

RESEARCH ARTICLE

Defining R-squared measures for mixed-effects location scale models

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Funding information

National Institute of Diabetes and
Digestive and Kidney Diseases,
Grant/Award Number: R01 DK125414

Ecological momentary assessment and other modern data collection technologies facilitate research on both within-subject and between-subject variability of health outcomes and behaviors. For such intensively measured longitudinal data, Hedeker et al extended the usual two-level mixed-effects model to a two-level mixed-effects location scale (MELS) model to accommodate covariates' influence as well as random subject effects on both mean (location) and variability (scale) of the outcome. However, there is a lack of existing standardized effect size measures for the MELS model. To fill this gap, our study extends Rights and Sterba's framework of R^2 measures for multilevel models, which is based on model-implied variances, to MELS models. Our proposed framework applies to two different specifications of the random location effects, namely, through covariate-influenced random intercepts and through random intercepts combined with random slopes of observation-level covariates. We also provide an R function, *R2MELS*, that outputs summary tables and visualization for values of our R^2 measures. This framework is validated through a simulation study, and data from a health behaviors study and a depression study are used as examples to demonstrate this framework. These R^2 measures can help researchers provide greater interpretation of their findings using MELS models.

KEYWORDS

EMA, mixed-effects location scale model, R-squared, standardized effect size

1 | INTRODUCTION

Modern data collection methods such as ecological momentary assessments (EMA) have allowed more detailed examination of subjects' heterogeneity both at the between-subject (BS) level (also known as level 2) and the within-subject (WS) level (also known as level 1).¹ Hedeker et al² extended the commonly used mixed-effects regression model (MRM) into the mixed-effects location scale (MELS) model that includes both random location effects and random scale effects. Random location effects refer to random subject effects on the mean of the response variable, and random scale effects refer to random subject effects on the WS variability of the response variable. While scale sometimes only refers to standard deviation, here it is on variance metric.

Increasingly, researchers are encouraged to report effect sizes in addition to P -values for their study results.³ Standardized effect size measures are of particular interest as they allow direct comparison of different models. However, to the best of our knowledge, there are no existing standardized effect size measures specifically for MELS models.

Standardized effect size measures have been developed for MRMs. Earlier pseudo- R^2 measures evaluate a model's reduction in residual variance from the null model.⁴ A problem with this approach is that it can result in negative values and thus become meaningless as shown by Snijders and Bosker.⁵ Snijders and Bosker⁵ then resolved this problem by constructing R^2 using model-implied variances. Recently, Rights and Sterba⁶ introduced a comprehensive framework of R^2 measures for multilevel models using model-implied variances that measure both the total variance of the response variable explained and level-specific variances of the response variable explained. While this work primarily discusses cross-sectional multilevel models, their later work accommodates specific features of longitudinal multilevel models.⁷ Rights and Sterba also introduced supplementary visualization and R functions to help researchers implement their proposed framework.

Our study extends Rights and Sterba's framework to two-level MELS models. We develop frameworks of R^2 measures for two forms of MELS models, which differ in their characterization of the random location effects. In the first form, the model includes random subject intercepts and allows their variance to be influenced by both subject-level and observation-level covariates. Alternatively, the second form includes random subject intercepts and slopes of observation-level covariates. For both forms, R^2 measures are constructed for both the location model and the scale model. We also develop an R function, *R2MELS*, that allows calculation and visualization of R^2 measures specifically for MELS models.

1.1 | Mixed-effects location-scale (MELS) models with random intercepts with covariate-influenced variance

To begin, we review the MELS model for a two-level continuous response variable y_{ij} ($i = 1, 2, \dots, N$ subjects, $j = 1, 2, \dots, n_i$ observations) proposed by Hedeker et al.²

$$y_{ij} = \beta_0 + x_{ij}^T \beta + v_i + \epsilon_{ij}, \quad (1)$$

where β_0 is the fixed intercept of the location model, x_{ij} is the $p \times 1$ vector of fixed location effect covariates, and β is the corresponding $p \times 1$ vector of fixed location effects. BS heterogeneity is included via random intercepts v_i , also recognized as the random location effects. ϵ_{ij} is the observation-level residuals and incorporates WS heterogeneity. v_i and ϵ_{ij} are assumed to be normally distributed with mean 0 and variances $\sigma_{v_i}^2$ and $\sigma_{\epsilon_{ij}}^2$, respectively.

$\sigma_{v_i}^2$ is further modeled in log-linear form to account for different BS heterogeneity at different values of covariates:

$$\sigma_{v_i}^2 = \text{Var}(v_i | u_{ij}) = \exp\left(\alpha_0 + u_{ij}^T \alpha\right), \quad (2)$$

where $\exp(\alpha_0)$ is the variance of v_i when the covariates equal zero, or if the covariates have no influence on the BS variance. u_{ij} is the vector of covariates influencing $\sigma_{v_i}^2$, which can contain both subject-level and observation-level covariates. α represents the vector of fixed effects associated with u_{ij} on $\sigma_{v_i}^2$.

For ϵ_{ij} , both heteroskedasticity at different covariate values and heteroskedasticity between subjects are allowed. The heteroskedasticity of ϵ_{ij} between subjects is included via a random scale effect ω_i , which is assumed to follow a normal distribution with a mean of 0 and a variance of σ_{ω}^2 . $\sigma_{\epsilon_{ij}}^2$ is again modeled in log-linear form to ensure positive variance values:

$$\sigma_{\epsilon_{ij}}^2 = \text{Var}(\epsilon_{ij} | w_{ij}) = \exp\left(\tau_0 + w_{ij}^T \tau + \omega_i\right), \quad (3)$$

where $\exp(\tau_0)$ is the value of $\sigma_{\epsilon_{ij}}^2$ when the covariates and the random scale effect equal zero, or when there is neither any covariate influencing $\sigma_{\epsilon_{ij}}^2$ nor subject heteroskedasticity of ϵ_{ij} . w_{ij} is the vector of covariates influencing $\sigma_{\epsilon_{ij}}^2$. Similar to u_{ij} , w_{ij} can contain both subject-level and observation-level covariates. τ is the vector of fixed effects associated with w_{ij} on $\sigma_{\epsilon_{ij}}^2$.

For convenience, Equation (2) can be rewritten in terms of standardized random location effects, denoted as θ_{1i} :

$$y_{ij} = \beta_0 + x_{ij}^T \beta + \sigma_{v_i} \theta_{1i} + \epsilon_{ij}, \tag{4}$$

and Equation (3) can also be written in terms of standardized random scale effects, denoted as θ_{2i} :

$$\sigma_{\epsilon_{ij}}^2 = \exp\left(\tau_0 + w_{ij}^T \tau + \sigma_{\omega} \theta_{2i}\right), \tag{5}$$

Since the random location effects v_i and random scale effects ω_i are not necessarily uncorrelated, θ_{1i} and θ_{2i} are assumed to follow the following bivariate normal distribution:

$$\begin{pmatrix} \theta_{1i} \\ \theta_{2i} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{v\omega} \\ \rho_{v\omega} & 1 \end{pmatrix}\right), \tag{6}$$

where $\rho_{v\omega}$ is the correlation between the random location effect v_i and the random scale effect ω_i .

1.2 | MELS models with random slopes of observation-level covariates

Instead of having random intercepts and allowing covariates to influence their variance, random location effects can be modeled by random intercepts with constant variance and random slopes of observation-level covariates:⁸

$$y_{ij} = \beta_0 + x_{ij}^T \beta + z_{ij}^T v_i + \epsilon_{ij}, \tag{7}$$

where we use z_{ij} to represent an $(r + 1) \times 1$ vector with the first element of 1 for the random intercept followed by r observation-level covariates for random slopes. v_i , the corresponding $(r + 1) \times 1$ vector of the random location effects, follows the following distribution:

$$v_i \sim \mathcal{N}(0, \Sigma_v). \tag{8}$$

The first element of v_i is the random intercept, and the following elements are r random slopes associated with covariates in z_{ij} . The variances of the random intercept and the random slopes, $\sigma_{v_0}^2, \sigma_{v_1}^2, \dots, \sigma_{v_r}^2$, are scalars not influenced by any covariates.

In this form of MELS models, the fixed intercept, fixed location effects, and variance of observation-level residuals are modeled the same way as in Section 1.1.

Again, we can express the random effects as standardized random effects for convenience. Namely,

$$v_i = \begin{pmatrix} \sigma_{v_0} \theta_{0i} \\ \sigma_{v_1} \theta_{1i} \\ \vdots \\ \sigma_{v_r} \theta_{ri} \end{pmatrix} \tag{9}$$

and

$$\omega_i = \sigma_{\omega} \theta_{\omega i}, \tag{10}$$

where $\theta_{0i}, \theta_{1i}, \dots, \theta_{ri}$, and $\theta_{\omega i}$ follow a multivariate normal distribution with mean 0, and their variance-covariance matrix Σ_{θ} is a $(r + 2) \times (r + 2)$ matrix given as:

$$\Sigma_{\theta} = \begin{pmatrix} 1 & \rho_{v_0 v_{1i}} & \dots & \dots & \rho_{v_0 \omega_i} \\ \rho_{v_0 v_{1i}} & 1 & \dots & \dots & \rho_{v_{1i} \omega_i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & \rho_{v_{ri} \omega_i} \\ \rho_{v_0 \omega_i} & \rho_{v_{1i} \omega_i} & \dots & \rho_{v_{ri} \omega_i} & 1 \end{pmatrix}, \tag{11}$$

where $\rho_{v_{pi}v_{qi}}$ ($p \neq q; p, q = 0, 1, \dots, r$) is the correlation between v_{pi} and v_{qi} , and $\rho_{v_{pi}\omega_i}$ is the correlation between v_{pi} ($p = 0, 1, \dots, r$) and ω_i .

2 | THE PROPOSED METHOD

2.1 | Decomposition of observation-level covariates

As mentioned, x_{ij} , u_{ij} , and w_{ij} can contain both subject-level covariates and observation-level covariates, and z_{ij} is solely composed of observation-level covariates. Since an observation-level covariate not centered at the subject level contains both BS variation and WS variation, and we assume that the covariates are multivariate normally distributed at each level, we first decompose each of x_{ij} , u_{ij} , w_{ij} , and z_{ij} into a BS component and a WS component before deriving their contribution to the variance of the response. For example,

$$x_{ij} = \bar{x}_i + (x_{ij} - \bar{x}_i), \tag{12}$$

where \bar{x}_i is the vector of subject-level fixed location effect covariates, which characterizes the BS components of observation-level fixed location effect covariates. Elements of \bar{x}_i are multivariate normally distributed with mean μ_x and variance Φ_x^b . Analogously, $(x_{ij} - \bar{x}_i)$ is the vector of WS components of observation-level fixed location effect covariates, and its elements are multivariate normally distributed with mean 0 and variance Φ_x^w .

Similarly, decomposition of covariates influencing the variance of the random intercepts (ie, the BS variance) is given by $u_{ij} = \bar{u}_i + (u_{ij} - \bar{u}_i)$, where $\bar{u}_i \sim \mathcal{N}(\mu_u, \Phi_u^b)$, and $(u_{ij} - \bar{u}_i) \sim \mathcal{N}(0, \Phi_u^w)$. Decomposition of covariates influencing the WS variance is similarly: $w_{ij} = \bar{w}_i + (w_{ij} - \bar{w}_i)$, where $\bar{w}_i \sim \mathcal{N}(\mu_w, \Phi_w^b)$, and $(w_{ij} - \bar{w}_i) \sim \mathcal{N}(0, \Phi_w^w)$. Finally, decomposition of covariates for random slopes is given by $z_{ij} = \bar{z}_i + (z_{ij} - \bar{z}_i)$, where $\bar{z}_i \sim \mathcal{N}(\mu_z, \Phi_z^b)$, and $(z_{ij} - \bar{z}_i) \sim \mathcal{N}(0, \Phi_z^w)$. The recurring superscript b in the variance matrices represents BS components, and w represents WS components. Note that if one includes a random slope for an observation-level covariate in x_{ij} , the same covariate will also occur in z_{ij} , and the decomposition of this variable will be the same in x_{ij} and z_{ij} .

2.2 | Variance partitioning

2.2.1 | Variance partitioning for MELS models with random intercepts with covariate-influenced variance

For the model described in Section 1.1,

$$\begin{aligned} \text{Var}(y_{ij}) &= \text{Var} \left(\beta_0 + x_{ij}^T \beta + v_i + \epsilon_{ij} \right) \\ &= \text{Var} \left((x_{ij} - \bar{x}_i)^T \beta + \bar{x}_i^T \beta + v_i + \epsilon_{ij} \right) \\ &= \text{Var}((x_{ij} - \bar{x}_i)^T \beta) + \text{Var}(\bar{x}_i^T \beta) + \text{Var}(v_i) + \text{Var}(\epsilon_{ij}) \end{aligned} \tag{13}$$

since $(x_{ij} - \bar{x}_i)$, \bar{x}_i , v_i , and ϵ_{ij} are independent.

The WS variance of the response variable explained by fixed location effects of WS components of observation-level covariates, denoted as $f1$, can be derived based on the property of the multivariate normal distribution:⁹

$$\text{Var}((x_{ij} - \bar{x}_i)^T \beta) = \beta^T \Phi_x^w \beta. \tag{14}$$

Similarly, the BS variance of the response variable explained by fixed location effects of subject-level covariates and BS components of observation-level covariates, denoted as $f2$, can be expressed as:

$$\text{Var}(\bar{x}_i^T \beta) = \beta^T \Phi_x^b \beta. \tag{15}$$

In applying these formulas, we substitute Φ_x^w and Φ_x^b with their sample estimators.

As shown in Appendix A, the variance of the random intercepts is

$$\text{Var}(v_i) = e^{\alpha_0 + \mu_u^T \alpha + \frac{\alpha^T (\Phi_u^w + \Phi_u^b) \alpha}{2}}, \tag{16}$$

denoted as v . Let $m = e^{\alpha_0 + \mu_u^T \alpha}$ represent the variance of the random intercepts at the mean (both on the BS level and the WS level) of all covariates influencing the variance of the random intercepts. Note that the mean of the WS component of a covariate is zero. Then, $v - m$ represents the variance of the random intercepts explained by covariates. To further decompose $v - m$, logarithmic transformations are taken:

$$\log(v) - \log(m) = \frac{\alpha^T \Phi_u^w \alpha}{2} + \frac{\alpha^T \Phi_u^b \alpha}{2}, \tag{17}$$

where $v1 = \frac{\alpha^T \Phi_u^w \alpha}{2}$ represents the log-transformed variance of the random intercepts explained by fixed effects of WS components of observation-level covariates. $\frac{\alpha^T \Phi_u^b \alpha}{2}$, denoted as $v2$, is the log-transformed variance of the random intercepts explained by fixed effects of subject-level covariates and BS components of observation-level covariates. We use sample estimators of μ_u , Φ_u^w , and Φ_u^b in place of these population parameters themselves in practice.

The variance of the observation-level residuals, also known as the scale of the response variable, is

$$\text{Var}(\epsilon_{ij}) = e^{\tau_0 + \mu_w^T \tau + \frac{\tau^T (\Phi_w^w + \Phi_w^b) \tau + \sigma_\omega^2}{2}}, \tag{18}$$

denoted as e . $e0 = e^{\tau_0 + \mu_w^T \tau}$ is the variance of the observation-level residuals at the mean (both on the BS level and the WS level) of all covariates in the scale model, which is also the unexplained scale of the response variable, and $e - e0$ is the variance of the observation-level residuals explained by covariates and the random scale effects. Similar to the decomposition of $v - m$, $e - e0$ can be further decomposed on the logarithmic scale:

$$\log(e) - \log(e0) = \frac{\tau^T \Phi_w^b \tau}{2} + \frac{\tau^T \Phi_w^w \tau}{2} + \frac{\sigma_\omega^2}{2}, \tag{19}$$

where $e1 = \frac{\tau^T \Phi_w^w \tau}{2}$ is the log-transformed scale of the response variable explained by fixed effects of WS components of observation-level covariates, $e2 = \frac{\tau^T \Phi_w^b \tau}{2}$ is the log-transformed scale of the response variable explained by fixed effects of subject-level covariates and BS components of observation-level covariates, and $d = \frac{\sigma_\omega^2}{2}$ is the log-transformed scale of the response variable explained by the random scale effects. When applied, the sample estimators of μ_w , Φ_w^b , and Φ_w^w are used in Equations 18 and 19.

For simplicity, here the coefficients β 's, α 's, and τ 's are assumed to be the same for the BS and WS components of covariates. However, in some cases, it may be desirable to allow the BS and the WS component of a covariate to have different effects. For this, one can substitute the corresponding coefficients of distinct BS and WS effects in the appropriate equations. Our R function described in Section 2.4 allows for this possibility. Users would need to specify the BS component and the WS component of a covariate as two distinct variables in their dataset and input their corresponding effect estimates into the function separately.

2.2.2 | Variance partitioning for MELS models with random slopes of observation-level covariates

Here, we decompose the variance of the response variable based on the model described in Section 1.2.

$$\begin{aligned} \text{Var}(y_{ij}) &= \text{Var}(\beta_0 + x_{ij}^T \beta + z_{ij}^T v_i + \epsilon_{ij}) \\ &= \text{Var}((x_{ij} - \bar{x}_i)^T \beta + \bar{x}_i^T \beta + (z_{ij} - \bar{z}_i)^T v_i + \bar{z}_i^T v_i + \epsilon_{ij}) \\ &= \text{Var}((x_{ij} - \bar{x}_i)^T \beta) + \text{Var}(\bar{x}_i^T \beta) + \text{Var}((z_{ij} - \bar{z}_i)^T v_i) + \text{Var}(\bar{z}_i^T v_i) + \text{Var}(\epsilon_{ij}) \end{aligned} \tag{20}$$

given the independence of $(x_{ij} - \bar{x}_i), \bar{x}_i, v_i,$ and ϵ_{ij} as well as the independence of $(z_{ij} - \bar{z}_i)$ and \bar{z}_i .

The interpretations and derivations for $\text{Var}((x_{ij} - \bar{x}_i)^T \beta)$ (f1), $\text{Var}(\bar{x}_i^T \beta)$ (f2), and $\text{Var}(\epsilon_{ij})$ (e) are the same as in Section 2.2.1, while the variance partitioning for the random location effects are given by, with $\text{Tr}(\cdot)$ representing the trace of a matrix:

$$\text{Var}((z_{ij} - \bar{z}_i)^T v_i) = \text{Tr}(\Phi_z^w \Sigma_v) \tag{21}$$

denoted as $v1$, which is the WS variance of the response variable explained by random slopes of WS components of observation-level covariates, and

$$\text{Var}(\bar{z}_i^T v_i) = \text{Tr}(\Phi_z^b \Sigma_v) + \mu_z^T \Sigma_v \mu_z. \tag{22}$$

Here, $\text{Tr}(\Phi_z^b \Sigma_v)$ is denoted as $v2$, which corresponds to the BS variance of the response variable explained by random slopes of BS components of observation-level covariates. Also, $\mu_z^T \Sigma_v \mu_z$ is denoted as m , which represents the BS variance of the response variable explained by the random intercepts at the mean of BS components of all covariates for random location effects. The derivations of Equation (21) and Equation (22) can be found in Rights and Sterba's work.⁷

2.3 | Defining R^2 measures

We develop R^2 measures for the total variance of the response variable, the level-specific variances of the response variable, and the scale of the response variable.

Table 1 details R^2 measures for the location part of the model described in Section 1.1, and Table 2 describes R^2 measures for the location part of the model presented in Section 1.2. R^2 measures for the scale model are illustrated in Table 3. The superscripts in parentheses denote the source(s) of variation. Also, the subscripts represent the denominators of the R^2 measures, meaning which part of the variance of the response variable that one is trying to explain. Namely, subscript t indicates total variance of the response variable and is calculated as $f1 + f2 + v + e$ for MELS models with random intercepts (with covariate-influenced variance), and $f1 + f2 + v1 + v2 + m + e$ for MELS models with random slopes of observation-level covariates. Subscript w and subscript b represent WS variance of the response variable and BS variance of the response variable, respectively. For the model described in Section 1.1, the model-implied WS variance of the response variable is $f1 + e$, and the model-implied BS variance of the response variable is $f2 + v$. For the Section 1.2 model, the WS variance of the response variable equals $f1 + v1 + e$ while the BS variance of the response variable is calculated as $f2 + v2 + m$. Lastly, the subscript s represents variance of the observation-level residuals, which is denoted as e in both specifications of MELS models.

The R^2 s defined can measure the variance of the response variable explained by single sources of variation. Namely, $R_t^{2(f1)}, R_t^{2(f2)}, R_t^{2(v1)}, R_t^{2(v2)}, R_t^{2(m)}, R_w^{2(f1)}, R_b^{2(f2)}, R_b^{2(v1)}, R_b^{2(v2)}$, and $R_b^{2(m)}$ in Table 1, $R_t^{2(f1)}, R_t^{2(f2)}, R_t^{2(v1)}, R_t^{2(v2)}, R_t^{2(m)}, R_w^{2(f1)}, R_w^{2(v1)}, R_b^{2(f2)}, R_b^{2(v2)}$, and $R_b^{2(m)}$ in Table 2, as well as $R_s^{2(e1)}, R_s^{2(e2)}$, and $R_s^{2(d)}$ in Table 3 are single-source R^2 measures. We also define R^2 s that measure joint effects of multiple parts of the models. Specifically, $f = f1 + f2$ represents the variance of the response variable explained by fixed location effects of both subject-level covariates and observation-level covariates, v in Table 1 represents the variance of the response variable explained by the random intercepts, and $v = v1 + v2$ in Table 2 represents the variance of the response variable explained by the random slopes of both the WS components and BS components of observation-level covariates.

Since the proportion of the total variance of the response variable that is BS can be of interest to researchers applying MELS models, we add $R_t^{2(f2v)}$ in Table 1 and $R_t^{2(f2v2m)}$ in Table 2. These two R^2 s measure the proportion of total variance of the response variable explained by BS location effects.

2.4 | Implementation in R

The commented code for an R function named *R2MELS* and descriptions of its arguments are provided in the Supporting Information. The function is developed based on Rights and Sterba's *r2MLMLong* function.⁷ Users input their parameter estimates of a MELS model, and the function will output two tables of variance partitioning results (one for

TABLE 1 Definitions and interpretations of R^2 measures for the location part of an MELS model with random intercepts with covariate-influenced variance

Definition	Coefficients ^a	Covariates ^a	Interpretation
R^2 Measures for total variance of the response variable			
$R_t^{2(f1)} = \frac{f1}{f1 + f2 + v + e}$	β	$(x_{ij} - \bar{x}_i)$	Proportion of total variance of the response variable explained by fixed location effects of WS components of observation-level covariates
$R_t^{2(f2)} = \frac{f2}{f1 + f2 + v + e}$	β	\bar{x}_i	Proportion of total variance of the response variable explained by fixed location effects of subject-level covariates and BS components of observation-level covariates
$R_t^{2(f)} = \frac{f1 + f2}{f1 + f2 + v + e}$	β	$(x_{ij} - \bar{x}_i), \bar{x}_i$	Proportion of total variance of the response variable explained by fixed location effects
$R_t^{2(v1)} = \frac{v1(v-m)}{f1 + f2 + v + e}$	α	$(u_{ij} - \bar{u}_i)$	Proportion of total variance of the response variable explained by fixed effects of WS components of observation-level covariates on the variance of the random intercepts
$R_t^{2(v2)} = \frac{v2(v-m)}{f1 + f2 + v + e}$	α	\bar{u}_i	Proportion of total variance of the response variable explained by fixed effects of subject-level covariates and BS components of observation-level covariates on the variance of the random intercepts
$R_t^{2(m)} = \frac{m}{f1 + f2 + v + e}$	α_0, α	\bar{u}_i	Proportion of total variance of the response variable explained by random intercepts at the mean of all covariates influencing the variance of the random intercepts
$R_t^{2(v)} = \frac{v}{f1 + f2 + v + e}$	α_0, α	$(u_{ij} - \bar{u}_i), \bar{u}_i$	Proportion of total variance of the response variable explained by random location effects
$R_t^{2(fv)} = \frac{f1 + f2 + v}{f1 + f2 + v + e}$	β, α_0, α	$(x_{ij} - \bar{x}_i), \bar{x}_i, (u_{ij} - \bar{u}_i), \bar{u}_i$	Proportion of total variance of the response variable explained by both fixed location effects and random location effects
$R_t^{2(f2v)} = \frac{f2 + v}{f1 + f2 + v + e}$	β, α_0, α	$\bar{x}_i, (u_{ij} - \bar{u}_i), \bar{u}_i$	Proportion of total variance of the response variable explained by BS location effects
R^2 Measures for BS variance of the response variable			
$R_b^{2(f2)} = \frac{f2}{f2 + v}$	β	\bar{x}_i	Proportion of BS variance of the response variable explained by fixed location effects of subject-level covariates and BS components of observation-level covariates
$R_b^{2(v1)} = \frac{v1(v-m)}{f2 + v}$	α	$(u_{ij} - \bar{u}_i)$	Proportion of BS variance of the response variable explained by fixed effects of WS components of observation-level covariates on the variance of the random intercepts
$R_b^{2(v2)} = \frac{v2(v-m)}{f2 + v}$	α	\bar{u}_i	Proportion of BS variance of the response variable explained by fixed effects of subject-level covariates and BS components of observation-level covariates on the variance of the random intercepts
$R_b^{2(m)} = \frac{m}{f2 + v}$	α_0, α	\bar{u}_i	Proportion of BS variance of the response variable explained by random intercepts at the mean of all covariates influencing the variance of the random intercepts
$R_b^{2(v)} = \frac{v}{f2 + v}$	α_0, α	$(u_{ij} - \bar{u}_i), \bar{u}_i$	Proportion of BS variance of the response variable explained by random location effects
R^2 Measures for WS variance of the response variable			
$R_w^{2(f1)} = \frac{f1}{f1 + e}$	β	$(x_{ij} - \bar{x}_i)$	Proportion of WS variance of the response variable explained by fixed location effects of WS components of observation-level covariates

^aThe coefficients and covariates refer to elements of the model needed to calculate the source of variation in the specific R^2 measure, that is, what is labeled in the parenthesized superscript.

TABLE 2 Definitions and interpretations of R^2 measures for the location part of an MELS model with random slopes of observation-level covariates

Definition	Coefficients ^a	Covariates ^a	Interpretation
R^2 Measures for total variance of the response variable			
$R_t^{2(f1)} = \frac{f1}{f1 + f2 + v1 + v2 + m + e}$	β	$(x_{ij} - \bar{x}_i)$	Proportion of total variance of the response variable explained by fixed location effects of WS components of observation-level covariates
$R_t^{2(f2)} = \frac{f2}{f1 + f2 + v1 + v2 + m + e}$	β	\bar{x}_i	Proportion of total variance of the response variable explained by fixed location effects of subject-level covariates and BS components of observation-level covariates
$R_t^{2(f)} = \frac{f1 + f2}{f1 + f2 + v1 + v2 + m + e}$	β	$(x_{ij} - \bar{x}_i), \bar{x}_i$	Proportion of total variance of the response variable explained by fixed location effects
$R_t^{2(v1)} = \frac{v1}{f1 + f2 + v1 + v2 + m + e}$	v_i	$(z_{ij} - \bar{z}_i)$	Proportion of total variance of the response variable explained by random slopes of WS components of observation-level covariates
$R_t^{2(v2)} = \frac{v2}{f1 + f2 + v1 + v2 + m + e}$	v_i	\bar{z}_i	Proportion of total variance of the response variable explained by random slopes of BS components of observation-level covariates
$R_t^{2(m)} = \frac{m}{f1 + f2 + v1 + v2 + m + e}$	v_i	\bar{z}_i	Proportion of total variance of the response variable explained by random intercepts at the mean of BS components of all covariates for random location effects
$R_t^{2(vm)} = \frac{v1 + v2 + m}{f1 + f2 + v1 + v2 + m + e}$	v_i	$(z_{ij} - \bar{z}_i), \bar{z}_i$	Proportion of total variance of the response variable explained by random location effects
$R_t^{2(fvm)} = \frac{f1 + f2 + v1 + v2 + m}{f1 + f2 + v1 + v2 + m + e}$	β, v_i	$(x_{ij} - \bar{x}_i), \bar{x}_i, (z_{ij} - \bar{z}_i), \bar{z}_i$	Proportion of total variance of the response variable explained by both fixed location effects and random location effects
$R_t^{2(f2v2m)} = \frac{f2 + v2 + m}{f1 + f2 + v1 + v2 + m + e}$	β, v_i	\bar{x}_i, \bar{z}_i	Proportion of total variance of the response variable explained by BS location effects
R^2 Measures for BS variance of the response variable			
$R_b^{2(f2)} = \frac{f2}{f2 + v2 + m}$	β	\bar{x}_i	Proportion of BS variance of the response variable explained by fixed location effects of subject-level covariates and BS components of observation-level covariates
$R_b^{2(v2)} = \frac{v2}{f2 + v2 + m}$	v_i	\bar{z}_i	Proportion of BS variance of the response variable explained by random slopes of BS components of observation-level covariates
$R_b^{2(m)} = \frac{m}{f2 + v2 + m}$	v_i	\bar{z}_i	Proportion of BS variance of the response variable explained by random intercepts at the mean of BS components of all covariates for random location effects
$R_b^{2(v2m)} = \frac{v2 + m}{f2 + v2 + m}$	v_i	\bar{z}_i	Proportion of BS variance of the response variable explained by random location effects
R^2 Measures for WS variance of the response variable			
$R_w^{2(f1)} = \frac{f1}{f1 + v1 + e}$	β	$(x_{ij} - \bar{x}_i)$	Proportion of WS variance of the response variable explained by fixed location effects of WS components of observation-level covariates
$R_w^{2(v1)} = \frac{v1}{f1 + v1 + e}$	v_i	$(z_{ij} - \bar{z}_i)$	Proportion of WS variance of the response variable explained by random slopes of WS components of observation-level covariates
$R_w^{2(f1v1)} = \frac{f1 + v1}{f1 + v1 + e}$	β, v_i	$(x_{ij} - \bar{x}_i), (z_{ij} - \bar{z}_i)$	Proportion of WS variance of the response variable explained by both fixed location effects and random slopes of WS components of observation-level covariates

^aThe coefficients and covariates refer to elements of the model needed to calculate the source of variation in the specific R^2 measure, that is, what is labeled in the parenthesized superscript.

TABLE 3 Definitions and interpretations of R^2 measures for the scale part of an MELS model

Definition	Coefficients ^a	Covariates ^a	Interpretation
$R_s^{2(e1)} = \frac{e1}{e1 + e2 + d} \frac{(e - e0)}{e}$	τ	$(w_{ij} - \bar{w}_i)$	Proportion of variance of observation-level residuals explained by WS components of observation-level covariates
$R_s^{2(e2)} = \frac{e2}{e1 + e2 + d} \frac{(e - e0)}{e}$	τ	\bar{w}_i	Proportion of variance of observation-level residuals explained by subject-level covariates and BS components of observation-level covariates
$R_s^{2(e1e2)} = \frac{e1 + e2}{e1 + e2 + d} \frac{(e - e0)}{e}$	τ	$(w_{ij} - \bar{w}_i), \bar{w}_i$	Proportion of variance of observation-level residuals explained by covariates
$R_s^{2(d)} = \frac{d}{e1 + e2 + d} \frac{(e - e0)}{e}$	σ_ω	N/A	Proportion of variance of observation-level residuals explained by random scale effects

^aThe coefficients and covariates refer to elements of the model needed to calculate the source of variation in the specific R^2 measure, that is, what is labeled in the parenthesized superscript.

TABLE 4 Generating parameters and mean parameter estimates from 500 simulations

Parameter	True value	Simulated values mean (SD)
β_0	1	1.000(0.069)
β_1	-0.5	-0.497(0.020)
β_2	2	1.999(0.024)
β_3	1	1.000(0.013)
β_4	-2	-2.000(0.007)
β_5	3	3.000(0.010)
α_0	0.1	0.098(0.103)
α_1	0.4	0.401(0.012)
τ_0	0.2	0.199(0.061)
τ_1	0.3	0.300(0.013)
τ_2	-0.1	-0.101(0.013)
τ_3	0.5	0.502(0.018)
σ_ω^2	0.7	0.697(0.076)
$\rho_{v\omega}$	0.1	0.106 (0.075)

the location part of the model, and the other for the scale part of the model), two tables of R^2 values (one for the location part of the model, and the other for the scale part of the model) as well as a stacked bar plot of the single-source R^2 values.

3 | SIMULATION STUDY

To assess the validity of the proposed method, we conducted a small simulation study. Specifically, we fitted MELS models to 500 simulated datasets (200 subjects in each sample, 50 observations of each subject). For each simulated dataset, we allowed for two subject-level covariates, $\begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} \right)$, and three observation-level covariates that vary purely within-subjects, $\begin{pmatrix} x_{3ij} \\ x_{4ij} \\ x_{5ij} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0.3 & 0.75 \\ 0.3 & 1.5 & 0.2 \\ 0.75 & 0.2 & 1 \end{pmatrix} \right)$.

TABLE 5 Theoretical values of R^2 measures and average simulated values from 500 simulations

R^2 Measure	Theoretical value	Simulated values mean (SD)
R^2 Measures for total variance of the response variable		
$R_t^{2(f1)} = \frac{f1}{f1 + f2 + v + e}$	0.661	0.663(0.015)
$R_t^{2(f2)} = \frac{f2}{f1 + f2 + v + e}$	0.203	0.201(0.017)
$R_t^{2(f)} = \frac{f1 + f2}{f1 + f2 + v + e}$	0.864	0.864(0.008)
$R_t^{2(v1)} = \frac{v-m}{f1 + f2 + v + e}$	0.007	0.007(0.001)
$R_t^{2(m)} = \frac{m}{f1 + f2 + v + e}$	0.041	0.041(0.004)
$R_t^{2(v)} = \frac{v}{f1 + f2 + v + e}$	0.048	0.048(0.005)
$R_t^{2(fv)} = \frac{f1 + f2 + v}{f1 + f2 + v + e}$	0.912	0.912(0.006)
$R_t^{2(f2v)} = \frac{f2 + v}{f1 + f2 + v + e}$	0.251	0.249(0.017)
R^2 Measures for BS variance of the response variable		
$R_b^{2(f2)} = \frac{f2}{f2 + v}$	0.809	0.806(0.023)
$R_b^{2(v1)} = \frac{v-m}{f2 + v}$	0.028	0.029(0.004)
$R_b^{2(m)} = \frac{m}{f2 + v}$	0.163	0.165(0.020)
$R_b^{2(v)} = \frac{v}{f2 + v}$	0.191	0.194(0.023)
R^2 Measures for WS variance of the response variable		
$R_b^{2(f1)} = \frac{f1}{f1 + e}$	0.883	0.883(0.008)
R^2 Measures for scale of the response variable		
$R_s^{2(e1)} = \frac{e1}{e1+d} \frac{(e-e0)}{e}$	0.231	0.232(0.009)
$R_s^{2(d)} = \frac{d}{e1+d} \frac{(e-e0)}{e}$	0.256	0.254(0.024)

The location model was specified as follows:

$$y_{ij} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3ij} + \beta_4 x_{4ij} + \beta_5 x_{5ij} + \sigma_{v_i} \theta_{1i} + \epsilon_{ij}. \quad (23)$$

For the variance of the random intercept v_i ,

$$\sigma_{v_i}^2 = \exp(\alpha_0 + \alpha x_{3ij}). \quad (24)$$

For the variance of the observation-level residuals,

$$\sigma_{\epsilon_{ij}}^2 = \exp(\tau_0 + \tau_1 x_{3ij} + \tau_2 x_{4ij} + \tau_3 x_{5ij} + \sigma_{\omega} \theta_{2i}), \quad (25)$$

and

$$\begin{pmatrix} \theta_{1i} \\ \theta_{2i} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{v\omega} \\ \rho_{v\omega} & 1 \end{pmatrix} \right). \quad (26)$$

The generating parameters and average estimates from SAS PROC NLMIXED are summarized in Table 4, and the corresponding theoretical values of R^2 measures and average simulated values can be found in Table 5. As can be seen, all parameter estimates were well recovered, and all average simulated R^2 values resemble their corresponding theoretical values. Namely, all differences between the average simulated values and their corresponding theoretical values are lower than 0.004.

TABLE 6 Parameter estimates of the MELS model on health behaviors data

Parameter	Estimate	SE	T-value	P-value
β_0	5.855	0.109	53.631	< 0.001
β_{NA}	-0.644	0.016	-39.900	< 0.001
β_{Day_c}	-0.077	0.015	-5.247	< 0.001
α_0	-0.376	0.176	-2.144	0.036
α_{Day_c}	0.139	0.031	4.531	< 0.001
τ_0	-0.603	0.093	-6.509	< 0.001
τ_{Day_c}	0.073	0.025	2.888	0.005
σ_ω	0.735	0.070	10.451	< 0.001
$\rho_{v\omega}$	-0.500	0.100	-5.009	< 0.001

4 | EXAMPLES

4.1 | Example 1: Health behaviors data

Flueckiger et al¹⁰ collected intensive longitudinal data on 72 first-year psychology students from the University of Basel regarding their sleep quality, physical activity, positive and negative affect, learning goal achievement, and examination grades. We fit a MELS model on these data to examine how negative affect (NA) of the students and survey day influenced their mean positive affect (PA), how survey day influenced the BS variance of PA, as well as how survey day influenced the WS variance of PA. PA and NA were measured on 7-point Likert scales in which 1 means “not at all” and 7 means “extremely”. We used a grand-mean-centered and scaled version of survey day (*Day_c*), for which 1 unit indicates 1 week.

The location model is as follows:

$$PA_{ij} = \beta_0 + \beta_{NA}NA_{ij} + \beta_{Day_c}Day_c_{ij} + \sigma_{v_i}\theta_{1i} + \epsilon_{ij}. \tag{27}$$

The variance of the random subject intercepts and the variance of the observation-level residuals are modeled as

$$\sigma_{v_i}^2 = \exp(\alpha_0 + \alpha_{Day_c}Day_c_{ij}) \tag{28}$$

and

$$\sigma_{\epsilon_{ij}}^2 = \exp(\tau_0 + \tau_{Day_c}Day_c_{ij} + \sigma_\omega\theta_{2i}), \tag{29}$$

respectively. The random effects θ_{1i} and θ_{2i} follow the same bivariate normal distribution as specified in Equation (6). The parameter estimates from SAS PROC NLMIXED are included in Table 6, and a visualization of the proposed R^2 measures for this example is shown in Figure 1.

The first three bars on the left of Figure 1 correspond to the total variance, the BS variance, and the WS variance of PA, respectively. Within the bars, the red blocks represent proportion of the variance of PA explained by fixed location effects of WS components of observation-level covariates, $(NA_{ij} - \overline{NA}_i)$ and $(Day_c_{ij} - \overline{Day_c}_i)$, while the orange blocks can be interpreted as the proportion of the variance of PA, that is, the variance of observation-level residuals. Blue blocks correspond to the BS variance of PA. Specifically, the darkest blue blocks indicate the proportion of the variance of PA explained by the effect of $(Day_c_{ij} - \overline{Day_c}_i)$ on the variance of the random intercepts, and the lightest blue represents the variance of the random intercepts at the means of both the BS component and the WS component of *Day_c*. The mid-blue blocks show the proportion of the variance of PA explained by the fixed location effects of the BS components of NA and *Day_c*. Grey blocks, which represent the proportion of Var(PA) explained by the variance of random intercepts explained by $\overline{Day_c}_i$, are too small and thus almost invisible in the plot.

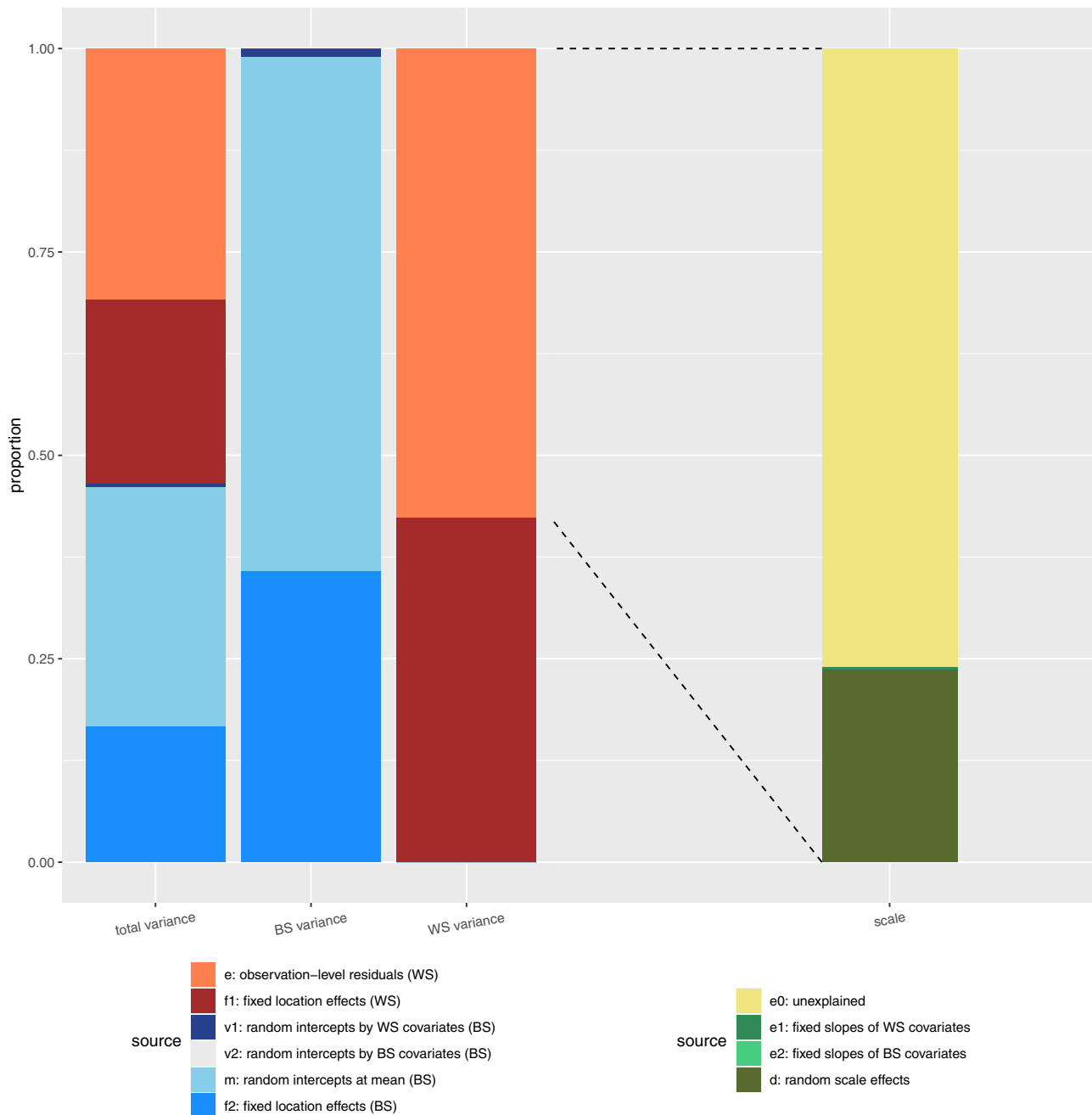


FIGURE 1 Variance partitioning for Example 1: application to health behaviors data

We can see that most (53.5%) of the variance of *PA* is within-subject, as represented by the red and orange blocks in the first column of Figure 1. For the WS variance of *PA* specifically, 42.3% is attributed to the fixed location effects of the WS components of *NA* and *Day_c*, which is visualized by the red block in the third column of Figure 1. In terms of the BS variance of *PA*, the random subject intercepts at the mean of both the BS component and the WS component of *Day_c*, which explain 63.1% of the BS variance of *PA* as indicated by the light blue portion of the middle column of Figure 1, are of particular importance. The R^2 measures for the scale model are summarized in the rightmost bar of Figure 1. As shown by the proportion of the bottom dark olive green block in this bar, 23.6% of the scale of *PA* is explained by random scale effects. Less than 0.5% of the scale of *PA* is explained by covariates, which is made clear by the almost negligible proportion of the two green blocks in the middle of this bar.

4.2 | Example 2: Depression study data

While the example in Section 4.1 presents a random intercepts model, in which its variance is modeled in terms of covariates, here we examine the application of our proposed R^2 framework to a MELS model with random slopes, as described in Section 1.2. The data are from Reisby et al's study¹¹ on the clinical responses of 66 depressed inpatients treated with anti-depressant medication. Here, we are interested in how patients' Hamilton depression score (*HamD*) changed following their weeks in the study (*week*). Additionally, *endog* is a subject-level dummy code that is coded as 1 if the patient had endogenous depression. The location model with response variable *HamD* and predictor *week*, controlled for *endog*, is specified as follows:

$$\begin{aligned} \text{HamD}_{ij} &= \beta_0 + v_{0i} + (\beta_{\text{week}} + v_{\text{week},i})\text{week}_{ij} + \beta_{\text{endog}}\text{endog}_i + \epsilon_{ij} \\ &= \beta_0 + \sigma_{v_0}\theta_{0i} + (\beta_{\text{week}} + \sigma_{v_{\text{week}}}\theta_{1i})\text{week}_{ij} + \beta_{\text{endog}}\text{endog}_i + \epsilon_{ij}, \end{aligned} \tag{30}$$

where $\sigma_{v_0}\theta_{0i}$ is the individual deviation from the average intercept β_0 , and $\sigma_{v_{\text{week}}}\theta_{1i}$ is the individual deviation from the average weekly change β_{week} .

The variance of the observation-level residuals is assumed to change along weeks and across subjects as well.

$$\sigma_{\epsilon_{ij}}^2 = \exp(\tau_0 + \tau_{\text{week}}\text{week}_{ij} + \sigma_{\omega}\theta_{\omega i}). \tag{31}$$

The distribution of the standardized random effects is given by:

$$\begin{pmatrix} \theta_{0i} \\ \theta_{1i} \\ \theta_{\omega i} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{v_{0i}v_{\text{week},i}} & \rho_{v_{0i}\omega_i} \\ \rho_{v_{0i}v_{\text{week},i}} & 1 & \rho_{v_{\text{week},i}\omega_i} \\ \rho_{v_{0i}\omega_i} & \rho_{v_{\text{week},i}\omega_i} & 1 \end{pmatrix} \right), \tag{32}$$

where $\rho_{v_{0i}v_{\text{week},i}}$ is the correlation between the random intercepts and the random slopes. Likewise, $\rho_{v_{0i}\omega_i}$ and $\rho_{v_{\text{week},i}\omega_i}$ are the correlations between the random intercepts and the random scale effects, and between the random slopes and the random scale effects, respectively.

Table 7 lists the parameter estimates for this model, and Figure 2 is the visualization of the proposed R^2 measures for this example. The meaning of each bar in Figure 2 can be interpreted as in Section 4.1, but for the variance of *HamD* instead of *PA*. The red blocks now represent the proportion of the variance of *HamD* explained by fixed location effects of the WS component of *week*. The orange blocks can be interpreted as the proportion of the variance of *HamD* that corresponds to observation-level residuals, and the mid-blue blocks indicate the proportion of the variance of *HamD* explained by fixed location effects of *endog* and the BS component of *week*. The newly added brown blocks correspond to the proportion of

TABLE 7 Parameter estimates of the MELS model on depression study data

Parameter	Estimate	SE	T-value	P-value
β_0	22.657	0.702	32.27	< 0.001
β_{week}	-2.357	0.201	-11.74	< 0.001
β_{endog}	1.533	0.924	1.66	0.102
τ_0	2.062	0.212	9.72	< 0.001
τ_{week}	0.100	0.067	1.51	0.137
σ_{ω}	0.538	0.116	4.65	< 0.001
σ_{v_0}	3.154	0.481	6.56	< 0.001
$\sigma_{v_{\text{week}}}$	1.379	0.174	7.92	< 0.001
$\rho_{v_{0i}v_{\text{week},i}}$	-0.207	0.180	-1.15	0.255
$\rho_{v_{0i}\omega_i}$	0.464	0.235	1.98	0.053
$\rho_{v_{\text{week},i}\omega_i}$	-0.485	0.213	-2.28	0.026

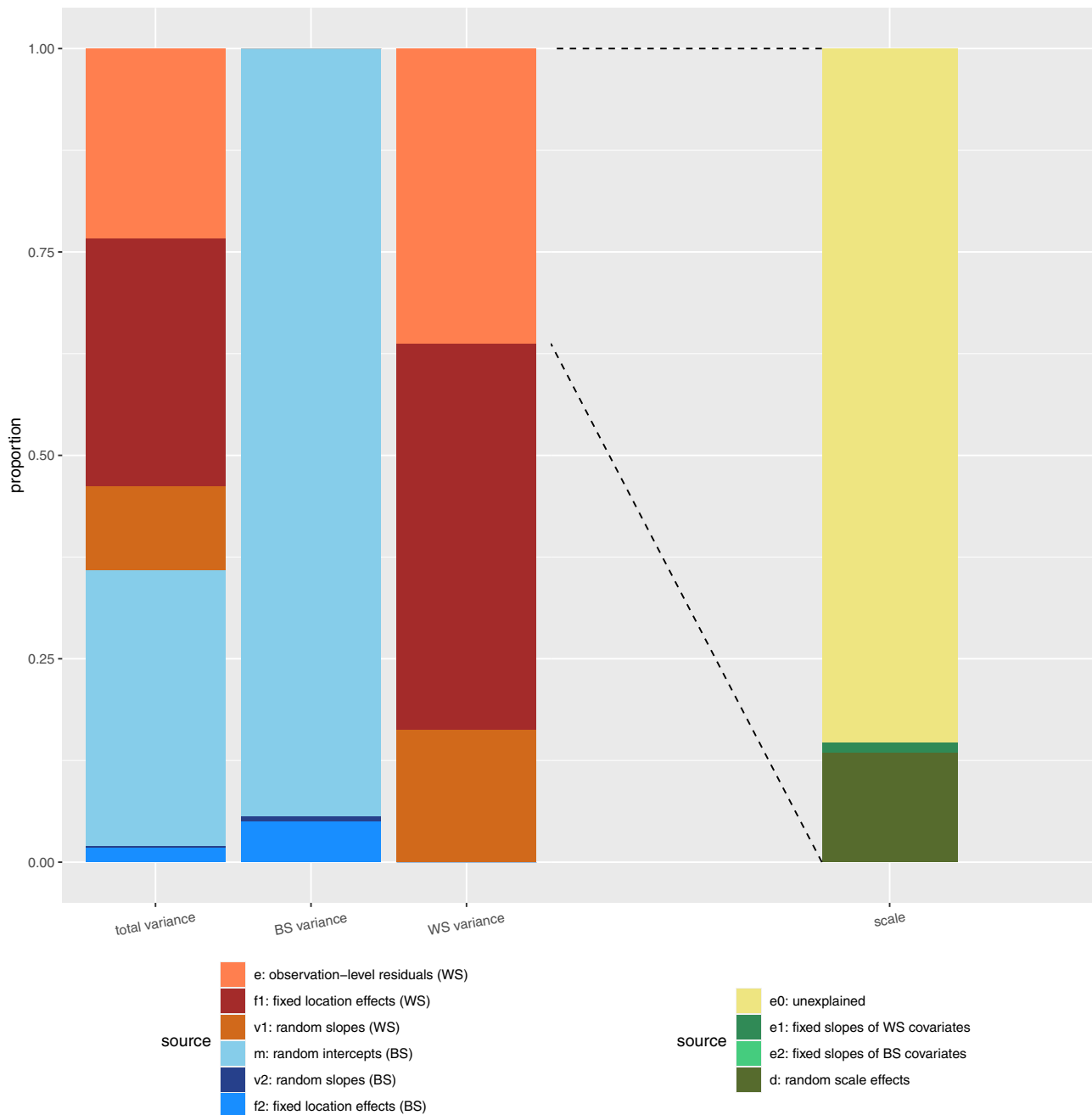


FIGURE 2 Variance partitioning for Example 2: application to depression study data

the variance of *HamD* explained by random slopes of WS components of *week*, and the dark blue blocks represent the proportion of the variance of *HamD* explained by random slopes of BS components of *week*. The light blue blocks still indicate the proportion of the variance of the response variable captured by the random intercepts but at the mean of $week_i$.

As represented by the total proportion of the blue blocks in the first column of Figure 2, 35.8% of the variance of *HamD* is between-subjects. While random slopes of BS components of *week* explain very little BS variance of *HamD*, as $week_i$ varies very little across subjects, the relative size of the light blue block in the second bar of Figure 2 shows that random subject intercepts at the mean of $week_i$ explain 94.3% of BS variance of *HamD*. Fixed location effects of the WS component of *week* explain the most (47.5%, the relative space of the red block in the third bar of Figure 2) WS variance of *HamD* while another 16.3% is attributed to random slopes of the WS component of *week* as indicated by the brown block in that

bar. 13.4% of the scale of *HamD* is explained by random scale effects while another 1.3% is explained by WS variation in *Week*. Less than 0.03% of the scale of *HamD* is explained by the BS component of *Week*.

5 | DISCUSSION

Our work extends Rights and Sterba's framework of defining R^2 measures for multilevel models^{6,7} to MELS models proposed by Hedeker et al² and Nordgren et al.⁸ The extended R^2 framework accommodates two special features of MELS modeling: (1) observation-level residual heteroskedasticity at different covariate values and across subjects; (2) inclusion of random location effects through either heteroskedastic random subject intercepts depending on covariates, or random subject intercepts and random subject slopes of observation-level covariates. We believe that this standardized effect size framework can facilitate the interpretation of MELS models and encourage wider use of this type of model.

In this article, our defined R^2 measures assume a two-level MELS model. Future work can extend these measures to three-level models, as developed by Lin et al¹² in which WS variation of the response variable is further divided into WS variation between waves and WS variation within waves. Also, the proposed R^2 measures can be expanded to other kinds of outcomes, for example, count outcomes and ordinal outcomes. An application of a MELS model for ordinal data was discussed by Hedeker et al.¹³ Furthermore, future research might take into consideration the autocorrelation of observation-level residuals. A recent development by Nestler¹⁴ extends the MELS models to include AR(1) autocorrelation influenced by subject-level covariates and a random subject effect for the autocorrelation.

Currently, our framework focuses on point estimates of R^2 measures. While reporting the point estimates is conventional for R^2 measures, researchers interested in coverage and confidence intervals of these measures can apply bootstrapping methods to calculate these quantities. There is no existing literature on bootstrapping in MELS models to our knowledge, but researchers can refer to Goldstein's discussion on bootstrapping in multilevel models.¹⁵

Lastly, this work focuses on defining R^2 's for a single model, and comparisons of R^2 's between different models are beyond the scope of this study. Researchers can refer to Rights and Sterba's recommendations on the use of R^2 differences in multilevel model comparisons for interpretation of differences between R^2 measures.¹⁶

ACKNOWLEDGEMENTS

This project is supported by National Institute of Diabetes and Digestive and Kidney Diseases (grant number R01 DK125414). The content of this study is solely the responsibility of the authors and does not represent the official views of National Institute of Diabetes and Digestive and Kidney Diseases.

DATA AVAILABILITY STATEMENT

The data used in the examples are publicly available. These data were derived from the following resources available in the public domain:

1. Health behaviors data: <https://dataverse.harvard.edu/dataset.xhtml?persistentId=doi:10.7910/DVN/27388>,
2. Depression study data: <https://hedeker.people.uic.edu/RIESBY.DAT.txt>.

Code used to generate data in the simulated study is included in the Supporting Information.

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SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

How to cite this article: Zhang X, Hedeker D. Defining R-squared measures for mixed-effects location scale models. *Statistics in Medicine*. 2022;41(22):4467-4483. doi: 10.1002/sim.9521

APPENDIX A. DERIVATIONS FOR VARIANCE OF RANDOM INTERCEPTS AND VARIANCE OF OBSERVATION-LEVEL RESIDUALS IN MELS MODELS WITH RANDOM INTERCEPTS WITH COVARIATE-INFLUENCED VARIANCE

By the law of total variance,

$$\begin{aligned}\text{Var}(v_i) &= E[\text{Var}(v_i|u_{ij})] + \text{Var}[E(v_i|u_{ij})] \\ &= E[\text{Var}(v_i|u_{ij})] + 0 \\ &= E[\text{Var}(v_i|u_{ij})].\end{aligned}\tag{A1}$$

Since

$$\begin{aligned}\text{Var}(v_i|u_{ij}) &= \exp(\alpha_0 + u_{ij}^T \alpha) \\ &= \exp(\alpha_0 + \bar{u}_i^T \alpha + (u_{ij} - \bar{u}_i)^T \alpha),\end{aligned}\tag{A2}$$

and \bar{u}_i^T follows a $\mathcal{N}(\mu_u, \Phi_u^b)$ distribution while $(u_{ij} - \bar{u}_i)$ follows a $\mathcal{N}(0, \Phi_u^w)$ distribution, $\text{Var}(v_i|u_{ij})$ follows a log-normal distribution whose natural logarithm follows a normal distribution with mean μ_v and variance σ_v^2 .

$$\mu_v = \alpha_0 + \mu_u^T \alpha,\tag{A3}$$

$$\begin{aligned}\sigma_v^2 &= \alpha^T \Phi_u^b \alpha + \alpha^T \Phi_u^w \alpha \\ &= \alpha^T (\Phi_u^b + \Phi_u^w) \alpha.\end{aligned}\tag{A4}$$

Therefore, by the expectation formula for log-normal variables,

$$\begin{aligned}
 E[\text{Var}(v_i|u_{ij})] &= \exp\left(\mu_v + \frac{\sigma_v^2}{2}\right) \\
 &= \exp\left(\alpha_0 + \mu_u^T \alpha + \frac{\alpha^T(\Phi_u^b + \Phi_u^w)\alpha}{2}\right),
 \end{aligned}
 \tag{A5}$$

that is,

$$\text{Var}(v_i) = \exp\left(\alpha_0 + \mu_u^T \alpha + \frac{\alpha^T(\Phi_u^b + \Phi_u^w)\alpha}{2}\right).
 \tag{A6}$$

Similarly, by the law of total variance,

$$\begin{aligned}
 \text{Var}(\epsilon_{ij}) &= E[\text{Var}(\epsilon_{ij}|w_{ij})] + \text{Var}[E(\epsilon_{ij}|w_{ij})] \\
 &= E[\text{Var}(\epsilon_{ij}|w_{ij})] + 0 \\
 &= E[\text{Var}(\epsilon_{ij}|w_{ij})].
 \end{aligned}
 \tag{A7}$$

And,

$$\begin{aligned}
 \text{Var}(\epsilon_{ij}|w_{ij}) &= \exp\left(\tau_0 + w_{ij}^T \tau + \omega_i\right) \\
 &= \exp\left(\tau_0 + \bar{w}_i^T \tau + (w_{ij} - \bar{w}_i)^T \tau + \omega_i\right),
 \end{aligned}
 \tag{A8}$$

where $\bar{w}_i \sim \mathcal{N}(\mu_w, \Phi_w^b)$, and $(w_{ij} - \bar{w}_i) \sim \mathcal{N}(0, \Phi_w^w)$. Hence, $\text{Var}(\epsilon_{ij}|w_{ij})$ is a log-normally distributed variable whose natural logarithm is normally distributed with mean

$$\mu_\epsilon = \tau_0 + \mu_w^T \tau,
 \tag{A9}$$

and variance

$$\begin{aligned}
 \sigma_\epsilon^2 &= \tau^T \Phi_w^b \tau + \tau^T \Phi_w^w \tau + \sigma_\omega^2 \\
 &= \tau^T (\Phi_w^b + \Phi_w^w) \tau + \sigma_\omega^2.
 \end{aligned}
 \tag{A10}$$

$$E[\text{Var}(\epsilon_{ij}|w_{ij})] = \exp\left(\tau_0 + \mu_w^T \tau + \frac{\tau^T(\Phi_w^b + \Phi_w^w)\tau + \sigma_\omega^2}{2}\right).
 \tag{A11}$$

Therefore,

$$\text{Var}(\epsilon_{ij}) = \exp\left(\tau_0 + \mu_w^T \tau + \frac{\tau^T(\Phi_w^b + \Phi_w^w)\tau + \sigma_\omega^2}{2}\right).
 \tag{A12}$$