)]} = \frac{g_2''(t_0)}{[g_2'(t_0)]^2},$$

or

$$P_1(g_2(t_0))g_2'(t_0) = -P_2(t_0).$$

(i) ⇔ (iii). It is immediate by taking into account that:

$$D(D_{g_2} g_1)(t_0) = D[(g_1 \circ g_2)'] = (g_1 \circ g_2)''.$$

This concludes the proof. □

4. Consistency, transitivity and additivity in intertemporal choice

In this section, we are going to more accurately present some concepts and properties which characterize the intertemporal choice by using both preferences and discount functions. In effect, let $X = [0, +\infty)$ be the set of all non-negative rewards, $D \subseteq \mathbb{R}$ the set of potential benchmarks (instants at which the decision is made) and $T = [0, +\infty]$ the set of all possible time intervals. Observe that we are considering a dynamic context in which the preference depends on the benchmark. Therefore, Definition 2 provides the most general setting in which the preferences of an individual (governed by his/her subjective perception of time) depend on the moment at which the decision is made.

Definition 2. A positive and continuous real-valued function

$$F : X \times D \times T \to X$$

such that

$$(x, d, t) \mapsto F(x, d, t)$$

is said to be a *dynamic discount function* if F is strictly increasing with respect to x, strictly decreasing with respect to t, and satisfies

1. $F(0, d, t) = 0$, for every $d \in D$ and every $t \in T$.
2. $F(x, d, 0) = x$, for every $x \in X$ and every $d \in D$.

$F(x, d, t)$ represents the amount equivalent at d of \$$x$ available at $d + t$.

Definition 3 is the stationary version of Definition 2, because the decision-making does not depend on the point of reference.

Definition 3. A positive and continuous real-valued function

$$F : X \times T \to X$$

such that

$$(x,t) \mapsto F(x,t)$$

is said to be a *stationary discount function* if F is strictly increasing with respect to x, strictly decreasing with respect to t, and satisfies

1. $F(0,t) = 0$, for every $t \in T$.
2. $F(x,0) = x$, for every $x \in X$.

$F(x,t)$ represents the amount equivalent at 0 of \$$x$ available at t.

Finally, Definition 4 sets the structure of the discount function to be employed in this paper, where time and amount behave as separated variables in the expression of the discount function.

Definition 4. A stationary (resp. dynamic) discount function $F(x,t)$ (resp. $F(x,d,t)$) is said to be *separable* if $F(x,t) = u(x)F(t)$ (resp. $F(x,d,t) = u(x)F(d,t)$), where u is a utility function, F (resp. $F(d,t)$) is strictly decreasing (resp. strictly decreasing with respect to t), and $F(0) = 1$ (resp. $F(d,0) = 1$, for every $d \in D$).

From now on, we will consider only separable, stationary (or dynamic) discount functions. In this case, we will refer to $F(t)$ (or $F(d,t)$) as the *unitary discount function*. However, before continuing with the presentation of this paper, we need the following definition.

Definition 5. An intertemporal choice with associated indifference relation $\sim$ is said to be *transitive* if, for every $x,y,z \in X$ $(x \le y \le z)$ and every $s,t,r \in T$ $(s \le t \le r)$, $(x,s) \sim (y,t)$ and $(y,t) \sim (z,r)$ implies $(x,s) \sim (z,r)$.

Basically, in the valuation process of an intertemporal choice, we can use either a fix point of reference or a variable benchmark. In the first case, the involved discount function is stationary whilst, in the second case, the discount function will be dynamic.

4.1. Case of a fix point of reference

In this case, if the fix point of reference is p, the intertemporal choice is valued with a given stationary discount function F_p, defined on the interval $[p,+\infty)$. Therefore, if $p \le s \le t \le r$, necessarily we must use the forward discount factors, derived from F_p. Thus, $(x,s) \sim_p (y,t)$ implies

$$\frac{u(x)}{u(y)} = \frac{F_p(t)}{F_p(s)}, \tag{13}$$

whilst $(y,t) \sim_p (z,r)$ implies

$$\frac{u(y)}{u(z)} = \frac{F_p(r)}{F_p(t)}. \tag{14}$$

By multiplying both sides of equations (13) and (14), one has:

$$\frac{u(x)}{u(z)} = \frac{F_p(r)}{F_p(s)} \tag{15}$$

from where $(x,s) \sim_p (z,r)$ and so the intertemporal choice is transitive.

4.2. Case of a variable benchmark

In this case, the intertemporal choice is valued with a dynamic discount function F, defined on $D \times T$. Thus, $(x,s) \sim (y,t)$ implies

$$u(x) = u(y)F(s,t-s), \tag{16}$$

whilst $(y,t) \sim (z,r)$ implies

$$u(y) = u(z)F(t,r-t). \tag{17}$$

By combining both equations (16) and (17), one has:

$$u(x) = u(z)F(s,t-s)F(t,r-t). \tag{18}$$

Definition 6 provides a condition equivalent to transitivity by using a discount function instead of preferences (see Proposition 1).

Definition 6. A dynamic discount function F is said to be *additive* if, for every $p \in D$ and every $a, b \in T$, the following equation holds:

$$F(p,a)F(p+a,b) = F(p,a+b),$$

provided that $p + a \in D$.

Obviously, one has the following statement.

Proposition 1. *An intertemporal choice with the indifference relation $\sim$ associated to the discount function F is transitive if, and only if, F is additive.*

Proof. It is obvious by taking $p = s$, $a = t - s$ and $b = r - t$, from which $a + b = r - s$. □

Now, we will try to relate all these concepts. Obviously, a dynamic discount function F gives rise to a family $\{F_p\}_{p\in D}$ of stationary discount functions, defined by $F_p(\cdot) = F(p,\cdot)$. Therefore, we can state Theorem 2 which means a bridge connecting the concepts of additivity, transitivity and consistency.

Theorem 2. *For a dynamic discount function, F, the following conditions are equivalent:*

(i) F is additive.
(ii) The intertemporal choice is transitive with respect to the indifference relation associated to F.
(iii) The intertemporal choice is transitive with respect to the indifference relation associated to F_p, for every $p \in D$.

Proof. (i) ⇒ (ii). It is obvious by taking into account Proposition 1.

(ii) ⇒ (iii). In effect, take into account that now $\frac{u(x)}{u(z)} = \frac{F_p(r)}{F_p(s)}$ is equivalent to $\frac{u(x)}{u(z)} = F(s,r)$.

(iii) ⇒ (i). In this case, $\frac{F_p(r)}{F_p(s)} = \frac{F_q(r)}{F_q(s)}$.

This concludes the proof of this theorem. □

The statements included in the former theorem can be particularized in the following corollary.

Corollary 1. *For a stationary discount function, F, the following conditions are equivalent:*

(i) F is additive.
(ii) The intertemporal choice is transitive with respect to the indifference relation associated to F.
(iii) The intertemporal choice with the indifference relation associated to F is consistent.
(iv) F is the exponential discount function.

As formerly indicated, from now on, we will deal with discount functions derived from the exponential discounting where time has been distorted by a given function $g(t)$, that is to say, $G(t) := \exp\{-kg(t)\}$. Alternatively, we could start from any stationary or dynamic discount function and distort its variable "time" with a given discount rate. However, this general case will not be considered here. By Corollary 1, it is obvious that the intertemporal choice governed by $G(t)$ is consistent if, and only if, $g(t) = t$, that is to say, the distorted time coincides with the calendar time.[6] Therefore, in the next section, we will consider those discount functions $G(t)$, with $g(t) \neq t$, which, consequently, describe inconsistent intertemporal choices. However, for the sake of simplicity, we will only consider the case of inconsistency consisting in decreasing impatience.

5. An analysis of decreasing impatience with time distortions

Definition 7 describes the main situation of intransitive (inconsistent) intertemporal choice (see, e.g., [33]).

Definition 7. A decision-maker exhibiting preferences $\preceq$ has *decreasing impatience* if, for every $s < t$, $k > 0$ and $0 < x < y$, $(x,s) \sim (y,t)$ implies $(x, s+k) \preceq (y, t+k)$.

In order to introduce the concepts of strongly and moderately decreasing impatience [36], we recall the following lemma.

[6] To the extent that distorted time is a construct to be considered at a mental/subjective/internal representational scale, an expression $g(t) = at$ $(a > 0)$ is more reasonable than $g(t) = t$ to represent "calendar time" because it allows for change of unit (to be attributed to the multiplicative constant). However, as formerly indicated, we will make a suitable change of variable in order to consider $g(t) = t$ as the calendar time.

Lemma 1 *([37]). A decision-making based on preferences $\succeq$ exhibits decreasing impatience if, and only if, for every (x,s) and (y,t) such that $(x,s)\sim(y,t)$ and $\tau>0$, there exists $\sigma=\sigma(x,y,s,t,\tau)$ $(0<\sigma<\tau)$ such that $(x,s+\sigma)\sim(y,t+\tau)$.*

Definitions 8 and 9, and Propositions 2 and 3 display the main types of decreasing impatience [36] and their characterizations.

Definition 8. A decision-maker exhibiting decreasing impatience has *strongly decreasing impatience* if $s\tau>t\sigma$.

The elasticity $\epsilon(t)$ of a discount function $F(t)$ is defined as [34]:

$$\epsilon(t):=-t\delta(t),$$

where $\delta(t)$ is the discount rate of $F(t)$. With this definition, we can state the following proposition.

Proposition 2. *A decision-maker exhibiting preferences $\succeq$ has strongly decreasing impatience if, and only if, the elasticity $\epsilon(t):=-t\delta(t)$ is increasing.*

Definition 9. A decision-maker exhibiting decreasing impatience has *moderately decreasing impatience* if $s\tau<t\sigma$.

Proposition 3. *A decision-maker exhibiting preferences $\succeq$ and decreasing impatience has moderately decreasing impatience if, and only if, $\epsilon(t)$ is decreasing.*

Now, we are going to characterize strongly decreasing impatience in order to facilitate generating subjective discount functions satisfying this property. In effect, let $F(t)=\exp\{-kt\}$, with $k>0$, the exponential discount function. It is well known that the instantaneous discount rate of F is constant and is given by $\delta_F(z)=k$. As indicated, our aim is to distort time with a function $g(t)$ such that the intertemporal choice described by $G(t):=\exp\{-kg(t)\}$ exhibits decreasing impatience, in which case:

$$\delta_G(z)=kg'(z)$$

and

$$\delta'_G(z)=kg''(z).$$

Therefore, it can be claimed that the intertemporal choice exhibits decreasing (resp. increasing) impatience if, and only if, $g(z)$ is increasing and concave (resp. convex). Moreover, if the decreasing impatience has to be strong, then, by Proposition 2, the elasticity of $G(t)$ must be increasing, that is to say,

$$\epsilon'_G(z)=-\delta_G(z)-z\delta'_G(z)\geq 0$$

or, equivalently,

$$\delta_G(z)+z\delta'_G(z)\leq 0.$$

By substituting $\delta_G(z)$ and $\delta'_G(z)$, one has:

$$g'(z)+zg''(z)\leq 0,$$

that is to say,

$$\frac{1}{z}\leq-\frac{g''(z)}{g'(z)}.$$

Taking into account that the two sides of the former inequality are positive, we can integrate them, between 1 and t, maintaining the same inequality:

$$\int_1^t\frac{1}{z}\mathrm{d}z\leq-\int_1^t\frac{g''(z)}{g'(z)}\mathrm{d}z,$$

which leads to the natural logarithm of z and $g'(z)$ in the left-hand and the right-hand side of the former inequality, respectively. Simple calculation results in:

$$g'(t)\leq\frac{g'(1)}{t}.$$

Once again, the two sides of the former inequality are positive whereby we can integrate between 1 and t, leading to:

$$g(t)\leq g(1)+g'(1)\ln t.$$

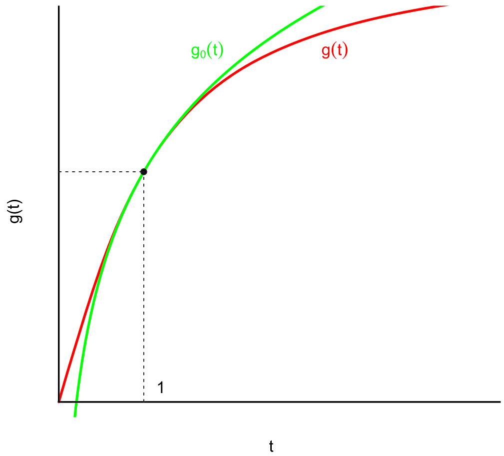

Fig. 12. Plotting the functions involved in Example 2.

The reasoning for $0 < t < 1$ is analogous but, in this interval, it is necessary to integrate twice between t and 1, leading to the same former inequality. Summarizing, we can claim the following statement which characterizes the time distortion of the exponential discounting when $F(t)$ is required to show strongly decreasing impatience.

Theorem 3. *A discount function $F(t)$ describes an intertemporal choice exhibiting strongly decreasing impatience if, and only if,*

$$g(t) \leq g(1) + g'(1) \ln t,$$

for every $t \in T$.

The discount function presented in Example 2 shows strongly decreasing impatience (see [37]). Observe that, in effect, it meets the criterion (necessary and sufficient condition) displayed in Theorem 3.

Example 2. The discount function $F(t) := \exp\{-\arctan t\}$ describes an intertemporal choice showing strongly decreasing impatience in the interval $[1, +\infty[$ (see Fig. 12, where $g_0(t) := g(1) + g'(1) \ln t$).

Analogously, we can enunciate the following theorem.

Theorem 4. *A discount function $F(t)$ describes an intertemporal choice exhibiting moderately decreasing impatience if, and only if,*

$$g(t) \geq g(1) + g'(1) \ln t,$$

for every $t \in T$.

Proof. The proof is analogous to that of Theorem 3 by changing the sign $\leq$ to $\geq$. □

The discount function presented in Example 3 shows moderately decreasing impatience (see [37]). Observe that, in effect, it meets the criterion (necessary and sufficient condition) displayed in Theorem 4.

Example 3. The discount function $F(t) := \frac{1}{1+kt}$, $k > 0$, describes an intertemporal choice showing moderately decreasing impatience (see Fig. 13, where again $g_0(t) := g(1) + g'(1) \ln t$).

6. Discussion

In the same way as our study, dos Santos and Martinez [38] understood inconsistency as the result of the time perception effect. These scholars, by using the special theory of relativity, address inconsistency as the result of a subjective time dilation and focus their study on a transformation of the time interval between the availability of two rewards. Also, Kim and Zauberman [9] considered the role of time perception in temporal discounting and demonstrate that human perception of anticipatory time (i.e., prospective duration of future time intervals which individuals have not experienced) is also nonlinearly scaled. More specifically, they found that participants' degree of hyperbolic discounting is positively associated with the level of contraction and negatively associated with diminishing sensitivity. A similar result was reached by Bradford et al. [39] who found two behavioral parameters affecting future

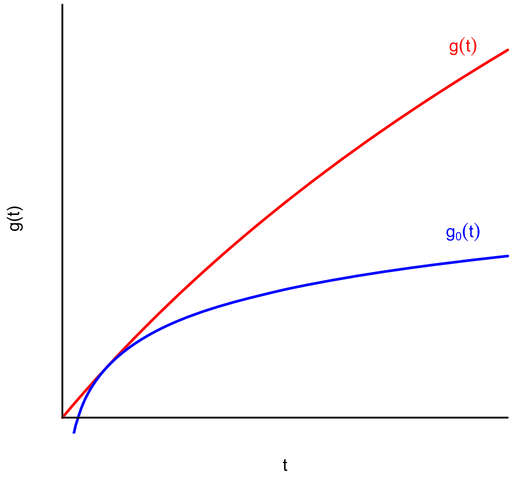

Fig. 13. Plotting the functions involved in Example 3.

events. The first is some component of hyperbolic discounting and the second factor is that non-constant discounting may also be a reflection of subjective time perception. Agostino et al. [40] showed that, on average, individuals perceive long-range time intervals not in a biased manner, but rather in a linear pattern. However, any deviation from exponential discounting in intertemporal choice to the compressed nature of subjective time must consider subjective time on an individual-participant basis.

A similar model was proposed by Thomas and Brown [41] and Thomas and Weaver [42], given by the following functional equation:

$$\tau(I) = \lambda f(t, I) + (1 - \lambda)g^{\star}(I),$$

where

- τ is the perceived duration of an interval containing certain information (I);
- $f(t, I)$ is a temporal information processor, or timer; and
- $g^{\star}(I)$ is non-temporal information processor.

The model by Thomas and Brown is also weighted and it is a distorted time approach, by using the terms of this manuscript, but it is not compound, and $g^{\star}$ is not even a function of duration.

Thus, the perceived duration is monotonically related to the weighted average of the amount of information encoded by the former two processors which function in parallel. In effect, according to Vasile [43], time perception could be analyzed in terms of neural mechanisms and networks, cognitive functions, consciousness (which implies cognition), and age. However, closely related to cognitions, emotions play an important role. On the other hand, Michon and Jackson [44] proposed that the principal attributes which determine the temporal information are the simultaneity and order of events. Lashley [45] argued the existence of "the logical and orderly arrangement of thought and action", that is to say, the "immediate switching in the nervous mechanism, without explicit consideration of what is already going on within the system".

7. Conclusion

This paper has dealt with those discount functions obtained by distorting the calendar time in the exponential discounting. The relevance of these discount functions derives from the widespread use of the exponential model which is the main reference of the Discounted Utility model, introduced by Samuelson [7]. Additionally, many individuals or groups of individuals perceive time differently from calendar time, making decisions in which the preferences are ruled by another discount function.

This way of obtaining discount functions starting from the exponential model has a noteworthy application in psychology because some personal traits, such as gender, ethnic, religion, health, etc., and the consumption of abuse substances, such as drugs, alcohol or tobacco, affect the perception of time and this leads to variations of impulsivity (impatience) and, consequently, to inconsistency in intertemporal choices.

The main contribution of this paper is the characterization of the time distortion which lead to moderately or strongly decreasing impatience as the main particular cases of inconsistency. The present manuscript is a theoretical work. Therefore, in a further research, we will administer a questionnaire to different groups of respondents in order to obtain an experimental time distortion $\hat{g}(t)$ with which derive an empirical discount function $\hat{F}(t) = \exp\{-k\hat{g}(t)\}$. In other words, the function fitting the data derived from this experiment will allow obtaining the underlying discount function to be used in the valuation of preferences. Thus, we will be capable of choosing between two dated rewards in the context of intertemporal choice.

CRediT authorship contribution statement

Salvador Cruz Rambaud: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Javier Sánchez García:** Writing – review & editing, Writing – original draft, Validation, Supervision, Formal analysis.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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