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A holistic framework to model student's cognitive process in mathematics education through fuzzy cognitive maps

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ABSTRACT

This study introduces a pioneering framework for modeling students' cognitive processes in mathematics education through Fuzzy Cognitive Maps (FCMs). By integrating key educational theories—Duval's Semiotic Representation Theory, Niss's Mathematical Competencies, Marton's Variation Theory, and the broad Engagement, Motivation, and Participation framework— the model offers a comprehensive and holistic understanding of students' cognitive landscapes. This research underscores the necessity of a multidimensional approach to capturing the intricate interplay of cognitive, affective, and behavioral factors in students' mathematical learning experiences. The novelty lies in its methodological innovation, employing FCMs to transcend traditional qualitative analyzes and facilitate quantitative insights into students' cognitive processes. This approach is particularly relevant in the current era dominated by digital learning environments and artificial intelligence, where real-time, automated analysis of student interactions is increasingly vital. The proposed FCM has been developed over the years with a data-driven approach; the concepts and relationships in it have been derived from the literature and refined by the author's experience in the field. Illustrated through case studies, the framework's utility is demonstrated in diverse contexts, highlighting how the quantitative data obtained are confirmed by qualitative approach: analyzing the impact of remote learning during the Covid-19 pandemic on student engagement and exploring Augmented Reality's role in enhancing mathematical conceptualization. These applications show the framework's adaptability and its potential to integrate new technologies in educational practices. However, the transition from qualitative to quantitative methodologies poses a challenge, given the prevalent use of qualitative approaches in mathematics education research. Additionally, the technological implementation of the FCM model in educational software presents practical hurdles, necessitating further development to ensure ease of integration and use in real-time educational settings. Future work will focus on bridging these methodological gaps and overcoming technological challenges to broaden the FCM model's applicability and enhance its contribution to advancing mathematics education.

1. Introduction

In the field of education, with a specific focus on mathematics, it is absolutely crucial to emphasize the profound importance of comprehending and accurately depicting students' cognitive processes [1,2]. Crafting thorough and intricate models that adeptly

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encapsulate the multifaceted dimensions of students' cognitive processes stands as a pivotal endeavor with far-reaching implications for enhancing pedagogical strategies and ultimately, elevating learning outcomes [3]. Indeed, the development of precise and all-encompassing models that elucidate the intricacies of students' thought patterns carries substantial merit. It serves as an indispensable tool for educators seeking to unravel the mysteries of how students approach mathematical conundrums, uncover their areas of proficiency, and pinpoint their challenges [4]. This, in turn, provides educators with the invaluable ability to provide personalized feedback tailored to support each student's educational journey, either directly or through an e-learning platform that can analyze a student's behavior and react accordingly [5].

One of the primary challenges in modeling student's behavior in mathematics education is accounting for the diverse range of factors that influence a student's learning experience. These factors include, but are not limited to, cognitive abilities, prior knowledge, motivation, learning styles, and socio-cultural backgrounds [6,7]. Capturing the nuances of such a complex process requires sophisticated models that can accommodate these variables and accurately predict student performance and learning trajectories. Another challenge lies in the availability and quality of data. To develop effective models of student behavior, researchers need access to large datasets that encompass various aspects of a student's learning experience [8]. However, obtaining this data can be difficult due to privacy concerns and logistical issues in collecting and storing large volumes of information. Furthermore, the quality of the data may vary, leading to potential biases or inaccuracies in the resulting models [9].

Despite these challenges, several approaches have been proposed and utilized by researchers to model student behavior. Some notable methods include: Bayesian Knowledge Tracing (BKT), Learning Factor Analysis (LFA) and Artificial Neural Networks (ANNs). BKT is a popular approach that models the probability of a student mastering a specific skill, given their history of correct and incorrect responses [10]. By updating these probabilities over time, BKT can provide insights into a student's learning progress and inform personalized instruction. LFA is a data-driven method that seeks to identify the optimal learning sequence for a given set of skills [11]. By analyzing how students perform on different tasks and the relationships between these tasks, LFA can determine the most effective instructional sequence to maximize learning gains. ANNs are a class of machine learning models that have been employed to model student behavior in various educational contexts [12]. By training these models on large datasets, researchers can identify patterns and relationships that may be difficult to detect using traditional statistical methods. Whereas, the authors in [13] delve into the core cognitive processes involved in mathematical investigation, such as specializing, conjecturing, justifying, and generalizing, but the exploration tends to be confined within the boundaries of mathematical investigation tasks, without a broader integration of these processes across diverse problem-solving contexts. Similarly the authors, in their study [14], provide a detailed examination of how math anxiety affects students' cognitive processes during problem-solving tasks, particularly in the context of pattern recognition. While their findings highlight the significant role of affective factors like anxiety in shaping cognitive engagement, the research predominantly concentrates on math anxiety within the narrow scope of pattern material problem-solving. This leaves an open avenue for a broader investigation into how a range of affective and cognitive factors might interact and influence problem-solving abilities in various mathematical areas. In their study, Ekawati et al. [15] delve into the cognitive processes of seventh-grade students as they tackle mathematical problems involving the concept of area conservation. The analysis uncovers that students engage in cyclical approaches throughout problem-solving, encompassing the development of solution strategies, the adjustment of these strategies in action, and the identification and correction of any errors that arise. These strategy cycles reflect significant reflection and adaptation in students' mathematical thinking processes. More precisely, the study highlights how students employ problem decomposition and the application of known formulas for area calculation, and how they react to inconsistencies or mistakes in their initial calculations. This iterative process mirrors the dynamic nature of mathematical thought and underscores the significance of adaptive problem-solving strategies. However, despite the detailed insights into students' cognitive pathways, Ekawati et al.'s work does not extend to integrating these observations into a comprehensive framework that maps the entire spectrum of cognitive processes involved in mathematical learning. While on one hand, the research provides nuanced understandings of the specific cognitive mechanisms underlying problem-solving related to area conservation, on the other hand, it leaves open the question of how these processes fit within a broader cognitive model of mathematics, thus encouraging further research in this area. The study by Campos et al. [16] investigates the relationship between cognitive processes and math performance among third-grade children. The researchers aimed to explore how working memory (WM), selective attention, and general intelligence (g factor) contribute to mathematical achievement in areas like arithmetic story problems and measurement skills. Their findings highlight significant correlations between math performance and several cognitive components, notably the executive visuo-spatial sketchpad and the g factor. This underscores the crucial role cognitive factors play in various mathematical domains. Specifically, the study reveals that the central executive component of WM is particularly influential in mathematical performance, suggesting that the ability to manage and manipulate information actively contributes to mathematical achievement.

To sum up, while previous research has made significant strides in elucidating the cognitive processes underpinning mathematical problem-solving and investigation, much of this literature has tended to examine these processes through a narrow lens, often isolating specific cognitive aspects without considering the broader, interconnected cognitive landscape students navigate. This segmented approach offers valuable insights into individual components of mathematical cognition but falls short of providing a comprehensive understanding of the intricate web of cognitive, affective, and behavioral factors that collectively influence students' mathematical learning experiences. Moreover, the predominant reliance on qualitative methodologies in the study of mathematics education, while invaluable for its depth of insight, poses limitations in contexts requiring real-time, automated interactions, such as in digital learning environments. As we venture further into the golden age of artificial intelligence, the necessity for research frameworks that can seamlessly integrate with AI-driven educational platforms becomes increasingly paramount. This is where the present study makes a groundbreaking contribution by bridging this critical gap.

In this article, an innovative approach to modeling students' cognitive processes in mathematics education through the use of Fuzzy Cognitive Maps (FCM) [17] is presented. The primary goal is to offer insights into the complex world of student cognition, paving the way for personalized learning experiences. In this work it is shown the potential of modeling students' cognitive processes for various purposes.

Firstly, a Fuzzy Cognitive Analysis can be proposed as a methodological framework that, starting from a qualitative analysis and appropriate categorization, attempts to give a quantitative answer to an educational phenomenon through the use of proposed model. An approach such as this can be considered a mixed methods approach. According to Johnson and Christensen [18], mixed methods research is a methodology wherein researchers integrate both qualitative and quantitative strategies (for instance, in terms of perspectives, data gathering, analysis, and inference methods) to achieve a comprehensive and multifaceted understanding while also validating findings. This approach may be applied within a single study or across a series of studies within a research program, with the integration occurring at various stages of the research process. The essence of mixed methods research lies in the strategic combination of qualitative and quantitative techniques at different phases of the research to enhance the depth and breadth of insights into the research questions. The challenge lies in effectively blending the two methodologies to complement each other without redundancy, devising ways to coalesce them seamlessly, and interpreting findings within a cohesive framework. The advantages of employing mixed methods include the concurrent development and verification of theories, the capability to address a wider array of research questions, and the enrichment of theoretical and practical knowledge. To demonstrate the strength and robustness of the proposed model as the focus of a mixed methods approach, two examples of the application of the implemented cognitive map to two distinct educational phenomenon will be presented in Section 4.

The analysis conducted through mixed methods is effective but not suitable for automated environments where immediate responses to student interactions with the learning setting are needed, such as in e-learning platforms. In these scenarios, conducting qualitative analysis in real-time swiftly is not feasible. Therefore, the proposed cognitive map emerges as a valuable tool, encapsulating experts' knowledge in modeling students' cognitive processes. This tool enables quick analysis of the learner's status, allowing for immediate responses, such as customizing educational content or providing tailored feedback to support the student's learning journey. The work of the author [19] present an example of applying the proposed cognitive map as a basis for generating automated feedback. In fact the work proposes an innovative approach within e-learning systems to mitigate high student dropout rates by enhancing student motivation and engagement. The core of the methodology is the utilization of Fuzzy Cognitive Maps (FCMs) to discern learners' current states, particularly in terms of their motivation and engagement, based on their interactions within the e-learning environment. The FCM assesses various behavioral indicators, such as completed sections, viewed videos, and forum activity, to gauge the learner's engagement and motivation levels. Once the learner's situation is understood through the FCM, the system dynamically generates adaptive feedback aimed at bolstering the student's motivation and engagement. This feedback could range from additional learning resources, encouragement messages, to personalized content adjustments, mirroring the role of a teacher who adapts their instructional approach based on the students' needs. The system's ability to provide timely and relevant feedback aims to create a more engaging and supportive learning experience, thereby reducing the likelihood of student dropout. This methodology exemplifies how the integration of FCMs within e-learning platforms can lead to a more responsive and adaptive learning environment, where feedback is not one-size-fits-all but tailored to each student's unique situation, thereby enhancing the overall effectiveness of online learning.

While the proposed approach can be generalized to accommodate students from various domains, it has been chosen to focus on mathematics education in undergraduate classes to meet the research purposes. In the following sections, results based on the proposed fuzzy cognitive map approach, showcasing its value in understanding and improving mathematics education, will be presented. To develop a comprehensive conceptual model of students' cognitive processes in mathematics, it has been drawn upon several well-established frameworks in mathematics education, which will be reported in Section 2; these include:

- Duvall's theory [20], which focuses on the conceptualization of mathematical concepts;
- Niss's framework [21], which delves into problem handling and mathematical thinking;
- Marton's theory [22], which emphasizes inquiry-based learning and understanding;
- Engagement, Motivation and Participation framework in the learning process widely discussed in the literature but accurately summarized by Capone and Lepore in [23].

By integrating these frameworks into the proposed fuzzy cognitive map, this work aims to offer a holistic and accurate representation of students' cognitive processes in mathematics education. The proposed Fuzzy Cognitive Map, in fact, has been developed over the years with a data-driven approach, furthermore, the concepts and relationships in it have been derived from the literature and refined by the author's experience in the field. Using the proposed FCM could lead to the development of more effective teaching strategies and, ultimately, improved learning outcomes for students.

Then, in case study section (Section 4), two distinct examples of FCMs developed within this research will be presented to illustrate the practical application of the proposed approach. Specifically, the first FCM was used to assess the influence of environmental changes on the academic progress of undergraduate mathematics students, while the second FCM was used to evaluate whether the conceptualization of specific mathematical concepts can be enhanced through the utilization of Augmented Reality (AR). These custom FCMs, despite serving different objectives and employing varying theoretical frameworks and numerical values for low-level concepts, are integral components of the same FCM which will be described in Section 3. The rules for value computation are adhered to by these individual maps, as they are a portion of the total map, demonstrating the flexibility of the approach to choose which part of the overall map to use according to the application context.

Table 1
Justification for the selection of educational theories and their integration into the FCM model.

Educational Theory/Framework	Justification for Selection	Integration into FCM Model
Duval's Semiotic Representation Theory	Focuses on cognitive processes involved in transitioning between mathematical representations, fundamental for understanding concepts.	Incorporates nodes for different semiotic registers and edges for cognitive transitions, aligning with Duval's emphasis on representation transitions.
Niss's Mathematical Competencies	Outlines essential competencies for mathematical proficiency, related to cognitive processes the FCM model aims to represent.	Represents each competency as a distinct node, with relationships depicting interactions contributing to mathematical proficiency.
Inquiry-based learning, Method of Varied Inquiry and Marton's Variation Theory	Emphasize learning through discernment of variation and invariance, critical for deep mathematical understanding.	Utilize nodes for mathematical concepts and their variations, with edges illustrating how variations affect understanding.
Engagement, Motivation, and Participation Framework	Integrates affective and behavioral dimensions impacting learning outcomes, providing a holistic view of learning experiences.	Includes nodes for engagement, motivation, and participation dimensions, interconnected with cognitive process nodes to illustrate their influence.

2. Theoretical framework

This section outlines the theoretical foundations underlying the cognitive process model of the student. Several key theories shaped the development of the model, spanning theories of learning, cognition, motivation, and engagement.

Subsection 2.1 presents Duvall's theory of semiotic representation, which illuminates how students construct and interpret representations of mathematical concepts. Subsection 2.2 discusses Niss' framework of mathematical competencies, framing the knowledge and skills required for mathematical proficiency. Subsection 2.3 explores three theories of learning: inquiry-based learning, the Method of Varied Inquiry, and Marton's Variation Theory. These theories model how students can achieve a deep understanding of concepts through systematic variation and reflection. Subsection 2.4 examines the framework of motivation and engagement, including how students' interest in and enjoyment of mathematics can influence their participation and persistence. Finally, subsection 2.5 introduces the theory of Fuzzy Cognitive Maps, which explains how students can develop complex cognitive representations of mathematical concepts.

The selection of Duval's Semiotic Representation Theory, Niss's Mathematical Competencies, Marton's Variation Theory, and the Engagement, Motivation, and Participation Framework as the foundational theories for the Fuzzy Cognitive Map model was motivated by their encompassing nature, which collectively addresses the spectrum of cognitive, affective, and behavioral dimensions critical to mathematics education. These theories provide a rich, empirically supported tapestry of insights into how students understand, engage with, and feel about mathematics, making them particularly suitable as a starting point for the FCM model.

Duval's theory illuminates the cognitive processes involved in transitioning between various mathematical representations, a fundamental aspect of understanding mathematical concepts. Niss's framework complements this by outlining the key competencies required for mathematical proficiency, offering a broad perspective on the skills essential for navigating mathematical content. Marton's Variation Theory adds depth by focusing on learning through discernment of variation and invariance, which is pivotal for achieving a deep understanding of mathematical concepts. Furthermore, the Engagement, Motivation, and Participation Framework integrates the affective and behavioral dimensions that significantly impact learning outcomes, providing a holistic view of students' learning experiences.

The integration of these theories provides a comprehensive framework that captures a wide range of factors influencing students' learning in mathematics. Their empirical support and widespread application in mathematics education lend a robust theoretical foundation to the study. Moreover, the synergy between these theories ensures a broad yet coherent coverage of essential learning dimensions without significant overlap, enhancing the model's efficiency.

While these theories form the core of the proposed FCM, it's important to note that the model is designed with flexibility in mind, allowing for the inclusion of additional theories as new insights emerge or as the scope of the research expands. This adaptability ensures that the FCM can evolve in response to emerging research findings or educational needs, maintaining its relevance and effectiveness in capturing the complex dynamics of students' cognitive processes in mathematics education. An overview of the educational theories selected for the foundation of the Fuzzy Cognitive Map (FCM) model, as well as their integration into the framework, is provided in Table 1.

2.1. Duval's theory of semiotic representation

Within the realm of cognitive psychology, the concept of representation holds significant importance in an individual's knowledge acquisition process. [24] emphasizes that knowledge mobilization relies on representation activity. To fully grasp the theory of registers of representation, it is essential to understand three core characteristics:

- As many distinct semiotic representations exist for a given mathematical object as there are semiotic registers employed in mathematics.
- Each unique semiotic representation of a mathematical object highlights specific properties of the object, rather than an exhaustive depiction of its features. The content explicitly stated corresponds to the representation.
- The content within semiotic representations should never be conflated with the mathematical objects they represent.

[20] asserts that comprehending a mathematical concept necessitates coordinating multiple (at least two) semiotic representation registers. Consequently, transitioning between representations is vital for understanding mathematical concepts. Two primary activities of semiotic representations are treatments and conversions.

Treatments involve transformations within the same register, such as performing calculations using a consistent notation system or completing figures based on connectivity or symmetry criteria. This highlights the intrinsic role of semiotic systems in mathematical processes. The treatments possible rely primarily on the specific register's semiotic transformation capabilities [20]. One criterion for selecting a semiotic register over another is its treatment power.

Conversions entail representation transformations that involve changing the register without altering the objects denoted. Epistemologically, conversions involve selecting a semiotic register that enables more efficient treatment, which cannot be reduced to a mere treatment. From a cognitive perspective, conversion fosters understanding of mathematical concepts. Generally, two distinct representations of an object do not share the same content.

Therefore, coordinating semiotic registers, managing multiple representations of the same concept, and transitioning between them are all necessary for recognizing a concept through one of its representations. This ability is crucial for mathematical comprehension, and it must be nurtured and supported in students, as it is not an inherent skill.

2.2. Niss's framework of mathematical competencies

The KOM-framework (KOM: Competencies and the Learning of Mathematics), as explained in Niss and Højgaard's article "Mathematical competencies revisited" [21] outlines eight distinct yet interrelated mathematical competencies: mathematical thinking, problem tackling, modeling, reasoning, representation, symbols and formalism, communication, and aids and tools. Each of these competencies comprises both a productive and an analytical aspect.

Niss's theory expands on these competencies, providing a more comprehensive understanding of the skills required for mathematical proficiency. The interconnected nature of these competencies is likened to the petals of a flower, where each competency maintains its unique identity while intersecting with others.

The aids and tools competency, in particular, involves two key components. On the one hand, it includes possessing knowledge of various mathematical tools and their properties, as well as understanding their potential applications and limitations in different contexts. On the other hand, it involves the reflective use of these aids, which demonstrates the student's ability to analyze and apply the tools effectively in various mathematical situations [21].

Niss's theory emphasizes the importance of developing these competencies in a holistic manner, as they collectively contribute to a student's overall mathematical proficiency. By examining each competency in detail, educators can better understand the specific skills and knowledge areas that need to be addressed in order to enhance students' mathematical understanding and performance.

Moreover, Niss's framework highlights the dynamic nature of mathematical competencies, suggesting that they can be refined and expanded upon through targeted instruction and practice. This perspective underscores the importance of fostering a growth mindset among students, encouraging them to continuously develop and refine their mathematical skills throughout their educational journey.

In summary, the KOM-framework and Niss's theory provide valuable insights into the various competencies required for mathematical proficiency. By exploring these competencies, educators and researchers can better understand the multifaceted nature of mathematical learning and develop more effective instructional strategies to support students' growth and success in mathematics.

2.3. Inquiry-based learning, method of varied inquiry and Marton's variation theory

Inquiry-based learning [25] is an educational approach that emphasizes the importance of students actively engaging in the learning process through questioning, exploring, and discovering new information. This method encourages students to take ownership of their learning and develop critical thinking skills by focusing on the process of inquiry, rather than solely on the acquisition of knowledge. In this context, the Method of Varied Inquiry (MVI) and Variation Theory can be seen as tools to support and enhance inquiry-based learning experiences.

The Method of Varied Inquiry (MVI), introduced by Arzarello in [26], is a model designed to assist students at all educational levels in examining a subject from multiple perspectives and gaining a deeper understanding. By facilitating their transition from 'natural' forms of argumentation to formal and mathematical reasoning, MVI aims to stimulate students' learning through the use of variation. This method is grounded in the theory of variation, which is closely related to inquiry-based learning.

Variation Theory [22] views learning as a change in the way a student perceives, experiences, and understands a relevant mathematical object through its critical and significant aspects. Central to this theory is the idea that students need to effectively manage novel situations arising during the teaching and learning processes in mathematics [27]. Based on this concept, Variation Theory portrays the students' learning process as a dynamic, science-like controlled experiment where one variable is changed to observe how another variable changes in response. As a result, student learning can only occur after experiencing variation.

Underpinning the Variation Theory is the assumption that teaching with variations on specific aspects of a phenomenon aids students in constructing mathematical meanings. In teaching situations that incorporate variations, learners are encouraged to engage in this type of learning process by considering a sequence of four patterns of variation [22]: contrast, generalization, separation, fusion.

By incorporating these principles, the Method of Varied Inquiry and Variation Theory provide valuable frameworks for supporting inquiry-based learning and promoting a deeper understanding of mathematical concepts.

2.4. Engagement, motivation and participation framework

Engagement, Motivation and Participation are broad concepts that have been studied extensively in the context of education and specifically in mathematics education. A deep literature review and application of these concepts which reflects the current state of art and the complex nature of these variables can be found in the research work of Capone and Lepore in [23]. These elements are considered crucial for creating a student-centered learning environment and reducing the dropout phenomenon, which is common in basic courses such as mathematics. STEM students often view mathematics as a sacrifice to be endured to proceed in their studies.

Motivation to learn is driven by expectations, goals, and emotions [28]. It can be intrinsic, extrinsic, or social [29,30]. Intrinsic motivation occurs when students take a course for its own sake, while extrinsic motivation is driven by external factors such as grades or certificates. Social motivation leads learners to take part in activities to meet others with similar interests.

Student engagement involves the time and energy students devote to educationally sound activities, policies, and practices that institutions use to induce students to take part in these activities [31]. Engagement includes active, passive, and disengaged participation, as well as behavioral, cognitive, and emotional engagement [32]. Behavioral engagement refers to involvement in learning tasks and environments, cognitive engagement refers to psychological investment in the learning process, and emotional engagement refers to affective reactions to learning tasks and environments [33].

Participation is another important factor for educational success, and it refers to the action of taking part in activities and projects [34,35]. Participation fosters mutual learning, and collaboration is a helpful tool used within the participatory culture as a desired educational outcome. Participation is fundamental to engage students in both in-person and online courses, and it is one of the most important aspects of student learning.

These definitions suggest that engagement extends beyond participation and includes students' interactions, assignments, and forum activities. A teacher cannot assume motivation behind a student's participation, and understanding the construct of engagement is an important step toward creating a valid measure of student engagement. By doing so, teachers can adapt their teaching methodologies accordingly and create a strong educational community that fosters emotional involvement and social activities.

In conclusion, motivation, engagement, and participation are crucial elements for creating a student-centered learning environment and reducing the dropout phenomenon. Teachers should understand the construct of engagement to create a valid measure of student engagement and adapt their teaching methodologies accordingly. Participation fosters mutual learning and collaboration, and it is fundamental to engage students in both in-person and online courses.

2.5. Fuzzy cognitive map

Fuzzy Cognitive Maps (FCMs) are a graphical representation and modeling technique that combines the principles of fuzzy logic and cognitive mapping to capture the complexity and uncertainty in dynamic systems [17]. FCMs have been extensively used in a wide range of applications, including social sciences, engineering, environmental management, and healthcare. The primary advantage of FCMs lies in their ability to model complex relationships among system components while accounting for the inherent ambiguity and imprecision that often exist in real-world scenarios.

An FCM is a directed graph consisting of nodes (concepts) and edges (causal relationships) between nodes. Each node represents a concept or variable in the system, and each edge denotes the causal influence between two nodes. The edges are assigned fuzzy weights, typically in the range of [-1, 1] and often described using fuzzy linguistic terms (e.g., low, high, very high, etc.), indicating the strength and direction of the causal relationships. Positive weights represent positive relationships (the increase of one concept leads to the increase of the other), while negative weights indicate negative relationships (the increase of one concept leads to the decrease of the other).

The system's state is represented by a vector of concept values, usually normalized to the range [0, 1]. The dynamic behavior of the system is modeled by updating the concept values based on the weighted influence of other concepts, typically using a sigmoid or hyperbolic tangent function to ensure that the updated values remain within the defined range. The iterative update process continues until the system reaches a stable state (equilibrium) or a predefined number of iterations is achieved.

FCMs have been employed in numerous applications to model complex systems and support decision-making processes. For example, in environmental management, FCMs have been used to analyze the impact of human activities on ecosystems and develop sustainable management strategies [36]. In healthcare, FCMs have been applied to model the progression of diseases and evaluate the effectiveness of various treatment options [37].

One notable advantage of FCMs in decision-making is their ability to incorporate expert knowledge and subjective judgments, making them particularly suitable for situations where data is scarce or uncertain. Additionally, FCMs can be easily combined with other computational techniques, such as optimization algorithms and machine learning, to enhance their predictive capabilities and support more informed decision-making. Future research in this area could explore the development of advanced learning algorithms for FCMs, the integration of FCMs with other computational methods, and the application of FCMs to new and emerging domains, further expanding their potential to contribute to our understanding and management of complex systems.

3. Building a FCM to model student's cognitive process in mathematics education

Before employing Fuzzy Cognitive Maps (FCMs), a comprehensive review of the scientific literature was undertaken to investigate which tool could best model student's cognitive process in mathematics education. This review assessed various fuzzy-type intelligence computing approaches suggested by the community to depict structures and behaviors, as well as diverse dynamic situations.

Compared to other fuzzy methodologies, FCMs offer certain benefits [38], such as being rooted in causal cognitive mapping, which enables efficient elicitation and capture of domain experts' knowledge in an intuitive format that can be easily managed and updated.

An FCM is typically generated through a collaborative process involving multiple experts. The author's approach is outlined as follows: the designed FCM is product of a consensus process in which four education experts participated. Drawing inspiration from one or more methodological frameworks commonly employed in mathematics education for describing current research, as reported in Section 2, each expert proposed an FCM identifying causal relationships and weights among concepts. Weights are denoted by linguistic terms such as no impact (0.00), very low (0.165), low (0.335), medium (0.50), almost high (0.665), high (0.835), and very high (1.00). The experts then combined the various maps to create a single FCM. In cases of disagreement regarding relationships and weights, the experts engaged in discussions to reach a consensus following the process showed in [39].

The proposed FCM is organized in multiple layers, as reported in Fig. 1. The thematic aspects chosen by the experts from the theoretical framework presented in Section 2 and from their own experience are represented in map nodes. For consistency, each node in the map is called, in this context, "concept", using the cognitive psychology mapping terminology [40]. The lowest level typically features concepts representing atomic variables and establishes a clear numerical correspondence with the observed phenomenon, which can be initialized in various ways. Obviously, being a fuzzy map, the value assigned to the concepts must be in the range [0,1]. One approach could be to employ an AI-based tool for interpreting data from audio-video streams. Alternatively, a human expert can estimate values based on qualitative observations during experimentation. Consistency is crucial. For instance, if a student is considered very happy, the variable "Happiness" is assigned a value of 1, while not happy enough corresponds to 0.375. Similarly, if a new student appears quite happy, the value assigned should be 0.750 to maintain consistency, rather than 1. Activation levels of these "leaf" concepts indicate the variable's value. When the value of one of these variables changes, other FCM concepts are influenced according to the causal relationships between them.

Middle layer generally contains nodes that make up the top-level concepts of the model they are intended to represent. They are an aggregation of the low-level concepts, which represent a coarser granularity and are functional in defining the detailed aspects of the theoretical frameworks considered. For example, if the high-level concept to be modeled is motivation, intermediate-level concepts contributing to its definition could include intrinsic motivation, extrinsic motivation, and social motivation. If the high-level concepts in the map were chosen taking into account mainly the theoretical framework chosen, the low and medium level concepts were chosen taking into account also these three factors: the study of the scientific literature regarding high level concepts; the gained experience by the author in recent years in using this technique to model the student's state; the feedback the author received from other researchers.

A list with all the concepts in the proposed map and their description is given in Table 5 in Appendix A, whereas here, the Table 2 reports some of the most relevant concepts in the map are described. The table basically summarizes different variables and their alignment with different theoretical frameworks and concepts. These variables intersect with different theories and concepts, including Niss's framework of Mathematical Competencies (KOM-framework), Inquiry-based learning, Marton's Variation theory, Dual's theory, and the broader Engagement, Motivation, and Participation framework, highlighting how they relate to students' cognitive processes in mathematics learning. It is essential to recognize that the foundations of all low-level concepts originate in the domain of mathematical learning. As these concepts aggregate to higher levels, they give rise to notions that are intricately linked to the specific subject area under consideration. For instance, the well established frameworks proposed by Niss and Duval are firmly rooted in the discipline of mathematics education. On the other hand, although high-level concepts such as motivation, commitment and participation may seem generic and applicable in different contexts, it is crucial to emphasize that their values derive from the aggregation of lower-level concepts exclusive to mathematics education. Therefore, at a high level, these concepts inherently belong to the domain of mathematics education. The calculated entities do not represent generic engagement, motivation and participation, but rather those specific to mathematics education.

The activation values of the concepts of the middle and final layers are computed, starting by the activation levels of the input layer, using the inference process of the FCM. Specifically, the activation level of the concept can be iteratively calculated, as reported in equation (1):

$$A_i^{k+1} = \mathrm{f}\left(A_i^k + \sum_{j=1, j\neq i}^{n} A_j^k w_{ji}\right) \tag{1}$$

where A_i^{k+1} is the activation value of concept C_i at time $k+1$, A_j^k is the activation level of the concept C_j at time k, w_{ji} is the weight between concept C_j and C_i, $f(\cdot)$ is a transformation function. In this work, a linear function for $f(\cdot)$ is used, as reported in equation (2):

$$A_i^{k+1} = \alpha\left(A_i^k + \sum_{j=1, j\neq i}^{n} A_j^k w_{ji}\right) \tag{2}$$

Here, α is a real-valued coefficient that influences the impact of the current activation level and the weighted sum of activations from other concepts. This approach facilitates the gradual refinement of activation values, with each repetition gradually refining the representation of concepts in the network, this mechanism embodies the essence of Fuzzy Cognitive Maps, wherein the relationships and interactions between concepts are molded iteratively to generate a coherent and dynamic representation of the system's cognitive processes. The use of a linear transformation function $f(\cdot)$ and the parameter α play important roles in governing the transmission of activation values throughout the network's layers, shaping the network's ability to capture complex relationships between concepts over time.

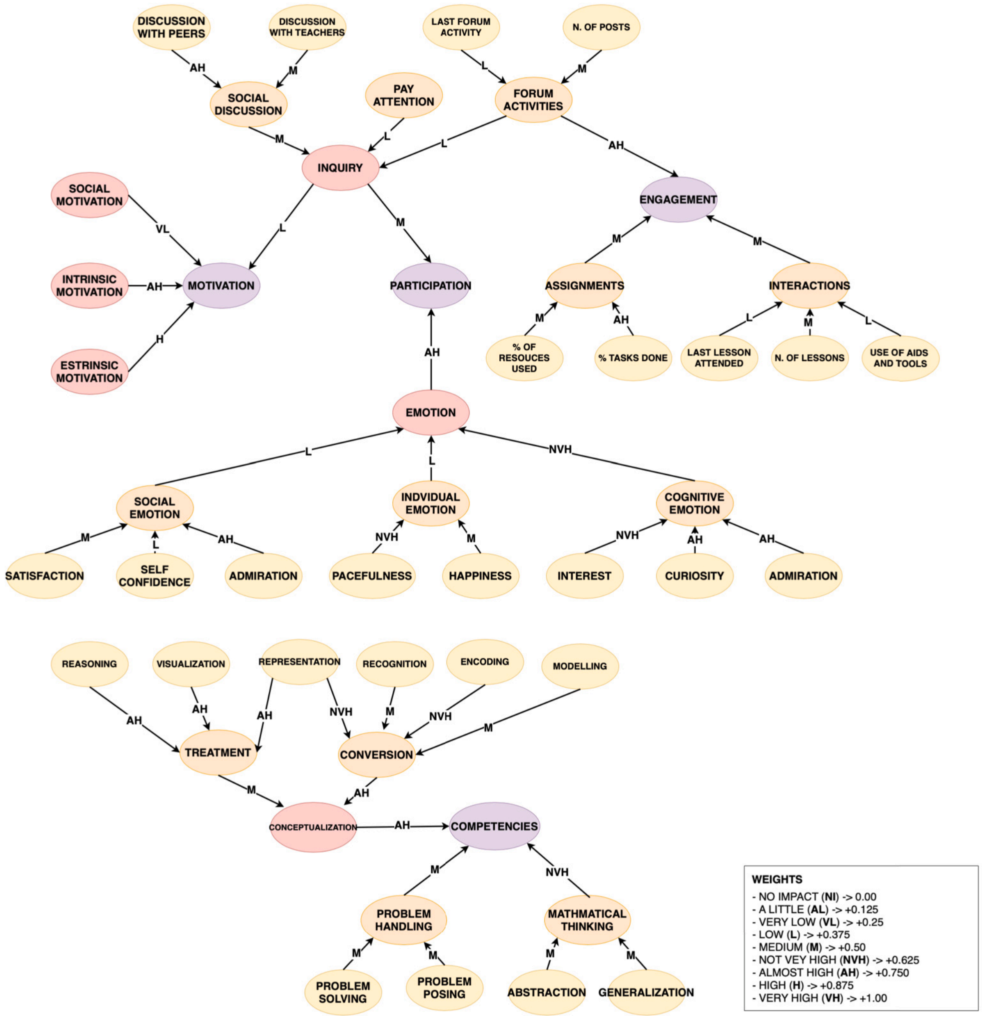

Fig. 1. The proposed FCM to model student's cognitive process.

Table 2
List of most of relevant concepts in the map with a short description and their inclusion in the frameworks. Full list is available in Appendix A.

Concept's name	Framework	Short Description
Competencies	KOM-framework	Development of knowledge, skills, and dispositions including critical thinking with symbols, aiding students' holistic mathematical understanding.
Conceptualization	Duval's theory	It refers to the cognitive process of understanding mathematical concepts through the use and interrelation of various semiotic representations such as symbols, graphs, diagrams, and natural language.
Emotion	Engagement, Motivation, and Participation framework	It includes the range of affective states that impact students' status during the learning process.
Participation	Engagement, Motivation, and Participation framework	It involves students taking an active role in their education by contributing, interacting, and collaborating with teachers and peers.
Motivation	Engagement, Motivation, and Participation framework	The psychological processes and factors that drive, direct, and sustain an individual's behavior toward a particular goal or desired outcome.
Engagement	Engagement, Motivation, and Participation framework	Active involvement, interest, and interactions of students in the process of learning mathematics.
Inquiry	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Student's active exploration and questioning to foster deeper understanding.

4. Using the FCM

In this section, to illustrate the practical application of the proposed approach, two distinct examples of FCMs developed within this research will be presented. Specifically, a first FCM designed to assess the influence of environmental changes on the academic

Table 3
Likert scales about statements of agreement, frequency, and satisfaction.

Levels	(1)	(2)	(3)	(4)	(5)
Agreement	Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree
Frequency	Never	Rarely	Sometimes	Often	Always
Satisfaction	Not at all Satisfied	Slightly satisfied	Moderately Satisfied	Very satisfied	Completely Satisfied

progress of undergraduate mathematics students, and a second FCM created to evaluate whether the utilization of Augmented Reality (AR) can enhance the conceptualization of specific mathematical concepts are shown. These custom FCMs, despite serving different objectives and employing varying theoretical frameworks and numerical values for low-level concepts, are integral components of the same over-arching FCM described in Section 3, adhering to identical rules for value computation. As the individual maps are a portion of the total map, it demonstrates the flexibility of the approach to choose which part of the overall map to use according to the application context. Each example will commence by elucidating the motivations behind employing these maps, followed by an examination of their structural aspects, and conclude by summarizing the key findings.

4.1. Evaluating the impact of changing environment on student's motivation, engagement and participation

In this initial example, conducted by the author in, the examination focuses on how changes in the learning environment affect students' motivation, engagement, and participation. The aim was to gain insights into patterns of student motivation, engagement, and participation, comparing the pre-Covid-19 era of blended learning with the pandemic period of distance learning through online platforms.

4.1.1. Context

Before proceeding with the details of the methodology, it is important to give a brief background related to the changing environment's impact on students. The Covid-19 pandemic served as a change in the learning environment that severally affects the school system. Students faced emotional stress and were frustrated from studying. Recent literature, i.e. Rutherford and colleagues [41], reported that students had low engagement levels with the use of technology for teaching during the pandemic. It was also highlighted that students experienced motivational changes: lower emotional cost and lower mathematic expectancy [42,43]. The study [44] was conducted with first-year engineering students enrolled in the Calculus II course during the academic years 2018/19, 19/20, and 20/21 at University of Salerno.

Specifically, the course was carried out during the second semester of the first year after students had attended and/or taken a Calculus I exam. The 2018/2019 academic session was conducted face-to-face, integrating conventional educational approaches with modern technological tools, such as augmented reality devices. Conversely, the academic years of 2019/2020 and 2020/2021 were conducted remotely. The course included 90 hours of lessons (six hours per week), divided into 54 hours of theory lessons and 36 hours of training; also 24 hours of exercises with the tutors, dividing the students into two sub-groups. The research team consisted of the course lecturer and two tutors. The following online resources have been used during Covid-19 pandemic: custom adaptive e-learning platform [45], Microsoft Teams, Doceri, Edmodo, the teacher's website, the teacher's YouTube channel, Geogebra AR. Some of these have already been used in previous courses as a support for face-to-face teaching. The contents of the course were: Linear Algebra, Differential equations, Functions of several variables, 3D Geometry, Curves, and Integral Curves, Double Integrals, Triple Integrals, Differential Forms, Surface and surface integrals, Function Series.

4.1.2. Methodology

The methodology employed in this study revolved around a singular case study aimed at identifying educational strategies to address the challenges posed by the shift to remote teaching due to the pandemic. This situation differed from prior research where the integration of technology in teaching was at the discretion of the educators [46,47]. In this scenario, educators were compelled to rely on Information and Communication Technologies (ICT) as the sole medium for instruction. Empirical data on the effectiveness of these teaching methods were gathered through direct observations of interactions between students and teachers mediated by technological tools. The analysis juxtaposed outcomes from exclusively online teaching with data from a preceding study conducted in a blended learning format during the 2018-2019 academic year. The investigative approach combined ideographic methods, utilizing qualitative tools such as student questionnaires, with nomothetic methods that analyzed quantitative data derived from test results, responses to Likert scale questionnaires, and student engagement with the e-learning platform. Students were surveyed to elicit their perceptions on encountered challenges, topics they found particularly difficult, and the significance of utilizing an e-learning system. The survey included hierarchical questions that required students to prioritize various aspects of a phenomenon using a Likert scale, alongside open-ended questions allowing for unrestricted expression, aiming to gauge the emotional responses of the students and gather insights on the strengths and weaknesses of the adopted pedagogical approach. The questionnaires were distributed via Google Forms, with multiple-choice items structured around a conventional five-point Likert scale, as reported in the Table 3. Additional qualitative data were extracted from the analysis of communications between students and between students and instructors within the e-learning platform. A custom Fuzzy Cognitive Map (FCM), reported in Fig. 2, used both qualitative and quantitative data to assess students' engagement, motivation, and participation according to the theoretical framework presented in the subsection 2.4. As can be noted this map is included in the overall map presented in section 3. Where-as the quantitative approach

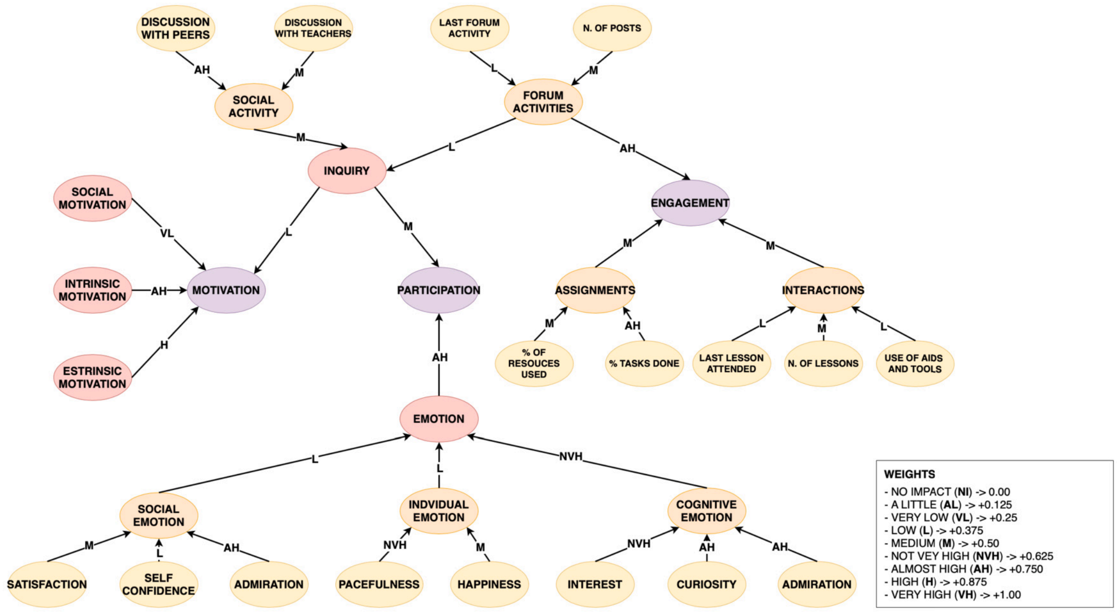

Fig. 2. FCM used to model students' engagement, motivation and participation.

identified instructional, educational, and training strategies that could be effective under specific conditions, while the qualitative approach provided insights into the underlying reasons for their effectiveness. The data on students' engagement, motivation, and participation computed by running the FCM reported in Fig. 2 were analyzed by comparing the results across the academic years 2018/2019, 2019/2020, and 2020/2021. Notably, blended learning was used in the academic year 2018/2019, while full distance learning was implemented in the academic years 2019/2020 and 2020/2021. The data on student engagement were gathered through tracking their interactions within the utilized platforms. The aspects concerning motivation were evaluated based on the responses collected from questionnaires distributed throughout the course. The facets associated with participation, emotional responses, and social involvement were identified through sentiment analysis conducted on video recordings of student webcams and by interpreting the responses to various questions within the questionnaire.

The three classes consisted of 131, 112, and 98 students, respectively. Cochran's formula (1963) was utilized to calculate the sample size for the experiment, as reported in equation (3).

$$n_0 = \frac{Z^2 pq}{e^2} \tag{3}$$

Where:

- e is the desired level of precision (i.e., the margin of error);
- p is the estimated proportion of the population that exhibits the attribute in question;
- q is $1 - p$;
- The Z value is obtained from a Z table and represents the abscissa of the normal curve that cuts off an area α at the tails ($1 - \alpha$ equals the desired confidence level, e.g., 95%);
- n_0 is the sample size.

In this experiment, the chosen parameters were: $\{Z = 2.33; p = 0.90; e = 0.10\}$ resulting in a sample size of 60 students, randomly selected each year.

4.1.3. Quantitative analysis

In this subsection, we delve into the analysis of student engagement, motivation, and participation, drawing comparisons across the academic years 2018/2019, 2019/2020, and 2020/2021. As previously stated, the 2018/2019 academic year incorporated a hybrid approach to learning, blending in-person and online elements, while the subsequent years, 2019/2020 and 2020/2021, transitioned to an entirely online format. As depicted in Fig. 3, the average values for the core concepts of the Fuzzy Cognitive Map (FCM) are presented. The data is segmented into three distinct student cohorts for comparative analysis: the "Blended Learning 2018/2019" cohort (illustrated in blue) encompasses students who participated in physical classroom settings and engaged with the designated e-learning platform during the 2018/2019 academic year. Conversely, the "Online Learning 2019/2020" cohort (highlighted in orange) consists of students who experienced the course exclusively online in the 2019/2020 academic year. Similarly, the "Online Learning

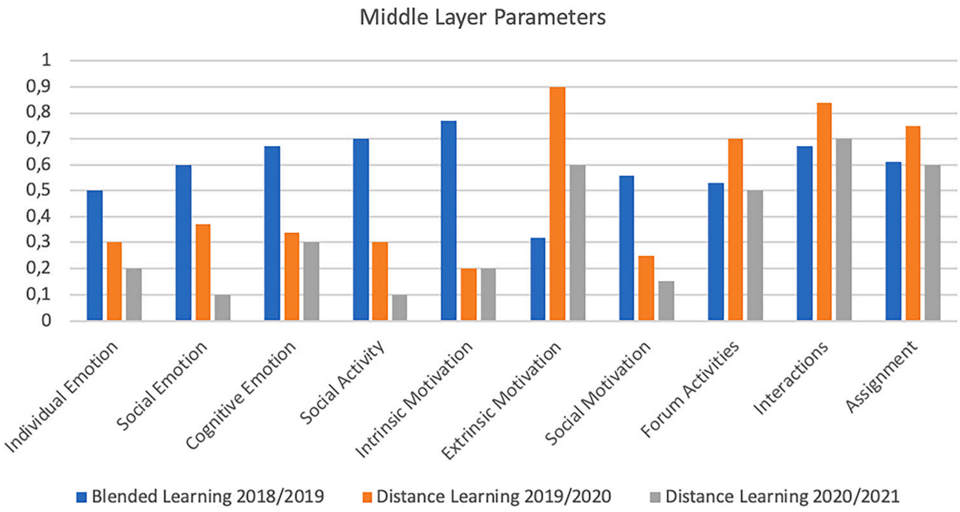

Fig. 3. Middle layer results computed through FCM.

2020/2021" cohort (represented in grey) refers to students who continued with the fully online learning model during the 2020/2021 academic year.

The first three parameters, Individual Emotion, Social Emotion, and Cognitive Emotion, reflect students' emotional states, such as Peacefulness, Happiness, Satisfaction, Self-confidence, Admiration, Interest, Curiosity, Enthusiasm, Pay Attention, and Discussion, as reported in Table 1. These parameters indicated a more positive emotional state during the blended learning approach in the 2018/2019 academic year. However, the emotional state of students in the 2019/2020 and 2020/2021 academic years was affected by the pandemic, resulting in a decline in these parameters, particularly in the second year of distance learning. Social Activity, which represents the level of engagement in terms of paying attention and participating in discussions, significantly dropped from blended learning to the second year of distance learning, despite efforts by teachers to encourage social interaction among students. The parameters related to motivation (Intrinsic Motivation, Extrinsic Motivation, and Social Motivation) exhibited substantial differences between the three academic years. In the 2018/2019 academic year, students attended the course and engaged in activities out of personal enrichment and the desire to share the learning experience with their peers. In contrast, the students in the 2019/2020 academic year were compelled to follow the online course due to external requirements for exam attendance. The situation worsened in the 2020/2021 academic year, where the motivation to take the exam was in-sufficient to sustain student participation. This was evident from the low attendance rates at the final exams: 66% in 2018/2019, 67% in 2019/2020, and 42% in 2020/2021. Parameters related to engagement (Forum Activities, Interactions, and Assignments) demonstrated consistent levels across the three years. Despite the shift to full distance learning, students in the 2019/2020 and 2020/2021 academic years displayed comparable levels of interaction with the e-learning platform as the 2018/2019 students, who used it as an additional tool alongside in-person activities.

The average levels of engagement, motivation, and participation were calculated using the FCM (Fig. 4), which summarized the data analysis of the middle-layer parameters. Despite the challenges and digital barriers of full distance learning, students during the first year of the pandemic demonstrated motivation (albeit extrinsic), engagement, and interaction with teachers through the e-learning platform. However, in the second year of the pandemic, student motivation appeared to decline, with reduced interest and participation in platform activities. Notably, during the first year of the pandemic (2019/2020), the engagement and motivation parameters, while modest, were higher than in the 2018/2019 academic year.

Furthermore, in addition to the utilization of FCM for quantitative evaluation, the study also employed data from Likert scale surveys completed by students at each course's conclusion for year-over-year comparison. The findings indicate that in the 2020-2021 academic year, 64% of students rated their frequency of engaging with digital tools for course activities as 4 or 5 on the Likert scale; this is in contrast to the 40% in 2018-2019 and 69.77% in 2019-2020. In the same academic year, 62.2% of students reported frequent to very frequent interactions (a Likert score of 4 or 5) within the e-learning platform's forum, a notable increase compared to 25% in 2018-2019. The 2019-2020 year saw the highest engagement, with 92% of students actively participating in the forum, which they felt helped to digitally replicate a collaborative study hall atmosphere where they could discuss solutions to the exercises introduced during lessons. Moreover, overarching observations revealed that the imposition of physical distance increased student engagement with the offered educational activities. A notable uptick was observed in student-to-student interactions via the educational social platform, surpassing rates from previous years. Specifically, the per-student average logins to the platform showed a marked increase, with an average of 12 logins in the 2018/2019 academic year surging to 57 in 2019/2020 and 52 in 2020/2021. Despite these changes, the average lecture attendance remained relatively stable, with 45 of 60 students attending in 2018/2019 compared to 42 of 60 in 2019/2020 and 40 of 60 in 2020/2021.

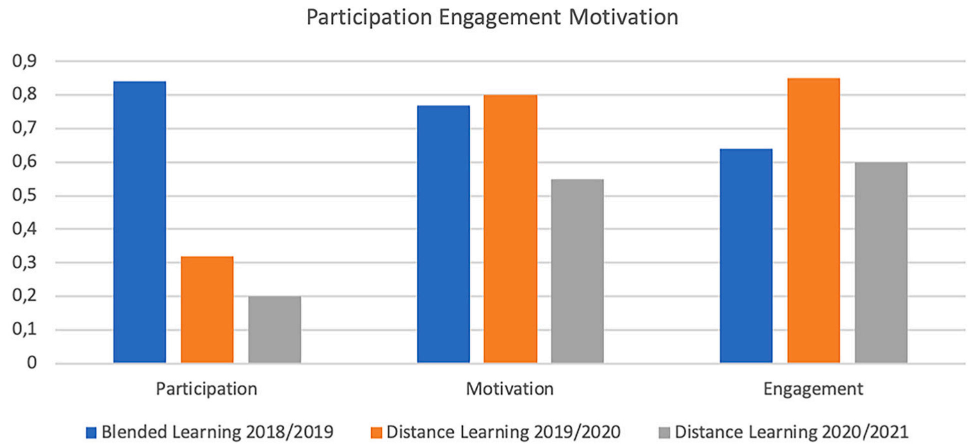

Fig. 4. FCM computed results, participation, motivation and engagement.

4.1.4. Qualitative analysis

The interpretation of quantitative data through the fuzzy cognitive map is echoed in the qualitative feedback gathered from the anonymous questionnaires distributed at the course's conclusion. These questionnaires, along with dialogues from the e-learning platform, have yielded insights into the degrees of participation, engagement, and motivation among the students. The inquiries focused on the students' perceptions of the learning activities and their interactions, their enthusiasm for attending classes and dedication to their studies, as well as their interactions with the e-learning system. In Table 4, a collection of the most noteworthy student responses to the questionnaire over the academic years 2018/2019, 2019/2020, and 2020/2021 is presented, highlighting the concepts of motivation, participation, and engagement. This information assists in conducting a qualitative analysis of the trends in students' motivation, participation, and engagement levels.

In considering motivation, the interactions with educators and the coursework were cited as significant motivators by students S1 and S2 from the academic year 2018/2019. This sentiment was echoed by a wide array of students. During 2019/2020, students S6 and S7 pointed out the challenges of adapting to mandatory online learning, a sentiment prevalent among the student body. The drive to engage in these courses seems to stem more from external pressures than from internal or communal desires, a finding supported by quantitative analyzes. Furthermore, student S13's response indicates that beyond aiming for good grades, there's a desire among students to develop their abilities for future success. For the 2020/2021 academic year, as demonstrated by students S14 and S15, confidence in returning to traditional classroom settings sharply declined, correlating with a marked decrease in overall student motivation when compared to previous years.

When addressing participation, students S4 and S5 from the 2018/2019 academic year emphasized the vital role that course content and peer interactions played in maintaining interest and a favorable outlook on their subjects. Students S10 and S11 noted that the shift to completely remote education adversely impacted their emotional wellbeing, which manifested in reduced focus and engagement. Similarly, students S8 and S9 from the 2019/2020 academic year expressed concerns over the loss of in-person dialogue, a factor that seemed to influence their emotional states negatively. The most profound impact on participation was noted in the 2020/2021 academic year, where students S16, S17, and S18's responses validated the findings from the Fuzzy Cognitive Map (FCM). Student S5's comments further highlighted the challenges of engaging with the course material without the benefit of peer and teacher interaction, emphasizing the emotional toll of the crisis.

Regarding engagement, student S3 from the 2018/2019 academic year pointed out the integral role of the e-learning platform in enriching theoretical understanding and offering regular opportunities for practice through supplementary materials. Echoing this sentiment, student S12 from 2019/2020 brought to light how digital resources were effectively integrated to bolster both theoretical and practical learning. Similarly, students S19 and S20 from 2020/2021 acknowledged the platform's structured approach in providing substantial digital resources for academic support. However, student S21 pointed out that despite the well-structured platform facilitating their studies, the prolonged crisis led to a decrease in engagement.

4.1.5. Discussion

This study employed a hybrid research methodology, blending the breadth and generalizability of quantitative methods with the depth and detail of qualitative insights. The quantitative component offered a broad view of the effectiveness of instructional strategies under various conditions, while the qualitative aspect shed light on the underlying mechanisms and contexts that make certain educational approaches successful.

Through this combined analysis, focusing on the key metrics of motivation, participation, and engagement, it was discovered that student attitudes towards the incorporation of digital tools in mathematics education shifted across the academic years 2018/2019, 2019/2020, and 2020/2021. During the 2018/2019 academic year, the integration of technology was seen as a beneficial enhancement to traditional learning methods. In contrast, the 2019/2020 academic year, marked by the onset of the pandemic, saw technology as a crucial lifeline that allowed students to continue their education and maintain connections with instructors and peers. However, by the 2020/2021 academic year, the perception of technology had evolved into being viewed as a barrier that disrupted the natural

Table 4
List of most noteworthy students' answers to the questionnaire across the years 2018/2019, 2019/2020 and 2020/2021, highlighting the covered concept.

Student	Year	Concept	Student's Answer
S1	2018/2019	Motivation	"I particularly appreciated the teacher's approach to the subject and the time dedicated to classroom exercises to clarify any doubts. The time set for the lessons certainly did not help in terms of attention but in most cases, however, the lesson went on without problems."
S2	2018/2019	Motivation	"It cannot be denied that the Calculus II course is complex, but I can say that the lessons were almost always clear, especially from the point of view of the exercises discussed in class with the teachers. Overall, it was a course that I enjoyed following, considering the difficulties inherent in the subject studied."
S6	2019/2020	Motivation	"Now I am distressed by everything around me; there is the fear of not being able to return to our lives and all this still does not seem true to me. I am not comfortable following the online lessons on the e-learning platform."
S7	2019/2020	Motivation	"I would never have taken a fully online course. I prefer contact with colleagues and teachers even if we have been enabled to study in the best possible way with the considerable volume of teaching material and exercises available on the platform."
S13	2019/2020	Motivation	"In this period of social isolation, studying has become a reason for me not to think about the current state but to have future perspectives."
S14	2020/2021	Motivation	"After almost two years of distance learning, there seems to be no end to this situation. I feel very unmotivated even to study."
S15	2020/2021	Motivation	"I often spend entire days on the computer. I get tired quickly and often don't feel like attending classes in their entirety."
S4	2018/2019	Participation	"The course was made very interesting thanks to the many examples of mathematical applications to physics."
S5	2018/2019	Participation	"I got hooked on the subject thanks to the high level of challenge presented by the exercises. Thanks to the support of colleagues and teachers, it was sustainable."
S8	2019/2020	Participation	"My concerns have concerned both not being able to compare with colleagues in class, perhaps even at the end of the lesson as I have always done before, and certainly also doing the same with the teacher."
S9	2019/2020	Participation	"The main concerns were the impossibility of having a direct confrontation with the teachers, poor communication between students and teachers, a greater detachment between students."
S10	2019/2020	Participation	"I solved the organizational aspect which is very important to me. However, I think I should still work on concentration as in this mode and being in a home environment, it tends to be inferior."
S11	2019/2020	Participation	"A piece of advice I would give is to divide the lessons with breaks, maybe every 45 minutes, to make them less heavy since it is frustrating to be at the PC all day."
S16	2020/2021	Participation	"Not being able to study on-site and deal with the professor and other students in person made me sad. The lack of socialization with the rest of the group completely turned my enthusiasm for studying."
S17	2020/2021	Participation	"Participating in face-to-face classes makes it much easier for me to learn. I find it difficult to take courses at a distance because of too many distractions and the lack of my classmates' presence supporting me in my studies."
S18	2020/2021	Participation	"It was very challenging to take the course at a distance because of the loss of what is student-class-teacher interaction. Better future teaching could keep together in-person and distance modes."
S3	2018/2019	Engagement	"The course was well organized with extra material usable by e-learning, helpful for carrying out the tests, with exhaustive and complete exercises carried out in the classroom. Enough exercises have been available online to keep practicing."
S12	2019/2020	Engagement	"My feelings are positive, the course worked well because in addition to the didactic material always available on the e-learning platform, the teacher was able to integrate the lessons with an interactive whiteboard, which is fundamental for this subject. Great to have also the recordings of the lessons available."
S19	2020/2021	Engagement	"The professors do their best to make us follow the course profitably, providing us with supplementary materials easily available on the e-learning platform."
S20	2020/2021	Engagement	"The feedback on the platform has been constructive in clarifying doubts, especially on more advanced topics."
S21	2020/2021	Engagement	"On the e-learning platform, we found all the material needed to take the exam, but I often didn't want to log on because I had already spent too many hours on the pc."

flow of classroom learning and isolated students within their own spaces. This notion of technology as a 'barrier' reflects the students' desire to navigate past these digital constraints to reclaim their social and academic lives in a more traditional setting.

The synthesis of the Fuzzy Cognitive Map (FCM) analysis with the qualitative data underscored the profound effects of the pandemic on the students' mental states and their overall educational journey. Initial findings indicated a degree of resilience and adaptability among students in the face of the pandemic's challenges. However, as the situation persisted, there was a noticeable decline in their motivation and engagement, highlighting the critical need to consider the emotional and communal dimensions of online learning. The study underscored the necessity for deliberate efforts to sustain and boost student motivation, engagement, and participation, especially during extended periods of remote education.

To conclude, in this first example, the FCM model served the following purpose: aiding teachers in assessing some aspects of students' cognitive status (i.e. engagement, motivation and participation) through quantitative analytics supported also by qualitative findings in order to assist students in mathematics learning facing a changing environment conditions.

4.2. Evaluating how the use of augmented reality impacts on student's conceptualization, inquiry and emotion

Whereas in the first example a portion of the overall map was presented to describe certain cognitive aspects of the students during the Covid-19 period, in this second example a different use of another portion of the overall map is presented to analyze the impact on the students of augmented reality used when studying mathematical objects. Specifically the study emphasized designing and adopting teaching strategies using GeoGebra AR to enhance students' conceptualization and evaluate the impact on students' inquiry and emotion. The case was to evaluate the impact of AR on student's conceptualization, inequiry and emotion, and therefore, base on Duval's theory of semiotic representations (subsection 2.1) the research hypothesis was that AR could assists students in seamlessly transitioning between diverse semiotic representations enhancing the process of mathematical conceptualization.

4.2.1. Context

Learning mathematics can be challenging due to the need to conceptualize abstract ideas without concrete references [48]. To understand mathematical concepts, learners should engage with one or more symbolic representations [49]. Mathematical concepts rely on representations since there are no tangible objects to directly observe. Therefore, the process of conceptualization relies on these representative forms. However, managing these representations can be complex as there is no direct connection to concrete objects for relating and transforming the representations. These difficulties are not limited to primary students but also affect secondary school and university students, as research in Mathematics Education suggests [48]. Several studies have explored how the meaningful integration of technology, such as Augmented Reality (AR), can help address these challenges [50]. Understanding the impact of digital artifacts on educational mediation is crucial in the context of learning, as it influences both the artifacts themselves and users' awareness of them, given the fundamental relationship between experience and conceptualization in educational processes.

In this study, presented here as second example of FCM usage, the investigation focuses on how students' transition between the different registers of semiotic representation to conceptualize the mathematical objects involved is fostered by AR. The exploration of how students' conceptualization of paraboloids is enhanced through the transition between different semiotic registers of representation, facilitated by the use of GeoGebra AR, forms the core of the research. A teaching activity involving thirty undergraduate students in mathematics at University of Bari was conducted to explore the characterization of paraboloids. All students were part of the same Calculus II class, from the University of Bari Mathematics course, and had previously taken courses in Calculus, Multivariable Calculus and Geometry. The experimentation was conducted a few weeks after starting the course during the Functions of several variables lectures. At that time, student attendance in the course was almost maximum having recorded only 1 drop out. In fact, there were 31 students enrolled but 30 students attending at the time of the experiment. The students worked in small groups (of five students each) with collective discussions led progressively by the authors.

4.2.2. Methodology

The activity utilized GeoGebra 3D and GeoGebra AR, dynamic geometry software, to guide students in transitioning between different registers of semiotic representation (2D and 3D graphs, algebraic expressions, equations). Through this transition, students were able to conceptualize the mathematical objects by investigating their mathematical characteristics. The teaching activity commenced with a preliminary task and proceeded with a series of tasks, comprising group work and subsequent collective discussions. The authors designed and implemented both the preliminary task and the subsequent task sequence.

Specifically, the teaching sequence began with an initial activity where learners were equipped solely with paper and pencil. They were tasked with linking certain contour lines and equations to their respective surfaces and then elucidating their reasoning. This opening activity was designed to prompt students to revisit some features of these mathematical entities. Following the initial activity, students engaged in three subsequent exercises, utilizing GeoGebra AR complemented by GeoGebra 3D. These exercises were crafted to guide students in understanding and defining paraboloids through a new lens. These exercises encouraged learners to engage with the mathematical concepts from an alternative viewpoint. In line with the objectives of our study, these tasks facilitated the exploration of semiotic representation manipulation and the experience of cognitive processes involved in the treatment and conversion of semiotic systems. In the first exercise, learners were instructed to alter the values of a control slider k and document the occurrences on both GeoGebra AR and GeoGebra 3D interfaces. The purpose of this exercise was to motivate students to correlate the adjustment of slider k values with the displacement of a z = k plane intersecting the paraboloids, thereby conceptualizing level curves as the intersection results. During the second exercise, the challenge for students was to deduce the equation of the 3D shape by analyzing level curves at varied k slider values. This exercise aimed to guide students towards discerning a link between the contour lines and the shape as the slider k varied, in both their graphical and analytical representations. In the final exercise, the task was to delineate the paraboloids by tweaking the parameters in their equations. Specifically, students examined the equation $z = ax^2 + by^2$, adjusting parameters a and b both individually and together, using sliders in GeoGebra AR and GeoGebra 3D platforms. Mathematically, this exercise sought to have students leverage the GeoGebra tools to geometrically and analytically define diverse paraboloids by altering the equation parameters. The goal was to aid students in articulating the attributes of various surface families as they experimented with parameter adjustments on sliders a and b, particularly how these changes impacted the surfaces and the resulting level curves formed by the intersection of these surface families with the plane $z = k$ as the k slider values varied.

During the group work, students were tasked with observing and uncovering specific characteristics of paraboloids within the same semiotic register or while transitioning between different registers. Their written responses were collected as data, while video recordings and transcriptions were also obtained. In addition to a qualitative analysis, a quantitative data analysis was conducted using the FCM reported in Fig. 5 based on Inquiry-based learning, Duval's Theory of semiotic representations and Marton's Variation Theory to identify the treatments and conversions performed by students, as well as the patterns of variation they created while

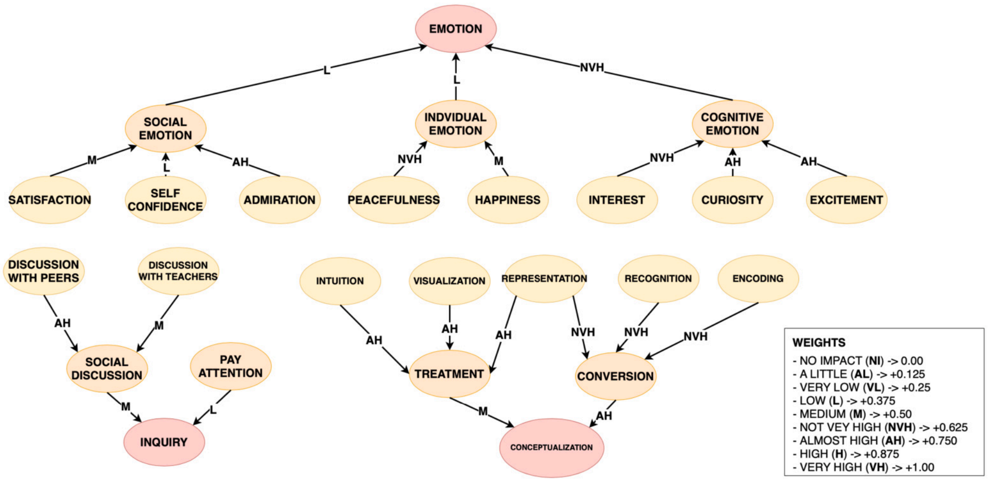

Fig. 5. FCM for student's emotion, inquiry and conceptualization evaluation.

employing GeoGebra AR to conceptualize and characterize paraboloids and level curves. As can be noted this map is included in the overall map presented in section 3.

For the experimentation, all 30 students were chosen without distinction and a comparison was made between two scenarios: the initial stage of the preliminary task using traditional tools like paper and pencil, and the activities conducted with the support of GeoGebra AR and GeoGebra 3D.

4.2.3. *Qualitative analysis*

In this subsection the focus is on dissecting the dialogue from the activity sessions, specifically looking through the lens of semiotics and variation theory.

During the initial phase, students were tasked with a foundational activity: without the aid of digital tools, they were to match level curves and equations with their respective paraboloid shapes using traditional methods of paper and pencil. This exercise set the stage for the more advanced tasks that would follow, incorporating GeoGebra AR and GeoGebra 3D to deepen their understanding of paraboloids.

Excerpt 1

S1: "On the graphical representation of the first exercise we have been very approximate and even now we are still unsure about the answers we gave..."

S2: "Actually it took a lot of work to recognize."

S1: "Whereas with GeoGebra you can rotate the image. Furthermore that is what helps you."

S3: "Seen from above the paraboloid cut from the plane already gives you an idea of the curves that must come in the plane."

From these reflections, it is clear that the absence of dynamic geometry tools like GeoGebra AR or GeoGebra 3D posed a challenge for the students in identifying the features of the paraboloids. The comparative ease brought by GeoGebra's interactive capabilities was noted, highlighting the software's utility in visualizing and understanding mathematical concepts. The subsequent discussions revolved around exercises involving the manipulation of a slider, observing the effects on both 2D and 3D representations, and deducing the analytical form of the surfaces by comparing different contour lines. These activities were designed to scaffold students' understanding of the relationships between two-dimensional and three-dimensional mathematical representations.

Excerpt 2

S5: "The relationship we have found is that in the 2D graph we have the drawing of the section of the paraboloid...that is the surface in 3D."

S4: "... the figure representing the intersection of the paraboloid with the plane $z=k$ is exactly the circumference drawn in the 2D figure. We actually made GeoGebra do the intersection between surfaces and found the circumference which is the same."

In these discussions, students articulated their observations regarding the interplay between 2D contour lines and their 3D counterparts, facilitated by the dynamic tools at their disposal. This process of exploration and deduction underscored the cognitive activities of transformation and conversion between different forms of semiotic representations, a core aspect of Duval's theory. Furthermore, drawing insights from Excerpt 2 and reflecting on the initial pair of tasks in the series, it becomes apparent that learners were able to engage with the concept of contrast variation, as framed by Marton's perspective. Notably, the transition from interacting with representations on both the 2D and 3D platforms allowed students to identify and describe the level curves as intersections occurring between the colored surface and the plane defined by $z = k$, in response to adjustments made to the slider k.

Excerpt 3

S8: "We have observed that for $k = 1$ the equation of the circumference which is obtained by intersecting the red surface with the blue plane $z = 1$ is $x^2 + y^2 = 1$; on the other hand for $k = 2$ we get $x^2 + y^2 = 2$ and so on... So we can generalize and say that we get $x^2 + y^2 = k$

depending on the k we chose ... If this is the equation of the intersecting curve then we can deduce that the surface in red has equation $x^2 + y^2 = z$ *which represents an elliptical paraboloid."*

The logical progression from specific observations to generalizations about the nature of the surfaces being studied was evident here. This transition from the graphical to the analytical representation, and vice versa, illustrates the depth of understanding that students were developing through these exercises. Similar to earlier observations, the act of contrasting level curves at diverse values of the slider k enabled students to familiarize themselves with the contrast variation pattern. Additionally, this scenario brought forth the emergence of a generalization pattern: as students pinpointed the equation governing the level curves with changing k values, they recognized the importance of the variable z in ascertaining the equation for the red surface. The design of the third task notably aimed to enrich students' understanding of the essential features that define both the graphical and analytical representations of paraboloid families.

Excerpt 4

S7: "If I move the slider with $a = 0$ *we have a parabolic surface with equation* $z = y^2$*... Whereas if we consider* $a = 1$*... we have a paraboloid... Whereas when* a *is negative we have hyperbolic paraboloids..."*

S7: "...the level curves for $a = 1$ *are concentric circumferences... While for the case of* $a = 0$*... level curves are parallel lines... Then the level curves for elliptic paraboloids with* $a > 1$ *are precisely ellipses... The last case when* a *is negative for hyperbolic paraboloids we have precisely the hyperbolas as level curves..."*

S2: "We have observed that if b is changed we always observed the same surfaces as before but... it is like if we had the axes inverted..."

S7: "In the first case where $a = b = 0$ *we have the plane* $z = 0$*. Instead we get an elliptic paraboloid if we take a greater than zero and b greater than zero."*

S5: "However we observed that it is only a 90° *rotation of the figure depending on the choice of* a *and* b*."*

In the final task, the focus shifted to the effects of varying parameters within the equations of paraboloids, providing students with a comprehensive view of the properties and behaviors of these complex surfaces. The dialogue from this phase further emphasized the value of GeoGebra tools in facilitating the cognitive processes of treatment and conversion between different semiotic systems, enhancing students' conceptual grasp of the mathematical concepts involved. Upon reviewing Excerpt 4, it became evident that a generalization pattern materialized as students adjusted the parameters a and b independently and noted alterations in the paraboloids' graphical depiction. This led to the categorization of various scenarios based on the resultant surface types and their associated level curves. In each case, students took into account the boundary conditions for the parameters a and b, as well as the orientation of the plane defined by $z = k$ intersecting the surface in question.

Throughout these discussions, the integral role of digital tools in bridging the gap between different representations of mathematical concepts became increasingly apparent, aligning with Duval's emphasis on the cognitive significance of navigating between semiotic registers.

4.2.4. *Quantitative analysis*

Seeking numerical validation of the insights gained from the qualitative analysis, the quantitative findings related to students' emotions, inquiries, and conceptual understanding are reported. These findings contrast the outcomes observed in two distinct settings: the initial phase of the foundational task conducted with paper and pencil, and the subsequent exercises facilitated by GeoGebra AR, enhanced with GeoGebra 3D support.

Fig. 6 illustrates the average input values for the middle-layer FCM concepts of the map reported in the Fig. 5. In fact, the graphs depict the comparison between data from students during the preliminary task without AR (represented in blue, "No-AR") and data from students using GeoGebra AR (represented in red, "AR"). The first two parameters, Pay Attention and Social Discussion, show similar results, with noticeable improvement when students used AR tools. Individual Emotion, Social Emotion, and Cognitive Emotion, which reflect students' emotional states (such as Peacefulness, Happiness, Satisfaction, Self-confidence, Admiration, Interest, Curiosity, and Excitement), indicate a more positive situation when students engaged with AR tools. Initially, the students' emotional state may have been affected by a lack of interest due to the performance of a standard task. However, the most significant difference is observed in the parameters of Treatment and Conversion. There was a considerable drop in these parameters during the "NO-AR" stage, with a difference of 50 percent.

Fig. 7 summarizes the average levels of Enquiry, Emotion, and Conceptualization calculated through the FCM execution, aligning with the analysis of the middle-layer parameters. Despite the task's difficulties, students displayed genuine enthusiasm when using AR tools and actively participated in the activities. They interacted with teachers and peers through social discussions and remained focused on the tasks (Inquiry parameter). AR was perceived as a collaborative experience. In contrast, during the "NO-AR" stage, students appeared less motivated, with lower levels of interest and emotional engagement in the tasks presented by the teachers (Emotion parameter). The most significant result relates to the Conceptualization parameter, with a 3-to-1 ratio favoring AR over the preliminary task. This quantitative finding show that AR tools facilitate students' transition between different semiotic representations, enhancing their mathematical conceptualization processes.

4.2.5. *Discussion*

In addition to qualitative analysis, a FCM analysis was conducted to further support the findings. The former was based on direct observation of the participants and a further interpretation of videos, which allowed to go into detail about verbal (discursive exchanges, oral reflections), non-verbal, proxemic, and interactional codes. The second was conducted by running the analysis algorithm with collected data fuzzified according to the implemented map model. Overall, the quantitative data analyzed through the FCM aligned with the qualitative results, highlighting the impact of AR on students' cognitive state and overall learning experience. The

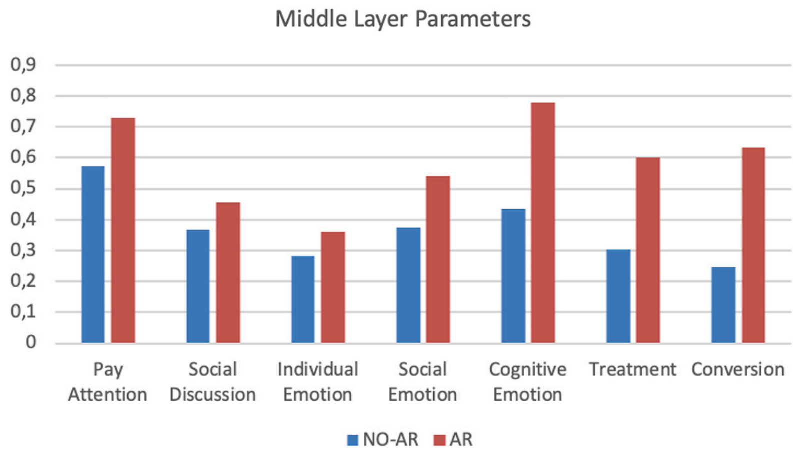

Fig. 6. Experiment results (middle layer) compared using FCM.

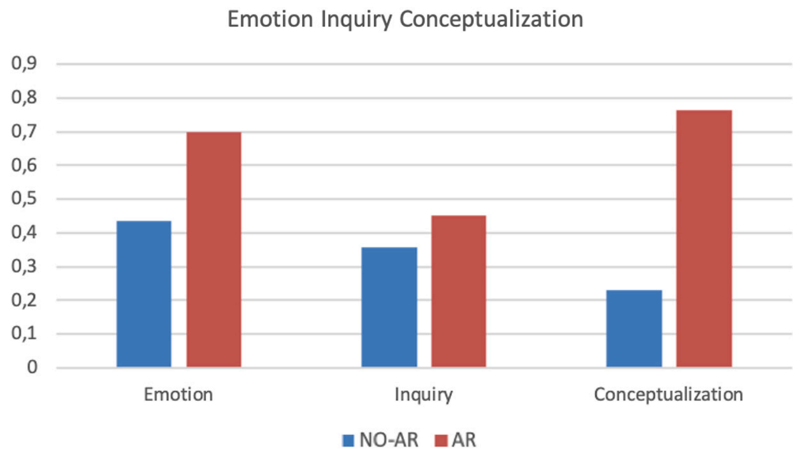

Fig. 7. Emotion, Inquiry and Conceptualization values using FCM.

findings suggested that, according with Duval's Semiotic Representation Theory the use of AR facilitated the students' manipulation between different representations of paraboloids and level curves. In particular, students represented level curves and surfaces in a given register of semiotic representation, treated them within the same register of semiotic representation, and converted them from one given register of semiotic representation to another. Furthermore, by analyzing data using Marton's Variation Theory, some of the patterns of variations emerging and stimulating the students while conceptualizing and characterizing the paraboloids were noticed.

In conclusion, the research underscored the significance of using and implementing teaching strategies that leverage Augmented Reality (AR) to elevate students' cognitive processes related to conceptualization, inquiry, and emotions represented by the proposed FCM model.

5. Conclusions and future works

This study underlines the significance of comprehensively modeling student's cognitive processes in the context of mathematics education. Through the innovative utilization of Fuzzy Cognitive Maps (FCMs), a customized model has been developed, encompassing well-established theories such as Duval's theory of semiotic representation, Niss's framework of Mathematical Competencies, Marton's Variation Theory, and the broader context of Engagement, Motivation, and Participation framework. This multifaceted model has enabled a holistic understanding of student's cognitive processes, crucial for refining teaching strategies and enhancing the educational experience. The presented FCM-based model has been successfully applied to multiple contexts, showing its versatility and adaptability. The FCM model not only exemplifies the essence of these frameworks but also offers educators a resourceful tool for conducting quantitative analyzes which can be enforced by a qualitative analysis. It should be noted that this map is only a proposal that can be modified according to the needs of the educator, but can also be extended with new concepts and relationships or only partially selected for use in specific contexts as demonstrated in the case studies. In fact, as evidence, this tailored FCM model includes the flexibility to cater to a range of educational contexts and objectives. Then, the versatility and adaptability of the FCM model have been showcased through its application across diverse educational scenarios, underlining its capacity not only to embody the core principles of the aforementioned theories but also to serve as a dynamic tool for educators. The model's inherent flexibility allows for modifications tailored to specific educational needs, extensions with new concepts, or selective usage within particular contexts, as demonstrated in the presented case studies.

Two different case studies exemplify its efficacy, highlighting how the quantitative data obtained through the execution of the FCMs are confirmed by the qualitative analysis. The first case study explored the impact of changing learning environments, especially during the transition to remote online teaching amid the Covid-19 pandemic. By combining a mixed-method approach with FCM analysis, the study revealed significant shifts in motivation, engagement, and participation levels across different academic years. This underscores the necessity of adaptive teaching strategies that effectively integrate digital tools and address evolving learning circumstances. In the second case study, the integration of Augmented Reality (AR) as an educational tool was assessed, revealing its potential to improve engagement, emotional states, and mathematical conceptualization. AR, aligned with Duval's semiotic representations, emerged as a channel for guiding students through various mathematical perspectives, augmenting their comprehension. In a broader context, both cases show the necessity of tailoring educational methodologies to suit the evolving needs of students and harnessing technology for optimal learning experiences. The use of FCMs as innovative tool alongside robust methodologies, provides a way to understand student's cognitive process including emotions, engagement levels and so on. This comprehensive insight paves the way for increase learning outcomes and supports the continued evolution of educational approaches in mathematics. In fact, through the use of customized FCM approach, educators can gather empirical evidence and conduct qualitative as well as quantitative analyzes, fostering a deeper understanding of student cognition. In essence, this approach summarizes a multipurpose tool that can be flexibly used by mathematics educators to push effective teaching, foster holistic understanding, and increase the overall educational landscape. As educational paradigms continue to evolve, this model holds the potential to refine teaching methodologies and enrich students' mathematical journeys.

To sum up, on one hand, the unique value of this research lies in its demonstration that quantitative findings can indeed align with and substantiate qualitative insights, thereby offering a robust framework capable of functioning within automated contexts. This alignment is crucial as it validates the feasibility of applying sophisticated analytical models, like Fuzzy Cognitive Maps (FCMs), in dynamic, AI-mediated educational settings. The utilization of FCMs to model and analyze the complex interplay of factors affecting students' engagement, motivation, and understanding in mathematics education represents a significant leap forward. It extends the applicability of cognitive modeling from the realm of theoretical exploration to practical, real-time educational interventions.

On the other hand, acknowledging the limitations of this study is crucial for appropriately contextualizing its contributions. While the Fuzzy Cognitive Map (FCM) model offers a novel approach to understanding students' cognitive processes in mathematics education, several constraints warrant attention:

1. Scope of Student Cognition Capture: Despite its comprehensive nature, the FCM model may not fully encapsulate the entirety of student cognition within mathematics education. Continuous refinement and expansion are necessary to ensure the model's adequacy in representing the multifaceted nature of mathematical learning.
2. Preference for Qualitative Analysis: Mathematics education research has traditionally favored qualitative methodologies for their depth of insight [51]. Introducing a model that incorporates quantitative analysis, like the FCM, might face resistance from those accustomed to purely qualitative approaches. It is essential to articulate the complementary benefits of integrating quantitative analysis.
3. Technological Implementation Challenges: Implementing the FCM model into educational software poses technological hurdles, including the need for sophisticated software capable of real-time data processing and feedback generation.
4. Generalizability Across Educational Contexts: The findings from applying the FCM model in specific contexts need further validation to ensure their generalizability across different educational settings and disciplines.
5. Integration with Artificial Intelligence: Future research should explore the integration of the FCM model with artificial intelligence to enable real-time, data-driven educational interventions. This integration presents both technological and methodological challenges that need addressing to ensure effective student support.

Future research should focus on broadening the theoretical base of the FCM model, enhancing its compatibility with qualitative research traditions, overcoming technological barriers to implementation, and rigorously testing its applicability in diverse educational environments. These efforts will help evolve the FCM model into a more adaptable and universally applicable tool in mathematics education research and practice. In addition, while our research has primarily focused on mathematics education, it is important to acknowledge the broader applicability of this framework. Future research could explore how cognitive processes vary across different academic disciplines such as science, humanities, or arts. Each of these fields has its unique cognitive demands and learning processes, and adapting the FCM framework could provide valuable insights into how students engage with and conceptualize knowledge in these areas. For instance, in science education, the framework could be used to model the cognitive processes involved in experimental design and hypothesis testing. In the humanities, it could help understand how students analyze texts and develop critical thinking skills. In the arts, it might be used to explore the interplay between creativity, technique, and expression. By extending this framework to other disciplines, a more comprehensive understanding of students' cognitive processes across the educational spectrum can be gained.

Another valuable avenue for enhancing this research involves the integration of advanced techniques, such as Large Language Models (LLMs). These models could significantly augment the qualitative analysis component of this research. Currently, the proposed framework has been implemented in an e-learning platform developed by the author [45,52] to guide students in their study paths through personalized feedback generated using the proposed cognitive process map as stated in the Introduction. However, exploring the potential of integrating ChatGPT [53], as LLMs, for educational purposes using the proposed model is absolutely fascinating and will be pursued in future research. The interest in ChatGPT among researchers is high, as evidenced by recent studies:

- Kasneci, Enkelejda, et al. [54] discuss the opportunities and challenges of large language models in education, emphasizing the potential of these models to create educational content, improve student engagement, and personalize learning experiences.
- Chen, Zihan, et al. [55] demonstrate the application of ChatGPT in financial forecasting, highlighting its capability to infer dynamic network structures from textual data and enhance predictive models.
- Adeshola, Ibrahim, and Adeola Praise Adepoju [56] analyze the impact of ChatGPT on education through a comprehensive study of social media data, indicating a generally positive reception and the potential for educational institutions to develop clear policies and guidelines to incorporate such technologies responsibly.

The integration of this proposed approach with ChatGPT to create a new educational agent will be crucial. This agent will be made available to researchers and educators, allowing them to use the framework directly for their experiments without needing specific technical skills. This integration aims to make advanced educational tools accessible and straightforward to implement, fostering innovative and effective educational practices.

In conclusion, this study has laid the groundwork for a multidimensional approach to understanding students' cognitive processes in mathematics education. By expanding the application of FCMs to other disciplines and incorporating advanced technologies like ChatGPT, the effectiveness of educational practices and support personalized learning experiences across diverse educational contexts can be further enhanced.

CRediT authorship contribution statement

Mario Lepore: Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Methodology, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Appendix A

Table 5
List of all concepts in the map with a short description and their inclusion in the frameworks.

Concept's name	Framework	Short Description
Problem Solving	KOM-framework	The process of addressing a mathematical problem by applying appropriate strategies and procedures to reach a solution. It involves analysis, planning, execution, and evaluation of solutions.
Problem Posing	KOM-framework	The ability to create and formulate challenging and meaningful mathematical problems.
Problem Handling	KOM-framework	The capacity to manage and deal with difficulties that may arise during the process of solving a mathematical problem. It includes adapting strategies, overcoming obstacles, and reflecting on the problem-solving process.
Abstraction	KOM-framework	The process of capturing essential characteristics while ignoring unnecessary details to create a simplified representation.
Generalization	KOM-framework	The process of extending concepts from specific cases to broader statements. It helps to recognize patterns, make connections, and apply knowledge in different contexts, fostering mathematical reasoning and problem-solving abilities.
Mathematical thinking	KOM-framework	A key component of mathematical competencies, as it encompasses the ability to think critically, reason logically, and solve problems effectively.
Competencies	KOM-framework	Development of knowledge, skills, and dispositions including critical thinking with symbols, aiding students' holistic mathematical understanding.
Reasoning	KOM-framework	Capacity to construct logical arguments, draw deductions, and justify mathematical solutions.
Conceptualization	Duval's theory	It refers to the cognitive process of understanding mathematical concepts through the use and interrelation of various semiotic representations such as symbols, graphs, diagrams, and natural language.
Visualization	Duval's theory	The act of creating a mental or physical image to understand mathematical ideas better.
Representation	Duval's theory/ KOM-framework	The use of symbols, graphs, or other forms to depict mathematical concepts.
Treatment	Duval's theory	The process of manipulating mathematical forms to solve problems or understand concepts.

(*continued on next page*)

Table 5 (*continued*)

Concept's name	Framework	Short Description
Recognition	Inquiry-based and Marton's Variation/ Duval's theory	It involves identifying patterns, structures, and variations to enhance mathematical understanding.
Encoding	Duval's theory/ KOM-framework	The process of translating real-world problems into mathematical language.
Modeling	KOM-framework	The ability to construct representations of real-world situations for effective problem-solving.
Conversion	Duval' theory	A process of transitioning between different semiotic representation while retaining the same object for enhanced understanding.
Satisfaction	Engagement, motivation and participation framework	Students' contentment and positive emotional response influenced mainly by their interests.
Self-confidence	Engagement, motivation and participation framework	Students' belief in their abilities, influencing their performance.
Admiration	Engagement, motivation and participation framework	Feeling of respect and positive regard towards mathematics.
Peacefulness	Engagement, motivation and participation framework	A sense of emotional serenity that influences cognitive processes in learning.
Happiness	Engagement, Motivation, and Participation framework	A positive emotional state.
Interest	Engagement, Motivation, and Participation framework	Students' attraction towards mathematics. It encompasses a student's willingness and desire to explore mathematical concepts and solve problems.
Curiosity	Engagement, Motivation, and Participation framework	A student's innate desire to explore, question, and seek answers about mathematical concepts and problems.
Social Emotion	Engagement, Motivation, and Participation framework	Feelings that students experience in social contexts related to math.
Individual Emotion	Engagement, Motivation, and Participation framework	It comprises the diverse range of personal emotional experiences.
Cognitive Emotion	Engagement, Motivation, and Participation framework	Feelings and mental processes that students experience while engaging with mathematical tasks, concepts, and problem-solving.
Emotion	Engagement, Motivation, and Participation framework	It includes the range of affective states that impact students' status during the learning process.
Participation	Engagement, Motivation, and Participation framework	It involves students taking an active role in their education by contributing, interacting, and collaborating with teachers and peers.
Extrinsic Motivation	Engagement, Motivation, and Participation framework	It involves students being motivated to learn or complete assignments because they expect to receive some form of external reinforcement or reward.
Intrinsic Motivation	Engagement, Motivation, and Participation framework	Intrinsic motivation involves students learning or pursuing knowledge primarily because they find the subject matter, the learning process, or the problem-solving aspects inherently enjoyable or interesting.
Social Motivation	Engagement, Motivation, and Participation framework	The influence of social factors, interactions, and relationships on an individual's motivation to pursue goals or participate in specific activities.
Motivation	Engagement, Motivation, and Participation framework	The psychological processes and factors that drive, direct, and sustain an individual's behavior toward a particular goal or desired outcome.
% of Resources Used	Engagement, Motivation, and Participation framework	Percentage of available educational resources that have been utilized or accessed by students
% of Task Done	Engagement, Motivation, and Participation theory	It reflects students' level of completion and progress in learning activities.
Last Session Attended	Engagement, Motivation, and Participation framework	The most recent meeting, class, or educational session that a student or participant has attended in an educational program or course.
N. of Lessons	Engagement, Motivation, and Participation framework	The count of instructional sessions or classes that a particular student has actively participated in or been present for within a specific educational program or course.
Use of Aids and Tools	KOM-framework	Students' effective utilization of tools, methods, and technology to enhance problem-solving and understanding.
Interaction	Engagement, Motivation, and Participation framework	The dynamic exchange and communication between students and educational resources as an integral part of the teaching and learning process for mathematics.
Assignments	Engagement, Motivation, and Participation framework	Completed tasks by the students to enhance their mathematical skills.
Engagement	Engagement, Motivation, and Participation framework	Active involvement, interest, and interactions of students in the process of learning mathematics.
Inquiry	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Student's active exploration and questioning to foster deeper understanding.
Social Discussion	Engagement, Motivation, and Participation framework	Collaborative interactions among students and/or with educators.
Pay Attention	Engagement, Motivation, and Participation framework	Student's active cognitive involvement, concentration, and focus during learning activities.
Forum Activities	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Student's active participation, communication, and collaboration in online platforms.
Discussion with Peers	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Student's interactive communication and collaboration with classmates.

Table 5 (*continued*)

Concept's name	Framework	Short Description
Discussion with Teachers	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Student's interactive communication with educators.
Last Forum Activity	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Latest student's activity in online discussions.
No of Posts	Inquiry-based learning, Method of Varied Inquiry, and Marton's Variation Theory	Total number of posts placed by the student in an online discussion.

References

[1] M. Hannula, P. Martino, M. Pantziara, Q. Zhang, F. Morselli, E. Heyd-Metzuyanim, G. Goldin, Attitudes, Beliefs, Motivation and Identity in Mathematics Education: An Overview of the Field and Future Directions, Springer Nature, 2016.
[2] D. McLeod, Research on affect in mathematics education: a reconceptualization, in: Handbook of Research on Mathematics Teaching and Learning, vol. 1, 1992, pp. 575–596.
[3] G. Skaggs, N. Bodenhorn, Relationships between implementing character education, student behavior, and student achievement, J. Adv. Acad. 18 (1) (2006) 82–114.
[4] S. Ritter, J. Anderson, K. Koedinger, A. Corbett, Cognitive tutor: applied research in mathematics education, Psychon. Bull. Rev. 14 (2007) 249–255.
[5] A. Jalal, M. Mahmood, Students' behavior mining in e-learning environment using cognitive processes with information technologies, Educ. Inf. Technol. 24 (2019) 2797–2821.
[6] L. Reyes, Affective variables and mathematics education, Elem. Sch. J. 84 (5) (1984) 558–581.
[7] J.L. Wilkins, X. Ma, Modeling change in student attitude toward and beliefs about mathematics, J. Educ. Res. 97 (1) (2003) 52–63.
[8] H. Jeong, G. Biswas, Mining Student Behavior Models in Learning-by-Teaching Environments, 2008.
[9] S. Tang, J. Peterson, Z. Pardos, Modelling student behavior using granular large scale action data from a mooc, arXiv preprint, arXiv:1608.04789, 2016.
[10] R. Pelánek, Bayesian knowledge tracing, logistic models, and beyond: an overview of learner modeling techniques, User Model. User-Adapt. Interact. 27 (2017) 313–350.
[11] H. Cen, K. Koedinger, B. Junker, Learning factors analysis–a general method for cognitive model evaluation and improvement, in: Intelligent Tutoring Systems: 8th International Conference, Proceedings 8, ITS 2006, Springer, Jhongli, Taiwan, 2006, pp. 164–175.
[12] A. Latham, K. Crockett, D. Mclean, Profiling student learning styles with multilayer perceptron neural networks, in: 2013 IEEE International Conference on Systems, Man, and Cybernetics, IEEE, 2013, pp. 2510–2515.
[13] J.B. Yeo, B.H. Yeap, Characterising the cognitive processes in mathematical investigation, Int. J. Math. Teach. Learn (2010).
[14] M.N. Ferdiansyah, R. Ekawati, Students' cognitive process in problem solving on pattern materials reviewed from math anxiety, J. Medives: J. Math. Educ. IKIP Veteran Semarang 5 (1) (2021) 137–150.
[15] R. Ekawati, A.W. Kohar, E.M. Imah, S.M. Amin, S. Fiangga, Students' cognitive processes in solving problem related to the concept of area conservation, J. Math. Educ. 10 (1) (2019) 21–36.
[16] I.S. Campos, L.S. Almeida, A.I. Ferreira, L.F. Martinez, G. Ramalho, Cognitive processes and math performance: a study with children at third grade of basic education, Eur. J. Psychol. Educ. 28 (2013) 421–436.
[17] B. Kosko, Fuzzy cognitive maps, Int. J. Man-Mach. Stud. 24 (1) (1986) 65–75.
[18] R.B. Johnson, L. Christensen, Educational Research: Quantitative, Qualitative, and Mixed Approaches, Sage Publications, 2019.
[19] G. D'Aniello, M. De Falco, M. Gaeta, M. Lepore, A situation-aware learning system based on fuzzy cognitive maps to increase learner motivation and engagement, in: 2020 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE, 2020, pp. 1–8.
[20] R. Duval, Registres de représentations sémiotiques et fonctionnement cognitif de la pensée, Ann. Didact. Sci. Cogn. 5 (1) (1993) 37–65.
[21] M. Niss, T. Højgaard, Mathematical competencies revisited, Educ. Stud. Math. 102 (2019) 9–28.
[22] F. Marton, U. Runesson, A. Tsui, The Space of Learning, Lawrence Erlbaum, Mahwah, NJ, 2004.
[23] R. Capone, M. Lepore, From distance learning to integrated digital learning: a fuzzy cognitive analysis focused on engagement, motivation, and participation during covid-19 pandemic, Technol., Knowl. Learn. 27 (4) (2022) 1259–1289.
[24] R. Duval, Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels, Peter Lang, Berne (Suiza), 1995.
[25] M. Pedaste, M. Mäeots, L. Siiman, T. Jong, S. Riesen, E. Kamp, E. Tsourlidaki, Phases of in-quiry-based learning: definitions and the inquiry cycle, Rev. Educ. Res. 14 (2015) 47–61.
[26] F. Arzarello, Basing on an inquiry approach to promote mathematical thinking in the classroom, in: B. Maj-Tatsis, M. Pytlak, E. Swoboda (Eds.), Inquiry Based Mathematical Education, Wydawnictwo Uniwersytetu Rzeszowskiego, 2016, pp. 9–20.
[27] F. Marton, M. Pang, On some necessary conditions of learning, J. Learn. Sci. 15 (2) (2006) 193–220, https://doi.org/10.1207/s15327809jls1502_2.
[28] C.P. Niemiec, R.M. Ryan, Autonomy, competence, and relatedness in the classroom: applying self-determination theory to educational practice, Theory Res. Educ. 7 (2) (2009) 133–144.
[29] M. Lepper, Motivational considerations in the study of instruction, Cogn. Instr. 5 (4) (1988) 289–309.
[30] A. Wigfield, J.S. Eccles, Expectancy–value theory of achievement motivation, Contemp. Educ. Psychol. 25 (1) (2000) 68–81.
[31] G. Kuh, What we're learning about student engagement from nsse: benchmarks for effective educational practices, Change: Mag. High. Learn. 35 (2) (2003) 24–32.
[32] J. Fielding-Wells, M. O'Brien, K. Makar, Using expectancy-value theory to explore aspects of motivation and engagement in inquiry-based learning in primary mathematics, Math. Educ. Res. J. 29 (2017) 237–254.
[33] J. Fredricks, P. Blumenfeld, A. Paris, School engagement: potential of the concept, state of the evidence, Rev. Educ. Res. 74 (1) (2004) 59–109.
[34] S. Harandi, Effects of e-learning on students' motivation, Proc., Soc. Behav. Sci. 181 (2015) 423–430.
[35] U. Bergmark, S. Westman, Student participation within teacher education: emphasising democratic values, engagement and learning for a future profession, High. Educ. Res. Dev. 37 (7) (2018) 1352–1365.
[36] U. Özesmi, S. Özesmi, Ecological models based on people's knowledge: a multi-step fuzzy cognitive mapping approach, Ecol. Model. 176 (1–2) (2004) 43–64.
[37] W. Froelich, E. Papageorgiou, M. Samarinas, K. Skriapas, Application of evolutionary fuzzy cognitive maps to the long-term prediction of prostate cancer, Appl. Soft Comput. 12 (12) (2012) 3810–3817.
[38] J. Aguilar, A survey about fuzzy cognitive maps papers, Int. J. Comput. Cogn. 3 (2) (2005) 27–33.

[39] K. Pérez-Teruel, M. Leyva-Vázquez, V. Estrada-Sentí, Mental models consensus process using fuzzy cognitive maps and computing with words, Ing. Univ. 19 (1) (2015) 173–188.
[40] S. Kaplan, Cognitive maps in perception and thought, Image Environ.: Cogn. Mapp. Spat. Behav. (1973) 63–78.
[41] T. Rutherford, K. Duck, J. Rosenberg, R. Patt, Leveraging mathematics software data to understand student learning and motivation during the covid-19 pandemic, J. Res. Technol. Educ. (2021) 1–38.
[42] L. Branchetti, R. Capone, M. Rossi, Distance–learning goes viral: redefining the teaching boundaries in the transformative pedagogy perspective, J. E-Learn. Knowl. Soc. 17 (2) (2021) 32–44.
[43] M. Chan, C. Sabena, D. Wagner, Mathematics education in a time of crisis—a viral pandemic, Educ. Stud. Math. 108 (1) (2021) 1–13.
[44] M. Lepore, R. Capone, The impact of changing environment on undergraduate mathematics students' status, Eur. J. Sci. Math. Educ. 11 (4) (2023) 672–689.
[45] R. Capone, M. De Falco, M. Lepore, The impact of covid-19 pandemic on undergraduate students: the role of an adaptive situation-aware learning system, in: 2022 IEEE Conference on Cognitive and Computational Aspects of Situation Management (CogSIMA), IEEE, 2022, pp. 154–161.
[46] K. De Witte, N. Rogge, Does ict matter for effectiveness and efficiency in mathematics education?, Comput. Educ. 75 (2014) 173–184.
[47] S. Ghavifekr, W.A.W. Rosdy, Teaching and learning with technology: effectiveness of ict integration in schools, Int. J. Res. Educ. Sci. 1 (2) (2015) 175–191.
[48] R. Duval, Registres de répresentations sémiotiques et fonctionnement cognitif de la pensée, Ann. Didact. Sci. Cogn. 5 (1993) 37–65.
[49] J. Godino, C. Batanero, Significado institucional y personal de los objetos matemáticos, Rech. Didact. Math. 14 (3) (1994) 325–355.
[50] A. Cahyono, Y. Sukestiyarno, M. Asikin, M. Ahsan, M. Ludwig, Learning mathematical modelling with augmented reality mobile math trails program: how can it work?, J. Math. Educ. 11 (2) (2020) 181–192, https://doi.org/10.22342/jme.11.2.10729.181-192.
[51] S.M. Parry, L.D. Knight, B. Connolly, C. Baldwin, Z. Puthucheary, P. Morris, J. Mortimore, N. Hart, L. Denehy, C.L. Granger, Factors influencing physical activity and rehabilitation in survivors of critical illness: a systematic review of quantitative and qualitative studies, Intensive Care Med. 43 (2017) 531–542.
[52] R. Capone, M. Lepore, Augmented reality to increase interaction and participation: a case study of undergraduate students in mathematics class, in: Augmented Reality, Virtual Reality, and Computer Graphics: 7th International Conference, Proceedings, Part II 7, AVR 2020, Springer, 2020, pp. 185–204.
[53] L. Floridi, M. Chiriatti, Gpt-3: its nature, scope, limits, and consequences, Minds Mach. 30 (2020) 681–694.
[54] E. Kasneci, K. Seßler, S. Küchemann, M. Bannert, D. Dementieva, F. Fischer, U. Gasser, G. Groh, S. Günnemann, E. Hüllermeier, et al., Chatgpt for good? On opportunities and challenges of large language models for education, Learn. Individ. Differ. 103 (2023) 102274.
[55] Z. Chen, L.N. Zheng, C. Lu, J. Yuan, D. Zhu, Chatgpt informed graph neural network for stock movement prediction, arXiv preprint, arXiv:2306.03763, 2023.
[56] I. Adeshola, A.P. Adepoju, The opportunities and challenges of chatgpt in education, Interact. Learn. Environ. (2023) 1–14.