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Multipartite entanglement criterion via generalized local uncertainty relations

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We study the detection of multipartite entanglement based on the generalized local uncertainty relations. A sufficient criterion for the entanglement of four-partite quantum systems is presented in terms of the local uncertainty relations. Detailed examples are given to illustrate the advantages of our criterion. The approach is generalized to general multipartite entanglement cases.

Quantum entanglement is a remarkable feature in quantum physics[1] and has attracted much attention in recent years. Entangled states are recognized as the essential resources in quantum information processing, with many experimental realizations[2,3] and applications in such as quantum algorithms[4], quantum teleportation[4,5], quantum cryptography[6]. Recently, it was shown that quantum entanglement is tightly connected to wave-particle duality, and it can create a wave-particle entangled state of two photons[7]. Detecting entanglement of multipartite systems is a fundamental problem in the theory of quantum entanglement. Separability criteria to determine whether a given state is separable or not are of crucial importance[8]. Enormous efforts have been dedicated to solve the separability problems[9–37]. Nevertheless, the characterization and quantification of multipartite entanglement are less understood than that of bipartite case, as multipartite states can be entangled in more different ways.

There have been many efficient entanglement criteria such as local uncertainty relations (LUR)[11,12], covariance matrix criterion (CMC)[13], computable cross-norm or realignment criterion (CCNR)[14], permutation separability criteria[15], criterion based on Bloch representations[17,18], entanglement witnesses[21], Bell-type inequalities criteria[22], and criterion based on quantum Fisher information[23]. Generally, these criteria are only necessary condition for separable states and have different advantages in detect different entanglements.

The LUR criterion, the symmetric CMC criterion and the realignment criterion are usually considered as complementary to the the positive partial transposition criterion. The main advantage of LUR criterion is that it allows us to detect the entanglement of quantum states without having to fully understand them, and it can detect bound entangled states more effectively.

Recently, based on the local sum uncertainty relations, some entanglement criteria have been proposed for both discrete and continuous variable bipartite systems and three-qubit systems[31–33]. Zhang et al. proposed a tighter form of the original LUR criterion to improve the range of entanglement detection[31], Akbari-Kourbolagh and Azhdargalam generalized the LUR criterion to the tripartite systems[33].

This paper is structured as follows. We start by introducing the entanglement criterion based on LUR for tripartite systems and generalize the entanglement criterion to four-partite quantum systems. Some detail examples are then given to illustrate the advantages of the criterion. Then, the entanglement criterion for N-partite systems ($N > 4$) is discussed. Brief discussion and summary are given at last.

Results

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ be an N-partite system with $\mathcal{H}_k$ the d_k-dimensional vector space associated with the k-th subsystem. An N-partite state $\rho \in \mathcal{H}$ is said to be separable if ρ can be written as

$$\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^N, \tag{1}$$

where ρ_i^k are density matrices of the subsystem $\mathcal{H}_k$, $0 \le p_i \le 1, \sum_i p_i = 1$.

In quantum theory, the observables of a quantum system are represented by a set of Hermitian operators $\{A_i\}$. The uncertainty principle shows that it is impossible to predict the measurement results of all observables of the system at the same time. The variance of A_i with respect to ρ is the uncertainty of an observable A_i, defining as $(\Delta A_i)^2_\rho = \langle A_i^2\rangle_\rho - \langle A_i\rangle^2_\rho$, where $\langle A_i\rangle_\rho = \mathrm{Tr}(\rho A_i)$ is the mean value. For a set of quantum observables $\{A_i\}$, there

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exits a constant U such that $\sum_i (\Delta A_i)^2_\rho \geq U$. This inequality gives a universally valid limitation of the measurement outcomes. Generally, it is difficult to determine the value U. For the case of Pauli matrices σ_x, σ_y and σ_z, one has $(\Delta\sigma_x)^2_\rho + (\Delta\sigma_y)^2_\rho + (\Delta\sigma_z)^2_\rho \geq 2$[32].

In Ref.[33], based on the local sum uncertainty relations, an entanglement criterion has been presented for tripartite systems.

Let $\{A_1^i\}$, $\{A_2^i\}$ and $\{A_3^i\}$ be the set of local observables associated to the subsystems $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, respectively. U_1, U_2, U_3 are lower bound of these local observables, such that $\sum_i \Delta(A_2^i)^2 \geq U_1$, $\sum_i \Delta(A_2^i)^2 \geq U_2$ and $\sum_i \Delta(A_3^i)^2 \geq U_3$. For any separable tripartite states, the following inequalities hold under any permutations of $\{1, 2, 3\}$[33]:

$$F_\rho^{12|3} \equiv \sum_i \Delta(A_1^i + A_2^i + A_3^i)^2_\rho - (U_1 + U_2 + U_3 + M_{12}^2 + M_{12|3}^2) \geq 0, \tag{2}$$

where $M_{12} = \sqrt{\sum_i \Delta(A_1^i)^2 - U_1} - \sqrt{\sum_i \Delta(A_2^i)^2 - U_2}$, $M_{12|3} = \sqrt{F_\rho^{12}} - \sqrt{\sum_i \Delta(A_3^i)^2 - U_3}$, $F_\rho^{12} = \sum_i \Delta(A_1^i + A_2^i)^2 - (U_1 + U_2 + M_{12}^2)$, A_1^i, A_2^i and A_3^i are the operators acting on the first, the second and the third subsystem with the rest subsystems as identity operators in the tripartite systems, respectively.

Generalizing the criterion (2) to four-partite systems, we consider the set of local observables $\{A_1^i\}$, $\{A_2^i\}$, $\{A_3^i\}$ and $\{A_4^i\}$ associated to the subsystems $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$, respectively. From the local sum uncertainty relations, there must exists lower bounds $U_j > 0$ for each nonsimultaneous observable $\{A_j^i\}$ for $j = 1, 2, 3, 4$. That is to say,

$$\sum_i \Delta(A_2^i)^2 \geq U_1, \quad \sum_i \Delta(A_2^i)^2 \geq U_2, \quad \sum_i \Delta(A_3^i)^2 \geq U_3, \quad \sum_i \Delta(A_4^i)^2 \geq U_4. \tag{3}$$

Then for four-partite quantum systems, we have the following conclusion.

Theorem 1 *For any four-partite separable states, the following inequalities hold simultaneously under any permutations of* $\{1, 2, 3, 4\}$,

$$\begin{aligned} F_\rho^{123|4} &= F - (M_{12}^2 + M_{12|3}^2 + M_{123|4}^2) \geq 0, \\ F_\rho^{12|34} &= F - (M_{12}^2 + M_{34}^2 + M_{12|34}^2) \geq 0, \end{aligned} \tag{4}$$

where $F = \sum_i \Delta(A_1^i + A_2^i + A_3^i + A_4^i)^2_\rho - \sum_{j=1}^4 U_j$, $M_{123|4} = \sqrt{F_\rho^{12|3}} - \sqrt{\sum_i \Delta(A_4^i)^2 - U_4}$, $M_{12|34} = \sqrt{F_\rho^{12}} - \sqrt{F_\rho^{34}}$.

Theorem 1 provides a necessary condition of separable four-partite states. The violations of the inequalities in (1) sufficiently imply entanglement. For the four-qubit W state, $\rho = |W_4\rangle\langle W_4|$ *with* $|W_4\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$. *Let* $A_1^1 = A_2^1 = A_3^1 = -A_4^1 = \sigma_x$, $A_1^2 = A_2^2 = A_3^2 = -A_4^2 = \sigma_y$ *and* $A_1^3 = A_2^3 = A_3^3 = -A_4^3 = \sigma_z$, *thus we get* $\sum_i \Delta(A_j^i)^2 \geq 2$, $M_{12} = 0$, $M_{34} = 0$, $M_{12|3} = \sqrt{3} - \sqrt{\frac{3}{4}}$, $M_{123|4} = \sqrt{\frac{27}{4} - M_{12|3}^2} - \sqrt{\frac{3}{4}}$ *and* $M_{12|34} = \sqrt{3}$, *which give rise to* $F_\rho^{123|4} = 3 - M_{12|3}^2 - M_{123|4}^2 < 0$ *and* $F_\rho^{12|34} = 0$, *which provide a violation for the inequalities (4). Therefore, the criterion identifies four-qubit W state is entangled. By taking use of Theorem 1, more generally states can be detected and we consider some detailed examples for mixed states below.*

Example 1 (Four-qubit W state mixed with white noise) We first consider $\rho_1 = \frac{p}{16}I + (1-p)|W_4\rangle\langle W_4|$, $0 \leq p \leq 1$. For this state, we choose $-A_1^1 = -A_2^1 = -A_3^1 = A_4^1 = \sigma_x$, $-A_1^2 = -A_2^2 = A_3^2 = A_4^2 = \sigma_y$ and $-A_1^3 = -A_2^3 = -A_3^3 = -A_4^3 = \sigma_z$, hence $\sum_i \Delta(A_j^i)^2 \geq 2$, $M_{12} = M_{34} = 0$, $M_{12|3} = \sqrt{3 - p^2} - \sqrt{1 - \frac{1}{4}(1-p)^2}$, $M_{123|4} = \sqrt{\frac{10p - 9p^2 + 11}{4} - M_{12|3}^2} - \sqrt{1 - \frac{1}{4}(1-p)^2}$ and $M_{12|34} = \sqrt{3 - p^2} - \sqrt{2p - p^2 + 1}$. Then, we get $F_{\rho_1}^{123|4} = 10p - 4p^2 - 2 - M_{12|3}^2 - M_{123|4}^2$ and $F_{\rho_1}^{12|34} = 10p - 4p^2 - 2 - M_{12|34}^2$. When $p \leq 0.3605$, $F_{\rho_1}^{123|4} \leq 0$, so the state ρ_1 violates one of the inequalities (4). Therefore, the four-partite LUR criterion identifies the ρ_1 as an entangled state, see Fig. 1. While, ρ_1 is detected based on the witness $\mathcal{W} = \frac{3}{4}I - |W_4\rangle\langle W_4|$ which is proposed in Ref.[27] when $p < 0.267$, see Fig. 2. That is to say our result detects better the entanglement than the criterion of Ref.[27].

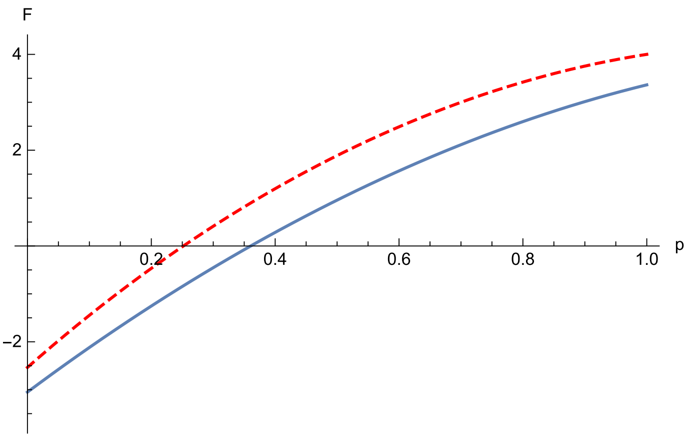

Figure 1. For the four-partite W state mixed with the white noise ρ_1. The the blue line stands for $F_{\rho_1}^{123|4}$ and the red dash line stands for $F_{\rho_1}^{12|34}$ in Theorem 1. We can see that when $p \leq 0.3605$, state ρ_1 violates one of the inequalities (4), hence ρ_1 is entangled for $p \leq 0.3605$.

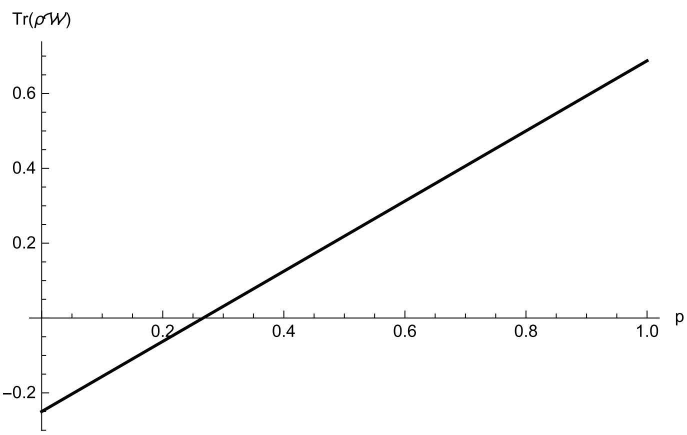

Figure 2. For the four-partite W state mixed with the white noise ρ_1. The the black line represents $\mathrm{Tr}(\rho_1 \mathcal{W})$ in Ref.[27]. We can see that ρ_1 is detected by the witness $\frac{3}{4}I - |W_4\rangle\langle W_4|$, thus ρ_1 is entangled for $p \leq 0.267$.

Example 2 (Four-qubit Dicke state mixed with white noise) Now, we take $\rho_2 = \frac{p}{16}I + (1-p)(|D_2^4\rangle\langle D_2^4|)$, $0 \leq p \leq 1$, where $|D_2^4\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle)$. For this state, we choose $-A_1^1 = -A_2^1 = A_3^1 = A_4^1 = \sigma_x$, $A_1^2 = A_2^2 = A_3^2 = -A_4^2 = \sigma_y$, $-A_1^3 = -A_2^3 = -A_3^3 = -A_4^3 = \sigma_z$. By direct calculations, we get $M_{12} = 0$, $M_{34} = 0$, $M_{12|3} = \sqrt{4-2p} - 1$, $M_{123|4} = \sqrt{\frac{35}{3} - \frac{26}{3}p - M_{123}^2} - 1$ and $M_{12|34} = \sqrt{4-2p} - \sqrt{2p}$, which yield $F_{\rho_2}^{123|4} = \frac{22}{3}(p-1) + \frac{2\sqrt{6}}{3}\sqrt{2p - 2 + 3\sqrt{4-2p}}$ and $F_{\rho_2}^{12|34} = 8p - 8 + 4\sqrt{\frac{4-p^2}{3}}$. When $p \leq 0.437$, $F_{\rho_2}^{12|34} \leq 0$, and $F_{\rho_2}^{123|4} \leq 0$ for $p \leq 0.543$. It can be seen, from Fig. 3, that the ρ_2 violate inequalities (4) for $p \leq 0.543$. Furthermore, comparing with the result in Ref.[27] which show that ρ_2 is entangled for $p < 0.356$ (see Fig. 4), the Theorem 1 also detects more entanglement.

For a more general case, we consider the set of local observable $\{A_1^i\}, \{A_2^i\}, \cdots, \{A_N^i\}$ associated to the subsystems $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_N$, respectively. Every local observable has a lower bound U_j $(j = 1, 2, \ldots, N)$ satisfies $\sum_i (A_j^i)^2 \geq U_j$. In order to simplify calculation, let i_N represent $\{A_{i_N}^i\}$ and the bi-partition index $(i_1 i_2 \cdots i_K | i_{K+1} \cdots i_N)$ is denoted as $k_1|k_0$, where $k_1 = i_1 i_2 \cdots i_K$ and $k_0 = i_{K+1} i_{K+2} \cdots i_N$, $\lceil \frac{N}{2} \rceil \leq K < N$ and $1 \leq i_1 < i_2 < \cdots < i_K \leq N$. For instance, if $N = 4$, hence $K = 2$, and $k_1|k_0 = \{12|34, 13|24, 14|23\}$, which represents three classes of bi-partition index of local observable set in N-body quantum system. Similar to the derivation of the Theorem 1, we obtain the following lemma and theorem.

Lemma 2 *For multipartite separable states, the following inequalities must hold:*

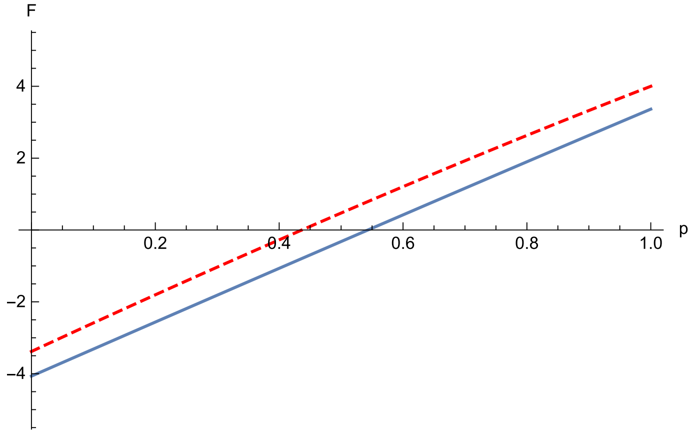

Figure 3. For the four-partite Dicke state D_2^4 mixed with the white noise ρ_2. The the blue line stands for $F_{\rho_2}^{123|4}$ and the red dash line stands for $F_{\rho_2}^{123|4}$ and the red dash line stands for $F_{\rho_2}^{12|34}$ in Theorem 1. When $p \leq 0.3605$, we can see that the state ρ_2 violates one of the inequalities (4), whence our criterion detects the entanglement of ρ_2 for $0 \leq p \leq 0.543$.

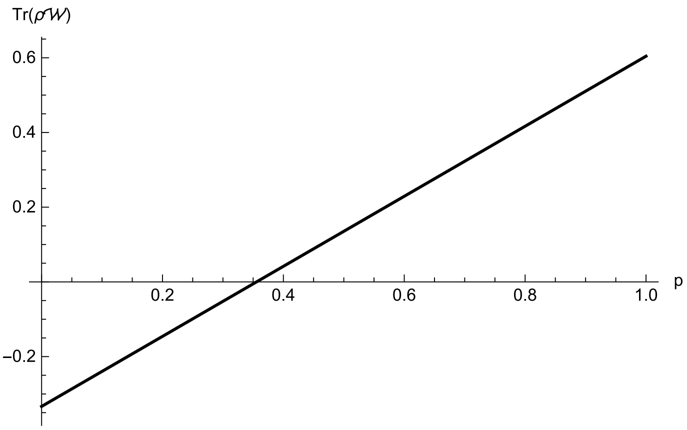

Figure 4. For the four-partite Dicke state D_2^4 mixed with the white noise ρ_2. The the black line stands for $\mathrm{Tr}(\rho\mathcal{W})$ in Ref.[27]. By using the witness $\mathcal{W}$, we can see that ρ_2 is entangled for $p \leq 0.356$.

$$\sqrt{F_\rho^{12\cdots N-1}}\sqrt{\sum_i \Delta(A_N^i)^2 - U_N} \pm \sum_i \left[\langle (A_1^i + \cdots + A_{N-1}^i) \otimes A_N^i\rangle - \langle A_1^i + \cdots + A_{N-1}^i\rangle\langle A_N^i\rangle\right] \geq 0, \quad (5)$$

and

$$\sqrt{F_\rho^{k_0}}\sqrt{F_\rho^{k_1}} \pm \sum_i \left[\langle (A_1^i + \cdots + A_K^i) \otimes (A_{K+1}^i + \cdots + A_N^i)\rangle - \langle A_1^i + \cdots + A_K^i\rangle\langle A_{K+1}^i + \cdots + A_N^i\rangle\right] \geq 0, \quad (6)$$

where $F_\rho^{12\ldots N-1} = \sum_i \Delta(A_1^i + A_2^i + \cdots + A_{N-1}^i)^2 - (\sum_{j=1}^{N-1} U_j + M_{12}^2 + M_{12|3}^2 + \cdots + M_{12\cdots N-2|N-1}^2)$, $F_\rho^{k_0} = \sum_i \Delta(A_1^i + A_2^i + \cdots + A_K^i)^2 - (\sum_{j=1}^{K} U_j + M_{12}^2 + M_{12|3}^2 + \cdots + M_{12\cdots K-1|K}^2), F_\rho^{k_1} = \sum_i \Delta(A_{K+1}^i + \cdots + A_N^i)^2 - (\sum_{j=K+1}^{N} U_j + M_{K+1K+2}^2 + \cdots + M_{K+1K+2\cdots N-1|N}^2)$.

Theorem 2 *For any multipartite separable states, the following inequalities hold under any permutations of the subsystems,*

$$F_\rho^{k_1|k_0} = F - (M_{i_1i_2}^2 + M_{i_1i_2i_3}^2 + \cdots + M_{i_1i_2\cdots i_K}^2 + M_{i_{K+1}i_{K+2}}^2 + \cdots + M_{i_{K+1}i_{K+2}\cdots i_N}^2 + M_{i_1i_2\cdots i_N}^2) \geq 0,$$

where

$$F=\sum_{i=1}^{N}\Delta(A_1^i+A_2^i+\cdots+A_N^i)_\rho^2-\sum_{j=1}^{N}U_j, \tag{7}$$

and

$$\begin{aligned} M_{k_1|k_0}&=\sqrt{F_\rho^{k_1|k_0}}-\sqrt{\sum_i \Delta(A_{i_N}^i)^2-U_{i_N}},\ \text{for } K=N-1,\\ M_{k_1|k_0}&=\sqrt{F_\rho^{k_1}}-\sqrt{F_\rho^{k_0}},\ \text{for } K<N-1. \end{aligned} \tag{8}$$

$A_{i_1}^i$ is an operator acting on the i_1-th subsystem $\mathcal{H}_{i_1}$ with the rest subsystems as identity operators in N-partite quantum systems.

Let us consider five-partite quantum systems to illustrate the theorem. In the case of $N=5$, we can have

$$\begin{cases} k_1\in\{123,124,125,134,135,145,234,235,245,345\}\ \text{and}\ k_0\in\{45,35,34,25,24,23,15,14,13,12\} & \text{K=3;}\\ k_1\in\{1234,1235,1245,1345,2345\}\ \text{and}\ k_0\in\{5,4,3,2,1\} & \text{K=4.}\end{cases}$$

Hence we have

$$\begin{aligned}
F_\rho^{1234|5}&=F-(M_{12}^2+M_{12|3}^2+M_{123|4}^2(M_{12|34}^2)+M_{1234|5}^2),\\
F_\rho^{1235|4}&=F-(M_{12}^2+M_{12|3}^2+M_{123|5}^2(M_{12|35}^2)+M_{1235|4}^2),\\
F_\rho^{1345|2}&=F-(M_{13}^2+M_{13|4}^2+M_{134|5}^2(M_{13|45}^2)+M_{1345|2}^2),\\
F_\rho^{2345|1}&=F-(M_{23}^2+M_{23|4}^2+M_{234|5}^2(M_{23|45}^2)+M_{2345|1}^2),\\
F_\rho^{1245|3}&=F-(M_{12}^2+M_{12|3}^2+M_{124|5}^2(M_{12|45}^2)+M_{1245|3}^2),\\
F_\rho^{123|45}&=F-(M_{12}^2+M_{12|3}^2+M_{45}^2+M_{123|45}^2),\quad F_\rho^{124|53}=F-(M_{12}^2+M_{12|4}^2+M_{53}^2+M_{124|53}^2),\\
F_\rho^{125|34}&=F-(M_{12}^2+M_{12|5}^2+M_{34}^2+M_{125|34}^2),\quad F_\rho^{134|52}=F-(M_{13}^2+M_{13|4}^2+M_{52}^2+M_{134|52}^2),\\
F_\rho^{135|24}&=F-(M_{13}^2+M_{13|5}^2+M_{24}^2+M_{135|24}^2),\quad F_\rho^{145|23}=F-(M_{14}^2+M_{14|5}^2+M_{23}^2+M_{145|23}^2),\\
F_\rho^{234|51}&=F-(M_{23}^2+M_{23|4}^2+M_{51}^2+M_{234|51}^2),\quad F_\rho^{235|41}=F-(M_{23}^2+M_{23|5}^2+M_{41}^2+M_{235|41}^2),\\
F_\rho^{245|13}&=F-(M_{24}^2+M_{24|5}^2+M_{13}^2+M_{245|13}^2),\quad F_\rho^{345|12}=F-(M_{34}^2+M_{34|5}^2+M_{12}^2+M_{345|12}^2),
\end{aligned} \tag{9}$$

where $F=\sum_i\Delta(A_1^i+A_2^i+\cdots+A_5^i)_\rho^2-\sum_{j=1}^{5}U_j$, $M_{1234|5}=\sqrt{F_\rho^{123|4}}-\sqrt{\sum_i\Delta(A_5^i)^2-U_5}$, $M_{123|45}=\sqrt{F_\rho^{12|3}}-\sqrt{F_\rho^{45}}$. $M_{2345|1}, M_{1345|2}, M_{1245|3}, M_{1235|4}, M_{124|53}, M_{125|34}, M_{134|52}, M_{135|24}, M_{145|23}, M_{234|51}, M_{235|41}, M_{245|13}, M_{345|12}$ have similar representations.

As a simple example, consider the five-qubit state $\rho=|W_5\rangle\langle W_5|$, with $|W_5\rangle=\frac{1}{\sqrt{5}}(|10000\rangle+|01000\rangle+|00100\rangle+|00010\rangle+|00001\rangle)$. Let $-A_1^1=A_2^1=-A_3^1=-A_4^1=A_5^1=\sigma_x$, $-A_1^2=-A_2^2=-A_3^2=A_4^2=A_5^2=\sigma_y$, $A_1^3=-A_2^3=-A_3^3=A_4^3=A_5^3=\sigma_z$. We have $U_1=U_2=U_3=U_4=U_5=2$, $M_{12}=M_{34}=0$, $M_{123}=0.2161$, $M_{123|4}=1.218$, $M_{12|34}=0$, $M_{1234|5}=0.2797$ and $M_{123|45}=0.8536$, which give rise to $F_\rho^{1234|5}=3-M_{123}^2-M_{1234}^2-M_{1234|5}^2<0$ and $F_\rho^{123|45}<0$, namely, the state is entangled.

Conclusion

We have generalized the LUR criterion for three qubit quantum systems to multiqubit quantum systems, and obtained new entanglement criteria for four-partite quantum systems as well as for general multipartite systems. By detailed examples we have shown that our criteria can detect better the entanglement than some existing criteria. It is further known that in certain situations they can provide a nonlinear refinement of linear entanglement witnesses[35], and it can be measured in experimental settings similar to those of entanglement witnesses. The effectiveness of the LUR criteria relies heavily on certain notions of information content of quantum states and choice of observables.

Quantum entanglement is fundamentally connected to the quantum steering, local uncertainty relations (LURs) are a common tool for entanglement detection, and the underlying idea can be directly generalized to steering detection[36].

The considered system here is closed systems with no decoherence effects taken into account. Also, it would be interesting to find criteria for open quantum systems, since realistic quantum systems inevitably interact with the environment. It would be also interesting if our approach may highlight further investigations on the k-separability[37] of multipartite systems and genuine multipartite entanglement detection.

Methods

Proof of the Theorem 1 By straightforward computation, we have

$$\begin{aligned}\sum_i \Delta(A_1^i + A_2^i + A_3^i + A_4^i)_\rho^2 &= \sum_i \Delta(A_1^i + A_2^i + A_3^i)^2 + \sum_i \Delta(A_4^i)^2 \\ &+ 2\sum_i \left[\langle (A_1^i + A_2^i + A_3^i) \otimes A_4^i\rangle - \langle A_1^i + A_2^i + A_3^i\rangle\langle A_4^i\rangle\right].\end{aligned}$$

Taking into account that for any tripartite separable states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$[33],

$$\sqrt{F_\rho^{12}}\sqrt{\sum_i \Delta(A_3^i)^2 - U_3} \pm \sum_i \left[\langle (A_1^i + A_2^i) \otimes A_3^i\rangle - \langle A_1^i + A_2^i\rangle\langle A_3^i\rangle\right] \geq 0, \tag{10}$$

where $F_\rho^{12} = \sum_i \Delta(A_1^i + A_2^i)^2 - (U_1 + U_2 + M_{12}^2)$, we obtain

$$\sum_i \Delta(A_1^i + A_2^i + A_3^i + A_4^i)_\rho^2 \geq U_1 + U_2 + U_3 + U_4 + M_{12}^2 + M_{12|3}^2 + M_{123|4}^2,$$

namely, $F_\rho^{123|4} \geq 0$. By relabeling the sub-indices, we have $F_\rho^{124|3} \geq 0$, $F_\rho^{134|2} \geq 0$ and $F_\rho^{234|1} \geq 0$, similarly. Concerning $F_\rho^{12|34}$, we have

$$\begin{aligned}\sum_i \Delta(A_1^i + A_2^i + A_3^i + A_4^i)_\rho^2 &= \sum_i \Delta(A_1^i + A_2^i)^2 + \sum_i \Delta(A_3^i + A_4^i)^2 \\ &+ 2\sum_i \left[\langle (A_1^i + A_2^i) \otimes (A_3^i + A_4^i)\rangle - \langle A_1^i + A_2^i\rangle\langle A_3^i + A_4^i\rangle\right].\end{aligned}$$

Since for any bipartite separable states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$, the following inequality holds[33],

$$\sqrt{\sum_i \Delta(A_2^i)^2 - U_1}\sqrt{\sum_i \Delta(A_2^i)^2 - U_2} \pm \sum_i \left[\langle A_1^i \otimes A_2^i\rangle - \langle A_1^i\rangle\langle A_2^i\rangle\right] \geq 0, \tag{11}$$

we get

$$\sum_i \Delta(A_1^i + A_2^i + A_3^i + A_4^i)_\rho^2 \geq U_1 + U_2 + U_3 + U_4 + M_{12}^2 + M_{34}^2 + M_{12|34}^2,$$

namely, $F_\rho^{12|34} \geq 0$. Similarly one can show that $F_\rho^{23|41} \geq 0$ and $F_\rho^{13|42} \geq 0$. □

Proof of the Theorem 2 We denote the length of k_0 as $|k_0|$. From above, one has $|k_0| + |k_1| = N$.

When $K = N - 1$, one has $|k_0| = 1$, by straightforward computation, we have

$$\begin{aligned}\sum_i \Delta(A_1^i + A_2^i + \cdots + A_N^i)_\rho^2 &= \sum_i \Delta(A_1^i + A_2^i + \cdots + A_{N-1}^i)^2 + \sum_i \Delta(A_N^i)^2 \\ &+ 2\sum_i \left[\langle (A_1^i + A_2^i + \cdots + A_{N-1}^i) \otimes A_N^i\rangle - \langle A_1^i + A_2^i + \cdots + A_{N-1}^i\rangle\langle A_N^i\rangle\right].\end{aligned}$$

By Lemma 2, for any multipartite separable states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$,

$$\begin{aligned}&\sqrt{F_\rho^{12\cdots N-1}}\sqrt{\sum_i \Delta(A_N^i)^2 - U_N} \\ &\pm \sum_i \left[\langle (A_1^i + A_2^i + \cdots + A_{N-1}^i) \otimes A_N^i\rangle - \langle A_1^i + A_2^i + \cdots + A_{N-1}^i\rangle\langle A_N^i\rangle\right] \geq 0,\end{aligned} \tag{12}$$

via calculation, we obtain

$$\sum_i \Delta(A_1^i + A_2^i + \cdots + A_N^i)_\rho^2 \geq \sum_{j=1}^N U_j + M_{12}^2 + M_{12|3}^2 + \cdots + M_{12\cdots N-1|N}^2,$$

namely, $F_\rho^{12\cdots N-1|N} \geq 0$. By relabeling the sub-indices, we have $F_\rho^{k_1|k_0} \geq 0$.

When $K < N - 1$, one has $|k_0| \geq 2$,

$$\sum_i \Delta(A_1^i + \cdots + A_N^i)^2_\rho = \sum_i \Delta(A_1^i + \cdots + A_K^i)^2 + \sum_i \Delta(A_{K+1}^i + \cdots + A_N^i)^2$$
$$+ 2\sum_i \Big[\langle (A_1^i + \cdots + A_K^i) \otimes (A_{K+1}^i + \cdots + A_N^i)\rangle - \langle A_1^i + \cdots + A_K^i\rangle\langle A_{K+1}^i + \cdots + A_N^i\rangle\Big].$$

By using Lemma 2, we get

$$\sum_i \Delta(A_1^i + A_2^i + \cdots + A_N^i)^2_\rho \geq \sum_{j=1}^N U_j + (M_{12}^2 + M_{12|3}^2 + \cdots + M_{12\cdots|K}^2 + M_{K+1K+2}^2 + \cdots + M_{K+1\cdots|N}^2 + M_{12\cdots|N}^2),$$

namely, $F_\rho^{12\cdots K|K+1K+2\cdots N} \geq 0$. By relabeling the sub-indices, one can show that $F_\rho^{k_0|k_1} \geq 0$. □

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The first and the second authors wrote the main manuscript text and all authors reviewed the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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