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Frobenius-norm-based measures of quantum coherence and asymmetry

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We formulate the Frobenius-norm-based measures for quantum coherence and asymmetry respectively. In contrast to the resource theory of coherence and asymmetry, we construct a natural measure of quantum coherence inspired from optical coherence theory while the group theoretical approach is employed to quantify the asymmetry of quantum states. Besides their simple structures and explicit physical meanings, we observe that these quantities are intimately related to the purity (or linear entropy) of the corresponding quantum states. Remarkably, we demonstrate that the proposed coherence quantifier is not only a measure of mixedness, but also an intrinsic (*basis-independent*) quantification of quantum coherence contained in quantum states, which can also be viewed as a normalized version of Brukner-Zeilinger invariant information. In our context, the asymmetry of N -qubit quantum systems is considered under local *independent* and *collective* $SU(2)$ transformations. Intriguingly, it is illustrated that the collective effect has a significant impact on the asymmetry measure, and quantum correlation between subsystems plays a non-negligible role in this circumstance.

In the 20th century, quantum mechanics, as the core of quantum physics, is undoubtedly one of the most profound scientific theories during the development process of modern science and philosophy. Notably, quantum coherence is one of the most remarkable and characteristic traits of quantum mechanics and also viewed as the critical resource for the emerging field of quantum technologies, such as quantum cryptography¹, quantum metrology² and quantum computation³. This intrinsic principle of quantum mechanics enforces its entire departure from classical lines of thought, similar to the notion of quantum entanglement⁴. In different occasions, the generalized concept of quantum coherence manifests itself as quantum superposition, quantum asymmetry or non-commutability. These specific notions have gotten extensive applications in quantum physics both theoretically and experimentally. Therefore, the characterization, quantification and application of quantum coherence are one of the central topics in quantum information science.

Although the investigation of coherence theory has a long history in classical and quantum optics^{5–9}, it does not provide an integrated and unified framework for quantitative study. In some contexts, even conflicting definitions have appeared with respect to the same concept^{10–15}, especially when the vectorial character of electromagnetic waves is involved. From the perspective of quantum information theory, a rigorous framework for the characterization and quantification of coherence has been formulated very recently, based on the quantum resource theory^{16,17}. On one hand, Baumgratz *et al.* defined the incoherent states and incoherent operations and further discussed the quantum coherence monotones in the constraint of a series of axiomatic-like prerequisites¹⁸, which is extremely similar to the approach adopted in quantum entanglement theory¹⁹. Within this framework, novel measures of quantum coherence have been proposed and the connections between coherence and other manifestations of quantum correlations have been carefully scrutinized^{20–31}. On the other hand, Marvian and his collaborators proposed the resource theory of asymmetry which included the former notion of quantum coherence as a special case^{32–37}. Moreover, the internal relations between these two approaches and their physical justifications have been further clarified lately^{38,39}.

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However, the story has never come to the end. The investigation of quantum coherence is now facing several crucial problems that need to be solved urgently. First of all, we should further identify the application scopes and physical meanings of existing measures of quantum coherence. In particular, the quantifiers presented in ref. 18 are established in the finite-dimensional setting and thus can not be directly applied to the infinite-dimensional systems. Moreover, explicit physical (or operational) meanings of existing measures of quantum coherence are to be pursued by the community, though partial results have been obtained^{31,40,41}. Secondly, as indicated by Marvian *et al.*, distinct approaches of quantifying quantum coherence will lead to different conclusions in a diverse set of tasks. For instance, the *speakeable* notions of coherence may not be suitable to characterize the consumed resource in the task of phase estimation^{37,38}. Moreover, Chitambar *et al.* also pointed out that all currently proposed *basis-dependent* theories fail to satisfy a criterion of physical consistency³⁹. In addition, the interactions between quantum coherence and other manifestations of quantum correlations still need to be uncovered^{22,23}. Finally, the quantitative relations between quantum coherence and other important quantities in quantum information are yet to be established^{42,43}.

In contrast to the resource-theory framework, we notice that other types of coherence or asymmetry measures are also of great significance in optical coherence theory and condensed matter physics. For example, a measure of *first-order coherence* $\mathcal{D} = \sqrt{2 \text{Tr} \rho^2 - 1}$ (for qubit subsystem) was exploited to introduce the concept of accessible coherence⁴⁴ and it was recently demonstrated that \mathcal{D} can be unveiled from hidden coherence in partially coherent light⁴⁵. Moreover, Fang *et al.* proposed a novel measure of the degree of symmetry by virtue of group theoretical approach and showed that this quantity can effectively detect the phenomena of accidental degeneracy and spontaneous symmetry breaking⁴⁶. In this work, we formulate two Frobenius-norm-based measures by further extending their previous work and more importantly, illustrate the clear physical meanings of these quantifiers. Through the establishment of relationship with other significant physical quantities (e.g., Brukner-Zeilinger information), we emphasize that the purity of quantum states is not only a measure of mixedness, but also a *basis-independent* quantification of quantum coherence contained in quantum states.

Results

Coherence measure based on Frobenius norm. In the famous work of Mandel and Wolf, they introduced an important measure depicting the *degree of polarization* for planar electromagnetic fields⁷

$$\mathcal{P}_2 = \sqrt{1 - \frac{4\det\Phi_2}{(\text{Tr}\Phi_2)^2}}, \quad (1)$$

where Φ_2 represents 2×2 equal-time coherence matrix at a given point \vec{r} and $[\Phi_2]_{ij} = \langle E_i^*(\vec{r}, t) E_j(\vec{r}, t) \rangle$, $i, j = x, y$. Note that this quantity characterizes the coherence between two mutually orthogonal components of the complex electric vector $\mathbf{E}(t) = [E_x(t), E_y(t)]T$. Be aware of the Hermiticity and non-negativity of the matrix Φ_2 , we can define a physically valid quantum state $\rho = \Phi_2/\text{Tr}\Phi_2$. Since the Bloch sphere is analogous to the Poincaré sphere, \mathcal{P}_2 can be rewritten as

$$\mathcal{P}_2 = \sqrt{1 - 4\det\rho} = \sqrt{2 \text{Tr} \rho^2 - 1} = \lambda_+ - \lambda_- = |\vec{s}|, \quad (2)$$

where we have used the Bloch representation $\rho = (1 + \vec{s} \cdot \vec{\sigma})/2$ and $\lambda_{\pm} = (1 \pm |\vec{s}|)/2$ are eigenvalues of ρ . It is apparent that \mathcal{P}_2 is equivalent to the degree of first-order coherence \mathcal{D} adopted in refs 44,45 and exactly equal to the length of the Bloch vector \vec{s} . In fact, in this case (e.g., for qubit system) ρ can also be recast as

$$\rho = \mathcal{P}_2 |\psi\rangle\langle\psi| + \frac{1 - \mathcal{P}_2}{2} \mathbb{I}_2, \quad (3)$$

with $|\psi\rangle\langle\psi| = (1 + \vec{s}_0 \cdot \vec{\sigma})/2$ and $\vec{s}_0 = \vec{s}/|\vec{s}|$.

Furthermore, Setälä *et al.* moved a step forward and extended the above 2D formalism into the formulation of the 3D degree of polarization coherence for arbitrary electromagnetic fields^{47,48}, that is

$$\mathcal{P}_3 = \sqrt{\frac{3}{2} \left[\frac{\text{Tr} \Phi_3^2}{(\text{Tr} \Phi_3)^2} - \frac{1}{3} \right]}, \quad (4)$$

where Φ_3 denotes the 3×3 coherence matrix at a given point \vec{r} and $[\Phi_3]_{ij} = \langle E_i^*(\vec{r}, t) E_j(\vec{r}, t) \rangle$, $i, j = x, y, z$. Similarly, by redefining $\rho = \Phi_3/\text{Tr}\Phi_3$, we obtain

$$\mathcal{P}_3 = \sqrt{\frac{3}{2} \left[\text{Tr} \rho^2 - \frac{1}{3} \right]}, \quad (5)$$

Actually, we observe that this formalism can be unified and generalized to arbitrary dimensional systems, motivated by the work of Luis⁴⁹, where a 4×4 density matrix was considered. The key idea is to quantify quantum coherence from the geometrical point of view, that is, the degree of coherence between distinct variables of optical fields is assessed as the distance between the coherence matrix and the identity matrix. Note that this geometrical approach is also employed in quantum resource theories^{16–19}. Here the identity matrix, or maximally mixed state in the sense of quantum information, is identified as the fully incoherent and completely unpolarized light^{7,49}. Therefore, we can construct the following general measure

$$\mathcal{C}(\rho) = \sqrt{\frac{d}{d-1}} \|\rho - \rho_\star\|_F, \tag{6}$$

where d is the dimension of the Hilbert space, $\rho_\star = \mathbb{I}_d/d$ denotes the maximally mixed state. and the Frobenius norm is given by $\|A\|_F = \sqrt{\text{Tr}(A^\dagger A)}$. Note that here the Frobenius norm is normalized to guarantee $\mathcal{C}(\rho) \in [0, 1]$ (see Methods section for more details). It is worth pointed out that this measure inherently possesses the following desirable properties:

- (i) When $d=2, 3, 4$, the coherence measure⁶ reduces to the existing quantifiers for 2D, 3D and 4D cases, where for 4D formalism, this measure was also termed as the *depolarization index*^{50,51}.
- (ii) No optimization is involved in $\mathcal{C}(\rho)$, which is in sharp contrast to the coherence measures proposed in ref. 18, though the optimization is rather straightforward there.
- (iii) $\mathcal{C}(\rho)$ is independent of the specific representation of ρ , that is, $\mathcal{C}(\rho)$ is *basis-independent*.
- (iv) $\mathcal{C}(\rho)$ is unitarily invariant, since $\mathcal{C}(U\rho U^\dagger) = \mathcal{C}(\rho)$ owing to the fact that the maximally mixed state ρ_\star is the *only* state that remains invariant under arbitrary unitary transformation. In fact, the property (iv) is equivalent to (iii).
- (v) $\mathcal{C}(\rho)$ has an analytical expression and explicit geometrical interpretation. As we shall see below, $\mathcal{C}(\rho)$ also has a clear operational meaning.

To further elaborate on the characteristics of $\mathcal{C}(\rho)$, we first present several observations where this coherence measure is involved.

Observation I: Straightforward calculation shows that the square of $\mathcal{C}(\rho)$ is directly proportional to the celebrated *Brukner-Zeilinger invariant information*⁵²

$$\mathcal{C}^2(\rho) = \frac{d}{d-1} (\text{Tr} \rho^2 - \text{Tr} \rho_\star^2) = \frac{d}{d-1} \left(\text{Tr} \rho^2 - \frac{1}{d} \right) = \frac{d}{d-1} \mathcal{I}_{BZ}(\rho), \tag{7}$$

where the Brukner-Zeilinger information \mathcal{I}_{BZ} is an operational notion defined as the sum of individual measures of information over a complete set of mutually complementary observables (MCO)⁵³

$$\mathcal{I}_{BZ}(\rho) \doteq \sum_{i=1}^m \sum_{j=1}^d \left(\text{Tr}(\Pi_{ij}\rho) - \frac{1}{d} \right)^2 = \text{Tr} \rho^2 - \frac{1}{d}. \tag{8}$$

Here $i = 1, \dots, m$ and $j = 1, \dots, d$ label complementary observables and their eigenvectors, respectively. A significant feature of \mathcal{I}_{BZ} is its independence of the specific choice of the measured set of MCO and this independence is also equivalent to the unitary invariance of $\mathcal{I}_{BZ}(U\rho U^\dagger) = \mathcal{I}_{BZ}(\rho)$, since the unitary transformations do not alter the eigenvalues of ρ and $\text{Tr} \rho^2$ only relies on them. Therefore, it is evident that $\mathcal{C}(\rho)$ (and \mathcal{I}_{BZ}) is a measure of *basis-independent* coherence contained in quantum states and quantifies the intrinsic randomness irrespective of the amount of entanglement between subsystems (if there are any). It is worth noting that the (global) purity does not solely determine the entanglement of a multi-partite state and this fact led to the investigation of the so-called *maximally entangled mixed states* for a given degree of purity^{54,55}. Moreover, all pure states should always be represented as a coherent superposition of a certain set of basis states rather than a classical mixing, and by our definition $\mathcal{C}(\rho) = 1$ for any pure state. This is reminiscent of the argument by von Neumann that the entropy of all pure states is defined to be 0 as a kind of normalization⁵⁶. Finally, since the Brukner-Zeilinger invariant information was successfully utilized in quantum teleportation⁵⁷, state estimation⁵⁸ and the violation of Bell's inequalities⁵⁹, $\mathcal{C}(\rho)$, as a renormalized version of \mathcal{I}_{BZ} , also plays an important role in all these quantum information tasks.

Observation II: The coherence measure $\mathcal{C}(\rho)$ is associated with entropy production problem of doubly-stochastic (bistochastic) quantum channels. By use of the quantum version of Kullback inequality (or a stronger Pinsker inequality⁶⁰)

$$S(\rho|\sigma) = \text{Tr}(\rho \ln \rho - \rho \ln \sigma) \geq \frac{1}{2} \text{Tr}^2 |\rho - \sigma| \geq \frac{1}{2} \|\rho - \sigma\|_F^2, \tag{9}$$

Streater proved the following theorem⁶¹:

Theorem 1. Let \mathcal{H} be a Hilbert space with $\dim \mathcal{H} = d < \infty$ and denote by $\mathcal{B}(\mathcal{H})_2$ the Hilbert space of operator on \mathcal{H} with scalar product $\langle A, B \rangle = \text{Tr}(A^\dagger B)$. If $T: \mathcal{B}(\mathcal{H})_2 \rightarrow \mathcal{B}(\mathcal{H})_2$ is a bistochastic channel which is ergodic and has spectral gap $\gamma \in [0, 1)$ (e.g., up to a scalar multiple, the identity matrix \mathbb{I} is the only fixed point of T in $\mathcal{B}(\mathcal{H})_2$, and the spectrum of the channel $T^\dagger \circ T$ is contained in the set $[0, 1 - \gamma] \cup \{1\}$), then for any density matrix ρ

$$S(T(\rho)) - S(\rho) \geq \frac{\gamma}{2} \|\rho - \rho_\star\|_F^2, \tag{10}$$

where $S(\rho) = -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy of ρ .

From Theorem 1, we find that two separated terms contribute to the lower bound of the entropy production. The first term $\|\rho - \rho_\star\|_F^2$ is just the unnormalized version of $\mathcal{C}(\rho)$, which characterizes the intrinsic coherence independent of the choice of a specific basis, while the spectral gap γ relies on representations of both T and ρ . To

further elucidate the role of the bistochastic channel T , we consider the thermodynamics problem raised in ref. 62, where the bistochastic channel was taken to be a projection measurement $\rho \rightarrow \Delta(\rho) := \sum_k \Pi_k \rho \Pi_k$. If $\{|l\rangle\}_l$ is the eigenbasis of ρ and $\{\Pi_k = |\phi_k\rangle\langle\phi_k|\}_k$ constitute another orthogonal basis, then γ is the second smallest eigenvalue of the matrix $\mathbb{I}_d - M^T M$, where M is a bistochastic matrix with the entries $M_{kl} = \langle \Pi_k | l \rangle = |\langle \phi_k | l \rangle|^2$. It is easy to show that γ is zero if $\{|l\rangle\}_l$ and $\{|\phi_k\rangle\}_k$ are the same, while $\gamma = 1$ if they are mutually unbiased. A deeper insight can be gained by noticing that once quantum thermodynamics is viewed as a resource theory, it can be cast as a *hybrid* of the resource theory of purity and the resource theory of asymmetry¹⁶. In particular, if $\{\Pi_k\}_k$ are the eigenvectors of an observable L , γ (e.g., second smallest eigenvalue of $\mathbb{I}_d - M^T M$) is a quantifier to characterize to what extent ρ fails to commute with L and $\gamma \neq 0$ if and only if ρ has some coherence over the eigenspaces of L . This line of thought coincides with that of Marvian *et al.*³⁶.

Observation III: For a square d -dimensional matrix A and $p \in [1, +\infty)$, the l_p norms (or vector norms) and Schatten- p norms are defined as⁶³

$$\|A\|_p := \left(\sum_{i,j=1}^d |A_{ij}|^p \right)^{1/p}, \quad \|A\|_p := (\text{Tr} |A|^p)^{1/p} = \left(\sum_{i=1}^r \sigma_i^p \right)^{1/p} = \|\vec{\sigma}(A)\|_p, \tag{11}$$

where $|A| = \sqrt{A^\dagger A}$, $r = \text{rank}(A)$ and $\vec{\sigma}(A)$ is used to refer to the vector of nonzero singular values $\vec{\sigma}(A) = (\sigma_1, \dots, \sigma_r)$. In addition to the monotonicity $\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|_\infty$ for $1 \leq p \leq q \leq \infty$, we can prove the following inequality by using Hölder's inequality⁶⁴

$$\|A\|_p \leq d^{\frac{1}{p} - \frac{1}{q}} \|A\|_q, \quad \forall q \geq p \geq 1. \tag{12}$$

Note that the monotonicity and the above inequality¹² is also satisfied by l_p norms, and the Schatten- p norms of A coincides with the ordinary (vector) l_p norms of $\vec{\sigma}(A)$. In particular, the Frobenius norm of A coincides with the corresponding l_2 norm. Therefore, $\mathcal{C}(\rho)$ gives a upper bound of l_1 norm coherence and trace-norm coherence discussed in refs 18 and 27 respectively

$$C_{l_1}(\rho) = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_1 \leq \|\rho - \rho_\star\|_1 \leq \sqrt{d^2} \|\rho - \rho_\star\|_2 = \sqrt{d(d-1)} \mathcal{C}(\rho), \tag{13}$$

$$C_{\text{tr}}(\rho) = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_{\text{Tr}} \leq \|\rho - \rho_\star\|_{\text{Tr}} \leq \sqrt{d} \|\rho - \rho_\star\|_F = \sqrt{d-1} \mathcal{C}(\rho). \tag{14}$$

where \mathcal{I} denotes the set of diagonal states defined in a pre-fixed basis. It is not surprising that the upper bounds^{13,14} are not tight in general though $C_{l_1}(\rho)$ and $C_{\text{tr}}(\rho)$ indeed achieve their maximum values with the closest state $\delta = \rho_\star$. In fact, $C_{l_1}^{\text{max}}(\rho) = d - 1$ and $C_{\text{tr}}^{\text{max}}(\rho) = 2(1 - 1/d)$, which is satisfied by the maximally coherent state¹⁸.

On the other hand, it is intuitive to assume that the intrinsic coherence measure $\mathcal{C}(\rho)$ may have a natural relationship with the resource theory of purity⁶⁵, due to the fact that $\mathcal{C}(\rho)$ is proportional to $\mathcal{I}_{\text{BZ}}(\rho) = \text{Tr} \rho^2 - 1/d$. Actually, $\mathcal{I}_{\text{BZ}}(\rho)$ provide a lower bound for the so-called *unique measure of information* introduced in ref. 66. By virtue of the quantum Kullback inequality⁹, we have

$$\mathcal{I}_{\text{BZ}}(\rho)/2 = \|\rho - \rho_\star\|_F^2/2 \leq S(\rho||\rho_\star) = \log_2 d - s(\rho), \tag{15}$$

where $I(\rho) = \log_2 d - s(\rho)$ is a unique measure for information, in the sense that $I(\rho) = \log_2 d - s(\rho)$ is equal to the optimal transition rate for mixed states to one qubit pure state π (i.e., $\pi = |0\rangle\langle 0|$)^{16,65}. However, since the Kullback inequality is not very sharp, this lower bound is rather loose, especially for the states with high purity.

Cohering (Purifying) power of quantum channels. With the framework of Baumgratz *et al.*¹⁸, we proposed the notion of cohering power of quantum channels (i.e., completely positive and trace preserving maps)²³, which has been further explored in ref. 67. In the context of this work, we can introduce a similar quantity to characterize the cohering (purifying) capacity of quantum channels

$$\mathcal{C}(\mathcal{E}) = \sqrt{\frac{d}{d-1}} \mathcal{E}\|\rho_\star\|_F - \rho_\star\|_F. \tag{16}$$

Complete positivity of $\mathcal{E}(\cdot)$ enables a Kraus decomposition

$$\mathcal{E}(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^{\dagger}, \quad \sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = \mathbb{I}_d. \tag{17}$$

where the second relation stems from the trace preserving property. Thus, the cohering power¹⁶ can be rewritten as

$$\mathcal{C}(\mathcal{E}) = \sqrt{\frac{1}{d(d-1)}} \text{Tr} \left[\left(\sum_{\mu} [K_{\mu}, K_{\mu}^{\dagger}] \right)^2 \right]. \tag{18}$$

It is observed that the operator $\sum_{\mu} [K_{\mu}, K_{\mu}^{\dagger}]$ is Hermitian and traceless. Moreover, $\mathcal{C}(\mathcal{E}) = 0$ if and only if $\mathcal{E}(\cdot)$ is a unital quantum channel, i.e., doubly-stochastic (bistochastic) completely positive map or $\mathcal{E}(\rho_{*}) = \rho_{*}$ ⁶⁸. Therefore, the cohering capacity $\mathcal{C}(\rho)$ can be recognized as a measure of *nonunitality* of quantum channels. Note that the nonunital channels also play a crucial role in the local creation of quantum correlations⁶⁹. Within the scope of resource theories^{16,17}, we *may* make a correspondence in our context: the only free state is the maximally mixed state ρ_{*} , the free operations are unital channels and the resources are states with $\mathcal{C}(\rho) > 0$. The justification of such a correspondence relies on the monotonicity of the function $\mathcal{C}(\rho)$ under unital channels. The following theorem proves that it is the case.

Theorem 2 For any unital (i.e, bistochastic) quantum channel, it holds that $\mathcal{C}(\mathcal{E}(\rho)) \leq \mathcal{C}(\rho)$.

Proof. Since $\mathcal{C}(\rho) \propto \text{Tr}\rho^2 - \text{Tr}\rho_{*}^2$, it is sufficient to prove $\text{Tr}(\mathcal{E}^2(\rho)) \leq \text{Tr}\rho^2$. Consider the Kraus decomposition of $\mathcal{E}(\cdot) = \sum_{\mu} K_{\mu}(\cdot)K_{\mu}^{\dagger}$ and we have

$$\begin{aligned} \text{Tr}(\mathcal{E}^2(\rho)) &= \text{Tr}\left(\sum_{\mu,\nu} K_{\mu}\rho K_{\mu}^{\dagger}K_{\nu}\rho K_{\nu}^{\dagger}\right) \\ &= \text{Tr}\left(\sum_{\mu,\nu} \rho K_{\mu}^{\dagger}K_{\nu}K_{\nu}^{\dagger}K_{\mu}\right) \\ &= \text{Tr}\left(\sum_{\mu,\nu} \rho A_{\mu\nu}\rho A_{\mu\nu}^{\dagger}\right), \end{aligned} \tag{19}$$

where we have defined $A_{\mu\nu} = K_{\mu}^{\dagger}K_{\nu}$. It is interesting to see that $\{A_{\mu\nu}\}$ also constitute a unital quantum channel $A(\cdot) = \sum_{\mu,\nu} A_{\mu\nu}(\cdot)A_{\mu\nu}^{\dagger}$ due to the identities

$$\sum_{\mu} A_{\mu\nu}^{\dagger}A_{\mu\nu} = \sum_{\mu} K_{\nu}^{\dagger}K_{\mu}K_{\mu}^{\dagger}K_{\nu} = K_{\nu}^{\dagger}K_{\nu}, \tag{20}$$

$$\sum_{\nu} A_{\mu\nu}A_{\mu\nu}^{\dagger} = \sum_{\nu} K_{\mu}^{\dagger}K_{\nu}K_{\nu}^{\dagger}K_{\mu} = K_{\mu}^{\dagger}K_{\mu}, \tag{21}$$

where the unitality of $\mathcal{E}(\cdot)$ is used. With the aid of Chauchy-Schwarz inequality for operators $|\text{Tr}(A^{\dagger}B)| \leq \|A\|_p \|B\|_q$ for $p^{-1} + q^{-1} = 1$ (e.g., $p = q = 2$ in our case), we finally obtain

$$\begin{aligned} \text{Tr}(\mathcal{E}^2(\rho)) &= \sum_{\mu,\nu} \text{Tr}(\rho A_{\mu\nu}\rho A_{\mu\nu}^{\dagger}) \leq \sum_{\mu,\nu} \sqrt{\text{Tr}(\rho A_{\mu\nu}A_{\mu\nu}^{\dagger}\rho)} \sqrt{\text{Tr}(A_{\mu\nu}\rho A_{\mu\nu}^{\dagger})} \\ &= \leq \frac{1}{2} \sum_{\mu,\nu} [\text{Tr}(\rho A_{\mu\nu}A_{\mu\nu}^{\dagger}\rho) + \text{Tr}(A_{\mu\nu}\rho A_{\mu\nu}^{\dagger})] \\ &= \text{Tr}\rho^2. \end{aligned} \tag{22}$$

where the unitality property of $A(\cdot)$ is applied.

We have attempted to check whether the unitality of quantum channels is also a necessary condition for the monotonicity of $\mathcal{C}(\rho)$. Note that for finite-dimensional Hilbert spaces, a dynamical semigroup is strictly purity-decreasing if and only if the Lindblad generator is unital⁷⁰. However, actually the Kraus or operator-sum representation is more general than the master equation approach³. Therefore, the necessity of unitality of operator-sum representation for strictly-decreasing purity is still an important unsolved problem. See Method section for more information.

Asymmetry measure based on Frobenius norm. In realistic physical scenarios, there are some states which are invariant under a set of symmetric operations or a given symmetric group. This fact tells us that a specific group of symmetric transformations assigns a specific type of symmetry to the underlying systems. A state ρ is called symmetric (or G -invariant) with respect to a finite or compact Lie group G if it satisfies^{34,36,71}

$$U_g(\rho) = U(g)\rho U^{\dagger}(g) = \rho, \forall g \in G, \tag{23}$$

where $U(g)$ denotes a unitary representation corresponding to the group element $g \in G$. We notice that this definition of symmetry is equivalent to other two criterions

$$\mathcal{G}_G(\rho) = \frac{1}{|G|} \sum_{g \in G} U(g)\rho U^{\dagger}(g) = \rho \stackrel{(i)}{\Leftrightarrow} \mathcal{U}_g(\rho) = \rho, \forall g \in G \stackrel{(ii)}{\Leftrightarrow} [U(g), \rho] = 0, \forall g \in G, \tag{24}$$

where the summation in $\mathcal{G}_G(\rho)$ will be replaced by integral over dg when a compact Lie group is considered. The criterion (i) induced a nature entropic measure of the asymmetry of ρ with respect to G ^{36,71}, (i.e., $A_G(\rho) = S(\mathcal{G}_G(\rho)) - S(\rho)$), which was also proved to be a measure of the quality of a quantum reference frame³³.

Based on the criterion (ii), it is realized that the commutator $[U(g), \rho]$ characterizes to what extent ρ is asymmetric with respect to G and some matrix norm of $[U(g), \rho]$ should be a reasonable measure of G -asymmetry³⁶.

Along this line of thought, a new quantifier of asymmetry has recently been proposed by Sun's group, which is defined as an average of the *fidelity deviations* of Hamiltonian H (or quantum state ρ) over a specific symmetric group $g \in G$ ⁴⁶

$$\mathcal{A}(G, H) = \frac{1}{4\|H_0\|_F^2} \int_g \| [U(g), H] \|_F^2 dg, \tag{25}$$

where $H_0 = H - \text{Tr}H/d$ is a re-biased version of H . $\mathcal{A}(G, H)$ is mathematically tractable and also has desirable properties (e.g., basis-independence), mainly owing to the geometric feature of the Frobenius norm⁶⁴. More importantly, it is has been demonstrated that $\mathcal{A}(G, H)$ can effectively detect some significant phenomena in condensed matter physics, such as accidental degeneracy and spontaneous symmetry breaking⁴⁶. Meanwhile, an analogous measure can be defined since we are more concerned with how to quantify the asymmetry (symmetry) of quantum states

$$\mathcal{A}(G, \rho) = \frac{1}{4\|\rho\|_F^2} \int_g \| [U(g), \rho] \|_F^2 dg, \tag{26}$$

$$\mathcal{S}(G, \rho) = 1 - \mathcal{A}(G, \rho) = \frac{1}{4\|\rho\|_F^2} \int_g \| \{U(g), \rho\} \|_F^2 dg. \tag{27}$$

With this formulation, we can investigate the asymmetry of any quantum state with respect to a specific symmetric group. Remarkably, while seemingly unrelated, there actually exists a close connection between $\mathcal{A}(G, \rho)$ and $\mathcal{C}(\rho)$. We first focus on the single qubit system $\rho = (1 + \vec{s} \cdot \vec{\sigma})/2$, where \vec{s} is the Bloch vector and can be parameterized as $\vec{s} = |\vec{s}|(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$ with polar and azimuthal angles (θ_0, ϕ_0) . Since the group of all linear unitary operations over a single qubit is equivalent to a $\mathbb{S}\mathbb{U}(2)$ algebra, here what interests us most is to investigate the performance of $\mathcal{A}(G, \rho)$ under the $\mathbb{S}\mathbb{U}(2)$ group. The group element of $\mathbb{S}\mathbb{U}(2)$ can be expressed as

$$R(\omega, \theta, \phi) = \exp\left(-i\frac{\omega}{2}\hat{n} \cdot \vec{\sigma}\right) = \begin{pmatrix} \cos\frac{\omega}{2} - i \sin\frac{\omega}{2} \cos\theta & -ie^{-i\phi} \sin\frac{\omega}{2} \sin\theta \\ -ie^{i\phi} \sin\frac{\omega}{2} \sin\theta & \cos\frac{\omega}{2} + i \sin\frac{\omega}{2} \cos\theta \end{pmatrix}, \tag{28}$$

where \hat{n} is a real unit vector and for simplicity we denote the triple as a vector $\vec{\omega} = \{\omega, \theta, \phi\}$. With the identity $\| [R(\vec{\omega}), \rho] \|_F^2 = 2 \text{Tr} \rho^2 - 2 \text{Tr} [\rho R(\vec{\omega}) \rho R(\vec{\omega})^\dagger]$ and $\|\rho\|_F^2 = \text{Tr} \rho^2$, the quantity $\mathcal{A}(G, \rho)$ for $\mathbb{S}\mathbb{U}(2)$ group can be explicitly calculated

$$\begin{aligned} \mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho) &= \frac{1}{4\|\rho\|_F^2} \int_{\vec{\omega}} \| [R(\vec{\omega}), \rho] \|_F^2 d\vec{\omega} \\ &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \int_{\vec{\omega}} \text{Tr} [\rho R(\vec{\omega}) \rho R(\vec{\omega})^\dagger] d\vec{\omega} \\ &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \sin^2\frac{\omega}{2} F(\omega, \theta, \phi) d\omega \\ &= \frac{1}{2} - \frac{1}{4 \text{Tr} \rho^2} = \frac{1}{2} - \frac{1}{2(1 + |\vec{s}|^2)}, \end{aligned} \tag{29}$$

where we have used the integral formula for a functional $F(\vec{\omega})$ over the $\mathbb{S}\mathbb{U}(2)$ group⁷²

$$\int d\vec{\omega} F(\vec{\omega}) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \sin^2\frac{\omega}{2} F(\omega, \theta, \phi) d\omega, \tag{30}$$

and in our case the functional $F(\omega, \theta, \phi) = \text{Tr} [\rho R(\vec{\omega}) \rho R(\vec{\omega})^\dagger]$ can be represented as

$$\begin{aligned} F(\omega, \theta, \phi) &= \frac{1}{16} \left\{ 8 + 3|\vec{s}|^2 + |\vec{s}|^2 \left[5 \cos\omega + 2 \sin^2\frac{\omega}{2} (\cos 2\theta_0 + \cos 2\theta \right. \right. \\ &\quad \left. \left. + 3 \cos 2\theta \cos 2\theta_0) \right] + 8|\vec{s}|^2 \sin^2\frac{\omega}{2} [\cos 2(\phi - \phi_0) \sin^2\theta \sin^2\theta_0 \right. \\ &\quad \left. + \cos(\phi - \phi_0) \sin 2\theta \sin 2\theta_0 \right\} \end{aligned} \tag{31}$$

As expected, the result²⁹ indicates that $\mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho)$ is a basis-independent quantity, that is, it does not rely on the initial orientation of \vec{s} , although $F(\omega, \theta, \phi)$ explicitly depend on θ_0 and ϕ_0 . Moreover, when $|\vec{s}|=0$, the initial state ρ is the maximally mixed state ρ_* and meanwhile $\mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho_*) = 0$. This observation coincides with the fact that for any symmetry and for any representation of the symmetry, the completely mixed state ρ_* is invariant under all symmetry transformations³⁴. Intriguingly, we build up a relationship between $\mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho)$ and $\mathcal{C}(\rho)$ for arbitrary single-qubit system

$$\mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho) = \frac{1}{2} - \frac{1}{2[1 + \mathcal{C}^2(\rho)]}. \tag{32}$$

Apparently, $\mathcal{A}(\mathbb{S}\mathbb{U}(2), \rho)$ is a monotonic increasing function of $\mathcal{C}(\rho)$, which implies that more intrinsic coherence signifies more asymmetry under the $\mathbb{S}\mathbb{U}(2)$ group. Such a simple but novel relation motivates us to inspect $\mathcal{A}(G, \rho)$ for N -qubit states by invoking the tensor product structure of Hilbert space.

An arbitrary N -qubit (pure or mixed) states can be written as

$$\rho = \frac{1}{2^N} \sum_{x_1, \dots, x_N=0}^3 T_{x_1, \dots, x_N} \sigma_{x_1}^1 \otimes \dots \otimes \sigma_{x_N}^N, \tag{33}$$

where $\sigma_{x_j}^j$ ($x_j = 1, 2, 3$) are the standard Pauli matrices in the Hilbert space of qubit j and σ_0^j is the corresponding identity operator. The set of real coefficients T_{x_1, \dots, x_N} with $x_j = 1, 2, 3$ constitutes the so-called correlation matrix T . To gain further insight into the property of $\mathcal{A}(G, \rho)$, we consider the symmetry group $G = \mathbb{S}\mathbb{U}(2) \otimes \dots \otimes \mathbb{S}\mathbb{U}(2)$, where the group element is $R_{\text{ind}} = R_1 \otimes \dots \otimes R_N$ and $R_j \in \mathbb{S}\mathbb{U}(2)$ are *independent* local unitary operations acting on qubit j . Similar to the single-qubit case²⁹, the key procedure is to calculate the average of $\text{Tr}[\rho R_{\text{ind}} \rho R_{\text{ind}}^\dagger]$ over $\mathbb{S}\mathbb{U}(2)^{\otimes N}$. Due to the tensor product structure of N -qubit states and the trace formula $\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$, we have the relation

$$\text{Tr}[\rho R_{\text{ind}} \rho R_{\text{ind}}^\dagger] = \frac{1}{4^N} \sum_{x_1, \dots, x_N=0}^3 \sum_{y_1, \dots, y_N=0}^3 T_{x_1, \dots, x_N} T_{y_1, \dots, y_N} \prod_{j=1}^N \text{Tr}(\sigma_{x_j}^j R_j \sigma_{y_j}^j R_j^\dagger). \tag{34}$$

Therefore, the problem reduces to the evaluation of the integral $\int_{\overline{\omega}} \text{Tr}(\sigma_{x_j}^j R_j \sigma_{y_j}^j R_j^\dagger) d\overline{\omega}$. In fact, we can prove that only when $x_j = y_j = 0$, this integral is nonzero, namely (see Methods section)

$$\int_{\overline{\omega}} \text{Tr}(\sigma_{x_j}^j R_j \sigma_{y_j}^j R_j^\dagger) d\overline{\omega} = \begin{cases} 2, & \text{if } x_j = y_j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

Finally, we obtain the asymmetry of N -qubit states under $\mathbb{S}\mathbb{U}(2)^{\otimes N}$ in a general form

$$\begin{aligned} \mathcal{A}(\mathbb{S}\mathbb{U}(2)^{\otimes N}, \rho) &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \frac{1}{4^N} \sum_{x_1, \dots, x_N=0}^3 \sum_{y_1, \dots, y_N=0}^3 T_{x_1, \dots, x_N} T_{y_1, \dots, y_N} \\ &\quad \times \prod_{j=1}^N \int_{\overline{\omega}} \text{Tr}(\sigma_{x_j}^j R_j \sigma_{y_j}^j R_j^\dagger) d\overline{\omega} \\ &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \frac{1}{4^N} T_{0 \dots 0} T_{0 \dots 0} 2^N = \frac{1}{2} - \frac{1}{2^{N+1} \text{Tr} \rho^2} \\ &= \frac{1}{2} - \frac{1}{2 + 2(2^N - 1)\mathcal{C}^2(\rho)} = \frac{1}{2} - \frac{1}{2 + 2^{N+1} \mathcal{I}_{\text{BZ}}(\rho)}, \end{aligned} \tag{36}$$

where the coefficient $T_{0 \dots 0} = \text{Tr} \rho = 1$. Obviously, this result reduces to the single-qubit case for $N=1$ and the maximally mixed state $\rho_* = \mathbb{I}_{2^N}/2^N$ still corresponds to the minimum value of $\mathcal{A}(G, \rho)$ $\mathcal{A}(G, \rho_*) = 0$. In addition to the unambiguous and monotonous relation between $\mathcal{A}(\mathbb{S}\mathbb{U}(2)^{\otimes N}, \rho)$ and $\mathcal{C}(\rho)$ (hence $\mathcal{I}_{\text{BZ}}(\rho)$), it is worthwhile to note that $\mathcal{A}(\mathbb{S}\mathbb{U}(2)^{\otimes N}, \rho)$ is independent of the correlation matrix T , more precisely, independent of the internal quantum correlations (e.g., entanglement) between subsystems.

Intuitively, one might be tempted to conjecture that *any* local symmetric transformation group G will lead to a *correlation-independent* measure of asymmetry. However, this is not the case. To illustrate this, let us take a closer look at local unitary transformations. Other than the independent local unitary transformations considered above, their *collective* counterparts also plays a crucial role in quantum information and computation⁷³. In this circumstance, the element of symmetry group is the tensor product of N *identical* unitary transformations $R(\overline{\omega})$, that is, $R_{\text{col}} = R \otimes \dots \otimes R$. Take two-qubit states for example and we obtain (see Methods section for more details)

$$\begin{aligned} \mathcal{A}(R \otimes R, \rho) &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \frac{1}{16} \sum_{x_1, x_2=0}^3 \sum_{y_1, y_2=0}^3 T_{x_1 x_2} T_{y_1 y_2} \\ &\quad \times \int_{\overline{\omega}} \text{Tr}(\sigma_{x_1}^1 R \sigma_{y_1}^1 R^\dagger) \text{Tr}(\sigma_{x_2}^2 R \sigma_{y_2}^2 R^\dagger) d\overline{\omega} \\ &= \frac{1}{2} - \frac{1}{2 \text{Tr} \rho^2} \frac{1}{16} \left(4 + \frac{4}{3} \sum_{i,j=1}^3 T_{ii} T_{jj} \right) \\ &= \frac{1}{2} - \frac{1}{8 \text{Tr} \rho^2} \left(1 + \frac{1}{3} \sum_{i,j=1}^3 T_{ii} T_{jj} \right). \end{aligned} \tag{37}$$

It is evident that the collective effect exhibits a significant influence on the asymmetry measure and $\mathcal{A}(R \otimes R, \rho)$ indeed relies on the diagonal entries of the correlation matrix T , which are basis-dependent quantities.

Discussion

In this work, we first formulate a coherence measure $\mathcal{C}(\rho)$ based on the Frobenius norm, which is defined from a geometric perspective. Remarkably, we have demonstrated that this quantity is not only a measure of mixedness, but also an intrinsic (*basis-independent*) quantification of quantum coherence contained in quantum states, which can also be viewed as a normalized version of Brukner-Zeilinger information⁵². To further illustrate this point, a comparison can be made between $\mathcal{C}(\rho)$ and the coherence measures proposed by Baumgratz *et al.*¹⁸. For example, for a single-qubit state $\rho = (1 + \vec{s} \cdot \vec{\sigma})/2$, those quantifiers are listed as follows

$$\mathcal{C}(\rho) = |\vec{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2}, \quad C_R(\rho) = h\left(\frac{1 + s_3}{2}\right) - h\left(\frac{1 + |\vec{s}|}{2}\right), \tag{38}$$

$$C_{I_1}(\rho) = C_{tr}(\rho) = \frac{1}{2}(|s_1 - is_2| + |s_1 + is_2|) = \sqrt{s_1^2 + s_2^2} = \sqrt{|\vec{s}|^2 - s_3^2}, \tag{39}$$

where $h(x) = -x \log x - (1 - x) \log (1 - x)$ is the binary entropy function and $C_R(\rho)$ is the relative entropy of coherence. From the above expressions, it is evident that $C_R(\rho)$, $C_{I_1}(\rho)$ and $C_{tr}(\rho)$ are all basis-dependent measures, since they all depend on s_3 , which is not a unitarily-invariant quantity. Moreover, the participation of s_3 precisely reflects the choice of the pre-fixed basis. In fact, we can present an alternative formulation of $\mathcal{C}(\rho)$ based on the eigenvalues λ_j of ρ , by noticing that $\lambda_j - 1/d$ is the corresponding eigenvalues of $\rho - \rho_*$, which is a Hermitian and traceless operator

$$\mathcal{C}(\rho) = \sqrt{\frac{d}{d-1} \sum_{j=1}^d \left(\lambda_j - \frac{1}{d}\right)^2} = \sqrt{\frac{1}{2(d-1)} \sum_{j,k=1}^d (\lambda_j - \lambda_k)^2}. \tag{40}$$

Note that $\mathcal{C}(\rho) = 1$ for any pure state. This result is just a consequence of the basis-dependent property of $\mathcal{C}(\rho)$: pure states can only be represented as a coherent superposition in *any* pre-fixed basis (without including itself as a base vector), and more precisely, any pure state can always be connected by a unitary transformation within a given dimension. In contrast to quantum correlations, $\mathcal{C}(\rho)$ is a *global* property of quantum states, which quantifies the intrinsic randomness irrespective of the amount of entanglement between subsystems. This is reminiscent of the argument by von Neumann that the entropy of all pure states is defined to be zero as a kind of normalization, but the entropy itself does *not* give a clue on the quantum correlations between subsystems of the (pure) states. Moreover, the current construction of coherence measure is not valid for infinite dimensional systems and a possible solution is to invoke the so-called extended (or unitized) Hilbert-Schmidt algebra instead⁷⁴.

Analogous to previous works^{23,67}, the cohering power $\mathcal{C}(\mathcal{E})$ of quantum channels is introduced in our context. It is demonstrated that $\mathcal{C}(\mathcal{E}) = 0$ if and only if $\mathcal{E}(\cdot)$ is a unital quantum channel, i.e., doubly-stochastic (bistochastic) completely positive map. Therefore, the cohering capacity $\mathcal{C}(\mathcal{E})$ can be recognized as a measure of *nonunitarity* of quantum channels. For instance, for some common types of decoherence processes (such as depolarizing and phase damping channels) $\mathcal{C}(\mathcal{E}) = 0$ since they are unital quantum channels. However, for dissipative channels (e.g., amplitude damping channel) $\mathcal{C}(\mathcal{E})$ is exactly equal to the damping parameter γ , which can be thought of as the probability of losing a photon (here we adopt the notations in ref. 3). Intrudingly, the spontaneous emission process is one of the strongest cohering channels, due to the fact that it maps any state into a certain pure state (e.g., the ground state of a system). As a comparison, it's worth noting that J. Wallman proposed a measure of coherence of quantum operations rather than the coherence of quantum states, where the role of purity is also highlighted⁷⁵. Based on the commutator between quantum state and the elements of a specific symmetric group G , we formulate an asymmetry measure $\mathcal{A}(G, \rho)$ by integrating the Frobenius norm of the commutator over this group. Two distinct situations are considered: local *independent* and *collective* $\mathbb{S}\mathbb{U}(2)$ transformations. For N -qubit quantum states, $\mathcal{A}(G, \rho)$ only relies on the purity and is actually equivalent to $\mathcal{C}(\rho)$ (or \mathcal{I}_{BZ}) under local independent $\mathbb{S}\mathbb{U}(2)$ transformations. However, for local collective $\mathbb{S}\mathbb{U}(2)$ transformations, $\mathcal{A}(G, \rho)$ also depends on the quantum correlations between subsystems. One important open question is to find a closed expression of $\mathcal{A}(G, \rho)$ for arbitrary N -qubit states (especially $N > 3$) in this case.

Methods

Normalization of $\mathcal{C}(\rho)$ and $\mathcal{C}(\mathcal{E})$. Here we only focus on $\mathcal{C}(\rho)$, because $\mathcal{E}(\cdot)$ is a completely positive and trace preserving map and hence $\mathcal{C}(\mathcal{E})$ is also a valid density matrix. Trace-preserving property renders $\text{Tr}(\mathcal{E}(\rho_*)) = \text{Tr}(\rho_*) = 1$, and we obtain

$$\|\mathcal{E}(\rho_*) - \rho_*\|_F^2 = \|\mathcal{E}(\rho_*)\|_F^2 - \frac{1}{d}. \tag{41}$$

Note that for $p \in [1, +\infty)$ and an arbitrary *non-negative* vector \vec{x} , the following inequality holds for vector p -norms⁷⁶

$$\|\bar{x}\|_p \leq \|\bar{x}\|_1^{1/p} \|\bar{x}\|_\infty^{(p-1)/p}. \tag{42}$$

Since the Schatten- p norms of an operator A is equal to its vector p -norm of its singular values (e.g., $\|A\|_p = \|\vec{s}(A)\|_p$), the above inequality also holds for Schatten- p norms. Therefore, we have

$$\|\mathcal{E}(\rho_\star) - \rho_\star\|_F^2 \leq \|\mathcal{E}(\rho_\star)\|_1 \|\mathcal{E}(\rho_\star)\|_\infty - \frac{1}{d} = \|\mathcal{E}(\rho_\star)\|_\infty - \frac{1}{d} \leq 1 - \frac{1}{d}, \tag{43}$$

where we have used $\|\mathcal{E}(\rho_\star)\|_1 = \text{Tr}[\mathcal{E}(\rho_\star)] = 1$ due to the positivity of $\mathcal{E}(\cdot)$ and the monotonicity $\|\mathcal{E}(\rho_\star)\|_\infty \leq \|\rho_\star\|_\infty$. It is quite clear that the equality is satisfied only if the quantum channel $\mathcal{E}(\cdot)$ maps the maximally mixed state ρ_\star to a pure state, which only possesses intrinsic (e.g., basis-independent) coherence in our context.

Necessity of unital channels. In ref. 70, it is proved that a dynamical semigroup is strictly purity-decreasing if and only if the Lindblad generator is unital. Here the quantum dynamic semigroup (in the Schrödinger picture) is a family of one-parameter linear trace-preserving maps $\{\mathcal{E}_\tau, \tau \geq 0\}$ satisfying an additional property except for complete positivity, namely⁷⁷

$$\mathcal{E}_{\tau_1} \mathcal{E}_{\tau_2} = \mathcal{E}_{\tau_1 + \tau_2}, \tag{44}$$

which is called semigroup condition or Markov property. However, we recall that a linear map \mathcal{E} is completely positive *if and only if* it admits a Kraus representation. Thus in our context we are dealing with a more general class of dynamical maps. Note that Nielsen and Chuang stated in their book that a quantum process described in terms of an operator-sum representation cannot necessarily be written down as a master equation, especially for non-Markovian dynamics³. In particular, it is worth pointing out that the technique used in ref. 70 does not work for our purpose. Therefore, the necessity of unitality of operator-sum representation for strictly-decreasing purity is still an important unsolved problem.

Moreover, though we present an elegant proof of the sufficient part, we can reach this conclusion from a different perspective. For $\alpha \in (0, 1) \cup (1, +\infty)$, the quantum α -divergence²⁸

$$D_\alpha(\rho||\sigma) = \frac{1}{1-\alpha} [\text{Tr}(\rho^\alpha \sigma^{1-\alpha}) - 1], \tag{45}$$

where we assume that $\text{supp}(\rho) \subset \text{supp}(\sigma)$. The monotonicity of $\mathcal{C}(\rho)$ under unital channels follows from the facts that $D_\alpha(\rho||\sigma)$ is monotone under completely positive and trace preserving maps in the range $\alpha \in (0, 2]$ ²⁸ and $D_2(\rho||\rho_\star) = (d-1)\mathcal{C}^2(\rho)$.

Integrals over independent and collective local unitary transformations. For independent local unitary transformations, the key task is to calculate the integral $\int_{\vec{\omega}} \text{Tr}(\sigma_x^j R_j \sigma_y^j R_j^\dagger) d\vec{\omega}$. For simplicity, we define $\mathbb{P}_{i,j} = \text{Tr}(\sigma_i R \sigma_j R^\dagger)$. A straightforward observation shows that the evaluations of $\mathbb{P}_{i,j}$ can be divided into four categories

$$\int_{\vec{\omega}} \mathbb{P}_{i,j} d\vec{\omega} = \begin{cases} 2, & \text{if } i = 0, j = 0 \\ 0, & \text{if } i = 0, j \neq 0 \\ 0, & \text{if } i \neq 0, j = 0 \\ 0, & \text{if } i \neq 0, j \neq 0 \end{cases} \tag{46}$$

The first three situations are evident but the last one is not immediately obvious. In order to calculate $\int_{\vec{\omega}} \mathbb{P}_{i,j} d\vec{\omega}$, the key point is that for any σ_j ($j = 1, 2, 3$), one can always find an operator $U \in \mathbb{SU}(2)$ such that $U\sigma_j U^\dagger = \sigma_3$. Moreover, this operator can be absorbed into the average over $\mathbb{SU}(2)$. Therefore, we only need to consider $\mathbb{P}_{3,3}$. Finally, we can obtain

$$\mathbb{P}_{3,3} = \text{Tr}(\sigma_3 R \sigma_3 R^\dagger) = 2 \cos^2 \theta + 2 \sin^2 \theta \cos \omega, \tag{47}$$

where we have used the identity

$$R(\vec{\omega}) = \exp\left(-i\frac{\omega}{2} \hat{n} \cdot \vec{\sigma}\right) = e^{-i\frac{\phi}{2}\sigma_3} e^{-i\frac{\theta}{2}\sigma_2} e^{-i\frac{\omega}{2}\sigma_3} e^{i\frac{\theta}{2}\sigma_2} e^{i\frac{\phi}{2}\sigma_3}, \tag{48}$$

and the commutation relations

$$e^{-i\frac{\theta}{2}\sigma_2} \sigma_3 e^{i\frac{\theta}{2}\sigma_2} = \cos \theta \sigma_3 + \sin \theta \sigma_1, \quad e^{-i\frac{\theta}{2}\sigma_2} \sigma_1 e^{i\frac{\theta}{2}\sigma_2} = \cos \theta \sigma_1 - \sin \theta \sigma_3. \tag{49}$$

Finally, for any $i, j = 1, 2, 3$ we have

$$\int_{\vec{\omega}} \mathbb{P}_{i,j} d\vec{\omega} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} \sin^2 \frac{\omega}{2} (2 \cos^2 \theta + 2 \sin^2 \theta \cos \omega) d\omega = 0. \tag{50}$$

However, the situation is much more complex when confronted with the collective local unitary transformations. Take two-qubit state for example, and in this case we should deal with the integral $\int_{\vec{\omega}} \mathbb{P}_{x_1, y_1} \mathbb{P}_{x_2, y_2} d\vec{\omega}$. Similar analysis leads to the fact

$$\int_{\vec{\omega}} \mathbb{P}_{x_1, y_1} \mathbb{P}_{x_2, y_2} d\vec{\omega} = \begin{cases} 4, & x_1 = x_2 = y_1 = y_2 = 0 \\ 0, & \exists x_j = 0 \text{ or } y_j = 0 \\ \frac{4}{3}, & x_1 = x_2 \neq 0, y_1 = y_2 \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad (51)$$

Special attention should be paid to the case $x_1 = x_2 \neq 0, y_1 = y_2 \neq 0$, that is

$$\int_{\vec{\omega}} \mathbb{P}_{x_1, y_1} \mathbb{P}_{x_2, y_2} d\vec{\omega} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} \sin^2 \frac{\omega}{2} \sin^2 (2 \cos^2 \theta + 2 \sin \theta \cos \omega)^2 d\omega = \frac{4}{3}. \quad (52)$$

We have also considered the three-qubit state and only present the result here

$$\begin{aligned} \int_{\vec{\omega}} \mathbb{P}_{x_1, y_1} \mathbb{P}_{x_2, y_2} \mathbb{P}_{x_3, y_3} d\vec{\omega} &= \frac{2^3}{3} \left(\delta_{x_1, 0} \delta_{x_2 \neq 0, x_3} \delta_{y_1, 0} \delta_{y_2 \neq 0, y_3} \right. \\ &\quad \left. + \delta_{x_2, 0} \delta_{x_1 \neq 0, x_3} \delta_{y_2, 0} \delta_{y_1 \neq 0, y_3} + \delta_{x_3, 0} \delta_{x_1 \neq 0, x_2} \delta_{y_3, 0} \delta_{y_1 \neq 0, y_2} \right) \\ &\quad + 2^3 \delta_{x_1, 0} \delta_{x_2, 0} \delta_{x_3, 0} \delta_{y_1, 0} \delta_{y_2, 0} \delta_{y_3, 0} + \frac{2^2}{3} \varepsilon_{x_1, x_2, x_3} \varepsilon_{y_1, y_2, y_3}, \end{aligned} \quad (53)$$

where δ_{ij} and ε_{ijk} denote the Kronecker delta and Levi-Civita symbol, respectively. However, the generalization to arbitrary N -qubit states is still missing, which is left as an open question for further study.

References

- Gisin, N., Ribordy, G., Tittel, W. & Zbinden, H. Quantum cryptography. *Rev. Mod. Phys.* **74**, 145 (2002).
- Giovannetti, V., Lloyd, S. & Maccone, L. Advances in quantum metrology. *Nat. Photonics* **5**, 222 (2011).
- Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, 2000).
- Schrödinger, E. Discussion of Probability Relations between Separated Systems. *Math. Proc. Cambridge Philos. Soc.* **31**, 555 (1935).
- Glauber, R. J. Coherent and Incoherent States of the Radiation Field. *Phys. Rev.* **131**, 2766 (1963).
- Sudarshan, E. C. G. Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams. *Phys. Rev. Lett.* **10**, 277 (1963).
- Mandel, L. & Wolf, E. *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 1995).
- Born, M. & Wolf, E. *Principles of Optics* (Cambridge Univ. Press, 1999).
- Wolf, E. *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge Univ. Press, 2007).
- Karczewski, B. Degree of coherence of the electromagnetic field. *Phys. Lett.* **5**, 191–192 (1963).
- Wolf, E. Unified theory of coherence and polarization of random electromagnetic fields. *Phys. Lett. A* **312**, 263–267 (2003).
- Tervo, J., Setälä, T. & Friberg, A. T. Degree of coherence for electromagnetic fields. *Opt. Express* **11**, 1137–1143 (2003).
- Setälä, T., Tervo, J. & Friberg, A. T. Complete electromagnetic coherence in the space-frequency domain. *Opt. Lett.* **29**, 328–330 (2004).
- Wolf, E. Comment on ‘Complete electromagnetic coherence in the space-frequency domain’. *Opt. Lett.* **29**, 1712 (2004).
- Setälä, T., Tervo, J. & Friberg, A. T. Reply to comment on ‘Complete electromagnetic coherence in the space-frequency domain’. *Opt. Lett.* **29**, 1713–1714 (2004).
- Horodecki, M. & Oppenheim, J. (Quantumness in the context of) resource theories. *Int. J. Mod. Phys. B* **27**, 1345019 (2013).
- Brandão, F. G. S. L. & Gour, G. Reversible Framework for Quantum Resource Theories. *Phys. Rev. Lett.* **115**, 070503 (2015).
- Baumgratz, T., Cramer, M. & Plenio, M. B. Quantifying Coherence. *Phys. Rev. Lett.* **113**, 140401 (2014).
- Vedral, V., Plenio, M. B., Rippin, M. A. & Knight, P. L. Quantifying Entanglement. *Phys. Rev. Lett.* **78**, 2275 (1997).
- Girolami, D. Observable Measure of Quantum Coherence in Finite Dimensional Systems. *Phys. Rev. Lett.* **113**, 170401 (2014).
- Shao, L.-H., Xi, Z., Fan, H. & Li, Y. Fidelity and trace-norm distances for quantifying coherence. *Phys. Rev. A* **91**, 042120 (2015).
- Streltsov, A., Singh, U., Dhar, H. S., Bera, M. N. & Adesso, G. Measuring Quantum Coherence with Entanglement. *Phys. Rev. Lett.* **115**, 020403 (2015).
- Yao, Y., Xiao, X., Ge, L. & Sun, C. P. Quantum coherence in multipartite systems. *Phys. Rev. A* **92**, 022112 (2015).
- Yuan, X., Zhou, H., Cao, Z. & Ma, X. Intrinsic randomness as a measure of quantum coherence. *Phys. Rev. A* **92**, 022124 (2015).
- Xi, Z., Li, Y. & Fan, H. Quantum coherence and correlations in quantum system. *Sci. Rep.* **5**, 10922 (2015).
- Cheng, S. & Hall, M. J. W. Complementarity relations for quantum coherence. *Phys. Rev. A* **92**, 042101 (2015).
- Rana, S., Parashar, P. & Lewenstein M. Trace-distance measure of coherence. *Phys. Rev. A* **93**, 012110 (2015).
- Rastegin, A. E. Quantum-coherence quantifiers based on the Tsallis relative α entropies. *Phys. Rev. A* **93**, 032136 (2016).
- Napoli, C. *et al.* Robustness of coherence: An operational and observable measure of quantum coherence. *Phys. Rev. Lett.* **116**, 150502 (2016).
- Chitambar, E. *et al.* Assisted Distillation of Quantum Coherence. *Phys. Rev. Lett.* **116**, 070402 (2016).
- Winter, A. & Yang D. Operational Resource Theory of Coherence. *Phys. Rev. Lett.* **116**, 120404 (2016).
- Gour, G. & Spekkens, R. W. The resource theory of quantum reference frames: manipulations and monotones. *New J. Phys.* **10**, 033023 (2008).
- Gour, G., Marvian, I. & Spekkens, R. W. Measuring the quality of a quantum reference frame: The relative entropy of frameness. *Phys. Rev. A* **80**, 012307 (2009).
- Marvian, I. & Spekkens, R. W. The theory of manipulations of pure state asymmetry: I. Basic tools, equivalence classes and single copy transformations. *New J. Phys.* **15**, 033001 (2013).
- Marvian, I. & Spekkens, R. W. Modes of asymmetry: The application of harmonic analysis to symmetric quantum dynamics and quantum reference frames. *Phys. Rev. A* **90**, 062110 (2014).
- Marvian, I. & Spekkens, R. W. Extending Noether’s theorem by quantifying the asymmetry of quantum states. *Nat. Commun.* **5**, 3821 (2014).

37. Marvian, I., Spekkens, R. W. & Zanardi, P. Quantum speed limits, coherence and asymmetry. arXiv:1510.06474.
38. Marvian, I. & Spekkens, R. W. How to quantify coherence: distinguishing speakable and unspeakable notions. arXiv:1602.08049.
39. Chitambar, E. & Gour, G. Are Incoherent Operations Physically Consistent?—A Critical Examination of Incoherent Operations. arXiv:1602.06969.
40. Singh, U., Bera, M. N., Misra, A. & Pati, A. K. Erasing Quantum Coherence: An Operational Approach. arXiv:1506.08186.
41. Bagan, E., Bergou, J. A., Cottrell, S. S. & Hillery, M. Relations between coherence and path information. *Phys. Rev. Lett.* **116**, 160406 (2016).
42. Singh, U., Bera, M. N., Dhar, H. S. & Pati, A. K. Maximally coherent mixed states: Complementarity between maximal coherence and mixedness. *Phys. Rev. A* **91**, 052115 (2015).
43. Bera, M. N., Qureshi, T., Siddiqui, M. A. & Pati, A. K. Duality of quantum coherence and path distinguishability. *Phys. Rev. A* **92**, 012118 (2015).
44. Kagalwala, K. H., Giuseppe, G. D., Abouraddy, A. F. & Saleh, B. E. Bell's measure in classical optical coherence. *Nat. Photonics* **7**, 72 (2013).
45. Svozilik, J., Vallés, A., Peřina, J. J. & Torres, J. P. Revealing Hidden Coherence in Partially Coherent Light. *Phys. Rev. Lett.* **115**, 220501 (2015).
46. Fang, Y. N., Dong, G. H., Zhou, D. L. & Sun, C. P. Quantification of Symmetry. *Commun. Theor. Phys.* **65**, 423–433 (2016).
47. Setälä, T., Kaivola, M. & Friberg, A. T. Degree of Polarization in Near Fields of Thermal Sources: Effects of Surface Waves. *Phys. Rev. Lett.* **88**, 123902 (2002).
48. Setälä, T., Shevchenko, A., T., Kaivola, M. & Friberg, A. T. Degree of polarization for optical near fields. *Phys. Rev. E* **66**, 016615 (2002).
49. Luis, A. Degree of coherence for vectorial electromagnetic fields as the distance between correlation matrices. *J. Opt. Soc. Am. A* **24**, 1063–1068 (2007).
50. Gil, J. J. & Bernabeu, E. Depolarization and polarization indices of an optical system. *Opt. Acta* **33**, 185–189 (1986).
51. Aiello, A. & Woerdman, J. P. Physical bounds to the entropy–depolarization relation in random light scattering. *Phys. Rev. Lett.* **94**, 090406 (2005).
52. Brukner, C. & Zeilinger, A. Operationally Invariant Information in Quantum Measurements. *Phys. Rev. Lett.* **83**, 3354 (1999).
53. Brukner, C. & Zeilinger, A. Conceptual inadequacy of the Shannon information in quantum measurements. *Phys. Rev. A* **63**, 022113 (2001).
54. Munro, W. J., James, D. F. V., White, A. G. & Kwiat, P. G. Maximizing the entanglement of two mixed qubits. *Phys. Rev. A* **64**, 030302 (2001).
55. Wei, T.-C., Nemoto, K., Goldbart, P. M., Kwiat, P. G., Munro, W. J. & Verstraete, F. Maximal entanglement versus entropy for mixed quantum states. *Phys. Rev. A* **67**, 022110 (2003).
56. Neumann, J. V. *Mathematical Foundations of Quantum Mechanics* (Springer, New York, 1932).
57. Lee, J. & Kim, M. S. Entanglement Teleportation via Werner States. *Phys. Rev. Lett.* **84**, 4236 (2000).
58. Rehacek, J. & Hradil, Z. Invariant Information and Quantum State Estimation. *Phys. Rev. Lett.* **88**, 130401 (2002).
59. Brukner, C., Żukowski, M. & Zeilinger, A. The essence of entanglement. arXiv:quant-ph/0106119.
60. Hiai, F., Ohya, M. & Tsukada, M. Sufficiency, KMS condition and relative entropy in von Neumann algebras. *Pacific J. Math.* **96**, 99–109 (1981).
61. Streater, R. F. Convergence of the quantum Boltzmann map. *Comm. Math. Phys.* **98**, 177–185 (1985).
62. Kammerlander, P. & Anders, J. Coherence and measurement in quantum thermodynamics. *Sci. Rep.* **6**, 22174 (2016).
63. Watrous, J. *Theory of Quantum Information* (University of Waterloo, Waterloo, 2011).
64. Bhatia, R. *Matrix Analysis* (Springer, 1997).
65. Horodecki, M., Horodecki, P. & Oppenheim, J. Reversible transformations from pure to mixed states and the unique measure of information. *Phys. Rev. A* **67**, 062104 (2003).
66. Horodecki, M. *et al.* Local Information as a Resource in Distributed Quantum Systems. *Phys. Rev. Lett.* **90**, 100402 (2002).
67. Mani, A. & Karimipour, V. Cohering and decohering power of quantum channels. *Phys. Rev. A* **92**, 032331 (2015).
68. Mendl, C. B. & Wolf, M. Unital Quantum Channels—Convex Structure and Revivals of Birkhoff's Theorem. *Commun. Math. Phys.* **289**, 1057–1086 (2009).
69. Streltsov, A., Kampermann, H. & Bruss, D. Behavior of Quantum Correlations under Local Noise. *Phys. Rev. Lett.* **107**, 170502 (2011).
70. Lidar, D. A., Shabani, A. & Alicki, R. Conditions for strictly purity-decreasing quantum Markovian dynamics. *Chem. Phys.* **322**, 82–86 (2006).
71. Vaccaro, J. A., Anselmi, F., Wiseman, H. M. & Jacobs, K. Tradeoff between extractable mechanical work, accessible entanglement, and ability to act as a reference system, under arbitrary superselection rules. *Phys. Rev. A* **77**, 032114 (2008).
72. Gilmore, R. *Lie Groups, Lie Algebras, and Some of Their Applications* (Wiley, New York, 1974).
73. Bourennane, M. *et al.* Decoherence-free quantum information processing with four-photon entangled states. *Phys. Rev. Lett.* **92**, 107901 (2004).
74. Minh, H., Biagio, M. & Murino, V. Log-Hilbert-Schmidt metric between positive definite operators on Hilbert spaces. In: *Advances in Neural Information Processing Systems* **27** (NIPS), pp. 388–396 (2014).
75. Wallman, J., Granade, C., Harper, R. & Flammia, S. T. Estimating the coherence of noise. *New J. Phys.* **17**, 113020 (2015).
76. Roga, W., Puchała, Z., Rudnicki, Ł. & Życzkowski, K. Entropic trade-off relations for quantum operations. *Phys. Rev. A* **87**, 032308 (2013).
77. Alicki, R. & Lendi, K. *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 2007).

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Author Contributions

Y.Y. and G.H.D. initiated the research project and established the main results under the guidance of C.P.S. and X.X. joined discussions and provided constructive suggestions. Y.Y. wrote the manuscript and all authors commented on the manuscript.

Additional Information

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