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New constructions of decision evaluation functions in three-way decision spaces based on uninorms

Zihang Jia[1,2] · Junsheng Qiao[1]

Abstract

In 2014, Hu introduced the concept of three-way decision spaces and axiomatic definition of decision evaluation functions. In three-way decision spaces, decision evaluation function satisfies minimum element axiom, monotonicity axiom and complement axiom. Since then, the research on construction method of decision evaluation functions from commonly used binary aggregation functions becomes a research hotspot. Meanwhile, uninorms, as one class of binary aggregation functions, have been successfully applied in various application problems, such as in decision making, image processing, data mining, etc. This paper continues to consider this research topic and mainly explores the new construction methods of decision evaluation functions based on uninorms. Firstly, we show two novel transformation methods from semi-decision evaluation functions to decision evaluation functions based on uninorms. Secondly, using known semi-decision evaluation functions, we give some new construction methods of semi-decision evaluation functions. Thirdly, we give some novel construction methods of decision evaluation functions and semi-decision evaluation functions related to fuzzy sets, interval-valued fuzzy sets, fuzzy relations and hesitant fuzzy sets. Based on them, decision maker can obtain more useful decision evaluation functions, thereby more choices can be used for realistic decision-making problems. Finally, we consider two real evaluation problems to illustrate the results obtained in this paper. The three-way decisions results of evaluation problem show that the construction method proposed in this paper is superior to some existing construction methods under some conditions.

Keywords (Semi-)three-way decision spaces · Three-way decisions · Uninorms · Decision evaluation functions

✉ Junsheng Qiao
jsqiao@nwnu.edu.cn; jsqiao@whu.edu.cn

Zihang Jia
zihangjia@outlook.com

[1] College of Mathematics and Statistics, Northwest Normal University, No. 967, Anning East Road, Lanzhou 730070, Gansu, People's Republic of China

[2] School of Mathematical Sciences, Dalian University of Technology, No. 2, Linggong Road, Dalian 116024, Liaoning, People's Republic of China

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1 Introduction

1.1 A short review of three-way decisions

The theory of three-way decisions, as an exploration of thinking in three (Yao 2012), was proposed by Yao (2009). The basic idea of three-way decisions arises from Pawlak's rough sets (Pawlak 1982; Pawlak 1991) and decision-theoretic rough sets (Yao 2003, 2007, 2008). In three-way decisions, there are three sorts of decision rules, that is, acceptance rules, uncertainty rules and rejection rules. According to these decision rules, for each object in a universe, it can be divided into positive region by acceptance rules, boundary region by uncertainty rules, and negative region by rejection rules. It is worth noting that three-way decisions is an extension of classical two-way decisions (Yao 2009, 2010, 2011). Under some conditions, three-way decisions is superior to the classical two-way decisions (Yao 2011). During the last 10 or more years, three-way decisions has met a fast development both in real application and theory. The researches on three-way decisions mainly focus on three aspects as follows.

The first aspect is the background researches on three-way decisions. These researches mainly focus on the generalization of Pawlak's rough sets, such as decision-theoretic rough sets (Yao 2003, 2007, 2008), game-theoretic rough sets (Azam and Yao 2014), fuzzy rough sets/rough fuzzy sets (Dubois et al. 1990), variable precision rough sets (Ziarko 1993), fuzzy covering-based rough sets (Yang and Hu 2016; Yang and Hu 2017), multi-granulation rough sets (Lin et al. 2013; Sun et al. 2021), and so on.

The second aspect involves the theoretical framework researches on three-way decisions. These studies mainly focus on construction and interpretation of decision evaluation functions (Yao 2010, 2011, 2012), value domain of decision evaluation functions (Yao 2012), the mode of three-way decisions (Yao 2012), three-way decision spaces (Hu 2014; Hu 2016; Hu et al. 2016; Hu 2017; Hu et al. 2017; Jia and Qiao 2020; Qiao and Hu 2018; Qiao and Hu 2020), trisecting-and-acting framework of three-way decisions (Yao 2015), and so on.

The third aspect refers to real application studies of three-way decisions. In real application, three-way decisions plays a key role in decision making (Jiang and Hu 2021; Li et al. 2020; Liang et al. 2015; Liang et al. 2016; Yao and Azam 2015), image processing (Yue et al. 2021), neural networks (Cheng et al. 2021), investment (Jiang and Hu 2021), incomplete fuzzy decision system(Zhan et al. 2021; Ye et al. 2021), clustering (Chu et al. 2020; Yu et al. 2021), classification (Subhashini et al. 2022), conflict analysis (Li et al. 2021; Lang et al. 2020; Lang and Yao 2021), active learning (Min et al. 2020), and so on.

1.2 A brief introduction of uninorms

Triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short) (Klement et al. 2000) are extensions of conjunctions and disjunctions. T-norms and t-conorms play a vital role in fuzzy logic. Uninorms, as a generalization of t-norms and t-conorms, was introduced by Yager and Rybalov (1996). Unlike t-norms with neutral element 1 and t-conorms with neutral element 0, the neutral element of uninorms can be taken each value of unit interval [0, 1]. Thus, t-norms and t-conorms are two special cases of uninorms. During the last 20 or more years, both theory and real application of uninorms were developing rapidly.

To be more specific, in theory, there are a lot of literatures which relate to various respects of uninorms, such as the structure of uninorms (Fodor et al. 1997; Li and Shi 2000), migrativity properties (Zhou and Yan 2021), conditional distributive equations (Zhang and Qin 2022), construction methods (Zong et al. 2020), and so on. In real application, uninorms plays an important role in decision making (Campanella and Ribeiro 2011; Yager and Rybalov 2011), image processing (Bustince et al. 2007; González-Hidalgo et al. 2015), data mining (Yan and Chen 2005), classification (Roy et al. 2020), neural networks (de Campos Souza and Lughofer 2021; de Campos Souza and Lughofer 2022a; de Campos Souza and Lughofer 2022b), and so on.

1.3 Motivation of this research

In 2014, after systemically researching on three-way decisions, Hu (2014) introduced the axiomatic definitions for decision condition, decision measurement and decision evaluation function and established three-way decision space on fuzzy lattice, i.e., a complete distributive lattice with a strong negation. Since then, in order to generalize the application range of three-way decision space, Hu (2016) extended the decision measurement domain of three-way decision space from fuzzy lattice to partially ordered set. Therefore, the existing three-way decisions become the special cases of three-way decision space, for instance, three-way decisions based on fuzzy sets (Hu 2014; Zadeh 1965), interval-valued fuzzy sets (Hu 2014), fuzzy relations (Qiao and Hu 2018; Jia and Qiao 2020), hesitant fuzzy sets (Hu 2016), shadowed sets (Jiang et al. 2022; Pedrycz 1998; Yao and Yang 2022), and so on. Therefore, three-way decision space is a useful tool to research three-way decisions.

In order to better illustrate our motivation, we review the concept of decision evaluation function as follows. Let X and Y be two universes. In this paper, $Map(X, Y)$ denotes the family of all mappings from X to Y. Let U and V be two nonempty universes. If we make decisions on universe U, then U is said to be a decision universe. If we define condition functions on universe V, then V is said to be a condition universe. Let $(P_C, \leq_{P_C}, N_{P_C}, 0_{P_C}, 1_{P_C})$ and $(P_D, \leq_{P_D}, N_{P_D}, 0_{P_D}, 1_{P_D})$ be two partially ordered sets with strong negations P_C and P_D.

Definition 1.1 (Hu 2016) Let V be a condition universe and U be a decision universe. Then a mapping $E : Map(V, P_C) \to Map(U, P_D)$ is said to be a decision evaluation function of U, if it satisfies the axioms as follows:

(E1)   Minimum element axiom

$$E((0_{P_C})_V) = (0_{P_D})_U.$$

(E2)   Monotonicity axiom

$$A \subseteq_{P_C} B \Rightarrow E(A) \subseteq_{P_D} E(B)$$

for each $A, B \in Map(V, P_C)$.

(E3)   Complement axiom

$$N_{P_D}(E(A)) = E(N_{P_C}(A))$$

for each $A \in Map(V, P_C)$.

On the one hand, decision evaluation function plays a key role in three-way decisions (Cabitza et al. 2017; Liu et al. 2016; Yao and Azam 2015). On the other hand, decision maker will obtain different decision results through different decision evaluation functions (Qiao and Hu 2018; Qiao and Hu 2020). Thus, in order to obtain reasonable decision results, there must be enough decision evaluation functions for decision maker to select. However, Hu found that a lot of useful functions only satisfy minimum element axiom and monotonicity axiom (Hu 2014; Hu 2016). Meanwhile, it has been pointed out that the complement axiom is necessary and important for decision evaluation functions in various previous researches (Liang and Liu 2014; Liang et al. 2015; Liang et al. 2016; Yao 2010, 2011; Zhao and Hu 2016). Therefore, it arises the following question.

Question 1: How to construct decision evaluation function as much as possible?

In order to response this problem, Hu (2017) introduced the concept of semi-decision evaluation function, i.e., decision evaluation function without complement axiom (see Fig. 1). By the transformation methods from semi-decision evaluation functions to decision evaluation functions proposed by Hu (2017), some useful semi-decision evaluation functions can be transformed to decision evaluation functions.

After that, since the special properties and practicability of aggregation functions, the research on construction methods of decision evaluation functions on the basis of some particular binary aggregation functions becomes a research hotspot. In 2018, Qiao and Hu (2018) studied the construction methods of semi-decision evaluation functions and decision evaluation functions based on t-norms and t-conorms. And then, in 2020, on the basis of overlap and grouping functions, Jia and Qiao (2020) researched the transformation methods from semi-decision evaluation functions to decision evaluation functions and the construction methods of semi-decision evaluation functions. Therefore, it is an effective way to construct decision evaluation functions by aggregation functions. However, in Hu (2017); Jia and Qiao (2020); Qiao and Hu (2018), the transformation methods from semi-decision evaluation functions to decision evaluation functions always assume that the strong negation $N_{P_D}(x) = 1 - x$ for all $x \in [0, 1]$. It limits the research on decision evaluation functions both in theory and application. Thus, it arises the following question.

Question 2: Whether there exists a construction method of decision evaluation functions such that the strong negation N_{P_D} of partially ordered set P_D is different from $N_{P_D}(x) = 1 - x$ for all $x \in [0, 1]$?

Therefore, in this paper, we mainly consider Questions 1 and 2. Firstly, in Sect. 1.2, it has been pointed that t-norms and t-conorms are two special cases of uninorms. Thus, in order to response Question 1, we mainly study how to construct semi-decision evaluation functions and decision evaluation functions based on uninorms. Representable uninorms, as a special
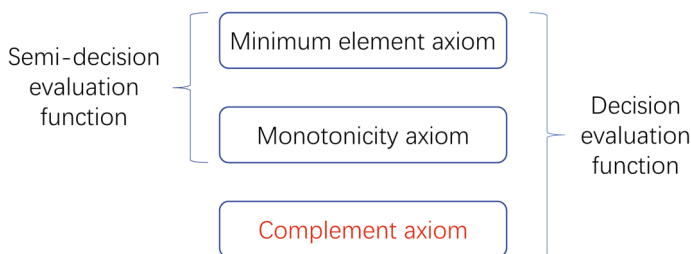

Fig. 1 The relationship between decision evaluation functions and semi-decision evaluation functions (Hu 2017)

class of uninorms (see Definition 2.8), is self-dual with respect to (w.r.t., for short) the strong negation derived from its additive generator (see Lemma 2.2). Therefore, in order to response Question 2, we study the construction methods of decision evaluation functions based on representable uninorms in this paper.

The main contributions of this paper are listed as follows.

(1) We propose a transformation method from semi-decision evaluation functions to decision evaluation functions based on uninorms. We also propose some novel construction methods of semi-decision evaluation functions based on uninorms. These methods provide some novel ways to construct decision evaluation functions and unify some t-norms and t-conorms-based methods proposed in previous works (Hu 2017; Qiao and Hu 2018).

(2) We propose a transformation method from semi-decision evaluation functions to decision evaluation functions and some construction methods of decision evaluation functions based on representable uninorms. These methods not only provide some novel ways to construct decision evaluation functions, but also extend the strong negation N_{P_D} from $N_{P_D}(x) = 1 - x$ for all $x \in [0, 1]$ to the strong negation derived from the additive generator of representable uninorms.

(3) We research the relationship among different decision evaluation functions.

(4) We analyse two real evaluation problems to illustrate the results obtained in this paper. Moreover, by a real evaluation problem of credit card applicants, we compare the uninorms-based transformation method with overlap and grouping functions-based transformation method. The comparison results show that uninorms-based transformation method is superior to overlap and grouping functions-based transformation method under some conditions.

From theoretical viewpoint, on the one hand, this paper is supplement of three-way decisions. On the other hand, the existing t-norms and t-conorms-based transformation methods and construction methods become the special cases of uniorms-based methods proposed in this paper. From applied viewpoint, these contributions of this paper can enlarge the potentiality of three-way decisions in solving some actual application problems, especially in areas where both uninorms and three-way decisions work together, such as decision making (Campanella and Ribeiro 2011; Jiang and Hu 2021; Li et al. 2020; Liang et al. 2015; Liang et al. 2016; Yager 2002; Yager and Rybalov 2011; Yao and Azam 2015), image processing (Bustince et al. 2007; González-Hidalgo et al. 2015; Yue et al. 2021), neural networks (Cheng et al. 2021; de Campos Souza and Lughofer 2021; de Campos Souza and Lughofer 2022a; de Campos Souza and Lughofer 2022b), and so on.

The paper is organized as follows. In Sect. 2, we review some basic notations and concepts. In Sect. 3, we give two transformation methods from semi-decision evaluation functions to decision evaluation functions derived from uninorms and representable uninorms, respectively. In Sect. 4, on the basis of uninorms, we give some new construction methods of semi-decision evaluation functions and decision evaluation functions related to known semi-decision evaluation functions, fuzzy sets, fuzzy relations, interval-valued fuzzy sets and hesitant fuzzy sets. In Sect. 5, we use two real examples to illustrate our results. In the last section, we summarize our work and show some future research directions.

2 Preliminaries

In this section, we recall some basic concepts and notations used in this paper.

Definition 2.1 (Hu 2016) Let $(P, \leq)$ be a partially ordered set with the maximum element 1_P and minimum element 0_P. A non-increasing mapping $N : P \to P$ is said to be a negation if $N(0_P) = 1_P$ and $N(1_P) = 0_P$. In particular, it is called involutive or strong, if $N \circ N = id_P$.

In this paper, $(P, \leq_P)$ denotes a partially ordered set with a strong negation N_P, the maximum element 1_P and minimum element 0_P. We always write it as $(P, \leq_P, N_P, 0_P, 1_P)$. In addition, we will use the following partially ordered sets and corresponding operations on them from Qiao and Hu (2018, 2020):

(1) Let $Y = [0, 1]$. Then, for each $A \in Map(X, [0, 1])$, A is said to be a fuzzy set of X (Zadeh 1965). For each $\lambda \in [0, 1]$, λ_X denotes a fuzzy set of X with membership function $\lambda_X(x) \equiv \lambda$.

(2) Let $Y = I^{(2)}$ where $I^{(2)}$ denotes the family of interval numbers over $[0, 1]$, i.e., $I^{(2)} = \{[\lambda^-, \lambda^+] : 0 \leq \lambda^- \leq \lambda^+ \leq 1\}$. Then, for each $A \in Map(X, I^{(2)})$, A is said to be an interval-valued fuzzy set of X (Hu 2014). Meanwhile, we always write an interval-valued fuzzy set A with membership function $A(x) = [A^-(x), A^+(x)]$ as $A = [A^-, A^+]$. For each interval-valued fuzzy set $A = [A^-, A^+]$, $A^{(m)}$ denotes a fuzzy set of X with membership function $A^{(m)}(x) = \frac{A^+(x) + A^-(x)}{2}$. $A^{(m)}$ is said to be the center of interval-valued fuzzy set A. The order relation on $I^{(2)}$ is defined as $[\lambda^-, \lambda^+] \leq [\mu^-, \mu^+]$ iff $\lambda^+ \leq \mu^+$ and $\lambda^- \leq \mu^-$. Moreover, for all $\alpha \in [0, 1]$, the notation $\overline{\alpha} = [\alpha, \alpha]$ is used. Let N be a strong negation on $[0, 1]$. Then the operations on $I^{(2)}$ are defined as: $N([\lambda^-, \lambda^+]) = [N(\lambda^+), N(\lambda^-)]$, $\quad [\lambda^-, \lambda^+] \vee [\mu^-, \mu^+] = [\lambda^- \vee \mu^-, \lambda^+ \vee \mu^+]$ $\quad$ and $[\lambda^-, \lambda^+] \wedge [\mu^-, \mu^+] = [\lambda^- \wedge \mu^-, \lambda^+ \wedge \mu^+]$.

(3) Let $Y = 2^{[0,1]} - \emptyset$. Then, for each $H \in Map(X, 2^{[0,1]} - \emptyset)$, H is said to be a hesitant fuzzy set of X (Hu 2016). Let N be a strong negation on $[0, 1]$. For each $\mathcal{A}, \mathcal{B} \in 2^{[0,1]} - \emptyset$, define the operations on $2^{[0,1]} - \emptyset$ as: $N(\mathcal{A}) = \{N(r) : r \in \mathcal{A}\}$, $\mathcal{A} \sqcap \mathcal{B} = \{a \wedge b : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $\mathcal{A} \sqcup \mathcal{B} = \{a \vee b : a \in \mathcal{A}, b \in \mathcal{B}\}$. The order relation on $2^{[0,1]} - \emptyset$ is defined as $\mathcal{A} \preceq \mathcal{B}$ iff $\mathcal{A} \sqcup \mathcal{B} = \mathcal{B}$ and $\mathcal{A} \sqcap \mathcal{B} = \mathcal{A}$.

Let $(P, \leq_P, N_P, 0_P, 1_P)$ be a partially ordered set and U be a universe. Then, for each $A \in Map(U, P)$ and $x \in U$, we denote $N_P(A)(x) = N_P(A(x))$. In addition, for $P = [0, 1]$, we denote $A^C(x) = 1 - A(x)$ for all $x \in [0, 1]$.

Further, for each $A, B \in Map(U, P)$, $A \subseteq_P B$ is defined as $A(x) \leq_P B(x)$ for all $x \in U$. It is distinct that $(Map(U, P), \subseteq_P)$ is a partially ordered set with a strong negation N_P, the maximum element $(1_P)_U$ and minimum element $(0_P)_U$ where $(0_P)_U$ and $(1_P)_U$ are defined as $(0_P)_U(x) = 0_P$ and $(1_P)_U(x) = 1_P$ for each $x \in U$. If $P = [0, 1]$, for convenience, the notations $\subseteq_P, N_P, (0_P)_U$ and $(1_P)_U$ are always written as $\subseteq, N, 0_U$ and 1_U, respectively.

Definition 2.2 (Hu 2017) Let V be a condition universe and U be a decision universe. Then a mapping $E : Map(V, P_C) \to Map(U, P_D)$ is said to be a semi-decision evaluation function of U, if it satisfies the axioms as follows:

(E1)   Minimum element axiom

$$E((0_{P_C})_V) = (0_{P_D})_U.$$

(E2)   Monotonicity axiom

$$A \subseteq_{P_C} B \Rightarrow E(A) \subseteq_{P_D} E(B)$$

for each $A, B \in Map(V, P_C)$.

For each semi-decision evaluation function $E : Map(V, P_C) \rightarrow Map(U, [0, 1])$, the notation $1_U - E((1_{P_C})_V)$ denotes $1 - E((1_{P_C})_V)(x)$ for each $x \in U$.

Definition 2.3 (Hu 2016) Let $(U, Map(V, P_C), P_D, E)$ be a three-way decision space, $\psi, \omega \in P_D$, $1_{P_D} \geq \psi > \omega \geq 0_{P_D}$ and $A \in Map(V, P_C)$. Then three-way decisions are defined as follows.

(1)   Acceptance region: $ACP_{(\psi, \omega)}(E, A) = \{x \in U : E(A)(x) \geq \psi\}$.
(2)   Rejection region: $REJ_{(\psi, \omega)}(E, A) = \{x \in U : E(A)(x) \leq \omega\}$.
(3)   Uncertain region: $UNC_{(\psi, \omega)}(E, A) = \left(REJ_{(\psi, \omega)}(E, A) \cup ACP_{(\psi, \omega)}(E, A)\right)^C$.

In the following, we review the concepts of t-norms, t-conorms, overlap functions, grouping functions and uninorms, respectively.

Definition 2.4 (Klement etal. 2000) A mapping $T : [0, 1]^2 \rightarrow [0, 1]$ (resp. $S : [0, 1]^2 \rightarrow [0, 1]$) is said to be a t-norm (resp. t-conorm) if it is commutative, associative, increasing and having 1 (resp. 0) as neutral element.

Moreover, a t-norm T is said to be positive if $T(u, v) = 0$, then either $u = 0$ or $v = 0$ and continuous if it is continuous in both arguments at the same time. If we take into account a t-norm T and a t-conorm S in the meantime, then they are always considered as a dual pair, that is, $T(x, y) = 1 - S(1 - x, 1 - y)$ for each $x, y \in [0, 1]$. We list some commonly used t-norms (see Fig. 2) from Klement et al (2000) as follows.

Example 2.1

(1)   The minimum t-norm $T_M : [0, 1]^2 \rightarrow [0, 1]$ is given, for any $u, v \in [0, 1]$, by
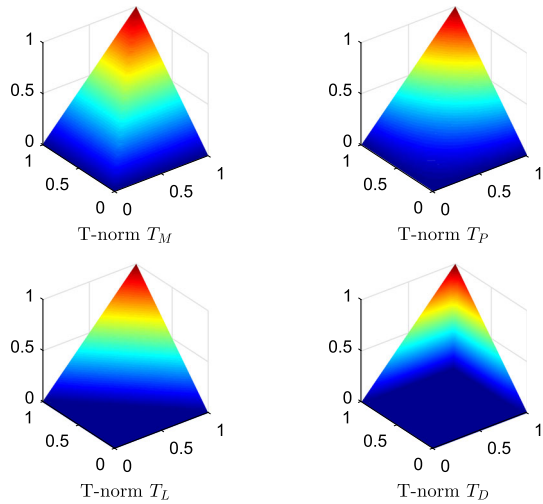
$$T_M(u, v) = \min\{u, v\}.$$

(2)   The product t-norm $T_P : [0, 1]^2 \rightarrow [0, 1]$ is given, for any $u, v \in [0, 1]$, by

$$T_P(u, v) = uv.$$

(3)   The Łukasiewicz t-norm $T_L : [0, 1]^2 \rightarrow [0, 1]$ is given, for any $u, v \in [0, 1]$, by

$$T_L(u, v) = \max(u + v - 1, 0).$$

Fig. 2 3D plots of four t-norms given in Example 2.1

T-norm T_M

T-norm T_P

T-norm T_L

T-norm T_D

(4) The drastic product t-norm $T_D : [0, 1]^2 \rightarrow [0, 1]$ is given, for any $u, v \in [0, 1]$, by

$$T_D(u, v) = \begin{cases} 0, & (u, v) \in [0, 1[^2, \\ \min(u, v), & \text{otherwise.} \end{cases}$$

Definition 2.5 (Bustince et al. 2010) A binary function $O : [0, 1]^2 \rightarrow [0, 1]$ is called an overlap function if, for each $u, v \in [0, 1]$, the following conditions hold:

(1) O is commutative;
(2) $O(u, v) = 0$ iff $uv = 0$;
(3) $O(u, v) = 1$ iff $uv = 1$;
(4) O is increasing;
(5) O is continuous.

In the following, we list some commonly used overlap functions (see Fig. 3) from Bedregal et al (2013).

Example 2.2

(1) Any positive and continuous t-norm is an overlap function.
(2) For any $p > 0$, the function $O_p : [0, 1]^2 \rightarrow [0, 1]$ given, for any $u, v \in [0, 1]$, by
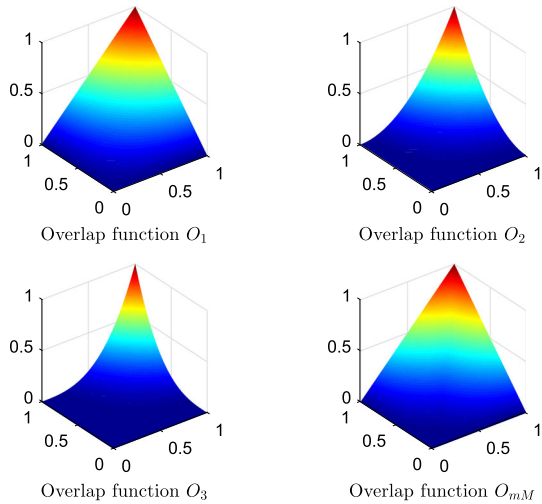
$$O_p(u, v) = u^p v^p$$

is an overlap function.
(3) The function $O_{mM} : [0, 1]^2 \rightarrow [0, 1]$ given, for any $u, v \in [0, 1]$, by

$$O_{mM}(u, v) = \min\{x, y\} \max\{x^2, y^2\}$$

is an overlap function.

Fig. 3 3D plots of the overlap functions given in Example 2.2

Overlap function O_1

Overlap function O_2

Overlap function O_3

Overlap function O_{mM}

Definition 2.6 (Bustince etal. 2012) A binary function $G : [0,1]^2 \to [0,1]$ is called a grouping function if, for each $u, v \in [0,1]$, the following conditions hold:

(1) G is commutative;
(2) $G(u,v) = 0$ iff $u = v = 0$;
(3) $G(u,v) = 1$ iff $u = 1$ or $v = 1$;
(4) G is increasing;
(5) G is continuous.

In the sequel, if we consider an overlap function O and a grouping function G at the same time, then they always represent a dual pair, that is, $O(x,y) = 1 - G(1 - x, 1 - y)$ for all $x, y \in [0,1]$.

Definition 2.7 (Yager and Rybalov 1996) A mapping $\mathcal{U} : [0,1]^2 \to [0,1]$ is said to be a uninorm if it is commutative, associative, increasing and having $e \in [0,1]$ as neutral element.

In Definition 2.7, if we take the neutral element of uninorm $\mathcal{U}$ as $e = 1$ (resp. $e = 0$), then $\mathcal{U}$ becomes a t-norm (resp. t-conorm). It is worth noting that Li and Shi (2000) have verified that $\mathcal{U}(1,0) \in \{0,1\}$ for all uninorms $\mathcal{U}$. In addition, a uninorm $\mathcal{U}$ is said to be disjunctive if $\mathcal{U}(1,0) = 1$ and conjunctive if $\mathcal{U}(1,0) = 0$.

Lemma 2.1 (*Fodor etal.* 1997) *Let $\mathcal{U}$ be a uninorm with neutral element $e \in]0,1[$. Then there exists a t-norm T_U and a t-conorm S_U such that*

$$\mathcal{U}(x,y) = \begin{cases} eT_{\mathcal{U}}(\dfrac{x}{e}, \dfrac{y}{e}), & (x,y) \in [0,e]^2, \\ e + (1-e)S_{\mathcal{U}}(\dfrac{x-e}{1-e}, \dfrac{y-e}{1-e}), & (x,y) \in [e,1]^2. \end{cases}$$

In addition, for each $(x,y) \in [0,e[\times]e,1]\cup]e,1] \times [0,e[$, one has that

Fig. 4 The structure of uninorms
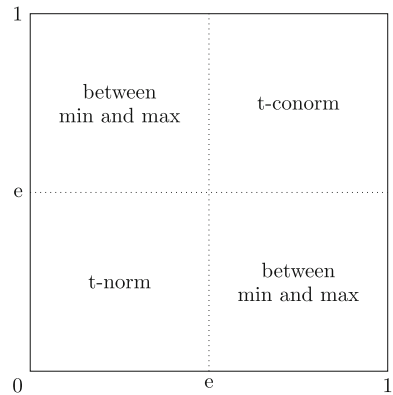
$$\min(x, y) \leq \mathcal{U}(x, y) \leq \max(x, y).$$

Lemma 2.1 demonstrates the structure of uninorms $\mathcal{U}$ (see Fig. 4). For each $(x, y) \in [0, e]^2$, the function of a uninorm likes a t-norm. For each $(x, y) \in [e, 1]^2$, the function of a uninorm likes a t-conorm.

Definition 2.8 (Fodor etal. 1997) Let $\mathcal{U}$ be a uninorm with neutral element $e \in]0, 1[$, if there exists a continuous strictly increasing function $h : [0, 1] \rightarrow [-\infty, +\infty]$ with $h(0) = -\infty, h(e) = 0, h(1) = +\infty$ such that

$$\mathcal{U}(x, y) = h^{-1}(h(x) + h(y))$$

for all $(x, y) \in [0, 1] \times [0, 1] \setminus \{(0, 1), (1, 0)\}$, then $\mathcal{U}$ is representable. The function h is said to be the additive generator of representable uninorm $\mathcal{U}$.

In this paper, the class of the representable uninorms are written as $\mathcal{U}_{rep}$.

Lemma 2.2 (*Fodor etal. 1997*) *Let $\mathcal{U}$ be a representable uninorm with neutral element $e \in]0, 1[$. Then, for each $x \in [0, 1]$, there exists a strong negation N given by $N(x) = h^{-1}(-h(x))$ with $N(e) = e$ such that*
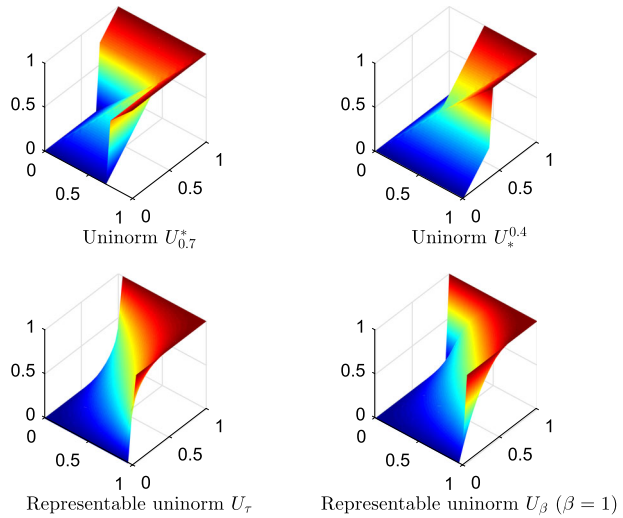
$$\mathcal{U}(x, y) = N(U(N(x), N(y)))$$

for each $(x, y) \in [0, 1] \times [0, 1] \setminus \{(0, 1), (1, 0)\}$.

For each $\mathcal{U} \in \mathcal{U}_{rep}$, the strong negation N on [0, 1] derived from the additive generator of $\mathcal{U}$ is always written as N^r. We list some commonly used uninorms (see Fig. 5) from Yager and Rybalov (1996) and Fodor et al (1997) as follows.

Example 2.3

(1) The uninorm $U_e^* : [0, 1]^2 \rightarrow [0, 1]$ with neutral element $e \in [0, 1]$ is given, for any $u, v \in [0, 1]$, by

Fig. 5 3D plots of four uninorms given in Example 2.3

Uninorm $U_{0.7}^*$

Uninorm $U_*^{0.4}$

Representable uninorm U_τ

Representable uninorm U_β ($\beta = 1$)

$$U_e^*(u, v) = \begin{cases} \min\{u, v\}, & (u, v) \in [0, e]^2, \\ \max\{u, v\}, & \text{otherwise.} \end{cases}$$

(2) The uninorm $U_*^e : [0, 1]^2 \to [0, 1]$ with neutral element $e \in [0, 1]$ is given, for any $u, v \in [0, 1]$, by

$$U_*^e(u, v) = \begin{cases} \max\{u, v\}, & (u, v) \in [e, 1]^2, \\ \min\{u, v\}, & \text{otherwise.} \end{cases}$$

(3) Take the additive generator as $h(x) = \ln\left(\frac{x}{1-x}\right)$, it follows that

$$\mathcal{U}_\tau(x, y) = \begin{cases} 0, & (x, y) \in \{(1, 0), (0, 1)\}, \\ \dfrac{xy}{(1 - x)(1 - y) + xy}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}_\tau$ is a representable uninorm with neutral element $e = 0.5$ and strong negation $N^r(x) = 1 - x$ for each $x \in [0, 1]$.

(4) For each $\beta > 0$, take the additive generator as $h_\beta(x) = \ln\left(-\frac{1}{\beta} \cdot \ln(1 - x)\right)$, it follows that

$$\mathcal{U}_\beta(x, y) = \begin{cases} 1, & (x, y) \in \{(1, 0), (0, 1)\}, \\ 1 - \exp\left(-\dfrac{1}{\beta} \cdot \ln(1 - x) \cdot \ln(1 - y)\right), & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}_\beta$ is a representable uninorm with neutral element $e = 1 - \exp(-\beta)$ and strong negation $N^r(x) = 1 - \exp\left(\frac{\beta^2}{\ln(1-x)}\right)$ for each $x \in [0, 1]$.

3 Transformation methods from semi-decision evaluation functions to decision evaluation functions on the basis of uninorms

In this section, we give two novel transformation methods from semi-decision evaluation functions to decision evaluation functions on the basis of uninorms and representable uninorms, respectively.

3.1 Transformation method on the basis of uninorms

In this subsection, if partially ordered set $P = [0, 1]$, then the corresponding strong negation N is given by $N(x) = 1 - x$ for each $x \in [0, 1]$.

Lemma 3.1 (*Yager and Rybalov* 1996) *Let $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$, then $\hat{\mathcal{U}}$, defined such that*

$$\hat{\mathcal{U}}(x, y) = 1 - \mathcal{U}(1 - x, 1 - y),$$

is a uninorm with neutral element $\hat{e} = 1 - e$.

In this paper, if we consider a uninorm $\mathcal{U}$ with neutral element $e \in [0, 1]$ and a uninorm $\hat{\mathcal{U}}$ with neutral element $\hat{e} \in [0, 1]$ in the meantime, then they always denote a dual pair, that is, $\hat{e} = 1 - e$ and $\hat{\mathcal{U}}(x, y) = 1 - \mathcal{U}(1 - x, 1 - y)$ for each $x, y \in [0, 1]$.

Theorem 3.1 *Let V be a condition universe, U be a decision universe, $E : Map(V, P_C) \to Map(U, [0, 1])$ be a semi-decision evaluation function of U, $\mathcal{U}$ and $\hat{\mathcal{U}}$ be two uninorms with neutral element e and $\hat{e}$ such that $1_U - E((1_{P_C})_V) \subseteq (e_U \cap \hat{e}_U)$. For each $A \in Map(V, P_C)$ and $x \in U$, take*

$$E_{(\mathcal{U}, \hat{\mathcal{U}}, N_{P_C})}(A)(x) = \frac{\mathcal{U}(1 - E(N_{P_c}(A))(x), E(A)(x))}{2}$$
$$+ \frac{\hat{\mathcal{U}}(1 - E(N_{P_c}(A))(x), E(A)(x))}{2}.$$

Then $E_{(\mathcal{U}, \hat{\mathcal{U}}, N_{P_C})}$ is a decision evaluation function of U.

Proof

(1) Minimum element axiom

    (i) Let $e, \hat{e} \in [0, 1]$ with $e \leq \hat{e}$. Then, it follows that

$$1_U - E((1_{P_c})_V) \subseteq (e_U \cap \hat{e}_U) \Rightarrow 1_U(x) - E((1_{P_c})_V)(x) \leq (e_U \cap \hat{e}_U)(x)$$
$$\Rightarrow 1 - E((1_{P_c})_V)(x) \leq e_U(x) \wedge \hat{e}_U(x)$$
$$\Rightarrow 1 - E((1_{P_c})_V)(x) \leq e,$$

for each $x \in U$. Therefore, it follows that

$$
\begin{aligned}
E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}((0_{P_C})_V)(x) &= \frac{\mathcal{U}(1 - E(N_{P_C}((0_{P_C})_V))(x), E((0_{P_C})_V)(x))}{2} \\
&\quad + \frac{\hat{\mathcal{U}}(1 - E(N_{P_C}((0_{P_C})_V))(x), E((0_{P_C})_V)(x))}{2} \\
&= \frac{\mathcal{U}(1 - E((1_{P_C})_V)(x), 0_U(x))}{2} \\
&\quad + \frac{\hat{\mathcal{U}}(1 - E((1_{P_C})_V)(x), 0_U(x))}{2} \\
&\leq \frac{\mathcal{U}(e, 0) + \hat{\mathcal{U}}(e, 0)}{2} \\
&= 0,
\end{aligned}
$$

for each $x \in U$.

(ii)   Let $e, \hat{e} \in [0, 1]$ with $e \geq \hat{e}$. Then, it can be verified similarly to above that $E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}((0_{P_C})_V)(x) = 0$ for each $x \in U$.

Therefore, one obtains that $E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}((0_{P_C})_V) = 0_U$.

(2) Monotonicity axiom

Let $A, B \in Map(V, P_C)$ with $A \subseteq_{P_C} B$. Then, one gets that

$$
\begin{aligned}
E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(A)(x) &= \frac{\mathcal{U}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2} \\
&\quad + \frac{\hat{\mathcal{U}}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2} \\
&\leq \frac{\mathcal{U}(1 - E(N_{P_C}(B))(x), E(B)(x))}{2} \\
&\quad + \frac{\hat{\mathcal{U}}(1 - E(N_{P_C}(B))(x), E(B)(x))}{2} \\
&= E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(B)(x),
\end{aligned}
$$

for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(A) \subseteq E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(B)$.

(3) Complement axiom

For each $A \in Map(V, P_C)$ and $x \in U$, it follows that

$$N(E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(A))(x) = 1 - \left(\frac{\mathcal{U}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2} \right.$$
$$\left. + \frac{\hat{\mathcal{U}}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2} \right)$$
$$= \frac{1 - \mathcal{U}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2}$$
$$+ \frac{1 - \hat{\mathcal{U}}(1 - E(N_{P_C}(A))(x), E(A)(x))}{2}$$
$$= \frac{\hat{\mathcal{U}}(E(N_{P_C}(A))(x), 1 - E(A)(x))}{2}$$
$$+ \frac{\mathcal{U}(E(N_{P_C}(A))(x), 1 - E(A)(x))}{2}$$
$$= E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(N_{P_C}(A))(x).$$

Therefore, one obtains that $N(E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(A)) = E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(N_{P_C}(A))$.

$\square$

Remark 3.1 In Theorem 3.1, if we take the neutral element of uninorm $\mathcal{U}$ as $e = 1$, then one concludes that $\hat{e} = 1 - e = 0$. Therefore, for each $x \in U$, it follows that

$$1_U - E((1_{P_C})_V) \subseteq (e_U \cap \hat{e}_U) \implies 1_U(x) - E((1_{P_C})_V)(x) \leq (e_U \cap \hat{e}_U)(x)$$
$$\implies 1 - E((1_{P_C})_V)(x) \leq (1_U(x) \wedge 0_U(x))$$
$$\implies E((1_{P_C})_V)(x) \geq 1 - (1 \wedge 0)$$
$$\implies E((1_{P_C})_V)(x) \geq 1.$$

Thus, one obtains that $E((1_{P_C})_V) = 1_U$. Then

$$E_{(\mathcal{U},\hat{\mathcal{U}},N_{P_C})}(A)(x) = \frac{T(1 - E(N_{P_C}(A))(x), E(A)(x))}{2}$$
$$+ \frac{S(1 - E(N_{P_C}(A))(x), E(A)(x))}{2}$$

becomes the transformation method given in Theorem 3.5 of Hu (2017). Therefore, Theorem 3.5 of Hu (2017) is a special case of Theorem 3.1. Theorem 3.1 generalizes Theorem 3.5 of Hu (2017) and includes more cases. In addition, Theorem 3.1 has the looser constraint conditions of semi-decision evaluation functions than Theorem 3.5 of Hu (2017).

Lemma 3.2 *(Jia and Qiao 2020) Let O be an overlap function, G be a grouping function, $E : Map(V, P_C) \to Map(U, [0, 1])$ be a semi-decision evaluation function of U and $E((1_{P_C})_V) = 1_U$. For each $x \in U$ and $A \in Map(V, P_C)$, take*

$$E_{(O,G,N_{P_C})}(A)(x) = \frac{O(E(A)(x), 1 - E(N_{P_C}(A))(x))}{2}$$
$$+ \frac{G(E(A)(x), 1 - E(N_{P_C}(A))(x))}{2}.$$

Then $E_{(O,G,N_{P_C})}$ is a decision evaluation function of U.

Remark 3.2 From Remark 3.1 and item (1) of Example 2.1, if we take uninorm $\mathcal{U}$ as a positive and continuous t-norm T, then the transformation method given in Theorem 3.1 becomes the transformation method given in Lemma 3.2. In addition, Theorem 3.1 has the looser constraint conditions of semi-decision evaluation functions than Lemma 3.2.

3.2 Transformation method on the basis of representable uninorms

Theorem 3.2 *Let V be a condition universe, U be a decision universe and E : $Map(V, P_C) \rightarrow Map(U, [0,1])$ be a semi-decision evaluation function of U satisfying the following conditions*:

(1) $E(A)(x) > 0$ if $A \neq (0_{P_C})_V$ for each $x \in U$;
(2) $E(A)(x) < 1$ for each $x \in U$ and $A \in Map(V, P_C)$.

Moreover, let $\mathcal{U}$ be a representable uninorm with neutral element $e \in]0,1[$ and strong negation N^r such that $N^r(E((1_{P_C})_V)) \subseteq e_U$. For each $A \in Map(V, P_C)$ and $x \in U$, take

$$E_{(\mathcal{U}_{rep},N_{P_C})}(A)(x) = \mathcal{U}(N^r(E(N_{P_C}(A))(x)), E(A)(x)).$$

Then $E_{(\mathcal{U}_{rep},N_{P_C})}$ is a decision evaluation function of U.

Proof

(1) Minimum element axiom
     Since $N^r(E((1_{P_C})_V)) \subseteq e_U$, it follows that

$$\begin{aligned}
E_{(\mathcal{U}_{rep},N_{P_C})}\big((0_{P_C})_V\big)(x) &= \mathcal{U}\big(N^r\big(E(N_{P_C}((0_{P_C})_V))(x)\big), E\big((0_{P_C})_V\big)(x)\big) \\
&= \mathcal{U}(N^r(E((1_{P_C})_V)(x)), 0_U(x)) \\
&\leq \mathcal{U}(e_U(x), 0) \\
&= 0,
\end{aligned}$$

     for each $x \in U$. Therefore, one obtains that $E_{(\mathcal{U}_{rep},N_{P_C})}((0_{P_C})_V) = 0_U$.
(2) Monotonicity axiom
     Let $A, B \in Map(V, P_C)$ with $A \subseteq_{P_C} B$. Then, one gets that

$$\begin{aligned}
E_{(\mathcal{U}_{rep},N_{P_C})}(A)(x) &= \mathcal{U}(N^r(E(N_{P_C}(A))(x)), E(A)(x)) \\
&\leq \mathcal{U}(N^r(E(N_{P_C}(B))(x)), E(B)(x)) \\
&= E_{(\mathcal{U}_{rep},N_{P_C})}(B)(x),
\end{aligned}$$

     for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U}_{rep},N_{P_C})}(A) \subseteq E_{(\mathcal{U}_{rep},N_{P_C})}(B)$.
(3) Complement axiom
     Firstly, for each $x \in U$ and $A \in Map(V, P_C)$, consider the following four cases.

(i) Let $E(A)(x) = 0$. Then one has that $A = (0_{P_C})_V$. Thus it follows that

$$
\begin{aligned}
N^r(E(N_{P_C}(A))(x)) &= N^r(E(N_{P_C}((0_{P_C})_V))(x)) \\
&= N^r(E((1_{P_C})_V)(x)) \\
&\leq e_U(x) \\
&< 1.
\end{aligned}
$$

Thus, one obtains that $N^r(E(N_{P_C}(A))(x)) \neq 1$.

(ii) Let $N^r(E(N_{P_C}(A))(x)) = 0$. Then one has that $E(N_{P_C}(A))(x) = 1$ which conflicts with the assumption.

(iii) Let $E(N_{P_C}(A))(x) = 0$. Then one has that $N_{P_C}(A) = (0_{P_C})_V$. Thus it follows that $A = (1_{P_C})_V$. Thus, one gets that

$$
N^r(E(A)(x)) = N^r(E((1_{P_C})_V)(x)) \leq e_U(x) < 1.
$$

Thus, one obtains that $N^r(E(A)(x)) \neq 1$.

(iv) Let $N^r(E(A)(x)) = 0$. Then one has that $E(A)(x) = 1$ which conflicts with the assumption.

According to above discussions, one concludes that $\{(N^r(E(N_{P_C}(A))(x)), E(A)(x)), (E(N_{P_C}(A))(x), \quad N^r(E(A)(x)))\} \subseteq [0,1] \times [0,1] \setminus \{(0,1),(1,0)\}$. Then, it follows that

$$
\begin{aligned}
N^r(E_{(\mathcal{U}_{rep}, N_{P_C})}(A))(x) &= N^r(\mathcal{U}(N^r(E(N_{P_C}(A))(x)), E(A)(x))) \\
&= \mathcal{U}(E(N_{P_C}(A))(x), N^r(E(A)(x))) \\
&= E_{(\mathcal{U}_{rep}, N_{P_C})}(N_{P_C}(A))(x).
\end{aligned}
$$

Therefore, one obtains that $N^r(E_{(\mathcal{U}_{rep}, N_{P_C})}(A)) = E_{(\mathcal{U}_{rep}, N_{P_C})}(N_{P_C}(A))$.

$\square$

4 Construction methods of semi-decision evaluation functions and decision evaluation functions based on uninorms

On the basis of the transformation methods from semi-decision evaluation functions to decision evaluation functions given in previous works (Hu 2017; Jia and Qiao 2020; Qiao and Hu 2018) and Sect. 3, a decision evaluation function can be obtained from known semi-decision evaluation functions. Therefore, in this section, in order to get more decision evaluation functions, we give some novel construction methods of semi-decision evaluation function related to known semi-decision evaluation functions, fuzzy sets, fuzzy relations, interval-valued fuzzy sets and hesitant fuzzy sets, respectively. At the same time, we give some novel construction methods of decision evaluation function based on representable uninorms. We also research the relationship between t-norms and t-conorms-based construction methods of semi-decision evaluation functions given in Qiao and Hu (2018) with uninorms-based construction methods of semi-decision evaluation functions proposed in this section.

4.1 Construction methods related to known semi-decision evaluation functions

Proposition 4.1 *Let V be a condition universe, U be a decision universe, $E_1, E_2 : Map(V, P_C) \to Map(U, [0,1])$ be two semi-decision evaluation functions of U and $\mathcal{U}$ be a uninorm with neutral element $e \in [0,1]$. For each $A \in Map(V, P_C)$ and $x \in U$, take*

$$E_{\mathcal{U}}(A)(x) = \mathcal{U}(E_1(A)(x), E_2(A)(x)).$$

Then $E_{\mathcal{U}}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 3.1 of Qiao and Hu (2018). □

Remark 4.1

(1) In Proposition 4.1, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$E_{\mathcal{U}}(A)(x) = T(E_1(A)(x), E_2(A)(x))$$

becomes the construction method given in Proposition 3.1 of Qiao and Hu (2018).
(2) In Proposition 4.1, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$E_{\mathcal{U}}(A)(x) = S(E_1(A)(x), E_2(A)(x))$$

becomes the construction method given in Proposition 3.2 of Qiao and Hu (2018).
(3) According to items (1) and (2), Propositions 3.1 and 3.2 of Qiao and Hu (2018) are special cases of Proposition 4.1 and unified by Proposition 4.1. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.1.

Proposition 4.2 *Let V be a condition universe, U be a decision universe, $E_1, E_2 : Map(V, P_C) \to Map(U, [0,1])$ be two semi-decision evaluation functions of U, $\mathcal{U}_i$ be a uninorm with neutral element $e_i \in [0,1]$ for $i = 1, 2, \cdots, n$ and $\lambda_i \in [0,1]$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \lambda_i = 1$. For each $A \in Map(V, P_C)$ and $x \in U$, take*

$$E_{(\mathcal{U},\lambda)}(A)(x) = \sum_{i=1}^n \lambda_i \mathcal{U}_i(E_1(A)(x), E_2(A)(x)).$$

Then $E_{(\mathcal{U},\lambda)}$ is a semi-decision evaluation function of U.

Proof It can be immediately derived from Proposition 4.1 . □

Remark 4.2 In Proposition 4.2, take $n = 2$ and the neutral element of uninorm $\mathcal{U}_1$ and $\mathcal{U}_2$ as $e_1 = 1$ and $e_2 = 0$, respectively. Then

$$E_{(\mathcal{U},\lambda)}(A)(x) = \lambda_1 T(E_1(A)(x), E_2(A)(x)) + \lambda_2 S(E_1(A)(x), E_2(A)(x))$$

becomes the construction method given in Proposition 3.3 of Qiao and Hu (2018). Therefore, Proposition 3.3 of Qiao and Hu (2018) is a special case of Proposition 4.2. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.2.

Proposition 4.3 *Let V be a condition universe, U be a decision universe, $E:$ $Map(V, P_C) \rightarrow Map(U, [0, 1])$ be a semi-decision evaluation functions of U, $N^{\star}$ be a negation on [0, 1] and $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$ such that $N^{\star}(E((1_{P_C})_V)) \subseteq e_U$. For each $A \in Map(V, P_C)$ and $x \in U$, take*

$$E_{(\mathcal{U}, N_{P_C})}(A)(x) = \mathcal{U}(N^{\star}(E(N_{P_C}(A))(x)), E(A)(x)).$$

Then $E_{(\mathcal{U}, N_{P_C})}$ is a semi-decision evaluation function of U.

Proof

(1) Minimum element axiom
     Since $N^{\star}(E((1_{P_C})_V)) \subseteq e_U$, it follows that

$$\begin{aligned}
E_{(\mathcal{U}, N_{P_C})}((0_{P_C})_V)(x) &= \mathcal{U}(N^{\star}(E(N_{P_C}((0_{P_C})_V))(x)), E((0_{P_C})_V)(x)) \\
&= \mathcal{U}(N^{\star}(E((1_{P_C})_V)(x)), (0_{P_D})_U(x)) \\
&\leq \mathcal{U}(e_U(x), 0) \\
&= 0,
\end{aligned}$$

     for each $x \in U$. Therefore, one obtains that $E_{(\mathcal{U}, N_{P_C})}((0_{P_C})_V) = 0_U$.
(2) Monotonicity axiom
     Let $A, B \in Map(V, P_C)$ with $A \subseteq_{P_C} B$. Then, one gets that

$$\begin{aligned}
E_{(\mathcal{U}, N_{P_C})}(A)(x) &= \mathcal{U}(N^{\star}(E(N_{P_C}(A))(x)), E(A)(x)) \\
&\leq \mathcal{U}(N^{\star}(E(N_{P_C}(B))(x)), E(B)(x)) \\
&= E_{(\mathcal{U}, N_{P_C})}(B)(x),
\end{aligned}$$

     for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U}, N_{P_C})}(A) \subseteq E_{(\mathcal{U}, N_{P_C})}(B)$.

     $\square$

Remark 4.3

(1) In Proposition 4.3, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$ and the negation $N^{\star}$ as $N^{\star}(x) = 1 - x$ for all $x \in [0, 1]$. Then

$$E_{(\mathcal{U}, N_{P_C})}(A)(x) = T(1 - E(N_{P_C}(A))(x), E(A)(x))$$

     becomes the construction method given in Proposition 3.4 of Qiao and Hu (2018).
(2) In Proposition 4.3, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$ and the negation $N^{\star}$ as $N^{\star}(x) = 1 - x$ for all $x \in [0, 1]$. Then, one gets that

$$1 = 1_U(x) - 0_U(x) = 1_U(x) - e_U(x) \leq E((1_{P_C})_V)(x) \leq 1,$$

     for each $x \in U$. Therefore, one obtains that $E((1_{P_C})_V) = 1_U$. Then

$$E_{(\mathcal{U}, N_{P_C})}(A)(x) = S(1 - E(N_{P_C}(A))(x), E(A)(x))$$

     becomes the construction method given in Proposition 3.5 of Qiao and Hu (2018).

(3) According to items (1) and (2), Propositions 3.4 and 3.5 of Qiao and Hu (2018) are special cases of Proposition 4.3 and unified by Proposition 4.3. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.3.

4.2 Construction methods related to fuzzy sets

Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be two nonempty finite sets. In the following, we always denote $(m \times n)$-dimensional fuzzy matrix $R = (r_{ij})$ given by $r_{ij} = R(x_i, y_i)$ as the fuzzy relation R under the $X \times Y$ (Fan 2000).

Proposition 4.4 *Let U be a condition universe and also be a decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$ and $P_C = P_D = [0, 1]$. For each $A \in Map(U, [0, 1])$ and $x \in U$, take $E_{(\mathcal{U},I)} : Map(U, [0, 1]) \rightarrow Map(U, [0, 1])$ as:*

$$E_{(\mathcal{U},I)}(A)(x) = \mathcal{U}(A(x), A(x)).$$

Then $E_{(\mathcal{U},I)}$ is a semi-decision evaluation function of U.

Proof

(1) Minimum element axiom
Since $\mathcal{U}(0, 0) = 0$, it follows that

$$E_{(\mathcal{U},I)}(0_U)(x) = \mathcal{U}(0_U(x), 0_U(x)) = 0,$$

for each $x \in U$. Therefore, one obtains that $E_{(\mathcal{U},I)}(0_U) = 0_U$.

(2) Monotonicity axiom
Let $A, B \in Map(U, [0, 1])$ with $A \subseteq B$. Then, one gets that

$$E_{(\mathcal{U},I)}(A)(x) = \mathcal{U}(A(x), A(x)) \leq \mathcal{U}(B(x), B(x)) = E_{(\mathcal{U},I)}(B)(x),$$

for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U},I)}(A) \subseteq E_{(\mathcal{U},I)}(B)$.

□

Remark 4.4

(1) In Proposition 4.4, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$E_{(\mathcal{U},I)}(A)(x) = T(A(x), A(x))$$

becomes the construction method given in Sect. 3 of Qiao and Hu (2018).

(2) In Proposition 4.4, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$E_{(\mathcal{U},I)}(A)(x) = S(A(x), A(x))$$

becomes the construction method given in Sect. 3 of Qiao and Hu (2018).

(3) The construction methods given in items (1) and (2) are special cases of Proposition 4.4 and unified by Proposition 4.4. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.4.

Fig. 6 Decision evaluation function $E_{(\mathcal{U}_{rep},I)}(A)$ and fuzzy set A
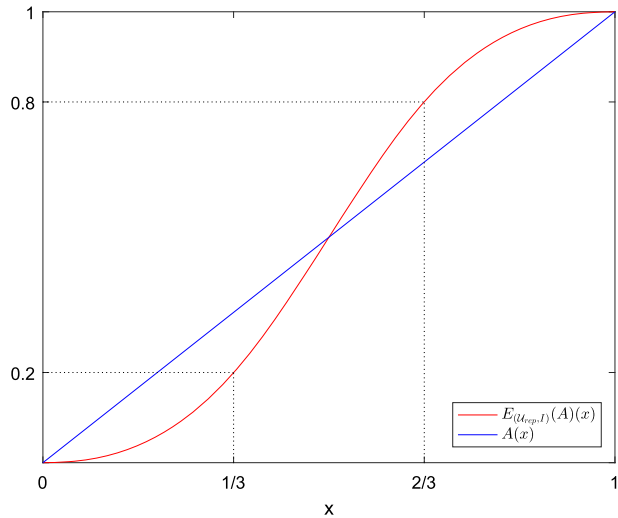
Theorem 4.1 *Let U be a condition universe and also be a decision universe, $\mathcal{U}$ be a representable uninorm with neutral element $e \in]0, 1[$ and $P_C = P_D = [0, 1]$. For each $A \in Map(U, [0, 1])$ and $x \in U$, take $E_{(\mathcal{U}_{rep},I)} : Map(U, [0, 1]) \to Map(U, [0, 1])$ as:*

$$E_{(\mathcal{U}_{rep},I)}(A)(x) = \mathcal{U}(A(x), A(x)).$$

Then $E_{(\mathcal{U}_{rep},I)}$ is a decision evaluation function of U.

Proof Items (E1) and (E2) of Definition 1.1 can be immediately verified from Proposition 4.4. We only verify item (E3) of Definition 1.1 as follows.

For each $x \in U$ and $A \in Map(U, [0, 1])$, it follows that

$$\begin{aligned}
N^r(E_{(\mathcal{U}_{rep},I)}(A))(x) &= N^r(\mathcal{U}(A(x), A(x))) \\
&= \mathcal{U}(N^r(A(x)), N^r(A(x))) \\
&= E_{(\mathcal{U}_{rep},I)}(N^r(A))(x).
\end{aligned}$$

Therefore, one obtains that $N^r(E_{(\mathcal{U}_{rep},I)}(A)) = E_{(\mathcal{U}_{rep},I)}(N^r(A))$. $\square$

Example 4.1 Let $U = [0, 1]$ and A be a fuzzy set of U with membership function $A(x) = x$. If we consider the additive generator $h(x) = \ln(\frac{x}{1-x})$, from the construction method discussed in Theorem 4.1, then we obtain a decision evaluation function $E_{(\mathcal{U}_{rep},I)}(A)$ of U (see Fig. 6) as follows:

$$E_{(\mathcal{U}_{rep},I)}(A)(x) = \frac{x^2}{2x^2 - 2x + 1}.$$

If we take $\psi = 0.8$ and $\omega = 0.2$, then three-way decision is given as follows.

(1) Acceptance region: $ACP_{(0.8,0.2)}(E_{(\mathcal{U}_{rep},I)}, A) = \{x | \frac{2}{3} \leq x \leq 1\}$.
(2) Rejection region: $REJ_{(0.8,0.2)}(E_{(\mathcal{U}_{rep},I)}, A) = \{x | 0 \leq x \leq \frac{1}{3}\}$.

(3)   Uncertain region: $UNC_{(0.8,0.2)}(E_{(\mathcal{U}_{rep},I)},A) = \{x| \frac{1}{3} <x< \frac{2}{3}\}$.

Lemma 4.1 *Let $\mathcal{U}$ be a uninorm with neutral element $e \in [0,1]$, X be a universe and $\lambda \in [0,1]$. For each $A \in Map(X, [0,e])$ and $x \in X$, one has that*

$$\mathcal{U}(A(x), \lambda) \leq \lambda.$$

Proof Since $A \in Map(X, [0,e])$, it follows that

$$\mathcal{U}(A(x), \lambda) \leq \mathcal{U}(e, \lambda) = \lambda,$$

for each $x \in X$. Therefore, one concludes that $\mathcal{U}(A(x), \lambda) \leq \lambda$. □

Proposition 4.5 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in]0,1]$, $P_C = [0,e]$, $P_D = [0,1]$ and $R \in Map(U \times V, [0,e])$ such that $\sum_{y \in V} R(x,y) \neq 0$ for each $x \in U$. For each $A \in Map(V, [0,e])$ and $x \in U$, take $E_{(\mathcal{U},R)} : Map(V, [0,e]) \rightarrow Map(U, [0,1])$ as:*

$$E_{(\mathcal{U},R)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A(y))}{\sum_{y \in V} R(x,y)}.$$

Then $E_{(\mathcal{U},R)}$ is a semi-decision evaluation function of U.

Proof Firstly, from Lemma 4.1, one obtains that $E_{(\mathcal{U},R)}$ is well defined.

(1)   Minimum element axiom
   Since $R \in Map(U \times V, [0,e])$, it follows that

$$E_{(\mathcal{U},R)}((0_{P_C})_V)(x) = \frac{\sum_{y \in V} \mathcal{U}(R(x,y), (0_{P_C})_V(y))}{\sum_{y \in V} R(x,y)}$$
$$\leq \frac{\sum_{y \in V} \mathcal{U}(e, 0)}{\sum_{y \in V} R(x,y)}$$
$$= 0,$$

   for each $x \in U$. Therefore, one obtains that $E_{(\mathcal{U},R)}((0_{P_C})_V) = 0_U$.

(2)   Monotonicity axiom
   Let $A, B \in Map(V, [0,e])$ with $A \subseteq_{P_C} B$. Then, one gets that

$$E_{(\mathcal{U},R)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A(y))}{\sum_{y \in V} R(x,y)}$$
$$\leq \frac{\sum_{y \in V} \mathcal{U}(R(x,y), B(y))}{\sum_{y \in V} R(x,y)}$$
$$= E_{(\mathcal{U},R)}(B)(x),$$

   for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U},R)}(A) \subseteq E_{(\mathcal{U},R)}(B)$.

   □

Remark 4.5 In Proposition 4.5, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Thus, one has that $P_C = [0, 1]$ and $R \in Map(U \times V, [0, 1])$. Then

$$E_{(\mathcal{U},R)}(A)(x) = \frac{\sum_{y \in V} T(R(x,y), A(y))}{\sum_{y \in V} R(x,y)}$$

becomes the construction method given in Proposition 4.2 of Qiao and Hu (2018). Therefore, Proposition 4.2 of Qiao and Hu (2018) is a special case of Proposition 4.5. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.5.

Proposition 4.6 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$, $P_C = P_D = [0, 1]$ and $R \in Map(U \times V, [0, e])$. For each $A \in Map(V, [0, 1])$ and $x \in U$, take $E'_{(\mathcal{U},R)} : Map(V, [0, 1]) \rightarrow Map(U, [0, 1])$ as:*

$$E'_{(\mathcal{U},R)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A(y))}{|V|}.$$

Then $E'_{(\mathcal{U},R)}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 4.5. □

Proposition 4.7 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}_1, \mathcal{U}_2$ be two uninorms with neutral element $e_1, e_2 \in [0, 1]$, $e = e_1 \wedge e_2$, $P_C = P_D = [0, 1]$ and $R \in Map(U \times V, [0, e])$. For each $A \in Map(V, [0, 1])$ and $x \in U$, take $E_{(\mathcal{U}_1,\mathcal{U}_2,R)} : Map(V, [0, 1]) \rightarrow Map(U, [0, 1])$ as:*

$$E_{(\mathcal{U}_1,\mathcal{U}_2,R)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}_1(\mathcal{U}_2(R(x,y), A(y)), A(y))}{|V|}.$$

Then $E_{(\mathcal{U}_1,\mathcal{U}_2,R)}$ is a semi-decision evaluation function of U.

Proof

(1) Minimum element axiom

   (i) Let $e_1, e_2 \in [0, 1]$ with $e_1 \leq e_2$, then one obtains that $e = e_1 \wedge e_2 = e_1$ and $R(x,y) \leq e \leq e_2$ for each $(x,y) \in U \times V$. Thus, it follows that

$$E_{(\mathcal{U}_1,\mathcal{U}_2,R)}((0_{P_C})_V)(x) = \frac{\sum_{y \in V} \mathcal{U}_1(\mathcal{U}_2(R(x,y), (0_{P_C})_V(y)), (0_{P_C})_V(y))}{|V|}$$
$$\leq \frac{\sum_{y \in V} \mathcal{U}_1(\mathcal{U}_2(e_2, 0), 0)}{|V|}$$
$$= 0,$$

   for each $x \in U$.
   (ii) Let $e_1, e_2 \in [0, 1]$ with $e_1 \leq e_2$, then it can be verified similarly to above that $E_{(\mathcal{U}_1,\mathcal{U}_2,R)}((0_{P_C})_V)(x) = 0$ for each $x \in U$.

Therefore, one obtains that $E_{(\mathcal{U}_1,\mathcal{U}_2,R)}((0_{P_C})_V) = 0_U$.

(2) Monotonicity axiom
     Let $A, B \in Map(V, [0,1])$ with $A \subseteq B$. Then, one gets that

$$
\begin{aligned}
E_{(\mathcal{U}_1, \mathcal{U}_2, R)}(A)(x) &= \frac{\sum_{y \in V} \mathcal{U}_1(\mathcal{U}_2(R(x,y), A(y)), A(y))}{|V|} \\
&\leq \frac{\sum_{y \in V} \mathcal{U}_1(\mathcal{U}_2(R(x,y), B(y)), B(y))}{|V|} \\
&= E_{(\mathcal{U}_1, \mathcal{U}_2, R)}(B)(x),
\end{aligned}
$$

for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U}_1, \mathcal{U}_2, R)}(A) \subseteq E_{(\mathcal{U}_1, \mathcal{U}_2, R)}(B)$.

$\square$

Remark 4.6 In Propositions 4.5–4.7, if we take each uninorm $\mathcal{U}$ as a conjunctive uninorm, then the condition $R \in Map(U \times V, [0, e])$ can be relaxed to $R \in Map(U \times V, [0, 1])$.

4.3 Construction methods related to interval-valued fuzzy sets

Proposition 4.8 *Let U be a condition universe and also be a decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$, $P_C = I^{(2)}$ and $P_D = [0, 1]$. Take $E_{(\mathcal{U}, \heartsuit)} : Map(U, I^{(2)}) \to Map(U, [0, 1])$ as:*

$$
E_{(\mathcal{U}, \heartsuit)}(A)(x) = \mathcal{U}(A^+(x), A^-(x))
$$

for each $x \in U$ and $A \in Map(U, I^{(2)})$, then $E_{(\mathcal{U}, \heartsuit)}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 4.3 of Qiao and Hu (2018).
$\square$

Remark 4.7

(1) In Proposition 4.8, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$
E_{(\mathcal{U}, \heartsuit)}(A)(x) = T(A^+(x), A^-(x))
$$

becomes the construction method given in Proposition 4.3 of Qiao and Hu (2018).

(2) In Proposition 4.8, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$
E_{(\mathcal{U}, \heartsuit)}(A)(x) = S(A^+(x), A^-(x))
$$

becomes the construction method given in Proposition 4.4 of Qiao and Hu (2018).

(3) According to items (1) and (2), Propositions 4.3 and 4.4 of Qiao and Hu (2018) are special cases of Proposition 4.8 and unified by Proposition 4.8. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.8.

Theorem 4.2 *Let U be a condition universe and also be a decision universe, $\mathcal{U}$ be a representable uninorm with neutral element $e \in]0, 1[$, $P_C = I^{(2)} \setminus \{[0, 1]\}$ and $P_D = [0, 1]$. Take $E_{(\mathcal{U}_{rep}, \heartsuit)} : Map(U, P_C) \to Map(U, [0, 1])$ as:*

$$E_{(\mathcal{U}_{rep},\heartsuit)}(A)(x) = \mathcal{U}(A^+(x), A^-(x))$$

for each $x \in U$ and $A \in Map(U, P_C)$, then $E_{(\mathcal{U}_{rep},\heartsuit)}$ is a decision evaluation function of U.

Proof Items (E1) and (E2) of Definition 1.1 can be immediately verified from Proposition 4.8. We only verify item (E3) of Definition 1.1 as follows.

For any $A \in P_C$, it follows that

$$\overline{0} = [0, 0] \leq A \leq [1, 1] = \overline{1}.$$

Therefore, $\overline{0}$ is the minimum element of P_C and $\overline{1}$ is the maximum element of P_C. Let $B = [0, 1]$, then one has that

$$N^r(B) = [N^r(1), N^r(0)] = [0, 1].$$

Therefore, $N^r(A) = [N^r(A^+), N^r(A^-)]$ is strong negation on P_C.

For each $A \in Map(V, P_C)$ and $x \in U$, consider the following two cases.

(1) Let $N^r(A^-(x)) = 0$. Then one has that $A^-(x) = 1$. Therefore, one concludes that $A^+(x) = 1$. Then one obtains that $N^r(A^+(x)) = 0 \neq 1$.
(2) Let $N^r(A^+(x)) = 0$. Then one has that $A^+(x) = 1$. Since $[0, 1] \notin P_C$, one obtains that $N^r(A^-(x)) \neq N^r(0) = 1$.

According to above discussions, one concludes that $\{(A^+(x), A^-(x)), (N^r(A^+(x)), N^r(A^-(x)))\} \subseteq [0, 1] \times [0, 1] \setminus \{(0, 1), (0, 1)\}$. Then, it follows that

$$\begin{aligned} N^r(E_{(\mathcal{U}_{rep},\heartsuit)}(A))(x) &= N^r(\mathcal{U}(A^+(x), A^-(x))) \\ &= \mathcal{U}(N^r(A^+(x)), N^r(A^-(x))) \\ &= E_{(\mathcal{U}_{rep},\heartsuit)}(N^r(A))(x). \end{aligned}$$

Therefore, one obtains that $N^r(E_{(\mathcal{U}_{rep},\heartsuit)}(A)) = E_{(\mathcal{U}_{rep},\heartsuit)}(N^r(A))$. $\qquad\square$

Example 4.2 Let $U = [0, 1]$, A be a interval-valued fuzzy set of U with membership function $A(x) = [0.1 + 0.8x, 0.9]$. If we consider the additive generator $h_1(x) = \ln(-\ln(1 - x))$, from the construction method discussed in Theorem 4.2, we obtain a decision evaluation function $E_{(\mathcal{U}_{rep},\heartsuit)}(A)$ of U (see Fig. 7) as follows:

$$E_{(\mathcal{U}_{rep},\heartsuit)}(A)(x) = 1 - \exp(-\ln(0.1) \cdot \ln(0.9 - 0.8x)).$$

If we take $\psi = 0.6$ and $\omega = 0.4$, then three-way decision is given as follows.

(1) Acceptance region: $ACP_{(0.6,0.4)}(E_{(\mathcal{U}_{rep},\heartsuit)}, A) = \{x | \frac{9}{8} - \frac{5}{4}\exp\left(-\frac{\ln(0.4)}{\ln(0.1)}\right) \leq x \leq 1\}$.
(2) Rejection region: $REJ_{(0.6,0.4)}(E_{(\mathcal{U}_{rep},\heartsuit)}, A) = \{x | 0 \leq x \leq \frac{9}{8} - \frac{5}{4}\exp\left(-\frac{\ln(0.6)}{\ln(0.1)}\right)\}$.
(3) Uncertain region:

$$UNC_{(0.6,0.4)}(E_{(\mathcal{U}_{rep},\heartsuit)}, A) = \{x | \frac{9}{8} - \frac{5}{4}\exp\left(-\frac{\ln(0.6)}{\ln(0.1)}\right) < x < \frac{9}{8} - \frac{5}{4}\exp\left(-\frac{\ln(0.4)}{\ln(0.1)}\right)\}.$$

Fig. 7 Decision evaluation function $E_{(\mathcal{U}_{rep}, \triangledown)}(A)$ and interval-valued fuzzy set A
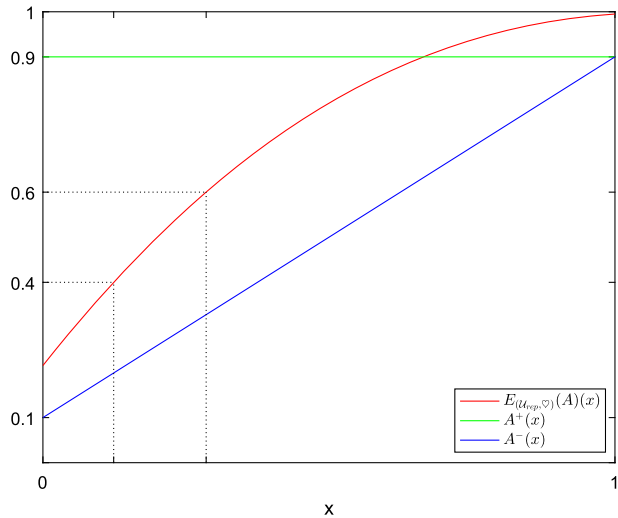
Let X be a universe and $\mu \in [0,1]$. In the following, we define $\mu^{(2)}$ as the family of interval numbers over $[0, \mu]$, i.e., $\mu^{(2)} = \{[\lambda^-, \lambda^+] : 0 \leq \lambda^- \leq \lambda^+ \leq \mu\}$. Then, for each $A \in Map(X, \mu^{(2)})$, A is also a interval-valued fuzzy set. For each $A \in Map(X, \mu^{(2)})$, there exists a fuzzy set $A^{(m)} \in Map(X, [0, \mu])$ with membership function $A^{(m)}(x) = \frac{A^-(x) + A^+(x)}{2}$ for each $x \in X$.

Proposition 4.9 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in]0,1]$, $P_C = e^{(2)}$, $P_D = [0,1]$ and $R \in Map(U \times V, [0, e])$ such that $\sum_{y \in V} R(x, y) \neq 0$ for each $x \in U$. Take $E_{(\mathcal{U}, \diamond)} : Map(V, e^{(2)}) \to Map(U, [0,1])$ as:*

$$E_{(\mathcal{U}, \diamond)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A^{(m)}(y))}{\sum_{y \in V} R(x,y)}$$

for each $x \in U$ and $A \in Map(V, e^{(2)})$, then $E_{(\mathcal{U}, \diamond)}$ is a semi-decision evaluation function of U.

Proof Firstly, from Lemma 4.1, one obtains that $E_{(\mathcal{U}, \diamond)}$ is well defined.

(1) Minimum element axiom
Since $R \in Map(U \times V, [0, e])$, it follows that

$$\begin{aligned}
E_{(\mathcal{U}, \diamond)}(\overline{0}_V)(x) &= \frac{\sum_{y \in V} \mathcal{U}(R(x,y), 0_V^{(m)}(y))}{\sum_{y \in V} R(x,y)} \\
&\leq \frac{\sum_{y \in V} \mathcal{U}(e, 0)}{\sum_{y \in V} R(x,y)} \\
&= 0,
\end{aligned}$$

for each $x \in U$. Therefore, one obtains that $E_{(\mathcal{U}, \diamond)}(\overline{0}_V) = 0_U$.

(2) Monotonicity axiom

Let $A, B \in Map(V, e^{(2)})$ with $A \subseteq_{P_C} B$. Then, one gets that

$$
\begin{aligned}
E_{(\mathcal{U}, \Diamond)}(A)(x) &= \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A^{(m)}(y))}{\sum_{y \in V} R(x,y)} \\
&\leq \frac{\sum_{y \in V} \mathcal{U}(R(x,y), B^{(m)}(y))}{\sum_{y \in V} R(x,y)} \\
&= E_{(\mathcal{U}, \Diamond)}(B)(x),
\end{aligned}
$$

for each $x \in U$. Therefore, one concludes that $E_{(\mathcal{U}, \Diamond)}(A) \subseteq E_{(\mathcal{U}, \Diamond)}(B)$.

$\square$

Remark 4.8 In Proposition 4.9, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then one obtains that $P_C = I^{(2)}$ and $R \in Map(U \times V, [0,1])$. Then

$$
E_{(\mathcal{U}, \Diamond)}(A)(x) = \frac{\sum_{y \in V} T(R(x,y), A^{(m)}(y))}{\sum_{y \in V} R(x,y)}
$$

becomes construction method given by Proposition 4.5 of Qiao and Hu (2018). Therefore, Proposition 4.5 of Qiao and Hu (2018) is a special case of Proposition 4.9. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.9.

Proposition 4.10 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0,1]$, $P_C = I^{(2)}$, $P_D = [0,1]$ and $R \in Map(U \times V, [0,e])$. Take $E_{(\mathcal{U}, \spadesuit)} : Map(V, I^{(2)}) \rightarrow Map(U, [0,1])$ as:*

$$
E_{(\mathcal{U}, \spadesuit)}(A)(x) = \frac{\sum_{y \in V} \mathcal{U}(A^{(m)}(y), R(x,y))}{|V|}
$$

for each $x \in U$ and $A \in Map(V, I^{(2)})$, then $E_{(\mathcal{U}, \spadesuit)}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 4.9. $\square$

Proposition 4.11 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in]0,1]$, $P_C = e^{(2)}$, $P_D = I^{(2)}$ and $R \in Map(U \times V, [0,e])$ such that $\sum_{y \in V} R(x,y) \neq 0$ for each $x \in U$. Take $E_{\mathcal{U}}^{\sharp} : Map(V, e^{(2)}) \rightarrow Map(U, I^{(2)})$ as:*

$$
E_{\mathcal{U}}^{\sharp}(A)(x) = \left[\frac{\sum_{y \in V} \mathcal{U}(R(x,y), A^{-}(y))}{\sum_{y \in V} R(x,y)}, \frac{\sum_{y \in V} \mathcal{U}(R(x,y), A^{+}(y))}{\sum_{y \in V} R(x,y)} \right]
$$

for each $x \in U$ and $A \in Map(V, e^{(2)})$, then $E_{\mathcal{U}}^{\sharp}$ is a semi-decision evaluation function of U.

Proof Firstly, from Lemma 4.1, one obtains that $E_{\mathcal{U}}^{\sharp}$ is well defined.

(1) Minimum element axiom

Since $R \in Map(U \times V, [0,e])$, it follows that

$$E_{\mathcal{U}}^{\sharp}(\overline{0}_V)(x) = \left[\frac{\sum_{y\in V}\mathcal{U}(R(x,y),\overline{0}_V^-(y))}{\sum_{y\in V}R(x,y)}, \frac{\sum_{y\in V}\mathcal{U}(R(x,y),\overline{0}_V^+(y))}{\sum_{y\in V}R(x,y)}\right]$$

$$\leq \left[\frac{\sum_{y\in V}\mathcal{U}(e,0)}{\sum_{y\in V}R(x,y)}, \frac{\sum_{y\in V}\mathcal{U}(e,0)}{\sum_{y\in V}R(x,y)}\right]$$

$$= \overline{0},$$

for each $x \in U$. Therefore, one obtains that $E_{\mathcal{U}}^{\sharp}(\overline{0}_V) = \overline{0}_U$.

(2) Monotonicity axiom

Let $A, B \in Map(V, e^{(2)})$ with $A \subseteq_{P_C} B$. Then, one gets that

$$E_{\mathcal{U}}^{\sharp}(A)(x) = \left[\frac{\sum_{y\in V}\mathcal{U}(R(x,y),A^-(y))}{\sum_{y\in V}R(x,y)}, \frac{\sum_{y\in V}\mathcal{U}(R(x,y),A^+(y))}{\sum_{y\in V}R(x,y)}\right]$$

$$\leq \left[\frac{\sum_{y\in V}\mathcal{U}(R(x,y),B^-(y))}{\sum_{y\in V}R(x,y)}, \frac{\sum_{y\in V}\mathcal{U}(R(x,y),B^+(y))}{\sum_{y\in V}R(x,y)}\right]$$

$$= E_{\mathcal{U}}^{\sharp}(B)(x),$$

for each $x \in U$. Therefore, one concludes that $E_{\mathcal{U}}^{\sharp}(A) \subseteq_{P_D} E_{\mathcal{U}}^{\sharp}(B)$. $\qquad\square$

Remark 4.9 In Proposition 4.11, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then, one concludes that $P_C = I^{(2)}$ and $R \in Map(U \times V, [0, 1])$. Then

$$E_{\mathcal{U}}^{\sharp}(A)(x) = \left[\frac{\sum_{y\in V}T(R(x,y),A^-(y))}{\sum_{y\in V}R(x,y)}, \frac{\sum_{y\in V}T(R(x,y),A^+(y))}{\sum_{y\in V}R(x,y)}\right]$$

becomes the construction method given by Proposition 4.6 of Qiao and Hu (2018). Therefore, Proposition 4.6 of Qiao and Hu (2018) is a special case of Proposition 4.11. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.11.

Remark 4.10 In Propositions 4.9 and 4.11, if take each uninorm $\mathcal{U}$ as a conjunctive uninorm, then the condition $R \in Map(U \times V, [0, e])$ can be relaxed to $R \in Map(U \times V, [0, 1])$.

Proposition 4.12 *Let V be a finite condition universe, U be a finite decision universe, $\mathcal{U}_1, \mathcal{U}_2$ be two uninorms with neutral elements $e_1, e_2 \in [0, 1]$, $e = e_1 \wedge e_2$, $P_C = P_D = I^{(2)}$ and $R \in Map(U \times V, [0, e])$. Take $E_{(\mathcal{U}_1,\mathcal{U}_2)}^{\sharp} : Map(V, I^{(2)}) \to Map(U, I^{(2)})$ as:*

$$E_{(\mathcal{U}_1,\mathcal{U}_2)}^{\sharp}(A)(x) = \left[\frac{\sum_{y\in V}\mathcal{U}_1(\mathcal{U}_2(R(x,y),A^-(y)),A^-(y))}{|V|},\right.$$

$$\left.\frac{\sum_{y\in V}\mathcal{U}_1(\mathcal{U}_2(R(x,y),A^+(y)),A^+(y))}{|V|}\right]$$

for each $x \in U$ and $A \in Map(V, I^{(2)})$, then $E^{\sharp}_{(\mathcal{U}_1, \mathcal{U}_2)}$ is a semi-decision evaluation function of U.

Proof It can be immediately derived from Propositions 4.7 and 4.11. □

4.4 Construction methods related to fuzzy relations

Proposition 4.13 *Let U be a nonempty finite universe, $U \times U$ be a condition universe and also be a decision universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$ and $P_C = P_D = [0, 1]$. For each $R \in Map(U \times U, [0, 1])$ and $(x, y) \in U \times U$, take $E_{(R,\mathcal{U})} : Map(U \times U, [0, 1]) \to Map(U \times U, [0, 1])$ as:*

$$E_{(R,\mathcal{U})}(R)(x, y) = \mathcal{U}(R(x, y), R(y, x)).$$

Then $E_{(R,\mathcal{U})}$ is a semi-decision evaluation function of $U \times U$.

Proof It can be checked in a similar way as that of Proposition 4.7 of Qiao and Hu (2018). □

Remark 4.11

(1) In Proposition 4.13, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$E_{(R,\mathcal{U})}(R)(x, y) = T(R(x, y), R(y, x))$$

becomes the construction method given in Proposition 4.7 of Qiao and Hu (2018).
(2) In Proposition 4.13, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$E_{(R,\mathcal{U})}(R)(x, y) = S(R(x, y), R(y, x))$$

becomes the construction method given in Proposition 4.8 of Qiao and Hu (2018).
(3) According to items (1) and (2), Propositions 4.7 and 4.8 of Qiao and Hu (2018) are special cases of Proposition 4.13 and unified by Proposition 4.13. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.13.

Theorem 4.3 *Let U be a nonempty finite universe, $U \times U$ be a condition universe and also be a decision universe, $P_C = P_D = [0, 1]$, $\mathcal{U}$ be a representable uninorms with element $e \in]0, 1[$. For each $(x, y) \in U \times U$ and $R \in Map(U \times U, [0, 1])$, take $E_{(R,\mathcal{U}_{rep})} : Map(U \times U, [0, 1]) \to Map(U \times U, [0, 1])$ as:*

$$E_{(R,\mathcal{U}_{rep})}(R)(x, y) = \mathcal{U}(R(x, y), R(x, y)).$$

Then $E_{(R,\mathcal{U}_{rep})}$ is a decision evaluation function of $U \times U$.

Proof It can be checked in a similar way as that of Theorem 4.1. □

Proposition 4.14 *Let U be a finite decision universe, $U \times U$ be a condition universe, $\mathcal{U}$ be a uninorm with neutral element $e \in [0, 1]$ and $P_C = P_D = [0, 1]$. For each $x \in U$ and $R \in Map(U \times U, [0, 1])$, take $E_{(R,U,\Diamond)} : Map(U \times U, [0, 1]) \to Map(U, [0, 1])$ as:*

$$E_{(R,U,\Diamond)}(R)(x) = \frac{\sum_{y\in U}\mathcal{U}(R(x,y),R(y,x))}{|U|}.$$

Then $E_{(R,U,\Diamond)}$ is a semi-decision evaluation function of U.

Proof

(1) Minimum element axiom
Since $\mathcal{U}(0,0) = 0$, it follows that

$$E_{(R,U,\Diamond)}(0_{U\times U})(x) = \frac{\sum_{y\in U}\mathcal{U}(0_{U\times U}(x,y),0_{U\times U}(y,x))}{|U|} = 0,$$

for each $x \in U$. Therefore, one obtains that $E_{(R,U,\Diamond)}(0_{U\times U}) = 0_U$.

(2) Monotonicity axiom
Let $R_1, R_2 \in Map(U \times U, [0,1])$ with $R_1 \subseteq R_2$. Then, one gets that

$$\begin{aligned}
E_{(R,U,\Diamond)}(R_1)(x) &= \frac{\sum_{y\in U}\mathcal{U}(R_1(x,y),R_1(y,x))}{|U|}\\
&\leq \frac{\sum_{y\in U}\mathcal{U}(R_2(x,y),R_2(y,x))}{|U|}\\
&= E_{(R,U,\Diamond)}(R_2)(x),
\end{aligned}$$

for each $x \in U$. Therefore, one concludes that $E_{(R,U,\Diamond)}(R_1) \subseteq E_{(R,U,\Diamond)}(R_2)$.

4.5 Construction methods related to hesitant fuzzy sets

Proposition 4.15 *Let U be a condition universe and also be a decision universe, $P_C = 2^{[0,1]} - \emptyset$, $P_D = [0,1]$ and $\mathcal{U}$ be a uninorm with neutral element of $e \in [0,1]$. For each $H \in Map(U, 2^{[0,1]} - \emptyset)$, $x \in U$ and $\lambda, \mu \in [0,1]$, take $E_{(H,\mathcal{U})} : Map(U, 2^{[0,1]} - \emptyset) \to Map(U, [0,1])$ as:*

$$E_{(H,\mathcal{U})}(H)(x) = \mathcal{U}(\lambda \sup H(x), \mu \inf H(x)).$$

Then $E_{(H,\mathcal{U})}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 4.9 of Qiao and Hu (2018). $\square$

Remark 4.12

(1) In Proposition 4.15, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$E_{(H,\mathcal{U})}(H)(x) = T(\lambda \sup H(x), \mu \inf H(x))$$

becomes the construction method given in Proposition 4.9 of Qiao and Hu (2018).

(2) In Proposition 4.15, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$E_{(H,\mathcal{U})}(H)(x) = S(\lambda \sup H(x), \mu \inf H(x))$$

becomes the construction method given in Proposition 4.10 of Qiao and Hu (2018).
(3) According to items (1) and (2), Propositions 4.9 and 4.10 of Qiao and Hu (2018) are special cases of Proposition 4.15 and unified by Proposition 4.15. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.15.

In this paper, we define the set $\mathcal{M}$ as

$$\mathcal{M} = \left\{ m \in 2^{[0,1]} - \emptyset : \inf m = 0, \sup m = 1 \right\}.$$

Theorem 4.4 *Let U be a condition universe and also be a decision universe, $P_C = (2^{[0,1]} - \emptyset) \setminus \mathcal{M}$, $P_D = [0,1]$ and $\mathcal{U}$ be a representable uninorm with neutral element of $e \in]0,1[$. For each $H \in P_C$ and $x \in U$, take $E_{(H,\mathcal{U}_{rep})} : Map(U, P_C) \rightarrow Map(U, [0,1])$ as:*

$$E_{(H,\mathcal{U}_{rep})}(H)(x) = \mathcal{U}(\sup H(x), \inf H(x)).$$

Then $E_{(H,\mathcal{U}_{rep})}$ is a decision evaluation function of U.

Proof Items (E1) and (E2) of Definition 1.1 can be verified from Proposition 4.15. We only verify item (E3) of Definition 1.1 as follows.

For each $H \in P_C$, it follows that

$$\{0\}_U \leq H \leq \{1\}_U.$$

Therefore, $\{0\}_U$ is the minimum element of P_C and $\{1\}_U$ is the maximum element of P_C. Suppose $K \in \mathcal{M}$, then, for each $x \in U$, one has hat

$$\inf N^r(K(x)) = N^r(\sup K(x)) = N^r(1) = 0,$$

and

$$\sup N^r(K(x)) = N^r(\inf K(x)) = N^r(0) = 1,$$

respectively. Thus, $N^r(H) = \{N^r(h) : h \in H\}$ is strong negation on P_C.

For each $H \in P_C$ and $x \in U$, consider the following two cases.

(1) Let $\inf N^r(H(x)) = 0$ and $\sup N^r(H(x)) = 1$, then one has that

$$\inf N^r(H(x)) = N^r(\sup H(x)) = 0$$

and

$$\sup N^r(H(x)) = N^r(\inf H(x)) = 1,$$

respectively. Therefore, one concludes that $\sup H(x) = 1$ and $\inf H(x) = 0$. It conflicts with assumptions.
(2) Let $\inf N^r(H(x)) = 1$ and $\sup N^r(H(x)) = 0$, then one has that
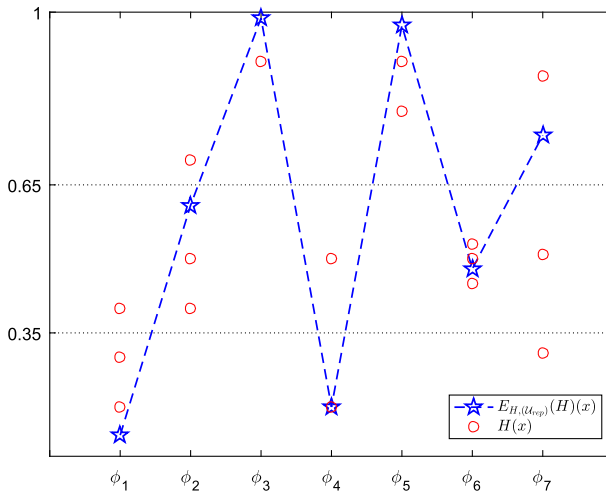
$$\inf N^r(H(x)) = N^r(\sup H(x)) = 1$$

Fig. 8 Decision evaluation function $E_{(H,\mathcal{U}_{rep})}(H)$ and hesitant fuzzy set H

and

$$\sup N^r(H(x)) = N^r(\inf H(x)) = 0.$$

Therefore, one concludes that $\sup H(x) = 0$ and $\inf H(x) = 1$. However, $\sup H(x) \geq \inf H(x)$. Thus, there not exist $\inf N^r(H(x)) = 1$ and $\sup N^r(H(x)) = 0$. According to above discussions, it follows that

$$
\begin{aligned}
E_{(H,\mathcal{U}_{rep})}(N^r(H))(x) &= \mathcal{U}(\sup N^r(H(x)), \inf N^r(H(x))) \\
&= \mathcal{U}(\sup_{h \in H(x)} N^r(h), \inf_{h \in H(x)} N^r(h)) \\
&= \mathcal{U}(N^r(\inf_{h \in H(x)} h), N^r(\sup_{h \in H(x)} h)) \\
&= N^r(\mathcal{U}(\inf H(x), \sup H(x))) \\
&= N^r(E_{(H,\mathcal{U}_{rep})}(H))(x).
\end{aligned}
$$

Therefore, one obtains that $E_{(H,\mathcal{U}_{rep})}(N^r(H)) = N^r(E_{(H,\mathcal{U}_{rep})}(H))$. $\qquad \square$

Example 4.3 Suppose $U = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7\}$ and $H = \frac{\{0.2, 0.3, 0.4\}}{\phi_1} + \frac{\{0.4, 0.5, 0.7\}}{\phi_2} + \frac{\{0.9\}}{\phi_3} + \frac{\{0.2, 0.5\}}{\phi_4} + \frac{\{0.8, 0.9\}}{\phi_5} + \frac{\{0.45, 0.5, 0.53\}}{\phi_6} + \frac{\{0.31, 0.51, 0.87\}}{\phi_7}$ is a hesitant fuzzy set of U. If we consider the additive generator $h(x) = \ln\left(\frac{x}{1-x}\right)$, from construction method discussed in Theorem 4.4, we have decision evaluation function $E_{(H,\mathcal{U}_{rep})}(H)$ of U (see Fig. 8) as:

$$E_{(H,\mathcal{U}_{rep})}(H) = \frac{0.143}{\phi_1} + \frac{0.609}{\phi_2} + \frac{0.988}{\phi_3} + \frac{0.200}{\phi_4} + \frac{0.973}{\phi_5} + \frac{0.480}{\phi_6} + \frac{0.750}{\phi_7}.$$

If we take $\psi = 0.65$ and $\omega = 0.35$, then three-way decision is given as follows:

(1) Acceptance region: $ACP_{(0.65, 0.35)}(E_{(H,\mathcal{U}_{rep})}, H) = \{\phi_3, \phi_5, \phi_7\}$.
(2) Rejection region: $REJ_{(0.65, 0.35)}(E_{(H,\mathcal{U}_{rep})}, H) = \{\phi_1, \phi_4\}$.
(3) Uncertain region: $UNC_{(0.65, 0.35)}(E_{(H,\mathcal{U}_{rep})}, H) = \{\phi_2, \phi_6\}$.

In this paper, we define the set $\mathcal{L}$ as $\mathcal{L} = \left\{ l \in 2^{[0,1]} - \emptyset : \inf l + \sup l = 1 \right\}$.

Theorem 4.5 *Let U be a condition universe and also be a decision universe, $P_C = (2^{[0,1]} - \emptyset) \setminus \mathcal{L}$, $P_D = [0,1]$ and $\mathcal{U}$ be a representable uninorm with neutral element of $e \in]0,1[$ and strong negation $N^r(x) = 1 - x$ on [0,1]. For each $H \in Map(U, P_C)$, $x \in U$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$, take $E'_{(H,\mathcal{U}_{rep})} : Map(U, P_C) \to Map(U, [0,1])$ as:*

$$E'_{(H,\mathcal{U}_{rep})}(H)(x) = \mathcal{U}(\lambda \sup H(x) + \mu \inf H(x), \lambda \inf H(x) + \mu \sup H(x)).$$

Then $E'_{(H,\mathcal{U}_{rep})}$ is a decision evaluation function of U.

Proof For each $H \in P_C$, it follows that

$$\{0\}_U \leq H \leq \{1\}_U.$$

Therefore, $\{0\}_U$ is the minimum element of P_C and $\{1\}_U$ is the maximum element of P_C. Suppose $H \in P_C$, then it follows that

$$
\begin{aligned}
\inf N^r(H(x)) + \sup N^r(H(x)) &= \inf_{h \in H(x)} N^r(h) + \sup_{h \in H(x)} N^r(h) \\
&= 1 - \sup_{h \in H(x)} h + 1 - \inf_{h \in H(x)} h \\
&= 2 - (\sup H(x) + \inf H(x)) \\
&\neq 1,
\end{aligned}
$$

for each $x \in U$. Thus, $N^r(H) = \{1 - h : h \in H\}$ is strong negation on P_C.

(1) Minimum element axiom
Since $\mathcal{U}(0,0) = 0$, it follows that

$$
\begin{aligned}
&E'_{(H,\mathcal{U})}(\{0\}_U)(x) \\
&= \mathcal{U}(\lambda \sup \{0\}_U(x) + \mu \inf \{0\}_U(x), \lambda \inf \{0\}_U(x) + \mu \sup \{0\}_U(x)) \\
&= 0,
\end{aligned}
$$

for each $x \in U$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$. Therefore, one obtains that $E'_{(H,\mathcal{U})}(\{0\}_U) = 0_U$.

(2) Monotonicity axiom
Let $K, H \in Map(U, P_C)$ with $K \subseteq_{P_C} H$. Then, one gets that

$$\inf K(x) \leq \inf H(x)$$

and

$$\sup K(x) \leq \sup H(x),$$

for each $x \in U$. Thus, it follows that

$$E'_{(H,\mathcal{U}_{rep})}(K)(x) = \mathcal{U}(\lambda \sup K(x) + \mu \inf K(x), \lambda \inf K(x) + \mu \sup K(x))$$
$$\leq \mathcal{U}(\lambda \sup H(x) + \mu \inf H(x), \lambda \inf H(x) + \mu \sup H(x))$$
$$= E'_{(H,\mathcal{U}_{rep})}(H)(x),$$

for each $x \in U$. Therefore, one concludes that $E'_{(H,\mathcal{U}_{rep})}(K) \subseteq E'_{(H,\mathcal{U}_{rep})}(H)$.

(3) Complement axiom

Consider the following two cases.

(i) $\lambda \sup H(x) + \mu \inf H(x) = 0$ and $\lambda \inf H(x) + \mu \sup H(x) = 1$.
(ii) $\lambda \sup H(x) + \mu \inf H(x) = 1$ and $\lambda \inf H(x) + \mu \sup H(x) = 0$.

These two cases can be unified by following expressions.

$$\lambda \sup H(x) + \mu \sup H(x) + \mu \inf H(x) + \lambda \inf H(x) = 1$$
$$\Longrightarrow (\lambda + \mu) \sup H(x) + (\mu + \lambda) \inf H(x) = 1$$
$$\Longrightarrow \sup H(x) + \inf H(x) = 1,$$

which is contradiction with $H \in Map(U, P_C)$.
According to above discussions, it follows that

$$E'_{(H,\mathcal{U}_{rep})}(N^r(H))(x)$$
$$= \mathcal{U}(\lambda \sup N^r(H)(x) + \mu \inf N^r(H)(x), \lambda \inf N^r(H)(x) + \mu \sup N^r(H)(x))$$
$$= \mathcal{U}(1 - \lambda \inf_{r \in H(x)} r - \mu \sup_{r \in H(x)} r, 1 - \lambda \sup_{r \in H(x)} r - \mu \inf_{r \in H(x)} r)$$
$$= 1 - \mathcal{U}(\lambda \inf_{r \in H(x)} r + \mu \sup_{r \in H(x)} r, \lambda \sup_{r \in H(x)} r + \mu \inf_{r \in H(x)} r)$$
$$= 1 - \mathcal{U}(\lambda \inf H(x) + \mu \sup H(x), \lambda \sup H(x) + \mu \inf H(x))$$
$$= N^r(E'_{(H,\mathcal{U}_{rep})}(H))(x).$$

Therefore, one obtains that $E'_{(H,\mathcal{U}_{rep})}(N(H)) = N(E'_{(H,\mathcal{U}_{rep})}(H))$.

□

Proposition 4.16 *Let U be a condition universe and also be a decision universe, $P_C = Finite(2^{[0,1]} - \emptyset)$, $P_D = [0,1]$ and $\mathcal{U}$ be a uninorm with neutral element of $e \in [0,1]$. For each $H \in Map(U, Finite(2^{[0,1]} - \emptyset))$ and $x \in U$, take*

$$E_{(H,\mathcal{U},\tau)}(H)(x) = \frac{\sum_{y \in H(x)} \mathcal{U}(y,y)}{|H(x)|}.$$

Then $E_{(H,\mathcal{U},\tau)}$ is a semi-decision evaluation function of U.

Proof It can be checked in a similar way as that of Proposition 4.11 of Qiao and Hu (2018).
□

Remark 4.13

(1)　In Proposition 4.16, take the neutral element of uninorm $\mathcal{U}$ as $e = 1$. Then

$$E_{(H,\mathcal{U},\tau)}(H)(x) = \frac{\sum_{y\in H(x)} T(y,y)}{|H(x)|}$$

becomes the construction method given in Proposition 4.11 of Qiao and Hu (2018).

(2)　In Proposition 4.16, take the neutral element of uninorm $\mathcal{U}$ as $e = 0$. Then

$$E_{(H,\mathcal{U},\tau)}(H)(x) = \frac{\sum_{y\in H(x)} S(y,y)}{|H(x)|}$$

becomes the construction method given in Proposition 4.12 of Qiao and Hu (2018).

(3)　According to items (1) and (2), Propositions 4.11 and 4.12 of Qiao and Hu (2018) are special cases of Proposition 4.16 and unified by Proposition 4.16. In addition, more novel semi-decision evaluation functions can be obtained from Proposition 4.16.

4.6 Summary of construction methods of decision evaluation functions

In Sect. 3, we show two novel construction methods of decision evaluation functions derived from different semi-decision evaluation functions (see Theorems 3.1 and 3.2). Furthermore, in Sects. 4.1–4.5, we show some novel construction methods of decision evaluations function (see Theorems 4.1–4.5) and semi-decision evaluation functions (see Propositions 4.1–4.16) related to existing semi-decision evaluation functions, fuzzy sets, interval-valued fuzzy sets, fuzzy relations and hesitant fuzzy sets, respectively.

In Remarks 3.1, 4.1–4.5, 4.7–4.9 and 4.11–4.13, we have pointed out that some existing construction methods of decision evaluation functions and semi-decision evaluation functions are special cases of the methods proposed in this paper. In addition, more novel decision evaluation functions and semi-decision evaluation functions can be obtained from these uninorms-based construction methods. Therefore, on the basis of uninorms, Sects. 3 and 4.1–4.5 fully answer the Question 1 proposed in Sect. 1.3.

Meanwhile, since representable uninorms are self-dual w.r.t. strong negation N^r, the construction methods of decision evaluation functions given in Theorems 3.2 and 4.1–4.4 extend strong negation N_{P_D} of partially ordered set P_D from $N_{P_D}(x) = 1 - x$ to N^r. Therefore, Theorems 3.2 and 4.1–4.4 answer the Question 2 proposed in Sect. 1.3.

5 Two illustrative examples

In this section, in order to illustrate the results obtained in this paper, we consider two real evaluation problems. Firstly, we propose an algorithm to describe the three-way decision process based on three-way decision space (see Algorithm 1). Secondly, we show the specific steps of three-way decisions based on three-way decision space derived from semi-decision evaluation functions. Thirdly, in Sect. 5.2, we consider an evaluation problem of investment projects. Finally, in Sect. 5.3, we compare different transformation methods by discussing an evaluation problem of credit card applicants.

5.1 The three-way decision methodology

Firstly, on the basis of three-way decision space, the three-way decision process of decision universe U can be simply described as Algorithm 1.

Algorithm 1 The algorithm of three-way decision based on three-way decision space.

Input A three-way decision space $(U, Map(V, P_C), P_D, E)$; $A \in Map(V, P_C)$; two parameters ψ and ω.

Output The decision evaluation values and three-way decision results of decision universe U.

1: **for all** x in U **do**
2:    **calculate:** the decision evaluation value of x as $E(A)(x)$.
3: **end for**
4: **for all** x in U **do**
5:    **if** $E(A)(x) \geq \psi$ **then** $x \in ACP_{(\psi,\omega)}(E, A)$.
6:    **else if** $E(A)(x) \leq \omega$ **then** $x \in REJ_{(\psi,\omega)}(E, A)$.
7:    **else** $x \in UNC_{(\psi,\omega)}(E, A)$.
8:    **end if**
9: **end for**
10: **return** The decision evaluation values and three-way decision result of decision universe U.

In Sect. 1.3, it has been pointed that a lot of useful functions are semi-decision evaluation functions. Therefore, in the following, we show the specific steps of three-way decision based on three-way decision space which is derived from semi-decision evaluation functions.

Input: A decision universe U; a condition universe V; partially ordered sets P_C; $A \in Map (V, P_C)$.

Output: The three-way decision result of decision universe U.

Step 1: Select a semi-decision evaluation function E_{semi} related to partially ordered set $Map(V, P_C)$.

Step 2: Select a transformation method from semi-decision evaluation functions to decision evaluation function and obtain the partially ordered set P_D.
Moreover, if E_{semi} satisfies the constraint conditions of the transformation method, then go to Step 3. Otherwise, repeat Step 2.

Step 3: Obtain a decision evaluation function E of U by transformation method and semi-decision evaluation function E_{semi}.

Step 4: Establish a three-way decision space $(U, Map(V, P_C), P_D, E)$.

Step 5: Give two parameters ψ and ω for three-way decision.

Step 6: Obtain the decision evaluation values and three-way decision results of decision universe U by Algorithm 1.

5.2 An evaluation problem of investment projects

In this subsection, we consider an real evaluation problem of investment projects (Qian and Shu 2018; Zhan et al. 2021). Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ be a set of ten

investment projects which are considering by venture capital company, $AT = \{C_1, C_2, C_3, C_4, FD\}$ be a set of attributes where C_1 denotes market venture, C_2 denotes management venture, C_3 denotes environment venture, C_4 denotes production venture and FD denotes the degree of preference for investment projects given by decision maker. The evaluation results of ten projects are listed in Table 1 where the symbol $*$ denotes that the venture factor of this project is unknown. The decision maker prefers to choose the investment project with higher value of attribute FD. The venture of investment project is higher if the value of venture factor attribute C_k $(k = 1, 2, 3, 4)$ is greater.

To begin with, we complete Table 1 by a pessimistic strategy, i.e., let the unknown values of venture factor $* = 4$. Then, for each investment project x_i $(i = 1, 2, \cdots, 10)$, we denote the value of venture attribute C_k $(k = 1, 2, 3, 4)$ of x_i as $x_i^{(k)}$. Then, by applying the formula

$$R(x_i, x_j) = 1 - \frac{1}{16} \sum_{k=1}^{4} |x_i^{(k)} - x_j^{(k)}|,$$

we get the fuzzy similarity relation of U on venture attributes C_1, C_2, C_3 and C_4 as:

$$R_v = \begin{bmatrix}
1 & \frac{5}{8} & \frac{1}{2} & \frac{7}{16} & \frac{13}{16} & \frac{3}{4} & \frac{7}{16} & 1 & \frac{11}{16} & \frac{9}{16} \\
\frac{5}{8} & 1 & \frac{3}{4} & \frac{13}{16} & \frac{11}{16} & \frac{5}{8} & \frac{11}{16} & \frac{5}{8} & \frac{13}{16} & \frac{13}{16} \\
\frac{1}{2} & \frac{3}{4} & 1 & \frac{13}{16} & \frac{9}{16} & \frac{5}{8} & \frac{15}{16} & \frac{1}{2} & \frac{11}{16} & \frac{11}{16} \\
\frac{7}{16} & \frac{13}{16} & \frac{13}{16} & 1 & \frac{5}{8} & \frac{7}{16} & \frac{3}{4} & \frac{7}{16} & \frac{3}{4} & \frac{5}{8} \\
\frac{13}{16} & \frac{11}{16} & \frac{9}{16} & \frac{5}{8} & 1 & \frac{11}{16} & \frac{1}{2} & \frac{13}{16} & \frac{7}{8} & \frac{5}{8} \\
\frac{3}{4} & \frac{5}{8} & \frac{5}{8} & \frac{7}{16} & \frac{11}{16} & 1 & \frac{11}{16} & \frac{3}{4} & \frac{11}{16} & \frac{13}{16} \\
\frac{7}{16} & \frac{11}{16} & \frac{15}{16} & \frac{3}{4} & \frac{1}{2} & \frac{11}{16} & 1 & \frac{7}{16} & \frac{5}{8} & \frac{3}{4} \\
1 & \frac{5}{8} & \frac{1}{2} & \frac{7}{16} & \frac{13}{16} & \frac{3}{4} & \frac{7}{16} & 1 & \frac{11}{16} & \frac{9}{16} \\
\frac{11}{16} & \frac{13}{16} & \frac{11}{16} & \frac{3}{4} & \frac{7}{8} & \frac{11}{16} & \frac{5}{8} & \frac{11}{16} & 1 & \frac{5}{8} \\
\frac{9}{16} & \frac{13}{16} & \frac{11}{16} & \frac{5}{8} & \frac{5}{8} & \frac{13}{16} & \frac{3}{4} & \frac{9}{16} & \frac{5}{8} & 1
\end{bmatrix}.$$

In the following, by the three-way decision steps given in Sect. 5.1, we use two decision evaluation functions derived from semi-decision evaluation functions to analyse this evaluation problem.

Firstly, by the construction method of semi-decision evaluation functions given in Proposition 4.6, we select two semi-decision evaluation functions as

$$E_{(\mathrm{semi}, T_M)}(FD)(x) = \frac{\sum_{y \in U} \min\{R_v(x, y), FD(y)\}}{|U|}$$

and

Table 1 A consistent incomplete ordered information system with fuzzy decision (Qian and Shu 2018)

U	C_1	C_2	C_3	C_4	FD
x_1	*	1	*	2	0.5
x_2	2	4	3	2	0.9
x_3	2	3	1	3	0.6
x_4	2	4	1	1	0.8
x_5	3	2	*	1	0.6
x_6	4	3	4	4	0.9
x_7	2	3	1	4	0.7
x_8	4	1	*	2	0.5
x_9	3	3	3	1	0.8
x_{10}	2	4	4	4	0.9

$$E_{(\mathrm{semi},T_P)}(FD)(x) = \frac{\sum_{y \in U} R_v(x,y) \cdot FD(y)}{|U|}$$

where $N_{P_C}(A) = A^C$. Then, we select the transformation method given by Theorem 3.2 where the representable uninorm is U_τ and obtain the partially ordered set $P_D = [0, 1]$. It can be verified that $E_{(\mathrm{semi},T_M)}$ and $E_{(\mathrm{semi},T_P)}$ satisfy the constraint conditions of Theorem 3.2. Therefore, we obtain two decision evaluation function of U as $E_{(\mathcal{U}_\tau,T_M)}$ and $E_{(\mathcal{U}_\tau,T_P)}$, respectively. After that, two corresponding three-way decision spaces are established. Further, we consider two parameters $\psi = 0.82$ and $\omega = 0.80$. Finally, by Algorithm 1, we obtain the decision evaluation values of U (see Fig. 9) as:

$$E_{(\mathcal{U}_\tau,T_M)}(FD) = \frac{0.7660}{x_1} + \frac{0.8468}{x_2} + \frac{0.8213}{x_3} + \frac{0.8171}{x_4} + \frac{0.8000}{x_5} + \frac{0.8173}{x_6}$$
$$+ \frac{0.8155}{x_7} + \frac{0.7660}{x_8} + \frac{0.8269}{x_9} + \frac{0.8377}{x_{10}}$$

and

$$E_{(\mathcal{U}_\tau,T_P)}(FD) = \frac{0.7747}{x_1} + \frac{0.8303}{x_2} + \frac{0.8179}{x_3} + \frac{0.8173}{x_4} + \frac{0.7978}{x_5} + \frac{0.8120}{x_6}$$
$$+ \frac{0.8207}{x_7} + \frac{0.7747}{x_8} + \frac{0.8173}{x_9} + \frac{0.8313}{x_{10}},$$

respectively. We also obtain the three-way decisions results of ten investment projects (see Table 2). In this following, on the basis of Table 2, we show some analyses of decision results.

(1) Investment projects x_2 and x_{10} are evaluated "acceptation" in two evaluation functions since x_2 and x_{10} have the highest degree of preference given by decision maker and lower venture compared with x_6.

(2) For investment projects x_1, x_5 and x_8, on the one hand, decision maker have lower degree of preference than others. On the other hand, we completed the incomplete values of Table 1 through a pessimistic strategy. Therefore, x_1, x_5 and x_8 are evaluated "rejection" in two evaluation functions.
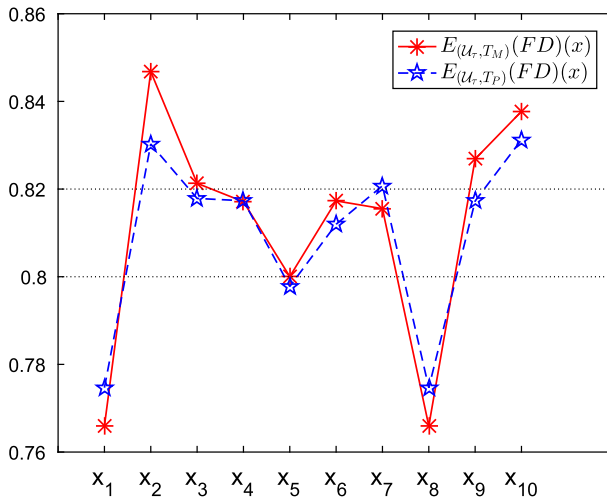
Fig. 9 The decision evaluation values of investment projects based on $E_{(\mathcal{U}_\tau, T_M)}$ and $E_{(\mathcal{U}_\tau, T_P)}$

Table 2 Three-way decision results of ten investment projects

	Acceptance region	Rejection region	Uncertain region
$E_{(\mathcal{U}_\tau, T_M)}$	$\{x_2, x_3, x_9, x_{10}\}$	$\{x_1, x_5, x_8\}$	$\{x_4, x_6, x_7\}$
$E_{(\mathcal{U}_\tau, T_P)}$	$\{x_2, x_7, x_{10}\}$	$\{x_1, x_5, x_8\}$	$\{x_3, x_4, x_6, x_9\}$

5.3 An evaluation problem of credit card applicants

In this subsection, we compare different transformation methods from semi-decision evaluation functions to decision evaluation functions. On the one hand, in Remark 3.1, we have pointed out that t-norms and t-conorms-based transformation method proposed in Hu (2017) is a special case of uninorm-based transformation method given in Theorem 3.1. On the other hand, the constraint conditions of semi-decision evaluation function of the transformation method given in Theorem 3.2 is more complex than others. Therefore, we only compare the overlap and grouping functions-based transformation method given in Jia and Qiao (2020) with the uninorms-based transformation method given in Theorem 3.1 by a real evaluation problem of credit card applicants (Hu 2017; Lin et al. 2012; Qiao and Hu 2018; Qiao and Hu 2020).

Suppose that $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ is a set of nine credit card applicants. In this evaluation problem, we consider two condition attributes of applicants, that is, education and salary. The values of attribute salary are $\{\text{low}, \text{middle}, \text{high}\}$ and the values of attribute education are $\{\text{good}, \text{better}, \text{best}\}$. The values of attributes of nine applicants are given by three specialists. We denote y_i $(i = 1, 2, 3)$ and n_i $(i = 1, 2, 3)$ as evaluation results given by specialists 1, 2 and 3, respectively. In addition, y_i denotes yes and n_i denotes no. The evaluation results of nine applicants are listed as Table 3. From Table 3, for the attributes "salary" and "education", three specialists give evaluation results, corresponding to attributive mean value, in Table 4. Moreover, the $x_i^{(k)}$ $(k = 1, 2, 3)$ denotes the attributive mean values in Table 4 for all applicants x_i $(i = 1, 2, \cdots, 9)$. Then, by applying the formula

Table 3 An evaluation information system (Hu 2017; Lin et al. 2012; Qiao and Hu 2018; Qiao and Hu 2020)

U	AT																	
Attributive value	Education									Salary								
	Best			Better			Good			High			Middle			Low		
	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
x_1	y_1	y_2	n_3	n_1	n_2	n_3	n_1	n_2	y_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3
x_2	n_1	y_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	n_2	n_3
x_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3
x_4	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3
x_5	y_1	y_2	n_3	n_1	y_2	n_3	n_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	y_1	n_2	n_3
x_6	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3
x_7	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	y_3	n_1	n_2	n_3	y_1	y_2	n_3	n_1	n_2	y_3
x_8	n_1	y_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	y_2	n_3
x_9	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3	n_1	n_2	n_3	n_1	n_2	n_3	y_1	y_2	y_3

Table 4 The corresponding attributive mean value of evaluation information system (Hu 2017; Qiao and Hu 2018; Qiao and Hu 2020)

U	AT					
Attributive mean value	Education			Salary		
	Best	Better	Good	High	Middle	Low
x_1	$\frac{2}{3}$	0	$\frac{1}{3}$	1	0	0
x_2	$\frac{1}{3}$	1	0	1	0	$\frac{1}{3}$
x_3	0	0	1	1	0	0
x_4	1	0	0	0	1	0
x_5	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	1	$\frac{1}{3}$
x_6	0	0	1	0	1	0
x_7	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$
x_8	$\frac{1}{3}$	1	0	0	1	$\frac{1}{3}$
x_9	0	0	1	0	0	1

$$R(x_i, x_j) = \exp\left\{ -\sum_{k=1}^{3} |x_i^{(k)} - x_j^{(k)}| \right\},$$

we obtain fuzzy similarity relations of U on salary attribute and education attribute, respectively, as:

$$R_s = \begin{bmatrix} 1 & 0.72 & 1 & 0.14 & 0.10 & 0.14 & 0.14 & 0.10 & 0.14 \\ 0.72 & 1 & 0.72 & 0.10 & 0.14 & 0.10 & 0.19 & 0.14 & 0.19 \\ 1 & 0.72 & 1 & 0.14 & 0.10 & 0.14 & 0.14 & 0.10 & 0.14 \\ 0.14 & 0.10 & 0.14 & 1 & 0.72 & 1 & 0.51 & 0.72 & 0.14 \\ 0.10 & 0.14 & 0.10 & 0.72 & 1 & 0.72 & 0.72 & 1 & 0.19 \\ 0.14 & 0.10 & 0.14 & 1 & 0.72 & 1 & 0.51 & 0.72 & 0.14 \\ 0.14 & 0.19 & 0.14 & 0.51 & 0.72 & 0.51 & 1 & 0.72 & 0.26 \\ 0.10 & 0.14 & 0.10 & 0.72 & 1 & 0.72 & 0.72 & 1 & 0.19 \\ 0.14 & 0.19 & 0.14 & 0.14 & 0.19 & 0.14 & 0.26 & 0.19 & 1 \end{bmatrix}$$

and

$$R_e = \begin{bmatrix} 1 & 0.19 & 0.26 & 0.51 & 0.51 & 0.26 & 0.72 & 0.19 & 0.26 \\ 0.19 & 1 & 0.10 & 0.19 & 0.19 & 0.10 & 0.14 & 1 & 0.10 \\ 0.26 & 0.10 & 1 & 0.14 & 0.26 & 1 & 0.19 & 0.10 & 1 \\ 0.51 & 0.19 & 0.14 & 1 & 0.26 & 0.14 & 0.72 & 0.19 & 0.14 \\ 0.51 & 0.19 & 0.26 & 0.26 & 1 & 0.26 & 0.37 & 0.19 & 0.26 \\ 0.26 & 0.10 & 1 & 0.14 & 0.26 & 1 & 0.19 & 0.10 & 1 \\ 0.72 & 0.14 & 0.19 & 0.72 & 0.37 & 0.19 & 1 & 0.14 & 0.19 \\ 0.19 & 1 & 0.10 & 0.19 & 0.19 & 0.10 & 0.14 & 1 & 0.10 \\ 0.26 & 0.10 & 1 & 0.14 & 0.26 & 1 & 0.19 & 0.10 & 1 \end{bmatrix}.$$

To begin with, let $A = \frac{0.9}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5} + \frac{0.2}{x_6} + \frac{0.7}{x_7} + \frac{0.3}{x_8} + \frac{0}{x_9}$ be a fuzzy set of U. Fuzzy set A denotes the middle-aged peoples of U. Then, by the construction method of semi-decision evaluation function given in Proposition 4.5, we select two semi-decision evaluation functions

$$E_{\text{semi}}^{\text{salary}}(A)(x) = \frac{\sum_{y \in V} T(R_s(x,y), A(y))}{\sum_{y \in V} R_s(x,y)}$$

and

$$E_{\text{semi}}^{\text{education}}(A)(x) = \frac{\sum_{y \in V} T(R_e(x,y), A(y))}{\sum_{y \in V} R_e(x,y)}$$

where T is the Łukasiewicz t-norm and $N_{P_C}(A) = A^C$. In the following, in order to obtain decision evaluation functions, we use two transformation methods from semi-decision evaluation functions to decision evaluation functions given in Theorem 3.1 and Lemma 3.2, respectively. It can be verified that $E_{\text{semi}}^{\text{salary}}$ and $E_{\text{semi}}^{\text{education}}$ satisfies the constraint condition of these two transformation methods.

Firstly, by the transformation method given in Theorem 3.1, we take uninorm $\mathcal{U}$ as $\mathcal{U}_{0.7}^*$ $\mathcal{U}_{0.4}^*$ and the product t-norm T_P, respectively. Then, from semi decision evaluations $E_{\text{semi}}^{\text{salary}}$ and $E_{\text{semi}}^{\text{education}}$ we obtain six decision evaluation functions as $E_{(\mathcal{U}_{0.7}^*, \mathcal{U}_*^{0.3}, C)}^{salary}$, $E_{(\mathcal{U}_{0.4}^*, \mathcal{U}_*^{0.6}, C)}^{salary}$,

Table 5 Decision evaluation values of attribute A of credit card applicants

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
$E^{salary}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$	0.869	0.790	0.869	0.564	0.597	0.564	0.575	0.597	0.152
$E^{salary}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$	0.869	0.790	0.869	0.564	0.597	0.564	0.575	0.597	0.365
$E^{salary}_{(T_P,S_P,C)}$	0.755	0.695	0.755	0.564	0.597	0.564	0.575	0.597	0.365
$E^{salary}_{(O_2,G_2,C)}$	0.654	0.609	0.654	0.531	0.549	0.531	0.536	0.549	0.440
$E^{salary}_{(O_3,G_3,C)}$	0.586	0.553	0.586	0.512	0.519	0.512	0.514	0.519	0.477
$E^{salary}_{(O_{mM},G_{mM},C)}$	0.734	0.671	0.734	0.553	0.582	0.553	0.564	0.582	0.374
$E^{education}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$	0.887	0.480	0.479	0.893	0.901	0.479	0.866	0.480	0.479
$E^{education}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$	0.887	0.480	0.479	0.893	0.724	0.479	0.866	0.480	0.479
$E^{education}_{(T_P,S_P,C)}$	0.751	0.480	0.479	0.756	0.724	0.479	0.761	0.480	0.479
$E^{education}_{(O_2,G_2,C)}$	0.648	0.490	0.490	0.652	0.620	0.490	0.661	0.490	0.490
$E^{education}_{(O_3,G_3,C)}$	0.581	0.496	0.496	0.584	0.559	0.496	0.592	0.496	0.496
$E^{education}_{(O_{mM},G_{mM},C)}$	0.733	0.483	0.482	0.739	0.711	0.482	0.738	0.483	0.482

$E^{salary}_{(T_P,S_P,C)}$, $E^{education}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$, $E^{education}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$ and $E^{education}_{(T_P,S_P,C)}$, respectively. It is worth noting that t-norm T_P is a special case of overlap function O_P.

Then, by the transformation method given in Lemma 3.2, we take overlap function O as O_2, O_3 and O_{mM}, respectively. Then, from semi decision evaluations E^{salary}_{semi} and $E^{education}_{semi}$, we obtain six decision evaluation functions as $E^{salary}_{(O_2,G_2,C)}$, $E^{salary}_{(O_3,G_3,C)}$, $E^{salary}_{(O_{mM},G_{mM},C)}$, $E^{education}_{(O_2,G_2,C)}$, $E^{education}_{(O_3,G_3,C)}$ and $E^{education}_{(O_{mM},G_{mM},C)}$, respectively.

Then, if we consider two parameters $\psi = 0.75$ and $\omega = 0.48$, by Algorithm 1, we obtain the decision evaluation values of U (see Table 5) and the three-way decisions results of U (see Table 6). In the following, on the basis of Tables 5 and 6, we show some analyses of these two transformation methods.

(1) For salary attribute, on the basis of semi-decision evaluation function E^{salary}_{semi}, we obtain six decision evaluation functions (see Fig. 10). Applicant x_9 are evaluated rejection in these six decision evaluation functions. This accords with his low salary evaluated by three specialists. Applicants x_4, x_5, x_6, x_7 and x_8 are evaluated uncertain in six decision evaluation function. However, applicants x_1 and x_3, as two middle-aged and high salary evaluated by three specialists people, are evaluated uncertain by decision evaluation functions $E^{salary}_{(O_2,G_2,C)}$, $E^{salary}_{(O_3,G_3,C)}$ and $E^{salary}_{(O_{mM},G_{mM},C)}$. It is not reasonable. Meanwhile, applicants x_1 and x_3 are evaluated acceptance by decision evaluation functions $E^{salary}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$, $E^{salary}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$ and $E^{salary}_{(T_P,S_P,C)}$. Therefore, under such conditions, the uninorms-based transformation method is superior to overlap and grouping functions-based transformation method.

(2) For education attribute, on the basis of semi-decision evaluation function $E^{education}_{semi}$, we obtain six decision evaluation functions (see Fig. 11). It is worth noting that all applicants are evaluated uncertain by decision evaluation functions $E^{education}_{(O_2,G_2,C)}$,

Table 6 Three-way decisions results of attribute A for evaluation of credit card applicants

	Acceptance region	Rejection region	Uncertain region
$E^{salary}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$	$\{x_1,x_2,x_3\}$	$\{x_9\}$	$\{x_4,x_5,x_6,x_7,x_8\}$
$E^{salary}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$	$\{x_1,x_2,x_3\}$	$\{x_9\}$	$\{x_4,x_5,x_6,x_7,x_8\}$
$E^{salary}_{(T_P,S_P,C)}$	$\{x_1,x_3\}$	$\{x_9\}$	$\{x_2,x_4,x_5,x_6,x_7,x_8\}$
$E^{salary}_{(O_2,G_2,C)}$	$\emptyset$	$\{x_9\}$	$\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$
$E^{salary}_{(O_3,G_3,C)}$	$\emptyset$	$\{x_9\}$	$\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$
$E^{salary}_{(O_{mM},G_{mM},C)}$	$\emptyset$	$\{x_9\}$	$\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$
$E^{education}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$	$\{x_1,x_4,x_5,x_7\}$	$\{x_2,x_3,x_6,x_8,x_9\}$	$\emptyset$
$E^{education}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$	$\{x_1,x_4,x_7\}$	$\{x_2,x_3,x_6,x_8,x_9\}$	$\{x_5\}$
$E^{education}_{(T_P,S_P,C)}$	$\{x_1,x_4,x_7\}$	$\{x_2,x_3,x_6,x_8,x_9\}$	$\{x_5\}$
$E^{education}_{(O_2,G_2,C)}$	$\emptyset$	$\emptyset$	U
$E^{education}_{(O_3,G_3,C)}$	$\emptyset$	$\emptyset$	U
$E^{education}_{(O_{mM},G_{mM},C)}$	$\emptyset$	$\emptyset$	U

Fig. 10 Six decision evaluation functions for salary attribute

$E^{education}_{(O_3,G_3,C)}$ and $E^{education}_{(O_{mM},G_{mM},C)}$. Decision maker can not obtain any useful information from the three-way decision results derived from $E^{education}_{(O_2,G_2,C)}$, $E^{education}_{(O_3,G_3,C)}$ and $E^{education}_{(O_{mM},G_{mM},C)}$. On the other hand, decision maker can obtain reasonable three-way decision results (see Table 6) from decision evaluation functions $E^{education}_{(\mathcal{U}^*_{0.7},\mathcal{U}^{0.3}_*,C)}$, $E^{education}_{(\mathcal{U}^*_{0.4},\mathcal{U}^{0.6}_*,C)}$ and $E^{education}_{(T_P,S_P,C)}$ which are derived from uninorms-based transformation method. Therefore, under such conditions, the uninorms-based transformation method is superior to overlap and grouping functions-based transformation method.
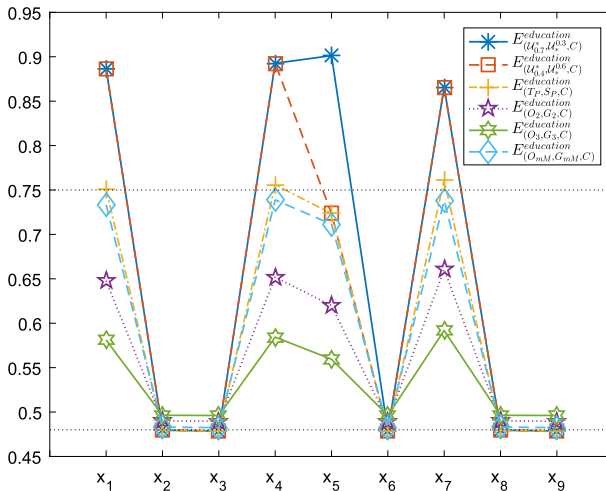
Fig. 11 Six decision evaluation functions for education attribute

According to above discussions, by using commonly used uninorms and overlap and grouping functions, the uninorms-based transformation method has better performance than overlap and grouping functions-based transformation method under some conditions. Meanwhile, in Remark 3.2, it has been pointed out that the uninorms-based transformation method has the looser constraint of semi-decision evaluation functions than overlap and grouping functions-based transformation method. Therefore, under some conditions, the uninorms-based transformation method is superior to overlap and grouping functions-based transformation method.

6 Conclusion

This article continues the work given by Jia and Qiao (2020) and Qiao and Hu (2018) and mainly discusses the construction methods of decision evaluation functions based on uninoms. For convenience, the uninorms-based construction methods of semi-decision evaluation functions and decision evaluation functions proposed in this paper can be called uni-construction methods. To be precise, this article obtains the following results:

- In order to construct more decision evaluation functions, this paper proposes two novel transformation methods from semi-decision evaluation functions to decision evaluation functions based on uninorms and representable uninorms, respectively. Meanwhile, on the basis of uninorms, this paper proposes some novel construction methods of semi-decision evaluation function and decision evaluation functions related to known semi-decision evaluation functions, fuzzy sets, interval-valued fuzzy sets, fuzzy relations and hesitant fuzzy sets, respectively.

- The representable uninorms-based construction methods of decision evaluation functions (see Theorems 3.2 and 4.1–4.4) generalize the strong negation N_{P_D} of partially ordered set P_D from $N_{P_D}(x) = 1 - x$ to the strong negation derived from the additive generator of representable uninorms.
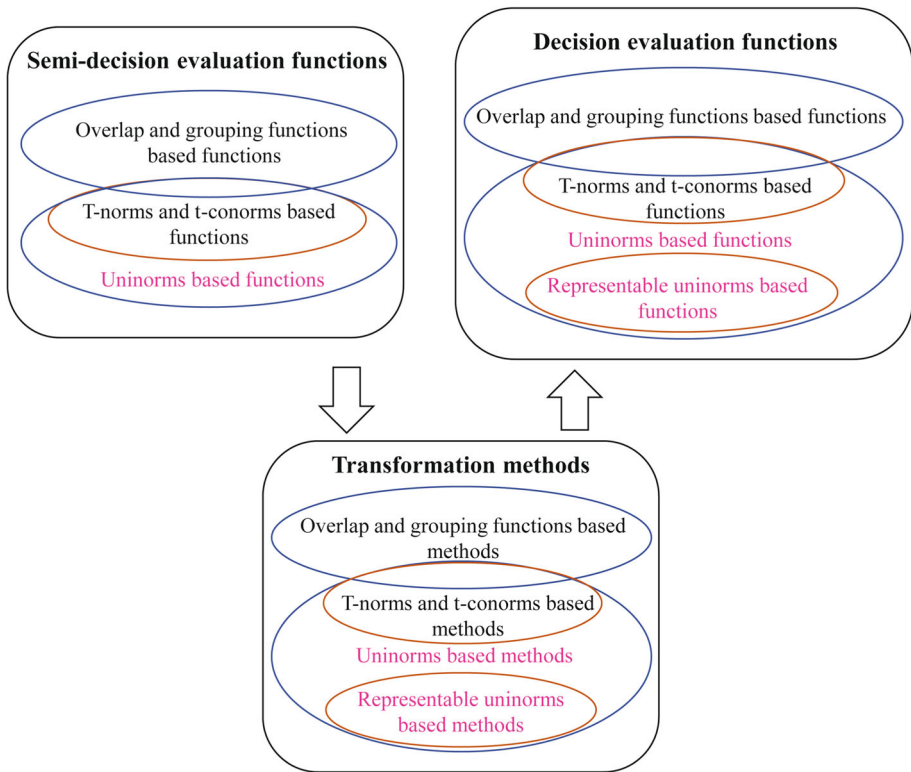
Fig. 12 The relationship among different decision evaluation functions

- The relationship among different decision evaluation functions are discussed in this paper (see Remarks 3.1,3.2, 4.1–4.5, 4.7–4.9 and 4.11–4.13). The Fig. 12 shows the relationship among different decision evaluation functions.

- This paper analyses two real evaluation problems to illustrate the results obtained in this paper. The corresponding three-way decision results are reasonable. Meanwhile, we compare the overlap and grouping functions-based transformation method with the uninorms-based transformation method. The comparison result shows that the uninorms-based transformation method is superior to the overlap and grouping functions based-transformation method under some conditions.

It is worth noting that this paper shows lots of methods to construct different decision evaluation functions for different partially ordered sets. Therefore, for different decision-making problems, the decision maker can choose corresponding methods to construct enough decision evaluation functions according to their decision-making environment. Furthermore, according to actual situation, decision maker can choose specific and appropriate decision evaluation functions to solve decision-making problems. For example, overlap functions have been successfully applied to fuzzy community detection problems (Gómez et al. 2016) and power quality diagnosis system (Nolasco et al. 2019). Therefore, for these problems, decision maker can preferentially choose overlap and grouping functions-based decision evaluation functions. Meanwhile, uninorms have been successfully used to neural networks (de Campos Souza and Lughofer 2021, 2022a, b) and classification

of COVID-19 markers (Roy et al. 2020). Thus, uninorms-based decision evaluation functions can be given priority to solve these problems.

In the later research, on the one hand, we will further research the application of decision evaluation functions in practical problems. On the other hand, from the theoretical viewpoint, there are still some unsolved problems about decision evaluation functions, such as:

- How to construct decision evaluation functions through other commonly used aggregation functions? For example, nullnorms, uni-nullnorms, and so on.
- The algebraic properties of decision evaluation functions have not been researched.
- In Theorem 3.2, the constraint conditions of semi-decision evaluation function are strong. Can the constraint conditions be relaxed?

Declarations

References

Azam N, Yao J (2014) Analyzing uncertainties of probabilistic rough set regions with game-theoretic rough sets. Int J Approx Reason 55:142–155

Bedregal B, Dimuro GP, Bustince H et al (2013) New results on overlap and grouping functions. Inf Sci 249:148–170

Bustince H, Pagola M, Barrenechea E (2007) Construction of fuzzy indices from fuzzy di-subsethood measures: application to the global comparison of images. Inf Sci 177:906–929

Bustince H, Fernandez J, Mesiar R et al (2010) Overlap functions. Nonlinear Anal: Theory Methods Appl 72 (3):1488–1499

Bustince H, Pagola M, Mesiar R et al (2012) Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. IEEE Trans Fuzzy Syst 20(3):405–415. https://doi.org/10.1109/TFUZZ.2011.2173581

Cabitza F, Ciucci D, Locoro A (2017) Exploiting collective knowledge with three-way decision theory: cases from the questionnaire-based research. Int J Approx Reason 83:356–370

Campanella G, Ribeiro R (2011) A framework for dynamic multiple-criteria decision making. Decis Support Syst 52:52–60

Cheng S, Wu Y, Li Y et al (2021) Twd-sfnn: three-way decisions with a single hidden layer feedforward neural network. Inf Sci 579:15–32

Chu X, Sun B, Li X et al (2020) Neighborhood rough set-based three-way clustering considering attribute correlations: an approach to classification of potential gout groups. Inf Sci 535:28–41

de Campos Souza PV, Lughofer E (2021) An evolving neuro-fuzzy system based on uni-nullneurons with advanced interpretability capabilities. Neurocomputing 451:231–251

de Campos Souza PV, Lughofer E (2022a) An advanced interpretable fuzzy neural network model based on uni-nullneuron constructed from n-uninorms. Fuzzy Sets Syst 426:1–26

de Campos Souza PV, Lughofer E (2022b) Efnn-nulluni: an evolving fuzzy neural network based on null-uninorm. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2022.01.010

Dubois D, Prade H, Barrenechea E (1990) Rough fuzzy sets and fuzzy rough sets. Int J Gen Syst 17:191–209

Fan Z (2000) On the convergence of a fuzzy matrix in the sense of triangular norms. Fuzzy Sets Syst 109:409–417

Fodor JC, Yager R, Rybalov A (1997) Structure of uninorms. Int J Uncertain Fuzziness Knowl-Based Syst 5:411–427

González-Hidalgo M, Massanet S, Mir A et al (2015) On the choice of the pair conjunction-implication into the fuzzy morphological edge detector. IEEE Trans Fuzzy Syst 23:872–884

Gómez D, Tinguaro Rodríguez J, Yáñez J et al (2016) A new modularity measure for fuzzy community detection problems based on overlap and grouping functions. Int J Approx Reason 74:88–107

Hu BQ (2014) Three-way decisions space and three-way decisions. Inf Sci 281:21–52

Hu BQ (2016) Three-way decision spaces based on partially ordered sets and three-way decisions based on hesitant fuzzy sets. Knowl-Based Syst 91:16–31

Hu BQ (2017) Three-way decisions based on semi-three-way decision spaces. Inf Sci 382–383:415–440

Hu BQ, Wong H, Yiu K (2016) The aggregation of multiple three-way decision space. Knowl-Based Syst 98:241–249

Hu BQ, Wong H, Yiu K (2017) On two novel types of three-way decisions in three-way decision spaces. Int J Approx Reason 82:285–306

Jia Z, Qiao J (2020) On decision evaluation functions in three-way decision spaces derived from overlap and grouping functions. Soft Comput 24:15159–15178

Jiang H, Hu BQ (2021) A novel three-way group investment decision model under intuitionistic fuzzy multi-attribute group decision-making environment. Inf Sci 569:557–581

Jiang C, Li Z, Yao J (2022) A shadowed set-based three-way clustering ensemble approach. Int J Mach Learn Cybern. https://doi.org/10.1007/s13042-022-01543-5

Klement E, Mesiar R, Pap E (2000) Triangular norms. Kluwer, Dordrecht

Lang G, Yao Y (2021) New measures of alliance and conflict for three-way conflict analysis. Int J Approx Reason 132:49–69

Lang G, Miao D, Fujita H (2020) Three-way group conflict analysis based on pythagorean fuzzy set theory. IEEE Trans Fuzzy Syst 28(3):447–461

Li Y, Shi Z (2000) Remarks on uninorms aggregation operators. Fuzzy Sets Syst 114:377–380

Li Z, Xie N, Huang D et al (2020) A three-way decision method in a hybrid decision information system and its application in medical diagnosis. Artif Intell Rev 53:4707–4736

Li X, Wang X, Lang G et al (2021) Conflict analysis based on three-way decision for triangular fuzzy information systems. Int J Approx Reason 132:88–106

Liang D, Liu D (2014) Systematic studies on three-way decisions with interval-valued decision-theoretic rough sets. Inf Sci 276:186–203

Liang D, Pedrycz W, Liu D et al (2015) Three-way decisions based on decision-theoretic rough sets under linguistic assessment with the aid of group decision making. Appl Soft Comput 29:256–269

Liang D, Liu D, Kobina A (2016) Three-way group decisions with decision-theoretic rough sets. Inf Sci 345:46–64

Lin G, Qian Y, Li J (2012) Nmgrs: neighborhood-based multigranulation rough sets. Int J Approx Reason 53:1080–1093

Lin G, Liang J, Qian Y (2013) Multigranulation rough sets: from partition to covering. Inf Sci 241:101–118

Liu D, Liang D, Wang C (2016) A novel three-way decision model based on incomplete information system. Knowl-Based Syst 91:32–45

Min F, Zhang SM, Ciucci D et al (2020) Three-way active learning through clustering selection. Int J Mach Learn Cybern 11:1033–1046

Nolasco DH, Costa FB, Palmeira ES et al (2019) Wavelet-fuzzy power quality diagnosis system with inference method based on overlap functions: case study in an ac microgrid. Eng Appl Artif Intell 85:284–294

Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 11:341–356

Pawlak Z (1991) Rough sets: theoretical aspects of reasoning about data. Kluwer, Dordrecht

Pedrycz W (1998) Shadowed sets: representing and processing fuzzy sets. IEEE Trans Syst Man Cybern B 28 (1):103–109

Qian W, Shu W (2018) Attribute reduction in incomplete ordered information systems with fuzzy decision. Appl Soft Comput 73:242–253

Qiao J, Hu BQ (2018) On transformations from semi-three-way decision spaces to three-way decision spaces based on triangular norms and triangular conorms. Inf Sci 432:22–51

Qiao J, Hu BQ (2020) On decision evaluation functions in generalized three-way decision spaces. Inf Sci 507:733–754

Roy S, Menapace W, Oei S et al (2020) Deep learning for classification and localization of covid-19 markers in point-of-care lung ultrasound. IEEE Trans Med Imaging 39(8):2676–2687

Subhashini L, Li Y, Zhang J et al (2022) Integration of semantic patterns and fuzzy concepts to reduce the boundary region in three-way decision-making. Inf Sci 595:257–277

Sun B, Tong S, Ma W et al (2021) An approach to mcgdm based on multi-granulation pythagorean fuzzy rough set over two universes and its application to medical decision problem. Artif Intell Rev. https://doi.org/10.1007/s10462-021-10048-6

Yager R (2002) Defending against strategic manipulation in uninorm-based multi-agent decision making. Eur J Oper Res 141:217–232

Yager R, Rybalov A (1996) Uninorm aggregation operators. Fuzzy Sets Syst 80:111–120

Yager R, Rybalov A (2011) Bipolar aggregation using the uninorms. Fuzzy Optim Decis Mak 10:59–70

Yan P, Chen G (2005) Discovering a cover set of arsi with hierarchy from quantitative databases. Inf Sci 173:319–336

Yang B, Hu BQ (2016) A fuzzy covering-based rough set model and its generalization over fuzzy lattice. Inf Sci 367–368:463–486

Yang B, Hu BQ (2017) On some types of fuzzy covering-based rough sets. Fuzzy Sets Syst 312:36–65

Yao Y (2003) Probabilistic approaches to rough sets. Expert Syst 20:287–297

Yao Y (2007) Decision-theoretic rough set models. In: Yao J, Lingras P, Wu W et al (eds) Rough sets and knowledge technology. RSKT 2007. Lecture notes in computer science, vol 4481. Springer, Heidelberg, pp 1–12

Yao Y (2008) Probabilistic rough set approximations. Int J Approx Reason 49:255–271

Yao Y (2009) Three-way decision: an interpretation of rules in rough set theory. In: Wen P, Li Y, Polkowski L et al (eds) Rough sets and knowledge technology. RSKT 2009. Lecture notes in computer science, vol 5589. Springer, Heidelberg, pp 642–649

Yao Y (2010) Three-way decisions with probabilistic rough sets. Inf Sci 180:341–353

Yao Y (2011) The superiority of three-way decisions in probabilistic rough set models. Inf Sci 181:1080–1096

Yao Y (2012) An outline of a theory of three-way decisions. In: Yao J, Yang Y, Słowiński R et al (eds) Rough sets and current trends in computing. Springer, Berlin, pp 1–17

Yao Y (2015) Rough sets and three-way decisions. In: Ciucci D, Wang G, Mitra S et al (eds) Rough sets and knowledge technology. Springer, Cham, pp 62–73

Yao J, Azam N (2015) Web-based medical decision support systems for three-way medical decision making with game-theoretic rough sets. IEEE Trans Fuzzy Syst 23:3–15

Yao Y, Yang J (2022) Granular rough sets and granular shadowed sets: three-way approximations in pawlak approximation spaces. Int J Approx Reason 142:231–247

Ye J, Zhan J, Sun B (2021) A three-way decision method based on fuzzy rough set models under incomplete environments. Inf Sci 577:22–48

Yu H, Chen L, Yao J (2021) A three-way density peak clustering method based on evidence theory. Knowl-Based Syst 211(106):532

Yue X, Chen Y, Yuan B et al (2021) Three-way image classification with evidential deep convolutional neural networks. Cogn Comput. https://doi.org/10.1007/s12559-021-09869-y

Zadeh L (1965) Fuzzy sets. Inf Control 8:338–353

Zhan J, Ye J, Ding W et al (2021) A novel three-way decision model based on utility theory in incomplete fuzzy decision systems. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2021.3078012

Zhang YM, Qin F (2022) Conditional distributivity equation of semi-uninorms over uninorms. Int J Approx Reason 142:290–300

Zhao X, Hu BQ (2016) Fuzzy probabilistic rough sets and their corresponding three-way decisions. Knowl-Based Syst 91:126–142

Zhou H, Yan X (2021) Migrativity properties of overlap functions over uninorms. Fuzzy Sets Syst 403:10–37

Ziarko W (1993) Variable precision rough set model. J Comput Syst Sci 46:39–59

Zong W, Su Y, Liu HW et al (2020) On the construction of uninorms by paving. Int J Approx Reason 118:96–111