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Utilization of Haar wavelet collocation technique for fractal-fractional order problem

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ABSTRACT

This work is devoted for establishing adequate results for the qualitative theory as well as approximate solution of "fractal-fractional order differential equations" (F-FDEs). For the required numerical results, we use Haar wavelet collocation (H-W-C) method which has very rarely utilized for F-FDEs. We establish the general algorithm for F-FDEs to compute numerical solution for the considered class. Also, we establish a result devoted to the qualitative theory via Banach fixed point result. A results devoted to Ulam-Hyers (U-H) stability are also included. Two pertinent examples are given along with the comparison and different norms of errors displayed in figures as well as tables.

1. Introduction

Fractal calculus is currently a popular area of study. This is because fractal calculus can more effectively describe phenomena involving complex geometry and irregular forms. The aforementioned calculus is also a relatively straightforward but incredibly effective tool for describing such phenomena in porous or hierarchical media. During Newton's lifetime, the area's foundation was set. Later, it developed into a fascinating issue in a variety of disciplines, such as mathematics and bio-engineering for porous media (see [1,2]). The groundwork for fractional calculus was also set at the same period. Fractal-fractional calculus is a later development that combines the fields of fractal and fractional calculus. The calculus discussed earlier has received a lot of focus, given that, where classic fractal-fractional calculus serves crucial roles in describing phenomena on the porous size scale, where mechanics becomes useless. In light of this significance, the aforementioned calculus was initiated. Actually, the field in question is rather recent and is capable of handling kinetics. Fractal fractional kinetics is the name given to the aforementioned operators. For additional background information and specifics on fractal derivatives (see [3]).

Sometimes, when describing a variety of real-world issues, it is important to understand how much data the system can hold. F-FDEs are being employed more frequently recently to deal with such descriptions. We quote [4] for a few important conclusions about fractal-fractional calculus. Here, we note that very good research has been done on FDEs in the area of fractional calculus devoted to existence theory, numerical analysis, and stability analysis (see [5]). He claimed that the area is a just emerging subject of study. There are several uses for the aforementioned operator in day-to-day activities. Fractal-fractional derivatives are increasingly

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used in modeling for applications involving material phase shift, according to research. In this paragraph, we note that authors [6] developed an HIV/AIDS model using a fractal-fractional derivative and a nonlocal kernel. The developers of [7] have created a kernel basis approach to compute numerical solutions to various F-FDEs in a similar manner. Researchers also created a numerical method for the fractal-fractional Klein-Gordon equation. Additionally, utilizing the aforementioned idea, authors explored the fractal-fractional Malkus waterwheel model. The authors explored the bright soliton behaviors of the excellent Boussinesq equation with fractal-fractional nonlinear kernels. Following the same steps, researchers have produced a thorough analysis of the fractal order cancer model including chemotherapeutic effects (see details [8–10]).

In order to establish qualitative results for the existence theory and approximation solution, authors have employed some methods from nonlinear analysis, fixed point theory, as well as numerical analysis. Researchers have employed perturbation techniques to explore some problems in order to compute solutions. Researchers have developed an effective technique for resolving the first-order hyper singular integral equations in replicating kernel spaces (see [11,12]). Similar to this, authors discovered a technique based on quasi-affine bi-orthogonal mappings to construct a numerical solution for weakly singular Volterra - Fredholm (V-F) issues, we refer [13]. Additionally, authors focused on using B-spline functions to numerically approximate nonlinear V-F integro-differential equations by using the fractional derivative of Atangana-Baleanu (A-B). The concept of fractal-fractional calculus was taken into consideration when developing a few new operators. Fractal-fractional operators combined with standard Caputo and A-B Caputo derivative were the names given to the novel operators that were suggested. F-FDEs, on the other hand, have not been adequately researched for both quantitative and qualitative findings. Recently published works that are pertinent to the existence theory and analytical findings are cited as [14].

For numerical or semi-analytical results, researchers have typically used standard numerical techniques such the Adam Bashforth methodology, Euler, RKM methods, and integral transform tools (see [15]). Since wavelets are effective numerical tools, they can be used in a variety of applied mathematics issues to approximate solutions. In the literature, various wavelets have been introduced. One of the wavelets that has recently seen the most use in the field of signal and image processing is the Haar wavelet. Haar developed the wavelet concept in 1908. Since his creation, researchers have used the aforementioned strategies to solve a wide range of issues. H-W-C approaches, the Legendre wavelet, the Hermit wavelet, etc. are some well-known wavelet-based techniques [16]. The wavelet methods mentioned above, however, have not been properly used to compute the numerical solution of F-FDEs.

The problems in question include two orders, one of which indicates the problem's order and the other of which indicates its fractal dimension. In order to close this gap, we used the H-W-C approach to create a numerical scheme for solving the following class of F-FDEs for the following at $t \in \mathfrak{T} = [0,1]$ as

$$\begin{aligned} {}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) =& a\mathrm{y}'(t) + b\mathrm{y}(t) \ + \mathrm{f}(t), \qquad \bar{\ell} \leq, \S \in (0,1], \\ \mathrm{y}(0) =& \wp, \end{aligned} \tag{1}$$

$$ {}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) = \mathrm{f}\,(t, \mathrm{y}(t)), \tag{2}$$

with $\mathrm{f} : \mathfrak{T} \to \mathbb{R}$, a, b are fixed real numbers. Here $\bar{\ell}$ is used for fractional order and § for fractal dimension. We expand the previously mentioned H-W-C approach to provide an algorithm that computes the numerical solution to the Eqn. (1) and Eqn. (2). Additionally, we offer certain examples with a graphical presentation for testimony. Also developed are certain conclusions supporting the existence of a solution following the definitions given in [17–19].

2. Fundamental and qualitative results

Here, we recollect some basic results from [17].

Definition 2.1. If y is defined on $(0,1)$ with order $\bar{\ell}$, such that y is fractal differentiable and continuous, then the said operator in Caputo sense over y with order $\bar{\ell}$ is described as

$$ {}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) = \begin{cases} \dfrac{1}{\Gamma(1-\bar{\ell})}\displaystyle\int_0^t \dfrac{dy}{d\pounds^{\S}}(t-\pounds)^{-\bar{\ell}}d\pounds, \quad 0<\bar{\ell},\S<1, \\ \displaystyle\int_0^t \dfrac{dy}{d\pounds}\dfrac{(t-\pounds)^{-\bar{\ell}}}{\Gamma(1-\bar{\ell})}d\pounds, \ \S=1, \ 0<\bar{\ell}<1, \\ \dfrac{d\mathrm{y}}{dt}, \ \S=1, \ \ \bar{\ell}=1, \end{cases} \tag{3}$$

where the fractal fractional derivative in Eqn. (3) is defined as

$$\frac{dy(\pounds)}{d\pounds^{\S}} = \frac{1}{\S\pounds^{\S-1}}\frac{dy(\pounds)}{d\pounds}.$$

Definition 2.2. In the same line, fractal-fractional integral is defined in Eqn. (4) as

$$
{}^{CFF}\mathrm{I}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t)=\begin{cases}\dfrac{\S}{\Gamma(\bar{\ell})}\displaystyle\int_0^t (t-\pounds)^{\bar{\ell}-1}\pounds^{\S-1}\mathrm{y}(\pounds)d\pounds, & 0<\bar{\ell},\S<1,\\ \displaystyle\int_0^t \dfrac{(t-\pounds)^{\bar{\ell}-1}}{\Gamma(\bar{\ell})}\mathrm{y}(\pounds)d\pounds, & 0<\bar{\ell}<1,\ \S=1,\\ \displaystyle\int_0^t \mathrm{y}(\pounds)d\pounds, & \bar{\ell}=1,\ \S=1.\end{cases} \tag{4}
$$

Let us write Eqn. (1) in Eqn. (5) as

$$
\begin{aligned}
{}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t)=&\phi(t,\mathrm{y},\mathrm{y}'), \qquad 0<\bar{\ell},\ \S\leq 1,\ \ t\in\mathfrak{I},\\
y(0)=&\wp,
\end{aligned} \tag{5}
$$

where $\phi:\mathfrak{I}\times\mathbb{R}^2\to\mathbb{R}$. The Banach space $\mathbb{B}=C(\mathfrak{I})$ with norm $\|\mathrm{y}\|_\infty=\max_{t\in[0,1]}\left\{|\mathrm{y}(t)|,|\mathrm{y}'(t)|\right\}$. The following assumption, we need to prove the required result.

(Q_1) For $\mathrm{y},\bar{\mathrm{y}}\in\mathbb{B}$, $\exists\ \mathbb{H}_\phi>0$, with

$$
|\phi(t,\mathrm{y},\mathrm{z})-\phi(t,\bar{\mathrm{y}},\bar{\mathrm{z}})|\leq \mathbb{H}_\phi\left[|\mathrm{y}|+|\mathrm{y}'|\right].
$$

In integral form, one may write Eqn. (5) also as

$$
\mathrm{y}(t)=\wp+\frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}\phi(\pounds,\mathrm{y}(\pounds),\mathrm{y}'(\pounds))d\pounds. \tag{6}
$$

Suppose define $\mathbb{Q}:\mathbb{B}\to\mathbb{B}$ via Eqn. (6) as

$$
\mathbb{Q}\mathrm{y}(t)=\wp+\frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}\phi(\pounds,\mathrm{y}(\pounds),\mathrm{y}'(\pounds))d\pounds. \tag{7}
$$

Theorem 2.3. *In view of hypothesis (Q_1), if the condition $\frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S-1)}<1$ holds, then problem Eqn. (1) has a unique solution.*

Proof. Let $\mathrm{y},\bar{\mathrm{y}}\in\mathbb{B}$, then using Eqn. (7), we have

$$
\begin{aligned}
\|\mathbb{Q}\mathrm{y}-\mathbb{Q}\bar{\mathrm{y}}\| &= \max_{t\in[0,1]}\left|\frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}[\phi(\pounds,\mathrm{y}(\pounds),\mathrm{y}'(\pounds))-\phi(\pounds,\bar{\mathrm{y}}(\pounds),\bar{\mathrm{y}}'(\pounds))]d\pounds\right|\\
&\leq \frac{\S}{\Gamma(\bar{\ell})}\int_0^1 \pounds^{\S-1}(1-\pounds)^{\bar{\ell}-1}\mathbb{H}_\phi[|\mathrm{y}-\bar{\mathrm{y}}|+|\mathrm{y}'-\bar{\mathrm{y}}'|]d\pounds\\
&\leq \frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S)}\left[\|\mathrm{y}-\bar{\mathrm{y}}\|+\|\mathrm{y}'-\bar{\mathrm{y}}'\|\right].
\end{aligned} \tag{8}
$$

In the same way, we have from Eqn. (6) by following the properties of aforesaid operators given in [17], the integral form given in as

$$
\mathrm{y}'(t)=\frac{\S}{\Gamma(\bar{\ell}-1)}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-2}\phi(\pounds,\mathrm{y}(\pounds),\mathrm{y}'(\pounds))d\pounds. \tag{9}
$$

Hence, we have from Eqn. (9) using $\mathrm{y},\bar{\mathrm{y}}\in\mathbb{B}$, that

$$
\|(\mathbb{Q}\mathrm{y})'-(\mathbb{Q}\bar{\mathrm{y}})'\|\leq\frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S-1)}\left[\|\mathrm{y}-\bar{\mathrm{y}}\|+\|\mathrm{y}'-\bar{\mathrm{y}}'\|\right]. \tag{10}
$$

Now from Eqn. (8) and Eqn. (10), one has

$$\|\mathbb{Q}\mathrm{y}-\mathbb{Q}\bar{\mathrm{y}}\|_\infty = \max_{t\in\mathfrak{I}}\left[\|\mathbb{Q}\mathrm{y}-\mathbb{Q}\bar{\mathrm{y}}\|, \|(\mathbb{Q}\mathrm{y})'-(\mathbb{Q}\bar{\mathrm{y}})'\|\right] \le \frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S-1)}\|\mathrm{y}-\bar{\mathrm{y}}\|_\infty.$$

Hence the result arrived for uniqueness. □

Remark 2.4. Let there exists a function ψ independent of y, such that

$$|\psi(t)| \le \varepsilon, \quad t\in[0,1].$$

The solution of

$$\begin{aligned} &{}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) = \phi(t,\mathrm{y},\mathrm{y}') + \psi(t), \qquad 0<\bar{\ell},\ \S\le 1,\ \ t\in\mathfrak{I}, \\ &\mathrm{y}(0)=\wp, \end{aligned} \tag{11}$$

can be deduced as

$$\begin{aligned} \mathrm{y}(t) &= \wp + \frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}\phi(\pounds,\mathrm{y}(\pounds),\mathrm{y}'(\pounds))d\pounds + \frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}\psi(\pounds)d\pounds \\ \mathrm{y}(t) &= \mathbb{Q}\mathrm{y}(t) + \frac{\S}{\Gamma(\bar{\ell})}\int_0^t \pounds^{\S-1}(t-\pounds)^{\bar{\ell}-1}\psi(\pounds)d\pounds. \end{aligned} \tag{12}$$

The solution Eqn. (12) of Eqn. (11) in view of Remark 2.4 yields

$$|\mathrm{y}(t)-\mathbb{Q}\mathrm{y}(t)| \le +\frac{\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S)}\varepsilon = \S_{\S,\bar{\ell}}\varepsilon. \tag{13}$$

Theorem 2.5. *The solution of Eqn. (1) is U-H stable if the condition* $1 > \frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S)}$ *holds.*

Proof. Assume y is general, and $\bar{\mathrm{y}}$ be at most one solution of Eqn. (1) respectively, then we derive the result of stability via Eqn. (13), and Remark 2.4.

$$\begin{aligned} \|\bar{\mathrm{y}}-\mathrm{y}\|_\infty &= \max_{t\in[0,1]}\left|\bar{\mathrm{y}}(t)-\mathbb{Q}\mathrm{y}(t)\right| \\ &\le \max_{t\in[0,1]}\left|\bar{\mathrm{y}}-\mathbb{Q}\bar{\mathrm{y}}(t)\right| + \max_{t\in[0,1]}\left|\mathbb{Q}\bar{\mathrm{y}}-\mathbb{Q}\mathrm{y}(t)\right| \\ &\le \S_{\S,\bar{\ell}}\varepsilon + \frac{\mathbb{H}_\phi\Gamma(\S+1)}{\Gamma(\bar{\ell}+\S)}\|\bar{\mathrm{y}}-\mathrm{y}\|_\infty. \end{aligned} \tag{14}$$

Hence, one has from Eqn. (14)

$$\|\bar{\mathrm{y}}-\mathrm{y}\|_\infty \le \frac{\S_{\S,\bar{\ell}}\varepsilon}{1-\mathbb{H}_\phi\S_{\S,\bar{\ell}}}.$$

Hence, the result of Eqn. (1) is U-H stable. □

3. Numerical scheme

H-W is a powerful tool to compute numerical solutions of many problems in applied analysis. It has been used very well to deal various problems of fractional calculus as well as of ordinary differential equations problems. Researchers have used the Haar wavelet method to compute numerical solutions of Lane-Emden equations with various boundary conditions in [18]. In addition, fractional order delay problems have been solved by the afore said scheme recently. For more frequent results can be read in [19] in which the said technique has been used to study various problems.

For Haar family, the scaling function on $[0,1)$ is given by $\flat_1(x)=1$, while the other terms of the said family can be expressed as

$$\flat_i(t) = \begin{cases} 1.0 & \text{with} \quad t\in[\bar{\ell}_1,\bar{\ell}_2), \\ -1.0 & \text{with} \quad t\in[\bar{\ell}_2,\bar{\ell}_3), \\ 0.0 & \text{otherwise}, \end{cases}$$

where $\bar{\ell}_1=\frac{\pounds}{d}$, $\bar{\ell}_2=\frac{1/2+\pounds}{d}$, and $\bar{\ell}_3=\frac{1+\pounds}{d}$, $d=2^j$, $j=0,1,\ldots,J$, $\pounds=0,1,\ldots,d-1$. The formula $i=d+\pounds+1$ is to calculate index i. We use the symbol

$$p_{i,1}(t) = \int_0^t \flat_i(x)dx, \tag{15}$$

and the value of the integral Eqn. (15) is

$$p_{i,1}(t) = \begin{cases} t - \bar{\ell}_1, \text{ if } t \in [\bar{\ell}_1, \bar{\ell}_2), \\ \bar{\ell}_3 - t, \text{ if } t \in [\bar{\ell}_2, \bar{\ell}_3), \\ 0 \text{ elsewhere.} \end{cases}$$

For H-W-C technique, $[a, b]$ can be discretized by

$$t_j = a + (b - a)\frac{j - 0.5}{2\mathrm{M}} \quad j = 1, 2, \ldots, 2\mathrm{M}, \tag{16}$$

where Eqn. (16) gives the collocation points (CPs) or nodal points.

Here, H-W-C method is extended for the solution of F-FDEs given in Eqn. (1). Consider $\frac{d\mathrm{y}(\pounds)}{d\pounds^{\S}}$ be square integrable function, then we can approximate in truncated series form as

$$\frac{d\mathrm{y}}{dt^{\S}} = \Sigma_{i=1}^{\aleph} a_i \flat_i(t), \tag{17}$$

we can also express Eqn. (17) as

$$\frac{1}{\S t^{\S-1}} \frac{d\mathrm{y}}{d\pounds} = \Sigma_{i=1}^{\aleph} a_i \flat_i(t). \tag{18}$$

Hence, from Eqn. (18), one has

$$\frac{d\mathrm{y}}{dt} = \S t^{\S-1} \Sigma_{i=1}^{\aleph} a_i \flat_i(t). \tag{19}$$

Integrate Eqn. (19) from 0 to t and after rearranging, we obtain the results given in Eqn. (20) as

$$\mathrm{y}(t) = \wp + \S \Sigma_{i=1}^{\aleph} a_i p_{i,1}(t). \tag{20}$$

Here, it should be kept in mind that if we take $\bar{\ell} = 1$, then from the definitions given Eqn. (3), and Eqn. (4), we will obtain usual fractal differential equations problem. We can then express the approximation as we do for usual problems. Therefore, we can express Eqn. (1) as

$$\frac{1}{\Gamma(1 - \bar{\ell})} \int_0^t \frac{d\mathrm{y}(\pounds)}{d\pounds^{\S}} (t - \pounds)^{-\bar{\ell}} d\pounds = a\mathrm{y}'(t) + b\mathrm{y}(t) + \mathrm{f}(t). \tag{21}$$

Applying Haar approximation to Eqn. (21), we get

$$\frac{1}{\Gamma(1 - \bar{\ell})} \int_0^t \Sigma_{i=1}^{\aleph} a_i \flat_i(\pounds)(t - \pounds)^{-\bar{\ell}} d\pounds = \left(a\Sigma_{i=1}^{\aleph} a_i \flat_i(t) + b\left(\wp + \S\Sigma_{i=1}^{\aleph} a_i p_{i,1}(t)\right) + \mathrm{f}(t)\right).$$

This implies

$$\frac{1}{\Gamma(1 - \bar{\ell})} \int_0^t \Sigma_{i=1}^{\aleph} a_i \flat_i(\pounds)(t - \pounds)^{-\bar{\ell}} d\pounds - \left(a\Sigma_{i=1}^{\aleph} a_i \flat_i(t) + b\S\Sigma_{i=1}^{\aleph} a_i p_{i,1}(t)\right) = (b\wp + \mathrm{f}(t)). \tag{22}$$

Putting nodal points t_j in Eqn. (22), one has the result given in Eqn. (23) as

$$\frac{1}{\Gamma(1 - \bar{\ell})} \int_0^{t_j} \Sigma_{i=1}^{\aleph} a_i \flat_i(\pounds)(t_j - \pounds)^{-\bar{\ell}} d\pounds - \left(a\Sigma_{i=1}^{\aleph} a_i \flat_i(t_j) + b\S\Sigma_{i=1}^{\aleph} a_i p_{i,1}(t_j)\right) = \left(b\wp + \mathrm{f}(t_j)\right). \tag{23}$$

On using the notion in Eqn. (23) as given by

$$\mho(i, j) = \frac{1}{\Gamma(1 - \bar{\ell})} \int_0^{t_j} \flat_i(\pounds)(t_j - \pounds)^{-\bar{\ell}} d\pounds - \left(a\flat_i(t_j) + b\S p_{i,1}(t_j)\right), \quad i, j = 1, 2, 3, \ldots, N. \tag{24}$$

By using Eqn. (24), we get

$$\Sigma_{i=1}^{\aleph} a_i \mho(i, j) = \left(b\wp + \mathrm{f}(t_j)\right).$$

Using the Lepik [19] method, the integral on the right side of equation Eqn. (24) is evaluated, and for $i = 1$, one has

$$\mho(j,1)=\frac{t_j^{1-\bar{\ell}}}{\Gamma(1-\bar{\ell})(1-\bar{\ell})}-a-bp_{1,1}(t_j),$$

likewise, we have for the remaining values of i

$$\mho(\ j,\ i)=\begin{cases}0,\ \text{for } t_j<\bar{\ell}_1,\\ \frac{1}{\Gamma(1-\bar{\ell})\,(1-\bar{\ell})}\left(\ t_j\ -\ \bar{\ell}_1\right)^{1-\bar{\ell}}-\left(a\flat_i(t_j)+b\S p_{i,1}(t_j)\right),\ \text{for } t_j\in[\bar{\ell}_1,\bar{\ell}_2],\\ \frac{1}{\Gamma(1-\bar{\ell})(1-\bar{\ell})}\left((t_j-\bar{\ell}_1)^{1-\bar{\ell}}-2(t_j-\bar{\ell}_2)^{1-\bar{\ell}}\right)-\left(a\flat_i(t_j)+b\S p_{i,1}(t_j)\right),\ \text{for } t_j\in[\bar{\ell}_2,\bar{\ell}_3],\\ \frac{1}{\Gamma(1-\bar{\ell})(1-\bar{\ell})}\left((t_j-\bar{\ell}_1)^{1-\bar{\ell}}-2(t_j-\bar{\ell}_2)^{1-\bar{\ell}}-(t_j-\bar{\ell}_3)^{1-\bar{\ell}}\right)\\ -\left(a\flat_i(t_j)+b\S p_{i,1}(t_j)\right),\ \text{for } t_j>\bar{\ell}_3.\end{cases}$$

In matrix notation, one can write

$$\mathbf{G}\boldsymbol{A}=\mathbf{C},$$

where

$$\mathbf{G}=[\mho_{ij}]_{N\times N},\quad \boldsymbol{A}=[a_i]_{N\times 1},\quad \mathbf{C}=[C_i]_{N\times 1},$$

and

$$C_j=\left(b\wp+\mathrm{f}(t_j)\right).$$

Thus a_i's is obtained as

$$\boldsymbol{A}=\mathbf{G}^{-1}\mathbf{C}.$$

By putting a_i in Eqn. (20) one can obtain the solution at nodal points.

For the nonlinear case a similar method can be developed for the solution of F-FDEs Eqn. (2). Using the adopted operator definition in Eqn. (2), one has

$$\frac{1}{\Gamma(1-\bar{\ell})}\int_0^t\frac{d\mathrm{y}(\pounds)}{d\pounds^{\S}}(t-\pounds)^{-\bar{\ell}}d\pounds=\mathrm{f}(t,\mathrm{y}).$$

Therefore

$$\frac{1}{\Gamma(1-\bar{\ell})}\int_0^t\frac{d\mathrm{y}}{d\pounds^{\S}}(t-\pounds)^{-\bar{\ell}}d\pounds=\left(f(t,\mathrm{y})\right),$$

applying H-W-C, one has

$$\frac{1}{\Gamma(1-\bar{\ell})}\int_0^t\sum_{i=1}^{\aleph}a_i\flat_i(\pounds)(t-\pounds)^{-\bar{\ell}}d\pounds=\left(\mathrm{f}(t,\wp+\S\sum_{i=1}^{\aleph}\wp_i p_{i,1}(t))\right),$$

yields the given system by utilizing CPs t_j.

$$\mathfrak{F}_j=\frac{1}{\Gamma(1-\bar{\ell})}\int_0^{t_j}\sum_{i=1}^{\aleph}a_i\flat_i(\pounds)(t_j-\pounds)^{-\bar{\ell}}d\pounds=\left(\mathrm{f}(t_j,\wp+\S\sum_{i=1}^{\aleph}\wp_i p_{i,1}(t_j))\right).$$

The *Broyden* method is used to solve this system. In addition, a_i unknown coefficients are deduced from solution.

Remark 3.1. Let y_{ap} be approximate and y_{ex} is the exact solution at $\aleph$ CPs, then maximum absolute error $\mathbb{E}_{cp}(\aleph)$ and root mean square error $\mathbb{M}_{cp}(\aleph)$ are computed in Eqn. (25), and Eqn. (26) as

$$\mathbb{E}_{cp}(\aleph)=\max\left|\mathrm{y}_{ex}(t)-\mathrm{y}_{ap}(t)\right|, \tag{25}$$

and

$$\mathbb{M}_{cp}(\aleph)=\sqrt{\frac{\S}{\aleph}\Sigma_{i=1}^{\aleph}\left(\mathrm{y}_{ex}(t_i)-\mathrm{y}_{ap}(t_i)\right)^2}. \tag{26}$$

Table 1
$\mathbb{E}_{cp}(\aleph)$ and $\mathbb{M}_{cp}(\aleph)$ errors for Experiment 4.1.

J	$\aleph = 2^{1+J}$	$\mathbb{E}_{cp}(\aleph)$	$\mathbb{M}_{cp}(\aleph)$
0	2	0.023959	0.022441
1	4	0.012017	0.008414
2	8	0.005354	0.003559
3	16	0.002259	0.001475
4	32	0.001004	6.264275e-004
5	64	5.262736e-004	2.993843e-004
6	128	3.496090e-004	1.786014e-004
7	256	2.855276e-004	1.354213e-004
8	512	2.625619e-004	1.201709e-004

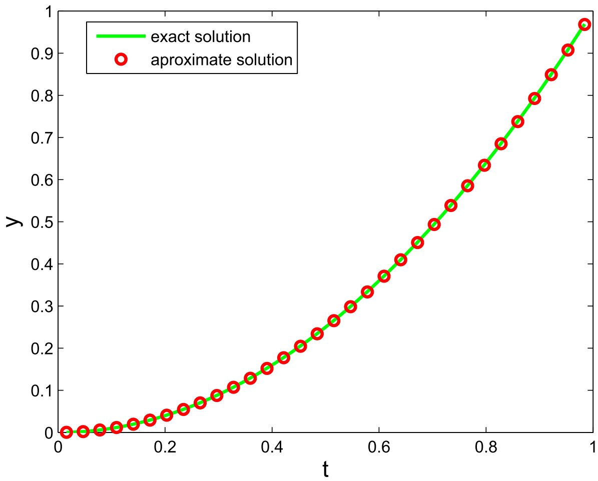

Fig. 1. Exact and numerical solution comparison at $\aleph = 32$, for Experiment 4.1 at $\bar{\ell} = 0.5,\ \S = 0.4$.

4. Numerical experiments

Current part is related to some numerical problems and their graphical presentation.

Example 4.1. Consider the following F-FDE

$$\begin{aligned} &{}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) - \mathrm{y}'(t) + \mathrm{y}(t) - t^2 = 0, \\ &\mathrm{y}(0) = 0. \end{aligned} \tag{27}$$

Let $y(t) = t^2$ be the exact solution of equation Eqn. (27) for $\bar{\ell} = 0.5$ and $= \S = 0.4$. We compute different norms in the Table 1 by using $\bar{\ell} = 0.5,\ \S = 0.4$. Here in Fig. 1, we present the comparison between exact and approximate solution.

Example 4.2. Consider the following F-FDE

$$\begin{aligned} &{}^{CFF}\mathbb{D}_{0,t}^{\bar{\ell},\S}\mathrm{y}(t) - \mathrm{y}'(t) - \frac{35\sqrt{\pi}t^2}{16}\sqrt{\mathrm{y}(t)} + 4t^3 = 0, \\ &\mathrm{y}(0) - 1 = 0. \end{aligned} \tag{28}$$

The exact solution at $\bar{\ell} = \S = 0.5$ of Eqn. (28) is given by

$$\mathrm{y}(t) = t^4.$$

The absolute error at different collocations points is displayed in Table 2. In addition, the comparison between exact and approximate solution in Fig. 2.

5. Conclusion

This work has been devoted to establish some existence and numerical results for classes of F-FDEs. We have used Banach theorem to prove the qualitative theory of existence of solution to the problem under our investigation. Also, a result devoted to stability

Table 2
$\mathbb{E}_{cp}(\aleph)$ and $\mathbb{M}_{cp}(\aleph)$ errors for Experiment 4.2.

J	$\aleph = 2^{1+J}$	$\mathbb{E}_{cp}(\aleph)$	$\mathbb{M}_{cp}(\aleph)$
0	2	0.085267	0.071418
1	4	0.003622	0.002625
2	8	7.235680e-004	3.818012e-004
3	16	2.976669e-004	1.781525e-004
4	32	2.869805e-005	1.857655e-005
5	64	6.004042e-006	4.902165e-006
6	128	1.289139e-006	1.172551e-006

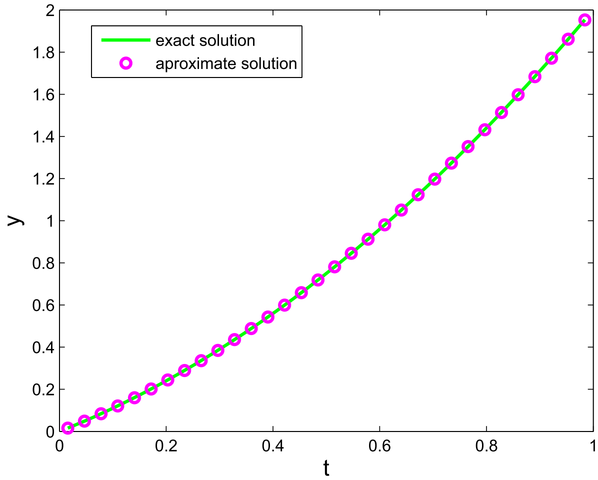

Fig. 2. Exact and numerical solution comparison at $\aleph = 32$, for Experiment 4.2 at $\bar{\ell} = 0.5$, $\S = 0.5$.

has been included. Further, on using H-W-C method, we have designed a scheme to deduce approximate results to the proposed problem. The results revealed that H-W-C technique can also be extended to F-FDEs to find the numerical solution to the mentioned problem. Some examples have treated by the said method and results been created graphically. In future, we will extend the aforesaid technique to study complex dynamical systems under fractal fractional derivatives of different type.

CRediT authorship contribution statement

Kamal Shah: Wrote the paper; analyzed and interpreted the data; contributed reagents, materials, analysis tools or data.
Rohul Amin: Wrote the paper; conceived and designed the experiments; performed the experiments.
Thabet Abdeljawad: Wrote the paper; contributed reagents, materials, analysis tools or data.

Declaration of competing interest

Authors declare that they have no conflict of interest.

Data availability statement

No data was used for the research described in the article.

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